INTRODUCTION TO GENERATION, MANIPULATION AND CHARACTERIZATION OF OPTICAL QUANTUM STATES

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ABSTRACT. Why do we need quantization to describe vision? What are the quadrature operators of the electromagnetic field? Is it possible to measure them? What are the characteristic functions useful for? In this short introduction we provide the theoretical tools needed to describe the generation, manipulation and characterization of optical quantum states and of the main passive (beam splitters) and active (squeezers) devices involved in the experiments, such as the Hong-Ou-Mandel interferometer and the continuous variable quantum teleportation. We also introduce the concept of operator ordering and the description of a system by means of the p-ordered characteristic functions. Then we focus on the quasi-probability distributions and, in particular, on the relation between the marginals of the Wigner function and the outcomes of the quadrature operator measurement. Finally, we introduce the balanced homodyne detection to measure the quadrature operator and homodyne tomography for measuring generic field operators also in the presence of non-unit quantum efficiency.

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1. Introduction

The development of optics is focused to understand and interpret the many optical phenomena observed (interference, diffraction,...). In the 19$^{\text{th}}$ century the investigation of optical phenomena was restricted to direct observation and photographic plates. The information available was about the spatial distribution of light: the information was restricted to averages and there was a lack of knowledge about the time resolved information. In order to describe the experimental results only the geometrical optics (coming from Greeks) and the wave optics (due to Huygens' ideas and Fresnel's mathematical models) were enough. When the laser was invented (and experimentally realized) it became possible to fully exploit the coherence properties of light (holography experiments), to perform the spectroscopic investigation of the (quantum) atom, the dynamics of chemical and biological processes and so on. Furthermore, the high intensities achievable with laser beams made possible the realization of nonlinear optical effects (generation of sum-, difference- and high-order frequencies).

Optics is one of the best testing grounds of Quantum Mechanics. Today, the investigation of optics, or, rather, quantum optics is based on *lasers*, as light sources, and on the *photoelectric effect*, as detection strategy: in this way it is possible to generate exotic quantum It is worth noting that by means of the photoelectric detectors it is possible to address individual events which generate photocurrents. All this unavoidably leads to the advent of the so-called statistical optics.

2. Classical waves and quadratures

An electromagnetic wave in isotropic insulating medium is described by the wave equation:

$$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = 0 \tag{1}$$

whose (monochromatic) solution can be written as:

$$\mathbf{E}(\mathbf{r},t) = E_0 \left[\alpha(\mathbf{r},t)\,\mathrm{e}^{-i\omega t} + \alpha^*(\mathbf{r},t)\,\mathrm{e}^{i\omega t}\right]\,\mathbf{p}(\mathbf{r},t), \tag{2}$$

where $\mathbf{r}$ and t represent the position vector and the time, respectively, E_0 is a suitable constant (with the correct physical dimension), $\alpha(\mathbf{r},t)$ is a complex amplitude function, ω is the frequency of the wave and $\mathbf{p}(\mathbf{r},t)$ the polarization vector. We can rewrite the amplitude $\alpha(\mathbf{r},t)$ as:

$$\alpha(\mathbf{r},t) = \alpha_0(\mathbf{r},t)\,\mathrm{e}^{i\phi(\mathbf{r},t)}, \tag{3}$$

where $\alpha_0(\mathbf{r},t)$ is the (dimensional) magnitude of the field and the phase term $\phi(\mathbf{r},t)$ determines the shape of the wave front. It is worth noting that the spatial distribution of $\phi(\mathbf{r},t)$ describes the curvature of the wave. In the case of a *plane wave* with wave vector $k = \omega/c$ and moving along the positive direction of z-axis we have:

$$\alpha(\mathbf{r},t) = \alpha_0\,\mathrm{e}^{ikz} \quad \Rightarrow \quad \mathbf{E}(\mathbf{r},t) = 2E_0\alpha_0\cos(kz-\omega t)\;\mathbf{p}(\mathbf{r},t). \tag{4}$$

In general, the phase function $\phi(\mathbf{r},t)$ describes the shape of the wave front as well as its absolute phase with respect to a reference. If we introduce the following *quadratures*:

$$x_1(\mathbf{r},t) = \alpha_0(\mathbf{r},t) + \alpha_0^*(\mathbf{r},t), \quad \text{and} \quad x_2(\mathbf{r},t) = -i\left[\alpha_0(\mathbf{r},t) - \alpha_0^*(\mathbf{r},t)\right], \tag{5}$$

we can explicitly write the absolute phase as:

$$\phi_0(\mathbf{r},t) = \tan^{-1}\left[\frac{x_2(\mathbf{r},t)}{x_1(\mathbf{r},t)}\right], \tag{6}$$

and Eq. (2) rewrites:

$$\mathbf{E}(\mathbf{r},t) = E_0\left[x_1(\mathbf{r},t)\,\cos\omega t + x_2(\mathbf{r},t)\,\sin\omega t\right]\,\mathbf{p}(\mathbf{r},t). \tag{7}$$

In Fig. 1 (a) we show the graphical representation of the classical wave in Eq. (7) in a given space point $\mathbf{r}$ at time t, which corresponds to a point in the two-dimensional x_1–x_2 plane. This kind of representation of the classical fields can be used to describe interference and diffraction phenomena.

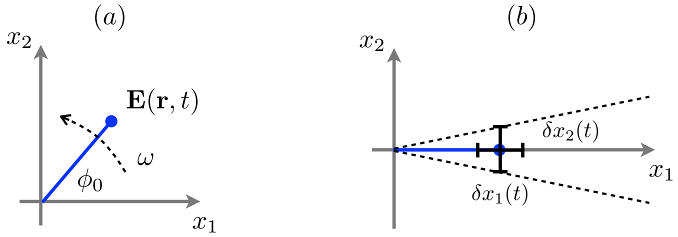

FIGURE 1. (a) Phasor representation of the classical wave in Eq. (7). (b) In the limit $\alpha_0 \gg \delta x_1(t)$ the fluctuations of the quadratures $\delta x_1(t)$ and $\delta x_2(t)$ correspond to amplitude and phase fluctuations (see the text for details).

In the presence of quadrature fluctuations, the complex amplitude $\alpha(t)$ of a classical field can be written as (without loss of generality we assume $\alpha_0 \in \mathbb{R}$):

$$\alpha(t) = \alpha_0 + \delta x_1(t) + i\delta x_2(t). \tag{8}$$

In the limit $\alpha_0 \gg \delta x_1(t), \delta x_2(t)$, it is clear from Fig. 1 (b) that $\delta x_1(t)$ leads to fluctuations of the amplitude of α_0, whereas $\delta x_2(t)$ is responsible of its phase fluctuations. Therefore, one usually refers to the quadrature x_1 as "amplitude" and to x_2 as "phase". We recall that phase and amplitude modulation can be used to encode signals into a classical electromagnetic wave.

Starting form the quadratures introduced in Eqs. (5), we can define the generic quadrature (since we are considering a particular point of the space at a given time, we drop the explicit dependence on $\mathbf{r}$ and t):

$$x_\theta = x_1 \cos\theta + x_2 \sin\theta. \tag{9}$$

In the presence of a classical field, the value of x_θ oscillates as a function of the quadrature phase θ.

3. Basics of electromagnetic field quantization

Modern light detectors are based on the photoelectric effect and, therefore, the introduction of the quantization of electromagnetic field is somehow natural. However, quantization is needed also to describe our vision process. Let's think of a starry night: if the vision process were based on a classical effect, the image of the stars would require many seconds to build up, that is the time needed to collect enough energy, like a 19^{th} century photographic plate. Since this is not the case (we see the faint stars!) and today we know that we need less than ten photons (depending on the frequency) to "detect" an object, we can conclude that a quantized description of light is needed.

