International Journal of Theoretical Physics (2019) 58:2914–2935
https://doi.org/10.1007/s10773-019-04174-9

Quantum Sensing of Curvature

Daniele Bonalda[1] · Luigi Seveso[1] · Matteo G. A. Paris[1]

Received: 16 March 2019 / Accepted: 27 May 2019 / Published online: 3 July 2019

Abstract

We address the problem of sensing the curvature of a manifold by performing measurements on a particle constrained to the manifold itself. In particular, we consider situations where the dynamics of the particle is quantum mechanical and the manifold is a surface embedded in the three-dimensional Euclidean space. We exploit ideas and tools from quantum estimation theory to quantify the amount of information encoded into a state of the particle, and to seek for optimal probing schemes, able to actually extract this information. Explicit results are found for a free probing particle and in the presence of a magnetic field. We also address precision achievable by position measurement, and show that it provides a nearly optimal detection scheme, at least to estimate the radius of a sphere or a cylinder.

Keywords Quantum sensing · Curvature

1 Introduction

In order to describe the kinematics and the dynamics of a physical system, from now on *a particle*, one should at first specify the manifold where the dynamics of the particle takes place, i.e. the manifold where the particle propagates. Depending on the nature of the system, this manifold may be flat or characterized by a curvature. In modelling a system, geometrical constraints are often postulated by looking at basic principles, or on the basis of general considerations. However, a question arises on whether it may be possible to obtain information about the manifold by a purely operational approach, i.e. by performing measurements on the system under investigation. Besides the fundamental interest, sensing the curvature has potential applications, e.g. due to the interest in two-dimensional curved systems, to describe physical effects such as Aharonov-Bohm oscillations [1], formation of Landau levels [2–6] and quantum Hall effect [7].

In this paper, we address the problem of probing a manifold by performing measurements on a particle constrained to move on the manifold itself. In particular, we focus on estimating the *curvature* of a manifold, and consider regimes where the dynamics of the particle is

✉ Matteo G. A. Paris
matteo.paris@fisica.unimi.it

1 Quantum Technology Lab, Dipartimento di Fisica Aldo Pontremoli, Università degli Studi di Milano, I-20133, Milano, Italy

quantum mechanical. To this aim, we employ ideas and tools from quantum estimation theory in order to quantify the information that is actually available according to the laws of quantum mechanics. In addition, we look for the optimal measurement, able to extract the maximum information about the curvature, as well as the optimal preparation of the particle, i.e. the preparation which is most sensitive to the curvature of the underlying manifold.

As a first step in this endeavour, we review the possible approaches to derive the Schrödinger equation for a particle constrained to a manifold, possibly subjected to an external field [8–16]. As we will see, there are at least two inequivalent approaches, one of which is more adherent to the physical situation we have in mind. We review the two approaches in order to establish our notation and, in particular, we discuss in some detail the differences and the similarities between the two methods, to illustrate the rationale behind our choice.

In order to quantify the available information about the curvature, that may be extracted by means of a measurement on the particle, we employ ideas and tools from quantum parameter estimation (QPE) theory [17–22]. QPE generalizes to the quantum case the problem of point estimation arising in classical statistics. There, the problem is to infer the value of a parameter by sampling from the population of a random variable, whose distribution depends on the parameter itself. An estimate of the parameter is built from the data sample using a point estimator, i.e. a parameter-space valued function of data, and the task is to optimise the estimation strategy with respect to a suitable figure of merit. Moving to the quantum mechanical case leads to the introduction of a quantum statistical model, i.e. a family of density operators, describing the possible states of a quantum system. An estimate of the parameter is here obtained by performing a measurement and then processing its outcomes via a suitable estimator. The main task is to make the optimal choice of both the measurement scheme and the estimator, in order to achieve the most precise determination of the parameter of interest. The central figures of merit in optimizing estimation of a parameter are the Fisher information (FI) and its quantum generalisation, the quantum Fisher information (QFI).

The paper is structured as follows. In Section 2 we review the possible approaches to quantization on a curved manifold, i.e. the *Lagrangian* approach based on the use of generalised coordinates, and the *Hamiltonian* one, where one considers the particle in $\mathbb{R}^3$, but forced to a two-dimensional manifold by a steep potential, which is constant on the surface and increases sharply in the normal direction. We illustrate the differences and the similarities between the two approaches by the specific examples of a particle constrained to a sphere, a cylinder, and a torus. In Section 3 we briefly review classical and quantum estimation theory, introducing the quantum Fisher information as a measure of information about a parameter contained in a family of quantum states. In Section 4 we analyze in some details the precision that may be achieved in estimating the curvature of a manifold by a free quantum probe, i.e. a particle that is affected only by the constraining potential forcing it to stay on the surface. In particular, we evaluate the quantum Fisher information for the radius of a sphere and a cylinder, and analyze its scaling properties with respect to time evolution and the radius itself. In Section 5 we consider a charged quantum probe and analyze the performances of estimation protocols in the presence of an external magnetic field. As we will see, the external field is a resource which allows one to estimate the radius by performing measurements on a stationary state, while without a field we need to measure the probe after a given time evolution. In Section 6 we study the Fisher Information for a position measurement, and show that it provides a nearly optimal detection scheme, at least for the sphere and the cylinder, i.e. the Fisher Information shows the same scaling of the QFI, which represents its upper bound. Section 7 closes the paper with some concluding remarks.

2 Schrödinger Equation for a Particle Constrained to a Manifold

The common procedure to quantize a Hamiltonian system consists in introducing canonical coordinates and substituting them with self-adjoint operators, satisfying the usual commutation relations. When the underlying space is Euclidean, no significant problem is encountered. The position and momentum operators are defined (in the position space representation) as follows:

$$Q\psi(q) = q\psi(q) \qquad P\psi(q) = -i\hbar\partial_q\psi(q). \tag{1}$$

The only subtlety in this case is related to operator ordering ambiguities. Indeed, it is well-known that the quantum Hamiltonian is not uniquely defined by its classical limit. That is, there exist in general multiple Hermitian operators, i.e. multiple functions of Q and P, which give rise to the same phase space function when Q and P are replaced by c numbers. When instead the underlying space is not Euclidean, but has a metric dependent on the coordinates, there is the additional problem that a naive quantization along the lines of (1) may lead to operators that are not self-adjoint, or do not satisfy the usual commutation relations. A known example is that of spherical coordinates, where the operator $i\hbar\partial_\theta$ is not self-adjoint.

This kind of problem may be solved in two different ways. The first approach is to use generalised coordinates, i.e. to assume that the only space existing is the manifold itself, equipped with its metric, and to quantize the system on the manifold itself [8]. The second approach [10] requires instead to consider the particle as living in $\mathbb{R}^3$, but forced to move only on the manifold by a steep potential, which is constant on the surface, whereas it increases sharply for every small displacement in the normal direction. Following this second approach, one can write the usual Schrödinger equation in the full Euclidean space, with the addition of the constraining potential term, and then separate it into two equations, corresponding to the dynamics in the normal direction and along the surface. It may be naively expected that the two methods always give rise to the same dynamics; however, this is not necessarily the case. In fact, there are examples in which the two procedures lead to different Hamiltonians and, in turn, to different dynamics.

2.1 Quantization: Lagrangian

Let us consider a particle of mass M, moving on a manifold with metric tensor g. The Christoffel symbols are defined as

$$\Gamma^{\rho}_{\mu\sigma} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\sigma\lambda} + \partial_\sigma g_{\mu\lambda} - \partial_\lambda g_{\mu\sigma}), \tag{2}$$

and allow one to build the following Riemann tensor:

$$R^{\sigma}_{\mu\rho\nu} = \partial_\rho\Gamma^{\sigma}_{\mu\nu} - \partial_\nu\Gamma^{\sigma}_{\mu\rho} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\rho\lambda} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda}. \tag{3}$$

The Ricci scalar is obtained by contracting over the indexes of Riemann tensor as follows:

$$R = g^{\mu\nu}R^{\lambda}_{\mu\lambda\nu}. \tag{4}$$

The classical Lagrangian of the particle is given by:

$$L = \frac{1}{2}Mg_{ij}\dot{q}^i\dot{q}^j + QA_i g^{ij}\dot{q}_j - QV, \tag{5}$$

where Q is a scalar constant, while A_i and V are functions of the q's, which may be interpreted as the vector and the scalar potentials respectively, describing the interaction between a particle with charge Q and an electromagnetic field. With this interpretation (5) is the classical Lagrangian of a single charged particle interacting with an electric and magnetic field. Upon quantizing the generalised coordinates q_j the following Schrödinger equation may be derived for the particle on the manifold [8]

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2M\sqrt{\det g}}\frac{\partial}{\partial q^i}(\sqrt{\det g}g^{ij}\frac{\partial\Psi}{\partial q^j}) + \xi\frac{\hbar^2}{M}R\Psi \\ &+\frac{iQ\hbar}{2M\sqrt{\det g}}\frac{\partial}{\partial q^i}(\sqrt{\det g}g^{ij}A_j)\Psi \\ &+\frac{iQ\hbar}{M}g^{ij}A_i\frac{\partial\Psi}{\partial q^j} + \frac{Q^2}{2M}g^{ij}A_iA_j\Psi + QV\Psi, \end{aligned} \tag{6}$$

where ξ is a free parameter, i.e. we may assign to ξ any real value and obtain the same classical theory in the limit $\hbar \to 0$. In the case of a free particle, the above Schrödinger equation reduces to

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M\sqrt{\det g}}\frac{\partial}{\partial q^i}(\sqrt{\det g}g^{ij}\frac{\partial\Psi}{\partial q^j}) + \xi\frac{\hbar^2}{M}R. \tag{7}$$

The above quantization procedure is completely general and works for every manifold without the need to embed it in an Euclidean space. However, this very feature, and the presence of the free parameter ξ, pose conceptual and practical problems. On the one hand, it is often the case that one has to study a particle living on a two-dimensional surface, knowing however that it is in reality embedded in the usual three-dimensional space. On the other hand, with this approach one would not be able to consider the effects of any normal field (e.g. a magnetic field normal to the surface), since an additional dimension is unavoidably required, which is not contemplated by the theory. This sort of problems can be solved by an alternative quantization [10, 11], which we are going to review in the following section.

