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UNITARY LOCAL INVARIANCE

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We address unitary local (UL) invariance of bipartite pure states. Given a bipartite state $|\Psi\rangle\rangle = \sum_{ij} \psi_{ij}\, |i\rangle_1 \otimes |j\rangle_2$ the complete characterization of the class of local unitaries $U_1 \otimes U_2$ for which $U_1 \otimes U_2|\Psi\rangle\rangle = |\Psi\rangle\rangle$ is obtained. The two relevant parameters are the rank of the matrix Ψ, $[\Psi]_{ij} = \psi_{ij}$, and the number of its equal singular values, i.e. the degeneracy of the eigenvalues of the partial traces of $|\Psi\rangle\rangle$.

Keywords: Unitary local invariance; entanglement.

Suppose one is given a bipartite pure state $|\Psi\rangle\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and asked for which (pairs of) unitaries the state is locally invariant, i.e.

$$U_1 \otimes U_2|\Psi\rangle\rangle = |\Psi\rangle\rangle. \tag{1}$$

This kind of invariance is closely related to the so-called environment-assisted invariance (envariance), which has recently been introduced[1,2] to understand the origin of the Born rule. More generally, unitary local (UL) invariance naturally arises whenever one investigates the possibility of *undoing* a local operation performed on a subsystem of a multipartite state by acting, though locally, on another subsystem. The somewhat related concept of twin observables has also been investigated in order to account for the invariance that can be observed in measurements performed on correlated systems.[3,4]

As we will see, any state $|\Psi\rangle\rangle$ is UL invariant for some pairs of unitaries and, as one may expect, UL invariance and entanglement properties of $|\Psi\rangle\rangle$ are somehow related. However, there are separable UL invariant states and, overall, the characterization of the class of unitaries leading to UL invariance is not immediate. In this paper, a complete characterization of the pairs of unitaries for which a given state $|\Psi\rangle\rangle$ is UL invariant is achieved in terms of the Schmidt decomposition of $|\Psi\rangle\rangle$, i.e. of the singular value decomposition of the matrix Ψ. We will show that the relevant parameter is the number of terms in the Schmidt decomposition of $|\Psi\rangle\rangle$ and, in particular, the number of (nonzero) equal Schmidt coefficients.

The main result of the paper is given by the ULI theorem, whereas Lemmas 1 and 2 contain preparatory results. More details on the relationship between UL invariance and entanglement are given at the end of the paper.

Let us start by establishing notation. Given a bases $\{|i\rangle_1 \otimes |j\rangle_2\}$ for the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ (with $\mathcal{H}_1$ and $\mathcal{H}_2$ generally not isomorphic), we can write any vector $|\Psi\rangle\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as[5]

$$|\Psi\rangle\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \psi_{ij} \, |i\rangle_1 \otimes |j\rangle_2 , \tag{2}$$

where ψ_{ij} are the elements of the matrix Ψ. The above notation induces a bijection among states $|\Psi\rangle\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and Hilbert–Schmidt operators

$$A = \sum_{ij} a_{ij} \, |i\rangle_{21}\langle j| \tag{3}$$

from $\mathcal{H}_1$ to $\mathcal{H}_2$. The following relations are an immediate consequence of the definitions (2) and (3):

$$A \otimes B |\Psi\rangle\rangle = |A \Psi B^T\rangle\rangle , \quad \langle\langle A|B\rangle\rangle = \mathrm{Tr}[A^\dagger B] \, , \tag{4}$$

$$\mathrm{Tr}_2[|A\rangle\rangle\langle\langle B|] = AB^\dagger , \quad \mathrm{Tr}_1[|A\rangle\rangle\langle\langle B|] = A^T B^* \tag{5}$$

where $A^{T,*,\dagger}$ denote transpose, conjugate and Hermitian conjugate respectively of the matrix A (and of the operator A with respect to the chosen basis). $\mathrm{Tr}_j[\cdots]$ denotes the partial trace over the Hilbert space $\mathcal{H}_j$ whereas $AB^\dagger$ and $A^T B^*$ in Eq. (5) are operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Using Eq. (4) the condition (1) for UL invariance can be rewritten as $\left|U_1 \Psi U_2^T\right\rangle\rangle = |\Psi\rangle\rangle$, thus leading to the matrix relation