In this section we briefly review the main steps to write the quantized Hamiltonian of the electromagnetic field. We consider a cavity with a roundtrip L, thus only the discrete optical modes whose frequencies satisfy the relation $\omega_l = 2\pi c\, l/L$, with $l = 0, 1, 2, \ldots$, are allowed. Upon introducing the quadratures $x_{1,l}$ and $x_{2,l}$ of each mode l, the Hamiltonian H of the classical fields can be written as (for the sake of simplicity we introduce the quantum constant $\hbar$):

$$H = \sum_l \frac{\hbar\omega_l}{4}\left(x_{1,l}^2 + x_{2,l}^2\right). \tag{10}$$

In order to quantize H, we define the canonical variables:

$$q_l = \sqrt{\frac{\hbar}{2\omega_l}}\, x_{1,l} \quad \text{and} \quad p_l = \sqrt{\frac{\hbar\omega_l}{2}}\, x_{2,l} \tag{11}$$

and the previous Hamiltonian becomes:

$$H = \frac{1}{2}\sum_l \left(p_l^2 + \omega_l^2 q_l^2\right). \tag{12}$$

The canonical quantization is obtained substituting to q_l and p_l the canonical operators $\hat{q}_l$ and $\hat{p}_l$, respectively, with commutation relation $[\hat{q}_l, \hat{p}_k] = i\hbar\,\delta_{l,k}$, that is:

$$\hat{H} = \frac{1}{2}\sum_l \left(\hat{p}_l^2 + \omega_l^2 \hat{q}_l^2\right). \tag{13}$$

We now introduce the annihilation and creation operators for the l-th mode, $\hat{a}_l$ and $\hat{a}_l^\dagger$, respectively, with $[\hat{a}_l, \hat{a}_k^\dagger] = \delta_{l,k}$, and define the following *quadrature operators*:

$$\hat{x}_{1,l} = \hat{a}_l + \hat{a}_l^\dagger \quad \text{and} \quad \hat{x}_{2,l} = -i\left(\hat{a}_l - \hat{a}_l^\dagger\right), \tag{14}$$

and the position- and momentum-like operators:

$$\hat{q}_l = \sqrt{\frac{\hbar}{2\omega_l}}\left(\hat{a}_l + \hat{a}_l^\dagger\right) \quad \text{and} \quad \hat{p}_l = -i\sqrt{\frac{\hbar\omega_l}{2}}\left(\hat{a}_l - \hat{a}_l^\dagger\right), \tag{15}$$

where $[\hat{x}_{1,l}, \hat{x}_{2,k}] = 2i\,\delta_{l,k}$. The quantum Hamiltonian $\hat{H}$ can be written as:

$$\hat{H} = \sum_l \hbar\omega_l\left(\hat{a}_l^\dagger\hat{a}_l + \frac{1}{2}\right), \tag{16}$$

that is the Hamiltonian of a set of quantum harmonic oscillators with frequencies ω_l. From now on we consider a single mode with frequency ω described by the field operator $\hat{a}$.

As in the classical case, we can define the generic quadrature operator as:

$$\hat{x}_\theta = \hat{x}_1 \cos\theta + \hat{x}_2 \sin\theta \tag{17}$$

$$= \hat{a}\,e^{-i\theta} + \hat{a}^\dagger\,e^{i\theta}. \tag{18}$$

We will see in the following that the expectation values of $\hat{x}_\theta$ can be quite different with respect to the case of the classical quadrature.

3.1. **Fock states.** The eigenstates of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ are the Fock states or number states $|n\rangle$, $n \in \mathbb{N}$, and the corresponding eigenvalues are the integer numbers n, namely:

$$\hat{n}|n\rangle = n|n\rangle. \tag{19}$$

We recall that the Fock states are a resolution of the identity operator, $\sum_n |n\rangle\langle n| = \hat{\mathbb{I}}$. The action of the annihilation and creation operators on a Fock state is:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{20}$$

respectively. It is worth noting that the *vacuum state* $|0\rangle$ is an eigenvector of the annihilation operator with zero eigenvalue. Sometimes it is useful to write $|n\rangle$ as a power of the creation operator applied to the vacuum state, namely:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n}}|0\rangle. \tag{21}$$

▫ **Exercise 1.** *Calculate the variance $\Delta^2(\hat{n})$ of the number operator $\hat{n}$ given the Fock state $|m\rangle$, namely:*

$$\Delta^2(\hat{n}) = \langle m|\hat{n}^2|m\rangle - \langle m|\hat{n}|m\rangle^2.$$

■

Let's turn our attention on the expectations of the quadrature operator $\hat{x}_\theta$. In the presence of a Fock state $|n\rangle$ it is easy to show that $\langle n|\hat{x}_\theta|n\rangle = 0$, $\forall n$ and $\forall\theta$. If we compare this result with the classical case, we can conclude that the number states are quite exotic states of light...

3.2. **Coherent states.** Light states with more familiar behavior are the eigenvectors of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}. \tag{22}$$

The states $|\alpha\rangle$ are called *coherent states* and are the closest approximation of the output state of a laser.

Exploiting the completeness relation $\sum_n |n\rangle\langle n| = \hat{\mathbb{I}}$ and the normalization condition $\langle\alpha|\alpha\rangle = 1$ we can find the photon number expansion of a coherent state (see Example 1).

▫ **Example 1.** *In this example we calculate the photon number statistics $p(n) = |\langle n|\alpha\rangle|^2$ of a coherent state $|\alpha\rangle$. We start using the completeness relation of Fock states:*

$$|\alpha\rangle \to \left(\sum_{n=0}^{\infty} |n\rangle\langle n|\right)|\alpha\rangle = \sum_{n=0}^{\infty}\langle n|\alpha\rangle|n\rangle.$$

Now we have:

$$\langle n|\alpha\rangle = \langle 0|\frac{(\hat{a})^n}{\sqrt{n!}}|\alpha\rangle = \langle 0|\alpha\rangle\frac{\alpha^n}{\sqrt{n!}},$$

where $\langle 0|\alpha\rangle \in \mathbb{C}$ and we used Eq. (21). Therefore, we can write:

$$|\alpha\rangle = \langle 0|\alpha\rangle\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

In order to find an explicit expression of $\langle 0|\alpha\rangle$ we recall that the normalization condition requires $\langle\alpha|\alpha\rangle = 1$, namely:

$$1 = \langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2} \Rightarrow |\langle 0|\alpha\rangle| = e^{-|\alpha|^2/2}.$$

In general $\langle 0|\alpha\rangle$ is a complex number, however a quantum state is defined up to an overall phase, thus we can put $\langle 0|\alpha\rangle = e^{-|\alpha|^2/2}$. Finally we have:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{23}$$

and the photon number distribution reads:

$$p(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}, \tag{24}$$

that is a Poisson distribution with:

$$\langle \hat{n}\rangle = |\alpha|^2 \quad \textit{and} \quad \Delta^2(\hat{n}) = |\alpha|^2,$$

as the reader can easily check. ■

If we calculate the expectation of the quadrature operator we find:

$$\langle \hat{x}_\theta\rangle = \langle\alpha|\hat{a}|\alpha\rangle \,\mathrm{e}^{-i\theta} + \langle\alpha|\hat{a}^\dagger|\alpha\rangle \,\mathrm{e}^{i\theta} = 2\,\Re\mathrm{e}\left[\alpha\,\mathrm{e}^{-i\theta}\right], \tag{25}$$

and if we put $\alpha = (x_1 + ix_2)/2$, with $x_1, x_2 \in \mathbb{R}$, we obtain:

$$\langle \hat{x}_\theta\rangle = x_1 \cos\theta + x_2 \sin\theta,$$

as for a classical wave [see Eq. (9)]. However, whereas in the classical case the uncertainty of the expectation of the quadrature is null, in the present case we have:

$$\Delta^2(\hat{x}_\theta) = \langle \hat{x}_\theta^2\rangle - \langle \hat{x}_\theta\rangle^2 = 1, \quad \forall\theta,$$

and, in particular:

$$\Delta^2(\hat{x}_1)\,\Delta^2(\hat{x}_2) = 1. \tag{26}$$

Since, in general, given two operators $\hat{A}$ and $\hat{B}$ we have:

$$\Delta^2(\hat{A})\,\Delta^2(\hat{B}) \geqslant \frac{1}{4}\left|\left\langle[\hat{A},\hat{B}]\right\rangle\right|^2, \tag{27}$$

and, in our case, $[\hat{x}_1, \hat{x}_2] = 2i$, we have found that coherent states are *minimun uncertainty states*.