2.2 Quantization: Constraining Hamiltonian

A more direct quantization procedure may be obtained by considering the particle as living in Euclidean space, but forced to stay on a thin layer of space around a surface by a steep potential [10]. Following this approach, there is no need to quantize directly on the curved space, because the particle actually lives in the full Euclidean space, where the Schrödinger equation is unambiguous. Due to the nature of the confining potential, the Schrödinger equation and the wave function may be factorized in a normal and a surface parts, provided that the confining potential depends on a squeezing parameter, and that the larger this parameter is, the thinner the allowed region normal to the surface. The Schrödinger equation for the surface part may be written as [10]

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2M}\left\{\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial q^i}\left(\sqrt{\det g}g^{ij}\frac{\partial\Psi}{\partial q^j}\right)\right. \\ &\left.+\left(\frac{1}{4}\mathrm{Tr}[\alpha]^2 - \det[\alpha]\right)\Psi\right\}, \end{aligned} \tag{8}$$

where α is a matrix, whose elements α_{ij} are the coefficients of the expansion of the derivatives of the normal versor on the tangent plane. In fact, it can be shown, and is also quite intuitive, that the derivative of the normal versor belongs to the tangent plane:

$$\frac{\partial n}{\partial q^j} = \sum_{j=1}^{2} \alpha_{ij} \frac{\partial \mathbf{r}}{\partial q^j}. \tag{9}$$

It is interesting to note that these coefficients can be expressed as a function of the metric g and of the second fundamental form h as follows

$$\begin{aligned} \alpha_{11} &= \frac{1}{\det[g]}(g_{12}h_{21} - g_{22}h_{11}) \quad \alpha_{12} = \frac{1}{\det[g]}(g_{21}h_{11} - g_{11}h_{21}) \\ \alpha_{21} &= \frac{1}{\det[g]}(g_{12}h_{22} - g_{22}h_{12}) \quad \alpha_{22} = \frac{1}{\det[g]}(g_{12}h_{21} - g_{11}h_{22}). \end{aligned} \tag{10}$$

If we now introduce the notation

$$V_s(q_1, q_2) := -\frac{\hbar^2}{2M}\left(\frac{1}{4}\mathrm{Tr}[\alpha]^2 - \det[\alpha]\right), \tag{11}$$

the quantity $V_s(q_1,q_2)$ may be interpreted as the surface potential due to the constraining. In terms of the mean curvature C and the Gaussian curvature K we may write

$$V_s(q_1, q_2) = -\frac{\hbar^2}{2M}(C^2 - K), \tag{12}$$

where

$$C = \frac{1}{2\det[g]}(g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}), \qquad K = \frac{\det[h]}{\det[g]}. \tag{13}$$

Besides the constraining potential, one may consider the particle subject to a scalar potential V and a vector potential A_i. Remarkably, it can be shown [14] that no coupling appears between the field and the surface curvature and that, with a proper choice of the gauge, the surface and the transverse dynamics are still factorized. For the surface part, we have

$$\begin{aligned} i\hbar\partial_t \Psi = &-\frac{\hbar^2}{2M\sqrt{\det g}} \frac{\partial}{\partial q^i}(\sqrt{\det g}\, g^{ij} \frac{\partial \Psi}{\partial q^j}) + V_s \Psi \\ &+ \frac{iQ\hbar}{2M\sqrt{\det g}} \frac{\partial}{\partial q^i}(\sqrt{\det g}\, g^{ij} A_j)\Psi \\ &+ \frac{iQ\hbar}{M} g^{ij} A_i \frac{\partial \Psi}{\partial q^j} + \frac{Q^2}{2M} g^{ij} A_i A_j \Psi + QV\Psi, \end{aligned} \tag{14}$$

with Q being the charge of the particle and V_s the surface potential. If we compare this equation with (6), we realize that they are quite the same except for the term containing the Ricci scalar, that in (14) is replaced by the term containing the surface potential.

2.2.1 Examples #1: the Sphere

Let us consider the surface of a sphere of radius λ, parametrized in spherical coordinates, i.e. the latitude $\theta \in [0,\pi]$ and the longitude $\phi \in [0,2\pi]$. The metric matrix elements are given by

$$g_{\theta\theta} = \lambda^2, \quad g_{\phi\phi} = \lambda^2 \sin^2\theta, \quad g_{\phi\theta} = g_{\theta\phi} = 0. \tag{15}$$

It follows that the Ricci scalar is given by $R = 2/\lambda^2$ and the surface is parametrized as $\mathbf{r}(\theta,\phi) = (\lambda \sin\theta \cos\phi, \lambda \sin\theta \sin\phi, \lambda \cos\theta)$. The normal vector is $n(\theta,\phi) = \mathbf{r}/\lambda$, and it is straightforward to see that

$$\frac{\partial n}{\partial(\theta,\phi)} = \frac{1}{\lambda}\frac{\partial \mathbf{r}}{\partial(\theta,\phi)}. \tag{16}$$

Upon comparing the above equation with (9), we see that:

$$\alpha_{\theta\theta} = \alpha_{\phi\phi} = 1/\lambda, \quad \alpha_{\theta\phi} = \alpha_{\phi\theta} = 0 \tag{17}$$

and it follows that $V_s = 0$. In this case, by choosing the parameter ξ in (6) equal to zero, the two quantization procedures lead to the same Schrödinger equation.

For a charged particle constrained to a sphere, and subject to the effect of a constant magnetic field $\mathbf{B}$ directed along the positive z-axis ($\theta = 0$), we have

$$\begin{aligned} i\hbar\partial_t\Psi &= -\frac{\hbar^2}{2M\lambda^2}\left[\partial_\theta^2\Psi + \cot\theta\,\partial_\theta\Psi + \frac{1}{\sin^2\theta}\partial_\phi^2\Psi\right] \\ &\quad + \frac{iQB\hbar}{2M}\partial_\phi\Psi + \frac{B^2Q^2\lambda^2\sin^2\theta}{8M}\Psi. \end{aligned} \tag{18}$$

2.2.2 Examples #2: the Cylinder

We now consider a particle on a cylinder. The points on the surface have coordinates ($x = \rho \cos\theta, y = \rho \sin\theta, z$), where $z \in \mathbb{R}$, $\theta \in [0,2\pi]$, and $\rho \in [0,+\infty]$. The surface of a cylinder of radius λ may be parametrised by the vector $\mathbf{r}(z,\theta) = (\lambda \cos\theta, \lambda \sin\theta, z)$, and the metric is given by

$$g_{zz} = 1, \quad g_{\theta\theta} = \lambda^2, \quad g_{z\theta} = g_{\theta z} = 0. \tag{19}$$

Since the metric does not depend on the coordinates (θ,z), it immediately follows that the Ricci scalar vanishes. In order to evaluate the α matrix let us consider the normal versor $n = (\cos\theta, \sin\theta, 0)$, together with its derivatives:

$$\frac{\partial n}{\partial z} = 0, \qquad \frac{\partial n}{\partial\theta} = \frac{1}{\lambda}\frac{\partial\mathbf{r}}{\partial\theta}. \tag{20}$$

It follows that:

$$\alpha_{\theta\theta} = 1/\lambda, \quad \alpha_{zz} = \alpha_{z\theta} = \alpha_{\theta z} = 0. \tag{21}$$

and the surface potential

$$V_s = -\frac{\hbar^2}{8M\lambda^2},$$

does not depend on the coordinates. Notice that in this case there is no possible choice of the parameter ξ making the Schrödinger equation (6) equal to that obtained from the constraining Hamiltonian approach, i.e. Equation (14). This happens because the Ricci scalar is zero, while the surface potential is not. On the other hand, the two Hamiltonians differ only by the presence of the surface potential, which is a constant, and therefore they give rise to the same dynamics.

If a magnetic field $\mathbf{B}$ is present, it will have a radial component, which we denote by B_1, and a normal component directed along the z-axis, which we denote by B_0. The radial component may be always chosen in order to have $\theta = 0$ without loss of generality. Such a magnetic field, in the gauge where the transverse component of the vector potential is zero ($A_\rho = 0$), may be obtained from the following vector potential: $(A_\theta, A_z, A_\rho) =$

$(\frac{1}{2}\lambda^2 B_0, \lambda B_1 \sin\theta, 0)$. It follows, after some intermediate steps, that the the Schrödinger equation is given by

$$\begin{aligned} i\hbar\partial_t\Psi &= -\frac{\hbar^2}{2M}\left(\frac{\partial_\theta^2\Psi}{\lambda^2} + \partial_z^2\Psi\right) + \frac{i\hbar Q B_0}{2M}\partial_\theta\Psi \\ &\quad + \frac{i\hbar\lambda Q B_1}{M}\sin\theta\partial_z\Psi \\ &\quad + \frac{\lambda^2 Q^2}{2M}\left(\frac{B_0^2}{4} + B_1^2\sin^2\theta\right)\Psi - \frac{\hbar^2}{8M\lambda^2}\Psi. \end{aligned} \tag{22}$$