$$U_1 \Psi = \Psi U_2^* . \tag{6}$$

The singular value decomposition of Ψ is given by $\Psi = S_1^T \Sigma S_2$, where S_1 and S_2 are unitary matrices of suitable dimension and Σ is the diagonal matrix $\Sigma = \mathrm{Diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots)$, where σ_j are the singular values of Ψ, i.e. the square roots of the eigenvalues of $\Psi^\dagger \Psi$; r is the rank of the matrix Ψ. The singular value decomposition of Ψ corresponds to the Schmidt decomposition of $|\Psi\rangle\rangle$:

$$\begin{aligned} |\Psi\rangle\rangle &= \left|S_1^T \Sigma S_2\right\rangle\rangle = \sum_{ij} \left(S_1^T \Sigma S_2\right)_{ij} |i\rangle_1 \otimes |j\rangle_2 \\ &= \sum_{ij} \sum_{kl} S_{1ki} \Sigma_{kl} S_{2lj} \, |i\rangle_1 \otimes |j\rangle_2 = \sum_k \sigma_k \, |\varphi_k\rangle_1 \otimes |\theta_k\rangle_2 , \end{aligned} \tag{7}$$

where $|\varphi_k\rangle_1 = \sum_i S_{1ki} |i\rangle_1$ and $|\theta_k\rangle_2 = \sum_l S_{2lj} |j\rangle_2$ are the Schmidt basis in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Let us now switch to the Schmidt basis, which will be employed for most of the paper. Bipartite pure states are thus represented by kets $|\Sigma\rangle\rangle$, where Σ is a diagonal matrix. The UL invariance relation (6) is rewritten as

$$R_1 \Sigma = \Sigma R_2^* \, , \tag{8}$$

where we have denoted by R_j $j = 1, 2$ the matrices corresponding to the unitary transformations in the new (Schmidt) basis.

The first step in the characterization of unitaries that leave invariant a given state $|\Psi\rangle\rangle$ is given by the following lemma.

Lemma 1. *Let* U_j, $j = 1, 2$ *be unitaries in* $\mathcal{H}_j$ *and* $|\Psi\rangle\rangle$ *a bipartite state on* $\mathcal{H}_1 \otimes \mathcal{H}_2$. *If* $U_1 \otimes U_2|\Psi\rangle\rangle = |\Psi\rangle\rangle$ *then* $[U_j, \varrho_j] = 0$, *where* ϱ_j *are the partial traces of* $|\Psi\rangle\rangle$, *i.e.* $\varrho_1 = \mathrm{Tr}_2[|\Psi\rangle\rangle\langle\langle\Psi|] = \Psi\Psi^\dagger$ *and* $\varrho_2 = \mathrm{Tr}_1[|\Psi\rangle\rangle\langle\langle\Psi|] = \Psi^T\Psi^* = (\Psi^\dagger\Psi)^T$.

Proof.

$$\begin{aligned} U_1^\dagger \varrho_1 U_1 &= \mathrm{Tr}_2[(U_1^\dagger \otimes \mathbb{I})|\Psi\rangle\rangle\langle\langle\Psi|(U_1 \otimes \mathbb{I})] \\ &= \mathrm{Tr}_2[(U_1^\dagger \otimes \mathbb{I})(U_1 \otimes U_2)|\Psi\rangle\rangle\langle\langle\Psi|(U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes \mathbb{I})] \\ &= \mathrm{Tr}_2[(\mathbb{I} \otimes U_2)|\Psi\rangle\rangle\langle\langle\Psi|(\mathbb{I} \otimes U_2^\dagger)] \\ &= \mathrm{Tr}_2[|\Psi\rangle\rangle\langle\langle\Psi|] = \varrho_1 . \end{aligned} \tag{9}$$

The proof that $U_2^\dagger \varrho_2 U_2 = \varrho_2$ and thus that $[U_2, \varrho_2] = 0$ goes along the same lines. As a consequence of Lemma 1, U_j and ϱ_j posses a common set of eigenvectors, which coincides with the Schmidt basis of $|\Psi\rangle\rangle$ in each Hilbert space.

From the above lemma and from Eq. (8) we can already draw some conclusions about the UL invariance properties of some particular class of quantum states. Let us first consider separable states. These states correspond to rank-one matrices $\Sigma = \sigma_1 \oplus \mathbf{0}$ and thus they are UL invariant under transformation $R_1 \otimes R_2$ if

$$R_1 = e^{i\phi} \oplus V_1 \quad R_2 = e^{-i\phi} \oplus V_2, \tag{10}$$

where ϕ is an arbitrary phase, and V_j, $j = 1, 2$ are arbitrary unitaries, each acting on the $(d_j - 1)$-dimensional null subspace of $\mathcal{H}_j$, corresponding to zero singular values. More generally, Eq. (8) indicates that each matrix R_j should be written as $R_j = W_j \oplus V_j$, where W_j acts on the r-dimensional subspace of $\mathcal{H}_j$ corresponding to the support of Σ, and V_j on the complementary $(d_j - r)$-dimensional null subspace. The rest of the paper is devoted to investigating the structure of W_j.