Two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal:

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\alpha^*)^n \beta^n}{n!} \tag{28}$$

$$= \exp\left(-\frac{|\alpha-\beta|^2}{2}\right) \exp\left(\frac{\alpha^*\beta - \alpha\beta^*}{2}\right), \tag{29}$$

and we have:

$$|\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2} \neq 0. \tag{30}$$

On the other hand, coherent states are *overcomplete*, in fact we have the following resolution of the identity:

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \hat{\mathbb{I}}. \tag{31}$$

▫ **Exercise 2.** *Prove Eq. (31). (Hint: use the completeness relation of the Fock states and the polar representation of complex numbers...)* ■

3.3. **Thermal states.** Up to now we have met two *pure* states, namely, the photon number states $|n\rangle$ and the coherent states $|\alpha\rangle$. Another quite common state is the thermal state, which is a *mixed* state, being a mixture of Fock states. The state of the radiation at thermal equilibrium is described by the density operator:

$$\hat{\varrho}_{\rm th}(N_{\rm th}) = \frac{1}{1+N_{\rm th}} \sum_{n=0}^{\infty} \left(\frac{N_{\rm th}}{1+N_{\rm th}}\right)^n |n\rangle\langle n|. \tag{32}$$

where $N_{\rm th} = \{\exp[\hbar\omega/(k_{\rm B}T)] - 1\}^{-1}$, ω being the radiation frequency, T the temperature and $k_{\rm B}$ the Boltzmann constant. It is straightforward to verify that the purity $\mathrm{Tr}[\hat{\varrho}^2_{\rm th}] < 1$ if $N_{\rm th} > 0$; if $N_{\rm th} = 0$ we have $\hat{\varrho}_{\rm th}(0) = |0\rangle\langle 0|$, that is the vacuum state. Of course, the photon number statistics is given by:

$$p(n) = \frac{1}{1+N_{\rm th}} \left(\frac{N_{\rm th}}{1+N_{\rm th}}\right)^n .$$

▫ **Exercise 3.** *Show that for a thermal state $\hat{\varrho}_{th}(N_{th})$ one has $\langle \hat{n}\rangle = Tr[\hat{\varrho}_{th}\,\hat{n}] = N_{th}$ and variance $\Delta^2(\hat{n}) = N_{th}(N_{th}+1)$.* ■

▫ **Exercise 4.** *Show that for a thermal state $\hat{\varrho}_{th}(N_{th})$ one obtains $\langle \hat{x}_\theta\rangle = 0$ and variance $\Delta^2(\hat{x}_\theta) = 2N_{th}+1$.* ■

4. Function of operators and ordering theorems

Given the Hermitian operator $\hat{A}$, such that $\hat{A}|\psi_n\rangle = A_n|\psi_n\rangle$, and a function $f(x)$ with Maclaurin expansion:

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{l!} f^{(l)}(0)\, x^l,$$

it follows:

$$f\left(\hat{A}\right)|\psi_n\rangle = f(A_n)|\psi_n\rangle \Rightarrow f\left(\hat{A}\right) = \sum_n f(A_n)|\psi_n\rangle\langle\psi_n|. \tag{33}$$

▫ **Example 2.** *A unitary operator $\hat{U}$ can be always written as $\hat{U} = \exp\left(i\hat{B}\right)$, where $\hat{B}$ is Hermitian. Therefore we have:*

$$\hat{U} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(i\hat{B}\right)^k .$$

■

Given two operators $\hat{A}$ and $\hat{B}$ we have the following theorems:

Theorem 1. *If* $[\hat{A}, \hat{B}] \in \mathbb{C}$, *we have:*

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp\left\{-\frac{1}{2}[\hat{A},\hat{B}]\right\}, \tag{34a}$$

$$= \exp(\hat{B})\exp(\hat{A})\exp\left\{\frac{1}{2}[\hat{A},\hat{B}]\right\}. \tag{34b}$$

■

Theorem 2.

$$e^{\hat{A}}\,\hat{B}\,e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots \tag{35}$$

■

Theorem 1 can be used in the case of Schrödinger evolution and in the presence of a Hamiltonian, which can be reduced to the form $\hat{H} \propto \hat{A} + \hat{B}$ where $[\hat{A}, \hat{B}] \in \mathbb{C}$. On the other hand, Theorem 2 is extremely useful in the case of the Heisenberg evolution of the operators under the action of a Hamiltonian $\hat{H}$. In this case $\hat{A} = -i\hat{H}t/\hbar$ and $\hat{B}$ is the operator under investigation.

□ **Example 3. (The displacement operator)** *The so-called displacement operator is:*

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right). \tag{36}$$

If we apply $\hat{D}(\alpha)$ *to the vacuum state and use Theorem 1 we have:*

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}\exp(\alpha\hat{a}^\dagger)\underbrace{\exp(\alpha^*\hat{a})|0\rangle}_{|0\rangle}$$

$$= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\alpha\hat{a}^\dagger\right)^n|0\rangle$$

$$= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

that is $\hat{D}(\alpha)|0\rangle = |\alpha\rangle$*: we have "displaced" the vacuum state and obtained a coherent state. We can associate the linear Hamiltonian* $\hat{H} = i\hbar(g\hat{a}^\dagger - g^*\hat{a})$ *with the displacement operator. In fact, we find the following evolution operator:*

$$\exp\left(-i\frac{\hat{H}}{\hbar}t\right) = \hat{D}(gt),$$

that is a displacement with amplitude $\alpha = gt$. ■

□ **Example 4.** *In this example we evaluate the Heisenberg evolution of the annihilation operator under the action of the displacement operator. We can apply the Theorem 2 with* $\hat{A} = \alpha^*\hat{a} - \alpha\hat{a}^\dagger$ *and* $\hat{B} = \hat{a}$*, since* $[\hat{A}, \hat{B}] = \alpha$ *only the first two terms of the r.h.s. of Eq. (35) survive, namely:*

$$\hat{D}^\dagger(\alpha)\,\hat{a}\,\hat{D}(\alpha) = \hat{a} + \alpha. \tag{37}$$

■

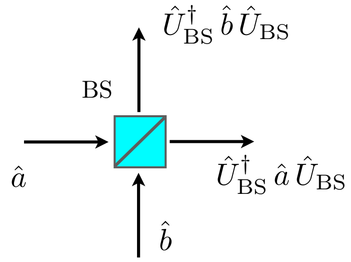

FIGURE 2. A beam splitter.