2.2.3 Examples #3: the Torus

In the following, we are not investigating in details quantum sensing on a torus. It is however of interest to briefly mention it, in order to present a specific example where the two quantization methods lead to unavoidably different Hamiltonians and dynamics. A torus is the surface generated by rotating a circle in three-dimensional space, about an axis coplanar with the circle. As such, it is specified by two parameters: the radius r of the circle and the distance R from the center of the circle to the center of the torus. A point on the surface of a torus is identified by two angles: $\theta \in [0,2\pi]$, which tells the angle on the circle, and $\phi \in [0,2\pi]$, which measures the angle around the center of the torus. The corresponding parametrization reads as follows:

$$x(\theta,\phi) = (R + r\cos\theta)\cos\phi,\ y(\theta,\phi) = (R + r\cos\theta)\sin\phi,\ z(\theta,\phi) = r\sin\theta, \tag{23}$$

and the metric is given by $g_{\theta\theta} = r^2$, $g_{\phi\phi} = (R + r\cos\theta)^2$, $g_{\theta\phi} = g_{\phi\theta} = 0$. The normal versor is thus $n = (\cos\theta\cos\phi, \cos\theta\sin\phi, \cos\theta)^T$, and the α matrix is given by

$$\alpha_{\theta\theta} = \frac{1}{r}, \quad \alpha_{\phi\phi} = \frac{\cos\theta}{R + r\cos\theta}, \quad \alpha_{\theta\phi} = \alpha_{\phi\theta} = 0. \tag{24}$$

The corresponding surface potential reads as follows

$$V_s(\theta,\phi) = -\frac{\hbar^2}{8M}\frac{R^2}{r^2(R + r\cos\theta)^2}. \tag{25}$$

The Ricci scalar of the torus is given by

$$R = \frac{2\cos\theta}{r(R + r\cos\theta)}, \tag{26}$$

and this means that the two procedures of quantization unavoidably lead to different Hamiltonians. In fact, there is no choice of the parameter ξ that can make the two Hamiltonians equal. Furthermore, the difference between the two Hamiltonians is not a constant, but rather it depends on the coordinates. For a free particle, the Hamiltonian obtained with a constraining potential leads to following Schrödinger equation:

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2M}\left[\frac{\partial_\theta^2}{r^2} - \frac{\sin\theta\partial_\theta}{r} + \frac{\partial_\phi^2}{(R + r\cos\theta)^2} + \left(\frac{R}{2r(R + r\cos\theta)}\right)^2\right]\Psi. \tag{27}$$

3 Quantum Parameter Estimation

Let us consider a family of quantum states ρ_λ, that are labeled by the values of a parameter of interest. In our case, the family consists of the possible states of a particle on the manifold, and the parameter corresponds to its curvature. We refer to the particle as a *quantum probe* for the parameter λ. In order to estimate λ, we perform the same measurement on repeated preparations of the quantum probe, and then suitably process the sample of outcomes. Let us denote by X the observable measured on the probe, and by $p(x|\lambda)$ the conditional distribution of the outcomes, assuming that the true value of the parameter is λ. We also assume to perform N independent repeated measurements on the probe. Once X has been chosen and the sample $\mathbf{x} = \{x_1,...,x_N\}$ has been collected, we process the data by an *estimator* $\lambda \equiv \lambda(\mathbf{x})$, i.e. a function from the space of data to the set of possible parameter values. The *estimated value* of the parameter is the average value of the estimator over data, i.e.

$$\overline{\lambda} = \int d\mathbf{x} p(\mathbf{x}|\lambda)\lambda(\mathbf{x}), \tag{28}$$

where $p(\mathbf{x}|\lambda) = \prod_{k=1}^{N} p(x_k|\lambda)$, owing to the independence of the M measurements. The *precision* of our estimation procedure is quantified by the variance of the estimator, i.e.:

$$V_\lambda \equiv \mathrm{Var}\lambda = \int d\mathbf{x} p(\mathbf{x}|\lambda) \left[\lambda(\mathbf{x}) - \overline{\lambda}\right]^2 . \tag{29}$$

The smaller V_λ, the more precise the estimation procedure.

For any (asymptotically) unbiased estimator, i.e. any estimator satisfying the condition $\overline{\lambda} \to \lambda$ for $N \gg 1$, there is a bound to the best achievable precision, given by the celebrated Cramér-Rao (CR) inequality:

$$V_\lambda \geq \frac{1}{N F_\lambda} \tag{30}$$

where F_λ is the so-called Fisher information (FI)

$$F_\lambda = \int dx p(x|\lambda) \left[\partial_\lambda \log p(x|\lambda)\right]^2 . \tag{31}$$

The most precise measurement to infer the value of λ is thus the measurement maximising the FI, where the maximisation is performed over all possible observables of the probe. To perform such maximisation analytically, one defines the symmetric logarithmic derivative L_λ(SLD) of the quantum statistical model, defined as the operator that satisfies the relation

$$\frac{(L_\lambda \rho_\lambda + \rho_\lambda L_\lambda)}{2} = \partial_\lambda \rho_\lambda . \tag{32}$$

Then, the quantum CR theorem states that the optimal quantum measurements are those corresponding to the spectral measure of the SLD, and consequently $F_\lambda \leq H_\lambda = \mathrm{Tr}[\rho_\lambda L_\lambda^2]$, where H_λ is usually referred to as the quantum Fisher information (QFI). The quantum CR inequality then states that:

$$V_\lambda \geq \frac{1}{N H_\lambda}, \tag{33}$$

which represents the ultimate bound to precision, i.e. a bound taking into account both the intrinsic (quantum), and extrinsic (statistical), source of fluctuations affecting the estimator.

Upon solving the eigensystem for the family of quantum states ρ_λ, we may write $\rho_\lambda = \sum_n \rho_n \left|\phi_n\right\rangle\left\langle\phi_n\right|$, where both the eigenvalues and the eigenvectors do, in general, depend on the parameter of interest. One can then prove that the QFI can be written in the following convenient form:

$$H_\lambda = \sum_n \frac{(\partial_\lambda \rho_n)^2}{\rho_n} + 2\sum_{n\neq m} \frac{(\rho_n - \rho_m)^2}{\rho_n + \rho_m} \left|\langle \phi_m|\partial_\lambda \phi_n\rangle\right|^2 \tag{34}$$

where the sum runs over the support of ρ_λ.The first term in (34) is the FI of the distribution of the eigenvalues ρ_n, whereas the second term is a positive definite, genuinely quantum, contribution, explicitly quantifying the potential quantum enhancement of precision. When the condition $F_\lambda = H_\lambda$ is met, the corresponding measurement is said to be *optimal*. If equality is satisfied in (33), the corresponding estimator is said to be *efficient*.

As it will be clear in the following, the family of states we are going to consider is made of pure states, $\rho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$, for which we have $\rho_\lambda^2 = \rho_\lambda$. From this, it follows that:

$$\partial_\lambda \rho_\lambda = \partial_\lambda \rho_\lambda^2 = (\partial_\lambda \rho_\lambda)\rho_\lambda + \rho_\lambda(\partial_\lambda \rho_\lambda). \tag{35}$$

Comparing this equation with the definition of the Symmetric Logarithmic Derivative (32), it immediately follows that for a pure quantum statistical model the SLD is given by

$$L_\lambda = 2\partial_\lambda \rho = 2(|\partial_\lambda \psi\rangle\langle\psi| + |\psi\rangle\langle\partial_\lambda \psi|). \tag{36}$$

Inserting this expression into the definition of the QFI, and using the fact that $\langle\partial_\lambda \psi_\lambda|\psi_\lambda\rangle$ is purely imaginary, we arrive at

$$H_\lambda = 4\left(\langle\partial_\lambda \psi_\lambda|\partial_\lambda \psi_\lambda\rangle - \left|\langle\partial_\lambda \psi_\lambda|\psi_\lambda\rangle\right|^2\right). \tag{37}$$

In the following, we are going to employ the above results to assess the performances of a quantum probe with the purpose of estimating the curvature of a manifold, e.g. the radius of a spherical or cylindrical surface. To this aim, let us denote by E_j and $|\phi_j\rangle$ the eigenvalues and the eigenvectors of the (time independent) particle Hamiltonian on the manifold, and write the generic pure state as $|\psi_\lambda\rangle = \sum_j c_j|\phi_j\rangle$. The state at time t may written as

$$|\psi_\lambda\rangle_t = \sum_j c_j e^{-\frac{it}{\hbar}E_j}|\phi_j\rangle, \tag{38}$$

where information about λ is encoded in the eigenvalues and the eigenvectors. As such, we have:

$$|\partial_\lambda \psi_\lambda\rangle_t = \sum_j c_j \partial_\lambda \left(e^{-\frac{it}{\hbar}E_j}\right)|\phi_j\rangle + \sum_j c_j e^{-\frac{it}{\hbar}E_j}|\partial_\lambda \phi_j\rangle. \tag{39}$$

Upon exploiting (38) and (39), we may then calculate $\langle\partial_\lambda \psi_\lambda|\partial_\lambda \psi_\lambda\rangle_t$ and $|\langle\psi_\lambda|\partial_\lambda \psi_\lambda\rangle_t|^2$ and, in turn, the QFI. In the case of a free particle on a sphere, we will explicitly calculate the QFI for a localized wave-packet, without needing to know its expansion on the Hamiltonian eigenstates. In other cases, e.g. a particle in the presence of a magnetic field, we will limit ourselves to compute the QFI for some relevant family of states, such as those obtained by a superposition of the ground state and a generic eigenstate of the system.

4 Sensing the Curvature by a Free Particle

In this section, we use the tools of quantum parameter estimation in order to assess the maximum precision that may be achieved in estimating the curvature of a manifold by a *free* quantum particle. By free, we mean that the particle is affected only by the constraining potential that forces it to stay on the surface. The rationale behind this approach is that quantum systems are inherently sensitive to the parameters of their Hamiltonians, which may be exploited to precisely estimate the values of those parameters by performing suitable measurements on them [23–32]. This idea has been recently exploited to develop a new approach to probe macroscopic systems, based on the quantification and optimisation of the information that can be extracted by an interacting quantum probe as opposed to a classical one. In particular, in this Section, we evaluate the Quantum Fisher Information of (37) for the radius of a sphere, either using a generic pure state or an initially localized wave packet as quantum probes. We also discuss the optimization of the probe's initial preparation. We then follow the same procedure for the case of a cylinder.