Let us first consider $|\Psi\rangle\rangle$ as a maximally entangled state; then Ψ is unitary with $\Psi = S_1^T \Sigma S_2$ and $\Sigma = \frac{1}{\sqrt{d}}\mathbb{I}_d$, $d = \min(d_1, d_2)$. Following Lemma 1 and Eq. (8), we have that $R_1 = R_2^*$, i.e. $|\Sigma\rangle\rangle$ is UL invariant for any transformation of the form $R \otimes R^*$ with R arbitrary unitary, i.e. $|\Psi\rangle\rangle$ is UL invariant for transformations $S_1^T R S_1^* \otimes S_2^T R^* S_2^*$. This relation, in turn, expresses isotropy of maximally entangled states.[6] If $|\Sigma\rangle\rangle$ has the form of a maximally entangled state immersed in a larger Hilbert space, then the same conclusion holds on the support of Σ. The two above statements, together with Lemma 1, can be summarized as a necessary and sufficient condition by the following lemma.

Lemma 2. *Let* $|\Sigma\rangle\rangle$ *be a rank* r *bipartite pure state in* $\mathcal{H}_1 \otimes \mathcal{H}_2$ *with* $\Sigma = \frac{1}{\sqrt{r}}\mathbb{I}_r$, *then* $U_1 \otimes U_2|\Sigma\rangle\rangle = |\Sigma\rangle\rangle$ *for* U_1, U_2 *if and only if* $U_1 = U_2^*$ *on the support of* Σ, *i.e. for*

$$U_1 = W \oplus V_1 \quad U_2 = W^* \oplus V_2, \tag{11}$$

where W, V_1 and V_2 are arbitrary unitaries on the corresponding r-dimensional, $(d_1 - r)$-dimensional and $(d_2 - r)$-dimensional subspaces.

Proof. Sufficiency: if $U_1 = W \oplus V_1$ and $U_2 = W^* \oplus V_2$ then Eq. (8) is automatically satisfied and $|\Sigma\rangle\rangle$, $\Sigma = \frac{1}{\sqrt{r}}\,\mathbb{I}_r$ is invariant under the action of $U_1 \otimes U_2$. Necessity: if $U_1 \otimes U_2 |\Sigma\rangle\rangle = |\Sigma\rangle\rangle$ then Lemma 1 assures that U_j and ϱ_j possess a common set of eigenvectors, which coincides with the Schmidt basis of $|\Psi\rangle\rangle$, i.e. the support of $|\Sigma\rangle\rangle$. This fact, together with Eq. (8), implies that $U_1 = U_2^*$ on the support of Σ. The thesis then follows by completing U_1 and U_2 in a unitary way on the remaining sectors of the Hilbert space. □

We are now ready to state the main result of the paper in the form of the following theorem.

Theorem (ULI). *Let $|\Psi\rangle\rangle$ be a bipartite pure state in $\mathcal{H}_1 \otimes \mathcal{H}_2$, with Ψ of rank r, and let r_k be the number of k-tuple of equal singular values, e.g. r_1 is the number of distinct singular values, r_2 the number of pairs and so on; $r_1 + 2r_2 + \cdots kr_k + \cdots = r$. Then $|\Psi\rangle\rangle$ is UL invariant, i.e $U_1 \otimes U_2 |\Psi\rangle\rangle = |\Psi\rangle\rangle$ if and only if $U_j = S_j^T R_j S_j^*$, $j = 1, 2$ with R_j given by*

$$\begin{aligned} R_1 &= e^{i\phi_1} \oplus \cdots \oplus e^{i\phi_{r_1}} \oplus D_1 \oplus \cdots \oplus D_{r_2} \\ &\quad \oplus T_1 \oplus \cdots \oplus T_{r_3} \oplus \cdots \oplus V_1, \end{aligned} \tag{12}$$