□ **Exercise 5.** *Calculate $\hat{D}^\dagger(\alpha)\,\hat{x}_\theta\,\hat{D}(\alpha)$ and, using the result, calculate $\langle\alpha|\hat{x}_\theta|\alpha\rangle$, where $|\alpha\rangle$ is a coherent state.* ■

5. THE BEAM SPLITTER

The beam splitter (BS) is one of the most common devices we can find in any quantum optics experiment. The schematic representation of a BS is given in Fig. 2. The Hamiltonian describing the interaction through a BS involves two input modes, $\hat{a}$ and $\hat{b}$, namely, $\hat{H} \propto \hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger$, and the corresponding evolution operator can be written as:

$$\hat{U}_{\rm BS}(\zeta) = \exp\left(\zeta\,\hat{a}^\dagger\hat{b} - \zeta^*\,\hat{a}\hat{b}^\dagger\right), \tag{38}$$

where $[\hat{a},\hat{b}] = 0$ and $\zeta = \phi\,{\rm e}^{i\theta}$. Since the two-boson operators $\hat{J}_+ = \hat{a}^\dagger\hat{b}$, $\hat{J}_- = \hat{a}\hat{b}^\dagger$ and $\hat{J}_3 = \frac{1}{2}[\hat{J}_+,\hat{J}_-] = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger)$ are a realization of the SU(2) algebra, we can rewrite Eq. (38) as follow:

$$\hat{U}_{\rm BS}(\zeta) = \exp\left[{\rm e}^{i\theta}\tan\phi\,\hat{a}^\dagger\hat{b}\right]\,(\cos^2\phi)^{(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})/2}\,\exp\left[-{\rm e}^{-i\theta}\tan\phi\,\hat{a}\hat{b}^\dagger\right], \tag{39a}$$

$$= \exp\left[-{\rm e}^{-i\theta}\tan\phi\,\hat{a}\hat{b}^\dagger\right]\,(\cos^2\phi)^{-(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})/2}\,\exp\left[{\rm e}^{i\theta}\tan\phi\,\hat{a}^\dagger\hat{b}\right], \tag{39b}$$

which can be useful to calculate the Schrödinger evolution of two states through a BS. Using the Theorem 2 it is easy to show that:

$$\hat{U}^\dagger_{\rm BS}(\zeta)\,\hat{a}\,\hat{U}_{\rm BS}(\zeta) = \hat{a}\cos\phi + \hat{b}\,{\rm e}^{i\theta}\sin\phi, \tag{40a}$$

$$\hat{U}^\dagger_{\rm BS}(\zeta)\,\hat{b}\,\hat{U}_{\rm BS}(\zeta) = \hat{b}\cos\phi - \hat{a}\,{\rm e}^{-i\theta}\sin\phi, \tag{40b}$$

that can be used to evaluate the Heisenberg evolution of some functions of the field operators. Due to the action of a BS on the two input modes, this kind of interaction is also called *two-mode mixing interaction*.

□ **Exercise 6.** *The state of a laser beam can be well approximated with a coherent state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ of its field mode $\hat{a}$, $\alpha \in \mathbb{R}$. Assuming that the mode $\hat{b}$ is initially in its vacuum state, calculate the evolution of the input state $|\alpha\rangle \otimes |0\rangle$ through a balanced BS, that is a BS with $\phi = \pi/4$ and $\theta = 0$. Do the same calculation with mode $\hat{b}$ excited in a coherent state $|\beta\rangle$, $\beta = |\beta|\,e^{i\varphi}$. (Hint: consider the Heisenberg evolution of the displacement operators.)* ■

▫ **Example 5. (Hong-Ou-Mandel effect)** *Let's assume that two single photons interacts through a balanced BS ($\phi = \pi/4$ and $\theta = 0$): what is the two-mode output state? We can write the two-mode input state of mode $\hat{a}$ and $\hat{b}$ as:*

$$|1\rangle|1\rangle = |1\rangle \otimes |1\rangle = \hat{a}^\dagger \hat{b}^\dagger |0\rangle.$$

The output state is thus given by $\hat{U}_{BS}|1\rangle|1\rangle$, with $\hat{U}_{BS} = \hat{U}_{BS}(\frac{\pi}{4})$. Exploining Eqs. (40) we have:

$$\hat{U}_{BS}|1\rangle|1\rangle = \hat{U}_{BS}\, \hat{a}^\dagger \hat{b}^\dagger\, \hat{U}_{BS}^\dagger \underbrace{\hat{U}_{BS}|0\rangle}_{|0\rangle},$$
$$= \left(\frac{\hat{a}^\dagger - \hat{b}^\dagger}{\sqrt{2}}\right)\left(\frac{\hat{b}^\dagger + \hat{a}^\dagger}{\sqrt{2}}\right)|0\rangle = \frac{|2\rangle|0\rangle - |0\rangle|2\rangle}{\sqrt{2}}.$$

Therefore if we put two photodetectors at the BS outputs, we never obtain coincidence counts that correspond to the output state $|1\rangle|1\rangle$. ■

6. Single-mode squeezing

When the value of a quadrature variance is less than the vacuum state one, in our case less than 1, we say that the state is "squeezed". Squeezing transformations correspond to Hamiltonians of the form $\hat{H} \propto (\hat{a}^\dagger)^2 + \hat{a}^2$ and the evolution operator can be written as:

$$\hat{S}(\xi) = \exp\left[\tfrac{1}{2}\xi(\hat{a}^\dagger)^2 - \tfrac{1}{2}\xi^* \hat{a}^2\right], \tag{41}$$

where $\xi = r\,e^{i\psi}$. If we define the operators $\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2$, $\hat{K}_- = \frac{1}{2}\hat{a}^2$ and $\hat{K}_3 = -\frac{1}{2}[\hat{K}_+, \hat{K}_-] = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \frac{1}{2})$, we obtain a boson realization of $SU(1,1)$ algebra. In particular, we can write:

$$\hat{S}(\xi) = \exp\left[\frac{\nu}{2\mu}(\hat{a}^\dagger)^2\right] \mu^{-(\hat{a}^\dagger \hat{a} + \frac{1}{2})} \exp\left[-\frac{\nu^*}{2\mu}\hat{a}^2\right], \tag{42}$$

where $\mu = \cosh r$ and $\nu = e^{i\psi}\sinh r$. The evolution of the annihilation operator under the action of the squeezing operator (41) reads:

$$\hat{S}^\dagger(\xi)\, \hat{a}\, \hat{S}(\xi) = \mu \hat{a} + \nu \hat{a}^\dagger. \tag{43}$$

▫ **Example 6. (Squeezed vacuum)** *If we apply the squeezing operator to the vacuum state, we obtain the squeezed vacuum:*

$$\hat{S}(\xi)|0\rangle = \frac{1}{\sqrt{\mu}} \sum_{n=0}^{\infty} \left(\frac{\nu}{2\mu}\right)^n \frac{\sqrt{(2n)!}}{n!} |2n\rangle,$$

which can be easily calculated using Eq. (42) and the expansion of the exponential. It is worth noting that the squeezed vacuum is a superposition of only even number states. ■

□ **Exercise 7.** *Let's assume that $\xi = r \in \mathbb{R}$. Show that:*

$$\begin{aligned} (i) &\quad \langle \hat{n} \rangle = \sinh^2 r; \\ (ii) &\quad \Delta^2(\hat{n}) = 2\sinh^2 r(\sinh^2 r + 1); \\ (iii) &\quad \langle \hat{x}_\theta \rangle = 0, \quad \forall \theta; \\ (iv) &\quad \Delta^2(\hat{x}_\theta) = e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta. \end{aligned}$$

□

Starting from the results of the Exercise 7 and focusing on the quadrature operators $\hat{x}_1$ and $\hat{x}_2$, obtained from $\hat{x}_\theta$ with $\theta = 0$ and $\theta = \pi/2$, respectively, we have:

(44) $$\Delta^2(\hat{x}_1) = \mathrm{e}^{2r} \quad \text{and} \quad \Delta^2(\hat{x}_2) = \mathrm{e}^{-2r},$$

namely, the variance of the quadrature $\hat{x}_2$ is below the vacuum level if $r > 0$. It is also worth noting that $\Delta^2(\hat{x}_1)\,\Delta^2(\hat{x}_2) = 1$ that is the squeezed vacuum is a minimum uncertainty state, but with different quadrature variances.