4.1 Free particle on a sphere

The free Hamiltonian for a particle on a sphere of radius λmaybe written as

$$\mathcal{H}_s = -\frac{\hbar^2}{2M\lambda^2}(\cot\theta\partial_\theta + \partial_\theta^2 + \frac{\partial_\phi^2}{\sin^2\theta}) \tag{40}$$

$$= \frac{J^2}{2M\lambda^2}, \tag{41}$$

where J denotes the angular momentum operator. The time independent Schrödinger equation $\mathcal{H}_s|\Psi\rangle = E|\Psi\rangle$ has finite and separable solutions for

$$(2EM\lambda^2)/(\hbar^2) = j(j+1),$$

with $j \in \mathbb{N}$. The eigenfunctions are the spherical harmonics, i.e.

$$|\Psi_{jm}\rangle = \iint d\theta d\phi \sin\theta Y_{jm}(\theta,\phi)|\theta,\phi\rangle \tag{42}$$

$$Y_{jm}(\theta,\phi) = (-1)^{\dfrac{|m|-m}{2}}\sqrt{\frac{2j+1}{4\pi}\frac{(j-|m|)!}{(j+|m|)!}}P_j^m(\cos\theta)e^{im\phi}, \tag{43}$$

where $P_j^m(x)$ are the Legendre polynomials with $m \in \mathbb{Z}$ and $-j \le m \le j$. The ket $|\theta,\phi\rangle$ in (42) denotes a localised state on the sphere, i.e.:

$$\langle\phi',\theta'|\theta,\phi\rangle = \delta(\phi'-\phi)\delta(\cos\theta'-\cos\theta) = \frac{1}{|\sin\theta|}\delta(\phi'-\phi)\delta(\theta'-\theta), \tag{44}$$

$$\langle\phi,\theta|\Psi_{jm}\rangle = Y_{jm}(\theta,\phi). \tag{45}$$

The corresponding eigenvalues are given by

$$E_{jm} = \frac{\hbar^2 j(j+1)}{2M\lambda^2}. \tag{46}$$

They do not depend on m and are $(2j+1)$ −degenerate. We notice that while the eigenvalues do depend on the curvature of the manifold, i.e. on the radius of the sphere, the eigenvectors do not. This means that preparing the particle in any given energy eigenstate and then performing a measurement cannot provide any information about the curvature. In order to see this more explicitly, let us remind that the spherical harmonics provide an orthonormal basis on the sphere. Thus, we may expand any state as

$$|\psi\rangle = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c_{jm} |\Psi_{jm}\rangle, \tag{47}$$

where the amplitudes c_{jm} do not depend on λ for any initial preparation of the probe particle. At the generic time t, the evolved state is given by

$$|\psi_\lambda\rangle = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c_{jm} e^{-\frac{it}{\hbar} E_{jm}} |\Psi_{jm}\rangle \quad . \tag{48}$$

We are now in the position of using (37) to evaluate the quantum Fisher information for the parameter λ, encoded in the generic evolved state $|\psi_\lambda\rangle$. In order to calculate the two terms involved in the QFI, we need to compute the derivative of the state. Using the shorthand $\sum_{mj} \equiv \sum_{j=0}^{\infty} \sum_{m=-j}^{j}$, we have

$$\begin{aligned} |\partial_\lambda \psi_\lambda\rangle &= \sum_{mj} c_{jm} e^{-\frac{it}{\hbar} E_{jm}} \left[-i \frac{t}{\hbar} \left(\partial_\lambda E_{jm} \right) \right] |\Psi_{jm}\rangle \\ &= \sum_{mj} c_{jm} e^{-\frac{it}{\hbar} E_{jm}} \left[\frac{i t \hbar j (j+1)}{M \lambda^3} \right] |\Psi_{jm}\rangle, \end{aligned} \tag{49}$$

where we have used the relation

$$\partial_\lambda E_{jm} = -\frac{2}{\lambda} E_{jm}. \tag{50}$$

Using the above equations, we arrive at

$$\langle \partial_\lambda \psi_\lambda | \partial_\lambda \psi_\lambda \rangle = \frac{t^2 \hbar^2}{M^2 \lambda^6} \sum_{mj} |c_{jm}|^2 j^2 (j+1)^2, \tag{51}$$

where we used the normalisation condition for the spherical harmonics. Following similar steps, we also have:

$$\langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle = i \frac{t \hbar}{M \lambda^3} \sum_{mj} |c_{jm}|^2 j (j+1), \tag{52}$$

$$\left| \langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle \right|^2 = \frac{t^2 \hbar^2}{M^2 \lambda^6} \left(\sum_{mj} |c_{lm}|^2 j (j+1) \right)^2 . \tag{53}$$

With the help of the preceding relations, we may now write down two expressions for the QFI, either in terms of the fluctuations of the Hamiltonian, or of the squared angular momentum:

$$\begin{aligned} H_\lambda &= 4 \frac{t^2 \hbar^2}{M^2 \lambda^6} \left[\sum_{mj} |c_{jm}|^2 j^2 (j+1)^2 - \left(\sum_{mj} |c_{lm}|^2 j (j+1) \right)^2 \right] && (54) \\ &= \frac{16 t^2}{\lambda^2 \hbar^2} \left(\left\langle \mathcal{H}_s^2 \right\rangle - \langle \mathcal{H}_s \rangle^2 \right) = \frac{16 t^2}{\lambda^2 \hbar^2} \left\langle \Delta \mathcal{H}_s^2 \right\rangle, && (55) \\ &= \frac{4 t^2}{M^2 \lambda^6 \hbar^2} \left(\left\langle J^4 \right\rangle - \left\langle J^2 \right\rangle^2 \right) = \frac{4 t^2}{M^2 \lambda^6 \hbar^2} \left\langle \left(\Delta J^2 \right)^2 \right\rangle, && (56) \end{aligned}$$

where, for a generic operator O, $\langle O\rangle$ stands for $\langle\psi_\lambda|O|\psi_\lambda\rangle$. Upon recalling the quantum Cramér-Rao inequality, we can conclude that longer time evolutions lead to a quadratic improvement in the achievable precision. This agrees with physical intuition, since time evolution leads to a spreading of the wave-function, i.e. it makes the particle *feel more* the effects of the curvature. Notice that at time $t=0$ the quantum Fisher information vanishes, i.e. a static measurement is completely uninformative in order to estimate the radius. Equation (54) also shows that the smaller the radius, the more precise the estimation procedure, as one may have intuitively expected. Notice that the dependence on λ is quite strong, i.e. the achievable precision increases quickly as the radius decreases. At the same time, the dependence of the QFI on the energy fluctuations tells us that, in order to optimally probe the curvature, we need the particle to be prepared in a superposition of energy eigenstates, i.e. that quantum effects represent a resource for the task of estimating the radius.

Let us now look for some specific particle initial preparations, in order to precisely estimate the radius, assuming that the overall mean energy $\bar{\mathrm{E}}$ of the particle is fixed. In order to maximise the QFI, see (54), optimal states should exhibit a broad spread in energy. We therefore expect them to be superpositions of different energy eigenstates with eigenvalues as far as possible from $\bar{\mathrm{E}}$. Since $\bar{\mathrm{E}}$ is the mean energy, we need superpositions of some eigenstates with energy larger than $\bar{\mathrm{E}}$, and some with smaller energy. For the sake of simplicity, and since the energy spectrum is bounded from below, we consider here only superpositions of two energy eigenstates, one of which is the ground state of the system, and seek for the second eigenstate in order to maximise the QFI. Candidate optimal states are thus of the form

$$|\varphi_{jm}\rangle = \cos\alpha|\Psi_{00}\rangle + \sin\alpha e^{i\beta}|\Psi_{jm}\rangle,$$

where the eigenstates of the free Hamiltonian $|\Psi_{jm}\rangle$ have been defined in (47), and $\alpha, \beta \in [0,2\pi)$ are the coefficients of the superposition. Since $E_{00}=0$ we have $\bar{\mathrm{E}} = E_{jm}\sin^2\alpha$ and

$$\left\langle\varphi_{jm}\left|\Delta\mathcal{H}_s^2\right|\varphi_{jm}\right\rangle = E_{jm}^2\sin^2\alpha\cos^2\alpha, \tag{57}$$

and, therefore,

$$H_\lambda = \frac{16t^2}{\hbar^2\lambda^2}\bar{\mathrm{E}}^2\frac{1}{\tan^2\alpha} \tag{58}$$

$$= \frac{4\hbar^2t^2}{M^2\lambda^6}j^2(j+1)^2\sin^2\alpha\cos^2\alpha. \tag{59}$$

The above expression for the QFI indicates that the best probing preparation is a superposition of the ground state and an energy eigenstate with the highest possible energy (angular momentum), compatible with experimental constraints. For fixed value of j, the optimal superposition is a balanced one, i.e. $\alpha=\pi/4$, whereas for fixed value of the mean energy, the optimal superposition is a strongly unbalanced one, i.e. a state $|\varphi_{jm}\rangle$ with $j\gg 1$ and small $\alpha\ll 1$ (while keeping fixed $\bar{\mathrm{E}}\simeq E_{jm}\alpha^2$).