$$\begin{aligned} R_2 &= e^{-i\phi_1} \oplus \cdots \oplus e^{-i\phi_{r_1}} \oplus D_1^* \oplus \cdots \oplus D_{r_2}^* \\ &\quad \oplus T_1^* \oplus \cdots \oplus T_{r_3}^* \oplus \cdots \oplus V_2, \end{aligned} \tag{13}$$

where S_j, $j = 1, 2$ are the unitaries entering the singular value decomposition of $\Psi = S_1^T \Sigma S_2$, $D_1, \ldots, D_{r_2}$ are arbitrary 2×2 unitary matrices, $T_1, \ldots, T_{r_3}$ arbitrary 3×3 unitary matrices, and so on. V_1 and V_2 are arbitrary unitaries in the null subspaces of $\mathcal{H}_j$ corresponding to zero singular values.

Proof. After moving to the Schmidt basis by $\Psi = S_1^T \Sigma S_2$, and according to the consideration made before Lemma 1, we can always write the ULI requirement as in Eq. (8), with $R_j = W_j \oplus V_j$, where the W_j are of rank r. Then, as a consequence of Lemma 1, each W_j can be decomposed into blocks acting on the eigenspaces of ϱ_j. Inside each eigenspace, whose dimension corresponds to the degeneracy of the eigenvalues of ϱ_j, i.e. to the multiplicity of each singular value of Ψ, the matrix Σ is proportional to the identity matrix. We can therefore apply Lemma 2, thus arriving at $W_1 = W_2^* = W$ with

$$W = e^{i\phi_1} \oplus \cdots \oplus e^{i\phi_{r_1}} \oplus D_1 \oplus \cdots \oplus D_{r_2} \oplus T_1 \oplus \cdots, \tag{14}$$

from which expressions (12) and (13) and, in turn, the thesis immediately follow. □

In conclusion, unitary local (UL) invariance of bipartite pure states has been addressed and the complete characterization of the class of local unitaries $U_1 \otimes U_2$ for which $U_1 \otimes U_2|\Psi\rangle\rangle = |\Psi\rangle\rangle$ has been obtained in terms of the singular values of the matrix Ψ. The explicit expression of the matrices U_1 and U_2 has been derived. Maximally entangled states are UL invariant under any transformation of the form $U \otimes U^*$ with arbitrary U whereas separable states are UL invariant for unitaries of the form $(e^{i\phi} \oplus V_1) \otimes (e^{-i\phi} \oplus V_2)$ with V_1 and V_2 acting on the null subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. In the general case, the two relevant parameters are the rank of Ψ and the number of equal singular values of Ψ, which determines the structure of the unitaries on the support.

Let us now go back to the connections between unitary local invariance and entanglement, in order to stress that the two concepts are not straightforwardly related each other. As already mentioned, separable states admit locally invariant transformations which, however, are just overall phase factors not observable either locally or globally. Consider the following, more instructive examples: a state with d slightly different Schmidt coefficients can be taken to be arbitrarily close to a maximally entangled state, so that the von Neumann entropy of the reduced states is approximately equal to $\log d$. The pairs of unitaries that leave it invariant are just opposite phase shifts diagonal in the two Schmidt bases. On the other hand, a state with only one Schmidt coefficient close to 1 and $d-1$ small and equal coefficients has an considerably larger set of locally invariant transformations, while the von Neumann entropy of its reduced states can be put arbitrarily close to zero. Thus, in general, UL invariance is not equivalent to entanglement — though for the pure bidimensional case it may become equivalent when supplemented by a suitable squeezing criterion.[7] Notice also that the structure of the locally invariant transformations can be used to evaluate the dimension of the sets of separable and maximally entangled states.[8]

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References

1. W. H. Zurek, *Phys. Rev. Lett.* **90**, 120404 (2003).
2. W. H. Zurek, quant-ph/0405161.
3. F. Herbut and F. N. Damnjanovic, *J. Phys. A* **33**, 6023 (2000).
4. F. Herbut, *J. Phys. A* **35**, 1691 (2002).
5. G. M. D'Ariano, P. Lo Presti and M. F. Sacchi, *Phys. Lett. A* **272**, 32 (2000).
6. M. Horodecki and P. Horodecki, *Phys. Rev. A* **59**, 4206 (1999).
7. A. R. Usha Devi, X. Wang and B. C. Sanders, *Quant. Inform. Proc.* **2**, 207 (2003).
8. M. M. Sinołęcka, K. Życzkowski and M. Kuś, *Acta. Phys. Pol. A* **33**, 2081 (2002).