□ **Exercise 8.** *Prove that the displaced squeezed state $\hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ is still a minimum uncertainty state.* ■

7. Two-mode squeezing

The two-mode counterpart of the single-mode squeezing corresponds to Hamiltonians of the form $\hat{H} \propto \hat{a}^\dagger \hat{b}^\dagger + \hat{a}\hat{b}$ and the evolution operator reads:

(45) $$\hat{S}_2(\xi) = \exp\left(\xi \hat{a}^\dagger \hat{b}^\dagger - \xi^* \hat{a}\hat{b}\right),$$

where $\xi = r\,\mathrm{e}^{i\psi}$. If we define the operators $\hat{K}_+ = \hat{a}^\dagger \hat{b}^\dagger$, $\hat{K}_- = \hat{a}\hat{b}$ and $\hat{K}_3 = -\frac{1}{2}[\hat{K}_+, \hat{K}_-] = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1)$, we obtain a two-boson realization of $\mathrm{SU}(1,1)$ algebra. As in the case of single-mode squeezing, we can write:

(46) $$\hat{S}_2(\xi) = \exp\left(\frac{\nu}{\mu}\hat{a}^\dagger \hat{b}^\dagger\right) \mu^{-(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1)} \exp\left(-\frac{\nu^*}{\mu}\hat{a}\hat{b}\right),$$

where $\mu = \cosh r$ and $\nu = \mathrm{e}^{i\psi}\sinh r$. The evolution of $\hat{a}$ and $\hat{b}$ under the action of $\hat{S}_2(\xi)$ leads to:

(47) $$\hat{S}_2^\dagger(\xi)\,\hat{a}\,\hat{S}_2(\xi) = \mu\hat{a} + \nu\hat{b}^\dagger \quad \text{and} \quad \hat{S}_2^\dagger(\xi)\,\hat{b}\,\hat{S}_2(\xi) = \mu\hat{b} + \nu^*\hat{a}^\dagger.$$

□ **Example 7. (Two-mode squeezed vacuum or twin-beam state)** *The reader can easily check that:*

(48) $$\hat{S}_2(\xi)|0\rangle = \frac{1}{\sqrt{\mu}} \sum_{n=0}^{\infty} \left(\frac{\nu}{\mu}\right)^n |n\rangle|n\rangle,$$

or, if we introduce the parameter $\lambda = e^{i\psi}\tanh r$:

(49) $$\hat{S}_2(\xi)|0\rangle = \sqrt{1 - |\lambda|^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle|n\rangle,$$

that is the two-mode squeezed vacuum or twin-beam state, since a measurement of the photon number on the two beams always leads to the same result. If we introduce $N = \sinh^2|\xi|$ *the TWB mean number of photons is* $\langle\hat{n}\rangle = 2N$ *and*

$$|\lambda|^2 = \frac{N}{N+1}.$$

■

It is worth noting that the twin-beam state (48) is a continuous-variable *maximally entangled state*. In fact, in the presence of a pure states of two subsystems $\hat{\varrho}_{\rm AB}$, entanglement can be quantified by the excess von Neumann entropy, namely, $\mathcal{E}(\hat{\varrho}_{\rm AB}) = \mathcal{S}(\hat{\varrho}_{\rm A}) + \mathcal{S}(\hat{\varrho}_{\rm A}) - \mathcal{S}(\hat{\varrho}_{\rm AB})$, where $\hat{\varrho}_{\rm A} = {\rm Tr}_{\rm B}[\hat{\varrho}_{\rm AB}]$ and $\hat{\varrho}_{\rm B} = {\rm Tr}_{\rm A}[\hat{\varrho}_{\rm AB}]$ are the density operators of the two subsystems and $\mathcal{S}(\hat{\varrho}) = -{\rm Tr}[\hat{\varrho}\log\hat{\varrho}]$ is the von Neumann entropy. In the case of the TWB (48), the two subsystems are described by two thermal states with the same mean photon number $N = \sinh^2|\xi|$. Since a thermal state maximizes the von Neumann entropy for a fixed energy N, the excess von Neumann entropy reaches its maximum: the TWB is maximally entangled.

□ **Exercise 9.** *Given the TWB* $\hat{S}_2(\xi)|0\rangle$ *explicitly write the expression of the density operators of the two subsystems and calculate the excess von Neumann entropy.* ■

□ **Exercise 10.** *Given the two squeezed states* $\hat{S}_{\hat{a}}(-r)|0\rangle$ *and* $\hat{S}_{\hat{b}}(r)|0\rangle$ *of modes* $\hat{a}$ *and* $\hat{b}$*, respectively, calculate the two-mode state obtained after their interference through a balanced BS. Assume* $r \in \mathbb{R}$ *and* $r > 0$. ■

8. The Fano factor

In order to classify the photon number distribution of an optical state, it is somehow convenient to use its width relative to a Poisson distribution. The Fano factor is defined as:

$$F = \frac{\Delta^2(\hat{n})}{\langle\hat{n}\rangle}, \qquad (50)$$

that is just the ration between the actual width of the photon number distribution and the width of a Poissonian with the same mean photon number. If $F > 1$ we are in the presence of *super-Poissonian* states; if $F < 1$ we have *sub-Poissonian* states. A typical example of sub-Poissonian state is given by the Fock states, for which we find $F = 0$. A coherent state has $F = 1$. The thermal state with $N_{\rm th}$ mean photons is super-Poissonian, since $F = N_{\rm th} + 1$. Also squeezed states are super-Poissonian and we have $F = 2(\langle\hat{n}\rangle + 1)$: therefore given a thermal state and a squeezed state with the same mean number of photons, the squeezed state has a Fano factor that is twice the thermal states one.

As a matter of fact, different quantum states may have the same Fano factor: both the coherent state $|\alpha\rangle$ and the mixed state $\hat{\varrho} = {\rm e}^{-|\alpha|^2}\sum_n (n!)^{-1}|\alpha|^{2n}|n\rangle\langle n|$ have the same Fano factor $F = 1$.

9. Operator ordering and number operator

We have three possible creation and annihilation ordering:

- *normal ordering* (creation operators on the left): $\left(\hat{a}^\dagger\right)^n \hat{a}^m$;
- *antinormal ordering* (creation operators on the right): $\hat{a}^m \left(\hat{a}^\dagger\right)^n$;
- *symmetric ordering* (balanced sum of all the possible combinations): for example, $[\hat{a}^\dagger \hat{a}]_S = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)$.

Depending on the problem and/or the system under investigation, one ordering could be preferred to the others. Given a function $f(\hat{a}, \hat{a}^\dagger)$ of the annihilation and creation operator, we use the notation $:f(\hat{a}, \hat{a}^\dagger):$ for the normal ordering, $\vdots f(\hat{a}, \hat{a}^\dagger) \vdots$ for the antinormal ordering and $[f(\hat{a}, \hat{a}^\dagger)]_S$ for the symmetric ordering. Is this section we focus only on the number operator $\hat{n}$, thus we have:

- $:\hat{n}: = \hat{a}^\dagger \hat{a}$;
- $\vdots \hat{n} \vdots = \hat{a}\hat{a}^\dagger$;
- $[\hat{a}^\dagger \hat{a}]_S = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) = \hat{a}^\dagger \hat{a} + \frac{1}{2}$.