Overall, the above requirements, are quite challenging from the practical point of view. Thus, we now concentrate our attention on a more realistic example, i.e. that of an initially localized wave packet on the sphere, which is then left free to evolve. Assuming the particle is initially localized in a generic point on the sphere, the state reads as follows:

$$|\psi_\kappa\rangle = \iint d\theta d\phi\sin\theta\,\psi_\kappa(\theta)|\theta,\phi\rangle, \tag{60}$$

where the wave-function $\psi_\kappa(\theta)$ is given in terms of the Von Mises distribution on the sphere, i.e.

$$\psi_\kappa(\theta) = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{\kappa}{\sinh\kappa}}\,e^{\frac{\kappa}{2}\cos\theta}. \tag{61}$$

The quantity κ is a concentration parameter. The larger k, the more concentrated the distribution about the origin. For vanishing κ one obtains the uniform distribution on the sphere. The mean energy of this state is given by

$$\bar{\mathrm{E}}_\kappa = \langle\psi_\kappa|\mathcal{H}|\psi_\kappa\rangle = \frac{\hbar^2}{4M\lambda^2}(\kappa\coth\kappa - 1), \tag{62}$$

which increases linearly with κ. Energy fluctuations are instead given by

$$\left\langle\psi_\kappa\left|\Delta\mathcal{H}^2\right|\psi_\kappa\right\rangle = \frac{\hbar^4}{16M^2\lambda^4}\left[1+\kappa^2\left(2-\coth^2\kappa\right)\right]. \tag{63}$$

leading to the following expressions for the QFI

$$H_\lambda = \frac{t^2\hbar^2}{M^2\lambda^6}\left[1+\kappa^2\left(2-\coth^2\kappa\right)\right], \tag{64}$$

$$= \frac{16t^2}{\hbar^2\lambda^2}\bar{\mathrm{E}}_\kappa^2\, g(\kappa), \tag{65}$$

$$g(\kappa) = \frac{1+\kappa^2\left(2-\coth^2\kappa\right)}{(1-\kappa\coth\kappa)^2}. \tag{66}$$

Upon comparing (65) with (59), we conclude that localized states show the same scaling of optimal superpositions with respect to the mean energy. Concerning localisation, we have

$$\begin{aligned} g(\kappa) &\simeq 1+\frac{2}{\kappa} \quad \kappa \gg 1 \\ g(\kappa) &\simeq 1+\frac{12}{\kappa^2} \quad \kappa \ll 1 \end{aligned} \tag{67}$$

which confirms the advantages of employing a quantum state (i.e. a coherent superposition) in order to probe the curvature.

4.2 Free particle on a cylinder

According to the results of Section 2.2.2, the free Hamiltonian for a particle constrained on an infinite cylinder of radius λ may be written as

$$\mathcal{H}_c = -\frac{\hbar^2}{2M}\left(\frac{\partial_\theta^2}{\lambda^2}+\partial_z^2+\frac{1}{4\lambda^2}\right), \tag{68}$$

$$= \frac{1}{2M}\left(\frac{J_z^2}{\lambda^2}+P_z^2-\frac{\hbar^2}{4\lambda^2}\right), \tag{69}$$

where $\theta \in [0,2\pi)$ is the angular coordinate, $z \in \mathbb{R}$ is the axial coordinate and J_z, P_z are the angular and the linear momentum operators along the z-axis, respectively. The corresponding

Schrödinger equation is separable in an angular and an axial part. The eigenvectors of $\mathcal{H}_c$ are given by the common eigenstates of the commuting operators L_z and P_z.

$$|\Phi_{km}\rangle = \iint d\theta dz \Phi_{km}(\theta, z)|\theta, z\rangle \tag{70}$$

$$\Phi_{km}(\theta, z) = \frac{1}{2\pi} e^{ikz} e^{im\theta} \tag{71}$$

where k is a real number, while m is an integer number in order to satisfy the boundary condition $\Phi(\theta + 2\pi, z) = \Phi(\theta, z)$. The corresponding eigenvalues are given by

$$E_{km} = \frac{\hbar^2}{2M}\left(k^2 + \frac{m^2}{\lambda^2} - \frac{1}{4\lambda^2}\right). \tag{72}$$

In (70), $|\theta, z\rangle$ denotes localized states, satisfying the relations:

$$\langle z, \theta|\Phi_{km}\rangle = \Phi_{km}(\theta, z) \tag{73}$$

$$\langle z', \theta'|\theta, z\rangle = \delta(z' - z)\delta(\theta' - \theta) \tag{74}$$

A generic preparation of a particle of the cylinder may be thus written as

$$|\psi\rangle = \sum_m \int dk c_m(k)|\Phi_{km}\rangle. \tag{75}$$

which evolves in time, acquiring a dependence on λ, according to:

$$|\psi_\lambda\rangle = \sum_m \int dk c_m(k) e^{-\frac{it}{\hbar}E_{km}}|\Phi_{km}\rangle. \tag{76}$$

In order to calculate the QFI, we need the derivative of the state with respect to the cylinder radius λ. From (72), we have

$$\partial_\lambda E_{km} = -\frac{\hbar^2}{M\lambda^3}\left(m^2 - \frac{1}{4}\right), \tag{77}$$

and therefore

$$|\partial_\lambda \psi_\lambda\rangle = \sum_m \int dk c_m(k) e^{-\frac{it}{\hbar}E_{km}}\left[\frac{it\hbar}{M\lambda^3}\left(m^2 - \frac{1}{4}\right)\right]|\Phi_{km}\rangle. \tag{78}$$

Following the same procedure as for the sphere, we arrive at:

$$\langle \partial_\lambda \psi_\lambda|\partial_\lambda \psi_\lambda\rangle = \frac{t^2\hbar^2}{M^2\lambda^6}\sum_m \int dk|c_m(k)|^2\left(m^2 - \frac{1}{4}\right)^2, \tag{79}$$

$$\langle \psi_\lambda|\partial_\lambda \psi_\lambda\rangle = i\frac{t\hbar}{M\lambda^3}\sum_m \int dk|c_m(k)|^2\left(m^2 - \frac{1}{4}\right), \tag{80}$$

$$\left|\langle \psi_\lambda|\partial_\lambda \psi_\lambda\rangle\right|^2 = \frac{t^2\hbar^2}{M^2\lambda^6}\left[\sum_m \int dk|c_m(k)|^2\left(m^2 - \frac{1}{4}\right)\right]^2. \tag{81}$$

The QFI is thus given by

$$\begin{aligned} H_\lambda &= \frac{4t^2\hbar^2}{M^2\lambda^6}\left\{\sum_m \int dk|c_m(k)|^2\left(m^2-\frac{1}{4}\right)^2 - \left[\sum_m \int dk|c_m(k)|^2\left(m^2-\frac{1}{4}\right)\right]^2\right\}, \\ &= \frac{4t^2}{M^2\lambda^6\hbar^2}\left\langle(\Delta J_z^2)^2\right\rangle. \end{aligned} \tag{82}$$

As it is apparent from (82), the QFI does not contain any terms depending on the linear momentum. This is intuitively correct, since monitoring the motion of the particle along the z-axis cannot provide any information about the curvature of the cylinder. On the other hand, the QFI is proportional to the variance of the squared angular momentum along the z-axis, which can be seen as the variance of the rotational energy of the particle. This confirms the results already obtained for the sphere. In order to see this more explicitly let us choose a probing particle which is not moving along the z-axis, i.e. $|c_m(k)|^2 = \delta(k)p_m$, where p_m, with $\sum_m p_m = 1$, is a generic distribution for the angular momentum. With this choice, (82) can be rewritten as:

$$H_\lambda = \frac{4t^2\hbar^2}{M^2\lambda^6}\left[\overline{m^4} - \left(\overline{m^2}\right)^2\right], \quad \overline{m^s} = \sum_m m^s p_m. \tag{83}$$

For a uniform distribution $p_m = 1/(2J)$, $m \in [-J,J]$, and for large J, we have

$$\overline{E} \simeq \frac{\hbar^2 J^2}{6M^2\lambda^2}, \qquad H_\lambda \simeq \frac{16t^2\hbar^2 J^4}{45M^2\lambda^6} \simeq \frac{64t^2}{5\hbar^2\lambda^2}\overline{E}^2. \tag{84}$$

For a probing particle prepared in a superposition $|\varphi_J\rangle = \frac{1}{\sqrt{2}}(|\Phi_{00}\rangle + |\Phi_{0J}\rangle)$ the scaling is the same, with a more favourable numerical factor:

$$\overline{E} \simeq \frac{\hbar^2 J^2}{4M^2\lambda^2}, \qquad H_\lambda = \frac{t^2\hbar^2 J^4}{M^2\lambda^6} \simeq \frac{16t^2}{\hbar^2\lambda^2}\overline{E}^2. \tag{85}$$

5 Sensing the Curvature by a Charged Particle in a Magnetic Field

In this section, we discuss sensing protocols involving a charged quantum probe which, besides being constrained to the surface, is subject to an external magnetic field. As we will see, the presence of an external field enhances precision, i.e. it represents a resource in the estimation of curvature.