▫ **Example 8.** *In this example we first consider a coherent state $|\alpha\rangle$. We find:*

$$\langle\alpha| :\hat{n}: |\alpha\rangle = |\alpha|^2, \quad \langle\alpha|\vdots \hat{n} \vdots|\alpha\rangle = |\alpha|^2 + 1, \quad \textit{and} \quad \langle\alpha|[\hat{n}]_s|\alpha\rangle = |\alpha|^2 + \tfrac{1}{2}.$$

The reader can easily check that:

$$\langle\alpha| :\hat{n}^k: |\alpha\rangle = |\alpha|^{2k},$$

which implies that for a coherent state the normal-ordered variance vanishes, $:\Delta^2(\hat{n}): = 0$. From this result follows that:

- $:\Delta^2(\hat{n}): = 0 \quad \Rightarrow$ *Poissonian;*
- $:\Delta^2(\hat{n}): > 0 \quad \Rightarrow$ *super-Poissonian;*
- $:\Delta^2(\hat{n}): < 0 \quad \Rightarrow$ *sub-Poissonian.*

In particular, in the presence of a thermal state $\hat{\varrho}_{th}(N_{th})$ we have $:\Delta^2(\hat{n}): = k!\, N_{th}^k$ and, thus, $:\Delta^2(\hat{n}): = N_{th}^2$. ■

▫ **Exercise 11.** *Calculate the normal-, anti-normal- and symmetric-ordered photon number variances in the case of a Fock state $|n\rangle$.* ■

10. Photon number statistics and moment generating function

In this section we address a useful method for studying the photon number statistics $p(n) = \langle n|\hat{\varrho}|n\rangle$ of a state $\hat{\varrho}$. The moment generating function is defined as:

$$M(\mu) = \sum_{n=0}^{\infty} (1-\mu)^n p(n), \quad (0 \leqslant \mu \leqslant 2). \tag{51}$$

Note that $M(0) = \sum_{n=0}^{\infty} p(n) = 1$, whereas:

$$\frac{(-1)^n}{n!} \frac{d^n}{d\mu^n} M(\mu)\bigg|_{\mu=1} = p(n).$$

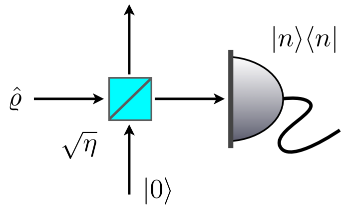

FIGURE 3. A photon number resolving detector with quantum efficiency η can be represented as an ideal photodetector and a BS with transmissivity η in front of it.

We also note that $M(2) = \sum_n p(2n) - \sum_n p(2n+1)$ corresponds to the difference between the probability that the photon number is even and the probability that it is odd. Furthermore, given $M(\mu)$ we can calculate the m-th moment of $p(n)$ as:

(52) $$\left[(\mu-1)\frac{d}{d\mu}\right]^m M(\mu)\Big|_{\mu=0} = \sum_{n=0}^{\infty} n^m p(n) \equiv \langle \hat{n}^m \rangle.$$

In the case of a coherent state $|\alpha\rangle$ we obtain the following expression for the moment generating function:

(53) $$|\alpha\rangle \quad \Rightarrow \quad M(\mu) = \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty} (1-\mu)^n \frac{|\alpha|^{2n}}{n!} = \mathrm{e}^{-\mu|\alpha|^2}.$$

In the presence of a thermal state we have:

(54) $$\hat{\varrho}_{\text{th}}(N_{\text{th}}) \quad \Rightarrow \quad M(\mu) = \frac{1}{1+N_{\text{th}}} \sum_{n=0}^{\infty} (1-\mu)^n \left(\frac{N_{\text{th}}}{1+N_{\text{th}}}\right)^n = \frac{1}{1+\mu N_{\text{th}}}.$$

It is interesting to note that in both the cases of the coherent state and of the thermal state the parameter μ of the moment generating function multiplies the mean energy, that is $|\alpha|^2$ or N_{th}, respectively. This is not the case if we consider, for example, a Fock state:

(55) $$|n\rangle \quad \Rightarrow \quad M(\mu) = (1-\mu)^n,$$

or the squeezed vacuum state:

(56) $$\hat{S}(\xi)|0\rangle \quad \Rightarrow \quad M(\mu) = \frac{1}{\sqrt{1+2\mu N - \mu^2 N}},$$

with $N = \sinh^2|\xi|$. In this last case we can easily see that $M(2) = 1$: this directly follows from the photon number statistics of the squeezed vacuum that involves only the even terms (see Example 6).

10.1. **Bernoulli sampling from non-unit efficiency photodetection.** A photon number resolving detector allows to directly measure the photon number distribution $p(n) = \langle n|\hat{\varrho}|n\rangle$ of an input state $\hat{\varrho}$ and is described by the projectors $|n\rangle\langle n|$ onto the photon number basis. However, a realistic detector has a non-unit quantum efficiency η, that can be seen as an overall loss of photons during the detection process. From the theoretical point of view, a real photodetector can be modeled as a BS with transmissivity η and an ideal photon number resolving detector, as sketched in Fig. 3. As a matter of fact, if we send a single photon state $|1\rangle$ to the realistic detector, η corresponds to the probability to detect it. What happens when we send a Fock state $|l\rangle$, $l > 1$? Let's focus on Fig. 3: the two-mode state just after the BS is:

$$\begin{aligned}\hat{U}_{\text{BS}}|l\rangle|0\rangle &= \frac{1}{\sqrt{l!}}\left(\sqrt{\eta}\,\hat{a}^{\dagger} - \sqrt{1-\eta}\,\hat{b}^{\dagger}\right)^{l}|0\rangle \\ &= \frac{1}{\sqrt{l!}}\sum_{k=0}^{l}\binom{l}{k}(-1)^{k}\sqrt{\eta^{l-k}(1-\eta)^{k}}\,(\hat{a}^{\dagger})^{l-k}(\hat{b}^{\dagger})^{k}|0\rangle \\ &= \sum_{k=0}^{l}(-1)^{k}\sqrt{\binom{l}{k}\eta^{l-k}(1-\eta)^{k}}\,|l-k\rangle|k\rangle \\ &= \sum_{m=0}^{l}(-1)^{l-m}\sqrt{\binom{l}{m}\eta^{m}(1-\eta)^{l-m}}\,|m\rangle|l-m\rangle.\end{aligned}$$

Therefore, the probability $P(m;\eta)$ to detect m photons, $m \leqslant l$, is given by:

$$P(m;\eta) = \binom{l}{m}\eta^{m}(1-\eta)^{l-m}. \tag{57}$$

Of course, if $\eta \to 1$ we have $P(m;\eta) \to p(m) = \delta_{l,m}$.

It is now clear that if we know the *actual* photon number statistics $p(m)$ of the state $\hat{\varrho}$, then the *detected* photon number statistics is:

$$P(m;\eta) = \sum_{l=0}^{\infty}\binom{l}{m}\eta^{m}(1-\eta)^{l-m}p(l), \tag{58}$$

and the corresponding moment generating function is:

$$M(\mu;\eta) = \sum_{m=0}^{\infty}(1-\mu)^{m}P(m;\eta) \tag{59}$$

$$= \sum_{l=0}^{\infty}(1-\eta\mu)^{l}p(l) \equiv M(\eta\mu), \tag{60}$$

that is the moment generating function of the actual photon distribution but with the substitution $\mu \to \eta\mu$.

□ **Exercise 12.** *Apply the result of Eq. (60) to the case of a coherent, thermal and squeezed state. What are the main differences? What is the physical insight?* ■

In Fig. 4 we show the effect of a non unit quantum efficiency on the photon number distribution of a coherent, a thermal and a squeezed vacuum state.

11. Characteristic functions

The use of characteristic functions or quasi-probability distributions allows to obtain a more complete statistical description of the field. They contain all the information necessary to reconstruct the density matrix of the state. The p-ordered characteristic function associated with the state $\hat{\varrho}$ is defined as:

$$\chi(\lambda, p) = \text{Tr}[\hat{\varrho}\hat{D}(\lambda)]\,e^{p|\lambda|^{2}/2}, \tag{61}$$

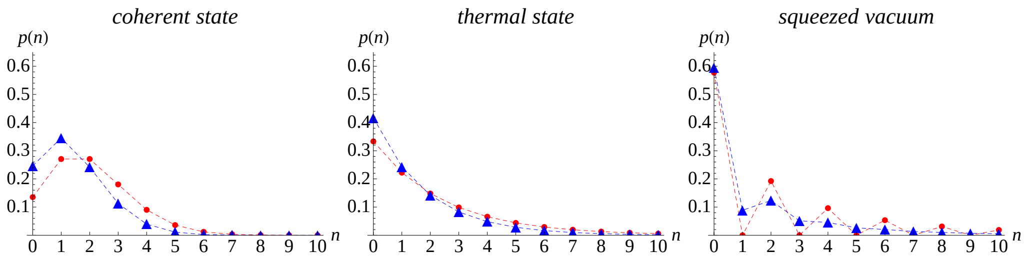

FIGURE 4. Photon number distribution of a thermal coherent state, thermal state and squeezed vacuum state in the ideal case ($\eta = 1$, circles) and in the presence of a non unit quantum efficiency ($\eta = 0.7$, triangles). In all the cases we set $\langle \hat{n} \rangle = 2.0$.

where $\hat{D}(\lambda) = e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}$ is the displacement operator. We recall that we can rewrite $\hat{D}(\lambda)$ also in the following three forms:

(62a) $$\hat{D}(\lambda) = e^{\lambda \hat{a}^\dagger}\, e^{-\lambda^* \hat{a}}\, e^{-|\lambda|^2/2},$$

(62b) $$\hat{D}(\lambda) = e^{-\lambda^* \hat{a}}\, e^{\lambda \hat{a}^\dagger}\, e^{|\lambda|^2/2},$$

(62c) $$\hat{D}(\lambda) = \sum_{n,m=0}^{\infty} \frac{\lambda^m (-\lambda^*)^n}{m!n!} \left[(\hat{a}^\dagger)^m \hat{a}^n \right]_s .$$

Therefore, according to the value of p, we may have the normal ordered characteristic function with $p = 1$:

(63) $$\chi(\lambda, 1) = \mathrm{Tr}[\hat{\varrho}\, e^{\lambda \hat{a}^\dagger}\, e^{-\lambda^* \hat{a}}], \quad \text{(normal ordered)}$$

the antinormal ordered characteristic function with $p = -1$:

(64) $$\chi(\lambda, -1) = \mathrm{Tr}[\hat{\varrho}\, e^{-\lambda^* \hat{a}}\, e^{\lambda \hat{a}^\dagger}], \quad \text{(antinormal ordered)}$$

and the symmetric ordered characteristic function with $p = 0$:

(65) $$\chi(\lambda, 0) = \mathrm{Tr}[\hat{\varrho}\, e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}], \quad \text{(symmetric ordered).}$$

The last form is simply referred to as characteristic function and one writes usually $\chi(\lambda) = \chi(\lambda, 0)$.

Starting from $\chi(\lambda, p)$ we can evaluate any p-ordered expectation value of a function of $\hat{a}^\dagger$ and $\hat{a}$, since:

(66) $$\left(\frac{\partial}{\partial \lambda} \right)^m \left(-\frac{\partial}{\partial \lambda^*} \right)^n \chi(\lambda, p) \Big|_{\lambda=0} = \langle (\hat{a}^\dagger)^m\, \hat{a}^n \rangle_p ,$$

where

$$\langle (\hat{a}^\dagger)^m\, \hat{a}^n \rangle_1 = \langle :(\hat{a}^\dagger)^m\, \hat{a}^n: \rangle, \quad \langle (\hat{a}^\dagger)^m\, \hat{a}^n \rangle_{-1} = \langle \vdots(\hat{a}^\dagger)^m\, \hat{a}^n\vdots \rangle,$$

and

$$\langle (\hat{a}^\dagger)^m\, \hat{a}^n \rangle_0 = \langle \left[(\hat{a}^\dagger)^m\, \hat{a}^n \right]_s \rangle.$$

We report also the so-called "Glauber formula" that allows to connect the density operator $\hat{\varrho}$ to its characteristic function, namely:

(67) $$\hat{\varrho} = \frac{1}{\pi} \int_{\mathbb{C}} d^2\lambda\, \chi(\lambda)\, \hat{D}^\dagger(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} d^2\lambda\, \mathrm{Tr}[\hat{\varrho}\, \hat{D}(\lambda)]\, \hat{D}^\dagger(\lambda) .$$

□ **Example 9.** *The p-ordered characteristic function of a Fock state $|n\rangle$ is:*

$$|n\rangle \to \chi(\lambda, p) = e^{(p-1)|\lambda|^2/2} L_n(|\lambda|^2)$$

where:

$$L_n(z) = \sum_{m=0}^{\infty} \binom{n}{m} \frac{(-z)^m}{m!}$$

are Laguerre polynomials.

□ **Example 10.** *From the previous example follows that the p-ordered characteristic function of a thermal state $\hat{\varrho}_{th}(N_{th})$ is given by:*

$$\hat{\varrho}_{th}(N_{th}) \to \chi_{th}(\lambda, p) = \exp\left[-\frac{1}{2}(1 + 2N_{th} - p)|\lambda|^2\right].$$

Indeed, if $N_{th} \to 0$ one has the p-ordered characteristic function of the vacuum state, namely:

$$|0\rangle \to \chi_{vac}(\lambda, p) = \exp\left[-\frac{1}{2}(1 - p)|\lambda|^2\right].$$

It is worth noting that the both $\chi_{th}(\lambda, p)$ and $\chi_{vac}(\lambda, p)$ are Gaussian*: the thermal and the vacuum state belong to the class of the so-called Gaussian states, that are states with Gaussian characteristic (or Wigner, as we will see) functions.*

Due to the very definition of the characteristic function, given $\chi(\lambda)$ of the state $\hat{\varrho}$, the characteristic function $\chi'(\lambda)$ of the state $\hat{U}\hat{\varrho}\hat{U}^\dagger$, where $\hat{U}$ is a unitary transformation, is given by:

$$\chi'(\lambda) = \mathrm{Tr}[\hat{U}\hat{\varrho}\hat{U}^\dagger\, \hat{D}(\lambda)] = \mathrm{Tr}[\hat{\varrho}\, \hat{U}^\dagger \hat{D}(\lambda)\hat{U}]. \tag{68}$$

In the case of a unitary transformation leading to a linear transformation of the field operators $\hat{a}$ and $\hat{a}^\dagger$, as in the case of the displacement or squeezing operators, it is straightforward to calculate $\chi'(\lambda)$. In fact, by using Eqs. (37) and (43), respectively, we have:

$$\begin{aligned} \hat{D}(\alpha)\hat{\varrho}\hat{D}^\dagger(\alpha) \to \chi'(\lambda) &= \mathrm{Tr}[\hat{\varrho}\, \hat{D}^\dagger(\alpha)\hat{D}(\lambda)\hat{D}(\alpha)] \\ &= \mathrm{e}^{\lambda\alpha^* - \lambda^*\alpha}\, \mathrm{Tr}[\hat{\varrho}\, \hat{D}(\lambda)] \\ &= \mathrm{e}^{\lambda\alpha^* - \lambda^*\alpha}\, \chi(\lambda). \end{aligned} \tag{69}$$

and (for the sake of simplicity we assume $r \in \mathbb{R}$):

$$\begin{aligned} \hat{S}(r)\hat{\varrho}\hat{S}^\dagger(r) \to \chi'(\lambda) &= \mathrm{Tr}[\hat{\varrho}\, \hat{S}^\dagger(r)\hat{D}(\lambda)\hat{S}(r)] \\ &= \mathrm{Tr}[\hat{\varrho}\, \hat{D}(\lambda \cosh r - \lambda^* \sinh r)] \\ &= \chi(\lambda \cosh r - \lambda^* \sinh r). \end{aligned} \tag{70}$$