5.1 Sphere in a Magnetic Field

In Section 2, we have discussed the Schrödinger equation for a particle of charge Q and mass M, constrained to the surface of a sphere with radius λ and subject to a magnetic field B directed along the positive z-axis. The corresponding, time-independent, Hamiltonian may be written as:

$$\mathcal{H}_{s\mathrm{B}} = \underbrace{-\frac{\hbar^2}{2M\lambda^2}\left[\partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right] + i\,\frac{QB\hbar}{2M}\partial_\phi}_{\mathcal{H}_{s\mathrm{B}}^{(0)}} + \underbrace{\frac{B^2Q^2\lambda^2\sin^2\theta}{8M}}_{\mathcal{H}_{s\mathrm{B}}^{(1)},}. \tag{86, 87}$$

where we have already emphasized the presence of two terms, the first one $\mathcal{H}_{s\mathrm{B}}^{(0)}$ containing terms up to the first power in the variable $y = QB$, and the second one, $\mathcal{H}_{s\mathrm{B}}^{(1)}$ which is of the order $O(y^2)$. Eigenvalues and eigenvectors of $\mathcal{H}_{s\mathrm{B}}$ may be found perturbatively, based on the observation that the eigenvectors of the free Hamiltonian $\mathcal{H}_s$, i.e. the vectors $|\Psi_{jm}\rangle$ of (42), are also eigenvectors of $\mathcal{H}_{s\mathrm{B}}^{(0)}$ with eigenvalues

$$E_{jm}^{(0)} = j(j+1)\frac{\hbar^2}{2M\lambda^2} - m\frac{QB\hbar}{2M}. \tag{88}$$

Up to order $O(y^2)$, we have $E_{jm} = E_{jm}^{(0)} + E_{jm}^{(1)}$, where

$$\begin{aligned} E_{jm}^{(1)} = \langle\Psi_{jm}|\mathcal{H}_{s\mathrm{B}}^{(1)}|\Psi_{jm}\rangle &= \lambda^2\frac{Q^2B^2}{8M}\iint d\theta d\phi \sin\theta Y_{jm}^*(\theta,\phi)\sin^2\theta Y_{jm}(\theta,\phi), \\ &= (-1)^m\frac{Q^2B^2}{4M}\frac{m^2+j^2+j-1}{(2j+3)(2j-1)}\lambda^2, \end{aligned} \tag{89}$$

and some standard identities involving associated Legendre polynomials have been employed. The corresponding eigenvectors $|\Xi_{jm}\rangle$ evaluate to:

$$|\Xi_{jm}\rangle = \frac{1}{\sqrt{\mathcal{N}}}\left(|\Psi_{jm}\rangle + \sum_{\kappa\mu\neq jm}\frac{\langle\Psi_{\kappa\mu}|\mathcal{H}_{s\mathrm{B}}^{(1)}|\Psi_{jm}\rangle}{E_{jm}^{(0)} - E_{\kappa\mu}^{(0)}}|\Psi_{\kappa\mu}\rangle\right), \tag{90}$$

where $\mathcal{N}$ is a normalisation factor and

$$E_{jm}^{(0)} - E_{\kappa\mu}^{(0)} = \frac{\hbar}{2M}\left\{\frac{\hbar}{\lambda^2}[j(j+1)-\kappa(\kappa+1)] + (m-\mu)QB\right\}, \tag{91}$$

$$\begin{aligned} \langle\Psi_{\kappa\mu}|\mathcal{H}_{s\mathrm{B}}^{(1)}|\Psi_{jm}\rangle &= \delta_{j\kappa}\delta_{m\mu}E_{jm}^{(1)} + \lambda^2\frac{Q^2B^2}{8M}\iint d\theta d\phi\sin\theta Y_{\kappa\mu}^*(\theta,\phi)\sin^2\theta Y_{jm}(\theta,\phi) \\ &= \delta_{j\kappa}\delta_{m\mu}E_{jm}^{(1)} + \lambda^2\frac{Q^2B^2}{8M}A_{jm}\left(\delta_{j,\kappa-2}+\delta_{j,\kappa+2}\right), \end{aligned} \tag{92}$$

$$A_{jm} = (-1)^{1+m}\sqrt{\frac{(j+m)(j-m)(j+m-1)(j-m-1)}{(2j+1)(2j-3)(2j-1)^2}}. \tag{93}$$

The perturbative ground state in the presence of a magnetic field is thus given by:

$$|\Xi_{00}\rangle = \frac{1}{\sqrt{1+g^2(\lambda)}}\left(|\Psi_{00}\rangle + g(\lambda)|\Psi_{20}\rangle\right), \tag{94}$$

$$g(\lambda) = \frac{Q^2B^2\lambda^4}{36\sqrt{5}\hbar^2}. \tag{95}$$

Remarkably, this is a λ-dependent superposition of the unperturbed eigenvectors, and thus, at variance with the case of a free particle, a measurement on the ground state of the system does provide information about the curvature of the sphere. The QFI may be evaluated using (37). To this aim we compute the derivative of the ground state and the following scalar product:

$$|\partial_\lambda\Xi_{00}\rangle = \frac{\partial_\lambda g}{\sqrt{1+g^2}}|\Psi_{20}\rangle - \left[\frac{g\partial_\lambda g}{1+g^2}\right]|\Xi_{00}\rangle, \tag{96}$$

$$\langle\Xi_{00}|\partial_\lambda\Xi_{00}\rangle = 0, \tag{97}$$

thus arriving at

$$H_\lambda = 4\left(\frac{\partial_\lambda g}{1+g^2}\right)^2 = \frac{9a^2\lambda^6}{(1+a^2\lambda^8)^2}, \qquad a = \frac{Q^2B^2}{36\sqrt{5}\hbar^2}. \tag{98}$$

The QFI vanishes both when $\lambda \to 0$, since $H_\lambda \simeq 9a^2\lambda^6for\lambda \ll 1$, and when $\lambda \to\infty$, since $H_\lambda \simeq 9/(a^2\lambda^{10})$ for $\lambda \gg 1$. It attains its maximum value $H_\lambda = \frac{45}{64}(135)^{\frac{1}{4}}\sqrt{a} \simeq 2.4\sqrt{a}$ for $\lambda = (3/5a^2)^{\frac{1}{8}} \simeq 0.94/a^{\frac{1}{4}}$, a value that may be changed by tuning the magnetic field (or the charge of the quantum probe). Overall, we have that the presence of the external field is a resource for curvature estimation. In particular, it allows to extract information even from measurements on the ground state of the system, i.e. a stationary state, without the need to measure the probe after a given time evolution.

5.2 Cylinder in a Radial Magnetic Field

In Section 2.2.2 we have written the Schröedinger equation for a particle of mass M, and electric charge Q, moving on the surface of a cylinder with radius λ immersed in a magnetic field (22). Let us now focus on the specific case where the magnetic field has only a radial component B_1, whereas the axial component is vanishing $B_0 = 0$. The Hamiltonian of the system may be written as

$$\mathcal{H}_{c\mathrm{B}} = \underbrace{-\frac{\hbar^2}{2M}\left[\frac{1}{\lambda^2}\left(\frac{1}{4}+\partial_\theta^2\right)+\partial_z^2\right]}_{\mathcal{H}_{c\mathrm{B}}^{(0)}} + \underbrace{i\,\frac{\lambda\hbar Q B_1}{M}\sin\theta\,\partial_z}_{\mathcal{H}_{c\mathrm{B}}^{(1)}} + \underbrace{\frac{\lambda^2Q^2B_1^2}{2M}\sin^2\theta}_{\mathcal{H}_{c\mathrm{B}}^{(2)}}, \tag{99, 100}$$

We intend to find the eigenvalues and eigenvectors of $\mathcal{H}_{c\mathrm{B}}$ perturbatively, at first order in the variable $y = QB_1$.Tothis aim, we neglect the effects of the term $\mathcal{H}_{c\mathrm{B}}^{(2)}$ and treat $\mathcal{H}_{c\mathrm{B}}^{(1)}$ as a perturbation to the unperturbed Hamiltonian $\mathcal{H}_{c\mathrm{B}}^{(0)}$. The unperturbed eigenvalues E_{km} and eigenvectors $|\Phi_{km}\rangle$ are those of the free Hamiltonian, and are given in (72) and (70) respectively. The eigenvalues are not changed by the perturbation since $\langle\Phi_{km}|\mathcal{H}_{c\mathrm{B}}^{(1)}|\Phi_{km}\rangle = 0$, whereas the perturbed eigenvectors $|\Upsilon_{km}\rangle$ are given by:

$$|\Upsilon_{km}\rangle = \frac{1}{\sqrt{\mathcal{N}}}\left(|\Phi_{km}\rangle + \sum_{\kappa\mu\neq km}\frac{\langle\Phi_{\kappa\mu}|\mathcal{H}_{c\mathrm{B}}^{(1)}|\Phi_{km}\rangle}{E_{km}-E_{\kappa\mu}}|\Phi_{\kappa\mu}\rangle\right), \tag{101}$$

where $\mathcal{N}$ is a normalisation factor and

$$E_{km} - E_{\kappa\mu} = \frac{\hbar^2}{2M}(k^2-\kappa^2) + \frac{\hbar^2}{2M\lambda^2}(m^2-\mu^2) \tag{102}$$

$$\langle\Phi_{\kappa\mu}|\mathcal{H}_{c\mathrm{B}}^{(1)}|\Phi_{km}\rangle = k\frac{\lambda\hbar QB_1}{M}\delta(k-\kappa)\left[\delta_{\mu-1,m}+\delta_{\mu+1,m}\right]. \tag{103}$$

The perturbed eigenstates in the presence of a magnetic field are thus:

$$|\Upsilon_{km}\rangle = \frac{1}{\sqrt{\mathcal{N}}}\left[|\Phi_{km}\rangle - \frac{2QB_1\lambda^3}{\hbar}\left(B_{km}|\Phi_{k,m+1}\rangle + B_{k,m-1}|\Phi_{k,m-1}\rangle\right)\right], \tag{104}$$

where

$$B_{km} = \frac{k}{1+2m}, \qquad \mathcal{N} = 1 + \frac{4Q^2B_1^2\lambda^6}{\hbar^2}\frac{2k^2(4m^2+1)}{(2m+1)^2(2m-1)^2}. \tag{105}$$

The above equations implies that the ground state is left unperturbed $|\Upsilon_{00}\rangle = |\Phi_{00}\rangle$, whereas the excited states are affected by the perturbations. In order to see the effect of the field on the achievable precision, let us evaluate the QFI for a generic preparation $|\Upsilon_{km}\rangle$, $k\neq 0$, of the particle. The derivative of the statistical model evaluates to:

$$|\partial_\lambda \Upsilon_{km}\rangle = -\frac{\partial_\lambda \mathcal{N}}{2\mathcal{N}}|\Upsilon_{km}\rangle - \frac{6QB_1\lambda^2}{\hbar}\left(B_{km}|\Phi_{k,m+1}\rangle + B_{k,m-1}|\Phi_{k,m-1}\rangle\right). \qquad (106)$$

The scalar products that are needed to compute the QFI may be found by a routine calculation. The resulting QFI is given by

$$H_\lambda = \frac{288a^2k^2(1+4m^2)\lambda^4}{(1-4m^2)^2 + 8a^2k^2(1+4m^2)\lambda^6}, \qquad a = \frac{QB_1}{\hbar}, \qquad (107)$$

which vanishes as $H_\lambda \propto \lambda^4$ for $\lambda \ll 1$ and as $H_\lambda \propto \lambda^{-2}$ for $\lambda \gg 1$,whereas it shows a maximum for an intermediate value of λ, which is a function of the external field and the particle charge. The lowest useful excited state is $|\Upsilon_{10}\rangle$, corresponding to $H_\lambda = 288a^2\lambda^4/(1 + 8a^2\lambda^6)$, which is maximized by $\lambda = (2a)^{-1/3}$, leading to $H_\lambda = 24(2a)^{2/3}$. Overall, we have that, as in the previous case, the presence of an external magnetic field is a resource, since it allows to acquire information on the radius even via a static measurement, i.e. even when the system is prepared in a stationary state.

6 Sensing the Curvature by Position Measurements

In the previous sections, we have evaluated under different scenarios the QFI, which, by means of the quantum Cramér-Rao theorem, sets the ultimate bound imposed by quantum mechanics to the precision of any estimation protocol aimed at characterising the curvature of a manifold. In this section, we turn our attention to position measurements, which represent the most natural choice to consider in realistic situations. For the sake of simplicity, and to maintain the section self-contained, we focus our attention to the case of a free particle, and assess the performance of position measurements in the estimation of the radius of either a sphere or a cylinder. To this aim, we evaluate the Fisher information, and compare it to the corresponding quantum Fisher information.

The measurement of position for a particle on a sphere or a cylinder is described by the following set of projectors:

$$\pi_s(\theta,\phi) = |\theta,\phi\rangle\langle\phi,\theta|, \qquad \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} d\phi \sin\theta\, \pi_s(\theta,\phi) = \mathbb{I}, \qquad (108)$$

$$\pi_c(z,\phi) = |z,\phi\rangle\langle\phi,z|, \qquad \int_{\mathbb{R}} dz \int_0^{2\pi} d\phi\, \pi_c(\theta,\phi) = \mathbb{I}. \qquad (109)$$

Let us start by focussing our attention on the case of the sphere. Given a generic preparation $|\psi\rangle = \sum_{jm} c_{jm}|\Psi_{jm}\rangle$ of the probe, the evolved state $|\psi_\lambda\rangle$ is given in (48). The position distribution at time t is thus:

$$p_t(\theta,\phi|\lambda) = \left|\langle\theta,\phi|\psi_\lambda\rangle\right|^2 = \left|\sum_{jm} c_{jm} e^{-i\frac{t}{\hbar}E_{jm}} Y_{jm}(\theta,\phi)\right|^2, \qquad (110)$$

and its derivative with respect to the radius:

$$\partial_\lambda p_t(\theta,\phi|\lambda) = \langle\theta,\phi|\left[|\partial_\lambda\psi_\lambda\rangle\langle\psi_\lambda| + |\psi_\lambda\rangle\langle\partial_\lambda\psi_\lambda|\right]|\theta,\phi\rangle$$
$$= \frac{\hbar t}{M\lambda^3}\left[i\left(\sum_{jm} c^*_{jm} e^{i\frac{t}{\hbar}E_{jm}} Y^*_{jm}(\theta,\phi)\right)\times\right.$$
$$\left.\times\left(\sum_{jm} c_{jm}\, j(j+1) e^{-i\frac{t}{\hbar}E_{jm}} Y_{jm}(\theta,\phi)\right) + c.c.\right] \qquad (111)$$

The Fisher information is given by

$$F_\lambda = \iint d\theta d\phi \sin\theta \frac{[\partial_\lambda p_t(\theta,\phi|\lambda)]^2}{p_t(\theta,\phi|\lambda)}, \qquad (112)$$

which, using (110) and (111), may be written as

$$F_\lambda = \frac{t^2}{\lambda^6}\frac{\hbar^2}{M^2}K_s(\lambda), \qquad (113)$$

where the function $K_s(\lambda)$ depends only weakly on λ, through the phase factor $e^{-i\frac{t}{\hbar}E_{jm}}$. For short time evolution, i.e. $t \ll (2M\lambda^2)/\hbar$, we have $e^{-i\frac{t}{\hbar}E_{jm}} \simeq 1$ and K becomes independent of λ, i.e.:

$$K_s \simeq \iint d\theta d\phi \sin\theta \frac{\left[i\sum_{\kappa\mu}\sum_{jm} c^*_{\kappa\mu} c_{jm}\, j(j+1) Y^*_{\kappa\mu}(\theta,\phi) Y_{jm}(\theta,\phi) + c.c.\right]^2}{\left|\sum_{jm} c_{jm} Y_{jm}(\theta,\phi)\right|^2}. \qquad (114)$$

Equation (113) implies that the Fisher information of position measurements shows the same scaling

$$F_\lambda \propto t^2/\lambda^6,$$

of the QFI in (56), i.e. position measurements provide a nearly optimal detection scheme for the curvature of a sphere. In order to illustrate the behaviour for longer evolution times, let us consider the ratio between the FI in (113) and the corresponding QFI of (56)

$$R_t(\lambda,\gamma) = \frac{F_\lambda}{H_\lambda} = \frac{K_s(\lambda)}{4\left(\Delta J^2\right)^2}. \qquad (115)$$

This quantity is bounded by the Cramér-Rao theorem, i.e. $0 \leq R_t(\lambda,\gamma) \leq 1$, with larger value of R corresponding to situations where the performance of position measurements is closer to the ultimate bound, thus providing a nearly optimal detection scheme. In Fig. 1 we show R for a particle initially prepared in the superposition $(\cos\gamma|\Psi_{00}\rangle + \sin\gamma|\Psi_{j0}\rangle)/\sqrt{2}$ as a function of γ, for different values of $t \gg 1$ and different values of j and λ. The upper panels are for $j = 1$ and the lower ones for $j = 2$. The left panels show the behaviour of R for $t = 10$ and the right ones for $t = 100$. In each panel, red circles illustrate results for $\lambda = 0.1$, blue squares for $\lambda = 1$, and black rhombi for $\lambda = 10$. As it is apparent from the plots, there always exists a value of γ for which R is considerably large, i.e. estimation by position measurement is nearly optimal. We also notice that R is not a monotone function of t, j and λ.

For the cylinder, measurements of position are described by the operators π_c in (108). However, no information about the curvature may be obtained from measurements of the axial coordinate, and the relevant information is encoded instead in the distribution of the angular coordinate θ. For a generic preparation of the probe on the cylinder $|\psi\rangle =$

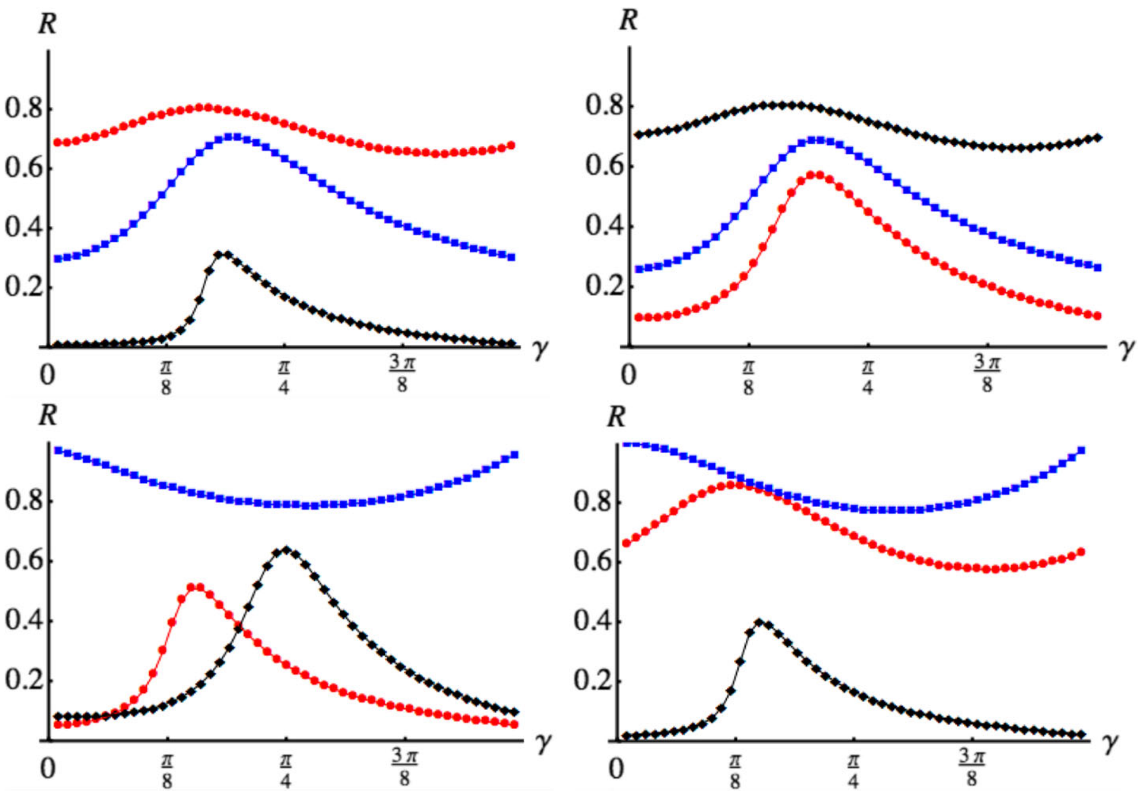

Fig. 1 The R quantity defined in (115) as a function of the superposition parameter γ. The upper panels are for $j = 1$ and the lower ones for $j = 2$. The left panels show the behaviour of R for $t = 10$ and the right ones for $t = 100$. In each panel, red circles illustrate results for $\lambda = 0.1$, blue squares for $\lambda = 1$, and black rhombus for $\lambda = 10$. As it is apparent from the plot, there always exists a value of γ for which R is non negligible

$\sum_m \int dk c_m(k)|\Phi_{km}\rangle$, the evolved state is given in (76), and the probability distribution of a θ-measurement is given by

$$q_t(\theta|\lambda) = \int dz \left|\langle z, \theta|\psi_\lambda\rangle\right|^2 = \frac{1}{2\pi}\sum_{mn}\gamma_{mn}e^{i(m-n)\theta}e^{-i\frac{t}{\hbar}(\epsilon_n - \epsilon_n)}, \tag{116}$$

where

$$\epsilon_n = \frac{\hbar^2}{2M\lambda^2}\left(m^2 - \frac{1}{4}\right), \qquad \gamma_{mn} = \int dk c_m(k)c_n^*(k). \tag{117}$$

Therefore, we have

$$\partial_\lambda q_t(\theta|\lambda) = \frac{t}{\lambda^3}\frac{i\hbar}{2M\pi}\sum_{mn}\gamma_{mn}\left(m^2 - n^2\right)e^{i(m-n)\theta}e^{-i\frac{t}{\hbar}(\epsilon_n - \epsilon_n)}, \tag{118}$$

so that the Fisher information provided by $q_t(\theta|\lambda)$ may be written as

$$F_\lambda = \frac{t^2}{\lambda^6}\frac{\hbar^2}{M^2}K_c(\lambda), \tag{119}$$

where $K_c(\lambda)$ is a function only weakly dependent on λ, as we have already seen in the case of the sphere. Equation (119) shows that the FI of $q_t(\theta|\lambda)$ scales as the corresponding QFI in (82), i.e. position measurements (actually θ-measurements) are nearly optimal for the purpose of detecting the curvature of a cylinder. As we have seen for the sphere, K_c becomes independent of λ for short times. Also the behaviour for long times is analogous to what we have seen for the sphere.

7 Conclusions

In conclusion, in this paper we have addressed the problem of estimating the curvature of a manifold by performing measurements on a quantum particle constrained to propagate on the manifold itself. In particular, we have focused on the case of two-dimensional manifolds embedded in three-dimensional Euclidean space. We have considered the quantum probe as living in the full Euclidean space, even if it is forced to remain within a thin layer of space around the surface by a steep potential. As a matter of fact, due to the nature of the confining potential, the Schrödinger equation and the wave function can be factorized into a normal and a surface components, the latter one providing a natural description of the dynamics on the given manifold.

Upon introducing tools from quantum estimation theory, we have first evaluated the ultimate bound to the estimation precision for a free probe, i.e. a probe subject only to the constraining potential, and have found universal scaling laws for the quantum Fisher information in terms of the time evolution and the radius. In particular, we have shown that a static measurement, i.e. a measurement performed right after the preparation of the probe, is of no use for the purpose of estimating the curvature. Rather, the probe should be left free to evolve on the manifold in order to acquire information about its curvature. We have then looked at the precision bound in the presence of an external field, showing that the field represents a resource, since it allows to exploit static measurements, e.g. on the ground state of the system, without the need to measure the probe after a given time evolution. Finally, we have considered the performance of position measurements, proving that the corresponding Fisher information exhibits the same scaling as the QFI with respect to the time of evolution and the radius. Thus, position measurements provide a nearly optimal detection scheme, at least when the unknown parameter is the radius of a sphere or a cylinder.

Our results, in addition to their fundamental interest, pave the way to applications based on the quantification and optimisation of the information, extracted via a quantum probe, about its ambient manifold. In particular, we foresee new developments in the design of optimal probing strategies aimed at estimating classical geometrical parameters by means of quantum probes, thus providing crucial ingredients for schemes of practical relevance.

Acknowledgements This work has been supported by SERB through project VJR/2017/000011. MGAP is member of GNFM-INdAM.

References

1. Bachtold, A., Strunk, C., Salvetat, J.-P., Bonard, J.-M., Forrò, L., Nussbaumer, T., Schönenberger, C.: Aharonov-bohm oscillations in carbon nanotubes. Nature (London) **397**, 673 (1999)
2. Aoki, H., Suezawa, H.: Landau quantization of electrons on a sphere. Phys. Rev. A **46**, R1163 (1992)
3. Greiter, M., Thomale, R.: Landau level quantization of Dirac electrons on the sphere. Ann. Phys. **394**, 33 (2018)
4. Ju, H., Vagner, I.D., Sundaram, B.: Electrons confined on the surface of a sphere in a magnetic field. Phys. Rev. B **46**, 9501 (1992)
5. Entin, M.V., Magarill, L.I.: Spin-orbit interaction of electrons on a curved surface. Phys. Rev. B **64**, 085330 (2001)
6. Cruz, P.C.S., Bernardo, R.C.S., Esguerra, J.P.H.: Energy levels of a quantum particle on a cylindrical surface with non-circular cross-section in electric and magnetic fields. Ann. Phys. **379**, 159 (2017)
7. Perfetto, E., Gonzàlez, J., Guinea, F., Bellucci, S., Onorato, P.: Quantum Hall effect in carbon nanotubes and curved graphene strips. Phys. Rev. B **76**, 125430 (2007)

8. DeWitt, B.S.: Dynamical theory in curved spaces. i. a review of the classical and quantum action principles. Rev. Mod. Phys. **29**, 377–397 (1957)
9. Jensen, H., Koppe, H.: Quantum mechanics with constraints. Ann. Phys. **63**, 586 (1971)
10. da Costa, R.C.T.: Quantum mechanics of a costrained particle. Phys. Rev. A **23**, 1982 (1981)
11. da Costa, R.C.T.: Constraints in quantum mechanics. Phys Rev A **25**, 2893 (1982)
12. Holland, P.R.: The Quantum Theory of Motion. Cambridge University Press, Cambridge (1993)
13. Destri, C., Maraner, P., Onofri, E.: On the definition of quantum free particle on curved manifolds. Nuovo Cim. **107**, 237 (1994)
14. Ferrari, G., Cuoghi, G.: Schrödinger equation for a particle on a curved surface in an electric and magnetic field. Phys. Rev. Lett. **100**, 230403 (2008)
15. Bernard, B.J., Lew Yan Voon, L.C.: Notes on the quantum mechanics of particles constrained to curved surfaces. Eur. J. Phys. **34**, 1235 (2013)
16. Shikakhwa, M.S., Chair, N.: Hamiltonian for a particle in a magnetic field on a curved surface in orthogonal curvilinear coordinates. Phys. Lett. A **380**, 2876 (2016)
17. Helstrom, C.W.: Cramèr-rao inequalities for operator-valued measures in quantum mechanics. Int. J. Theor. Phys. **8**, 361 (1973)
18. Helstrom, C.W.: Estimation of a displacement parameter of a quantum system. Int. J. Theor. Phys. **11**, 357 (1974)
19. Fujiwara, A., Nagaoka, H.: Quantum Fisher metric and estimation for pure state models. Phys. Lett. A **201**, 119 (1995)
20. Braunstein, S.L., Caves, C.M.: Statistical distance and the geometry of quantum states. Phys. Rev. Lett. **72**, 3439 (1994)
21. Paris, M.G.A.: Quantum estimation for Quantum Technology. Int. J. Quantum Inf. **07**, 125 (2009)
22. Seveso, L., Rossi, M.A.C., Paris, M.G.A.: Quantum metrology beyond the quantum Cramér-Rao theorem. Phys. Rev. A **95**, 012111 (2017)
23. Invernizzi, C., Paris, M.G.A., Pirandola, S.: Optimal detection of losses by thermal probes. Phys. Rev. A **84**, 022334 (2011)
24. Smirne, A., Cialdi, S., Anelli, G., Paris, M.G.A., Vacchini, B.: Quantum probes to assess correlations in a composite system. Phys. Rev. A **88**, 012108 (2013)
25. Benedetti, C., Buscemi, F., Bordone, P., Paris, M.G.A.: Quantum probes for the spectral properties of a classical environment. Phys. Rev. A **89**, 032114 (2014)
26. Paris, M.G.A.: Quantum probes for fractional Gaussian processes. Phys. A **413**, 256 (2014)
27. Benedetti, C., Paris, M.G.A.: Characterization of classical Gaussian processes using quantum probes. Phys. Lett. A **378**, 2495 (2014)
28. Rossi, M.A.C., Paris, M.G.A.: Entangled quantum probes for dynamical environmental noise. Phys. Rev. A **92**(R), 010302 (2015)
29. Tamascelli, D., Olivares, S., Benedetti, C., Paris, M.G.A.: Characterization of qubit chains by Feynman probes. Phys. Rev. A **94**, 042129 (2016)
30. Seveso, L., Paris, M.G.A.: Can quantum probes satisfy the weak equivalence principle. Ann. Phys. **380**, 213 (2017)
31. Bina, M., Grasselli, F., Paris, M.G.A.: Continuous-variable quantum probes for structured environments. Phys. Rev. A **97**, 012125 (2018)
32. Benedetti, C., Salari Sehdaran, F., Zandi, M.H., Paris, M.G.A.: Quantum probes for the cutoff frequency of Ohmic environments. Phys. Rev. A **97**, 012126 (2018)