11.1. **Trace rule for the characteristic functions.** Given two operators $\hat{A}$ and $\hat{B}$ and the corresponding characteristic functions $\chi_{\hat{A}}(\lambda)$ and $\chi_{\hat{B}}(\lambda)$, we have:

$$\mathrm{Tr}[\hat{A}\hat{B}] = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\lambda\, \chi_{\hat{A}}(\lambda)\, \chi_{\hat{B}}(-\lambda). \tag{71}$$

12. QUASI-PROBABILITY DISTRIBUTIONS

An alternative to the characteristic functions is given by the quasi-probability distributions, which are similar to phase-space distributions. A quasi-probability is a real-valued (though it could be negative), normalized function and the moments of products of $\hat{a}^\dagger$ and $\hat{a}$ are calculated evaluating suitable integrals as in the case of an actual probability distribution. The p-ordered quasi-probability distribution $W(\alpha, p)$ can be defined as the two-dimensional Fourier transform of the corresponding p-ordered characteristic function:

$$W(\alpha, p) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda\, \chi(\lambda, p)\, e^{\alpha\lambda^* - \alpha^*\lambda}. \tag{72}$$

We also recall that:

$$\frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda\, e^{\alpha\lambda^* - \alpha^*\lambda} = \delta^{(2)}(\alpha), \tag{73}$$

where $\delta^{(2)}(\alpha)$ is the two-dimensional Dirac's delta function. From the definition (72) follows that $W(\alpha, p)$ is normalized:

$$\int_{\mathbb{C}} d^2\alpha\, W(\alpha, p) = \int_{\mathbb{C}} d^2\lambda\, \chi(\lambda, p) \underbrace{\frac{1}{\pi^2} \int_{\mathbb{C}} d^2\alpha\, e^{\alpha\lambda^* - \alpha^*\lambda}}_{\delta^{(2)}(\lambda)} = \chi(0, p) = \mathrm{Tr}[\hat{\varrho}] = 1.$$

Furthermore, we have:

$$\int_{\mathbb{C}} d^2\alpha\, W(\alpha, p)\, (\alpha^*)^m \alpha^n = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda \int_{\mathbb{C}} d^2\alpha\, \chi(\lambda, p)\, (\alpha^*)^m \alpha^n\, e^{\alpha\lambda^* - \alpha^*\lambda},$$

but:

$$(\alpha^*)^m \alpha^n\, e^{\alpha\lambda^* - \alpha^*\lambda} = \left(-\frac{\partial}{\partial\lambda}\right)^m \left(\frac{\partial}{\partial\lambda^*}\right)^n e^{\alpha\lambda^* - \alpha^*\lambda},$$

thus, we have:

$$\begin{aligned}
\int_{\mathbb{C}} d^2\alpha\, W(\alpha, p)\, (\alpha^*)^m \alpha^n &= \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda\, \chi(\lambda, p) \left(-\frac{\partial}{\partial\lambda}\right)^m \left(\frac{\partial}{\partial\lambda^*}\right)^n \int_{\mathbb{C}} d^2\alpha\, e^{\alpha\lambda^* - \alpha^*\lambda}, \\
&= \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\lambda\, \chi(\lambda, p) \left(-\frac{\partial}{\partial\lambda}\right)^m \left(\frac{\partial}{\partial\lambda^*}\right)^n \delta^{(2)}(\lambda) \\
&= \quad \vdots \quad \textit{(integration by parts)} \quad \vdots \\
&= \left(\frac{\partial}{\partial\lambda}\right)^m \left(-\frac{\partial}{\partial\lambda^*}\right)^n \chi(\lambda, p)\Big|_{\lambda=0} = \langle (\hat{a}^\dagger)^m\, \hat{a}^n \rangle_p
\end{aligned}$$

as in the case of Eq. (66).

Given $q < p$, we have $\chi(\lambda, q) = \chi(\lambda, p)\,\mathrm{e}^{-(p-q)|\lambda|^2/2}$, and we obtain the following relation between $W(\alpha, q)$ and $W(\alpha, p)$:

$$W(\alpha, q) = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\lambda\,\chi(\lambda, p)\,\mathrm{e}^{-(p-q)|\lambda|^2/2}\,\mathrm{e}^{\alpha\lambda^*-\alpha^*\lambda}$$

$$= \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\lambda\,\underbrace{\int_{\mathbb{C}} \mathrm{d}^2\beta\,W(\beta, p)\,\mathrm{e}^{\lambda^*\beta-\lambda\beta^*}}_{\chi(\lambda,p)}\,\mathrm{e}^{-(p-q)|\lambda|^2/2}\,\mathrm{e}^{\alpha\lambda^*-\alpha^*\lambda}$$

$$= \frac{2}{\pi(p-q)}\int_{\mathbb{C}} \mathrm{d}^2\beta\,W(\beta, p)\,\exp\left[-\frac{2|\alpha-\beta|^2}{(p-q)}\right], \quad (q<p). \tag{74}$$

Equation (74) allows to pass from one ordering to the other. Note that as q decreases the peaks in the quasi-probability distribution become broader.

We have seen that there is a strict relation between the characteristic functions and the quasi-probability distributions. However, $W(\alpha, p)$ can be obtained directly from the density operator $\hat{\varrho}$, without passing through the characteristic function formalism, namely:

$$W(\alpha, p) = \mathrm{Tr}[\hat{\varrho}\,\hat{D}(\alpha)\,\hat{T}(p)\,\hat{D}^\dagger(\alpha)], \quad (p<1) \tag{75}$$

where:

$$\hat{T}(p) = \frac{2}{\pi(1-p)}\ :\!\exp\left(-\frac{2}{1-p}\,\hat{a}^\dagger\hat{a}\right)\!:, \tag{76}$$

therefore we have:

$$W(\alpha, p) = \frac{2}{\pi(1-p)}\sum_{n=0}^{\infty}\left(-\frac{1+p}{1-p}\right)^n \langle n|\hat{D}^\dagger(\alpha)\,\hat{\varrho}\,\hat{D}(\alpha)|n\rangle. \tag{77}$$

It is worth noting that $\langle n|\hat{D}^\dagger(\alpha)\,\hat{\varrho}\,\hat{D}(\alpha)|n\rangle$ is the photon number distribution of the state $\hat{\varrho}$ after a displacement of amount $-\alpha$.

12.1. **Glauber-Sudarshan *P*-representation.** The so-called P-representation allows to write a density operator as:

$$\hat{\varrho} = \int_{\mathbb{C}} \mathrm{d}^2\alpha\,P(\alpha)\,|\alpha\rangle\langle\alpha|, \tag{78}$$

where $|\alpha\rangle$ are coherent states and $P(\alpha)$ is called P-function. If we put $p = 1$ into Eq. (72) we obtain:

$$W(\alpha, 1) = P(\alpha). \tag{79}$$

12.2. **The Wigner function.** The Wigner function $W(\alpha)$, introduced by E. P. Wigner, is given by Eq. (72) with $p = 0$, namely, $W(\alpha, 0) = W(\alpha)$. Furthermore, we can write:

$$W(\alpha) = \mathrm{Tr}[\hat{\varrho}\,\hat{D}(\alpha)\,\hat{\Pi}(p)\,\hat{D}^\dagger(\alpha)], \tag{80}$$

where we introduced the parity operator $\hat{\Pi} = (-1)^{\hat{a}^\dagger\hat{a}}$. Note that $\hat{D}(\alpha)\,\hat{\Pi}(p)\,\hat{D}^\dagger(\alpha) = \hat{D}(2\alpha)\,\hat{\Pi}(p) = \hat{\Pi}(p)\,\hat{D}^\dagger(2\alpha)$

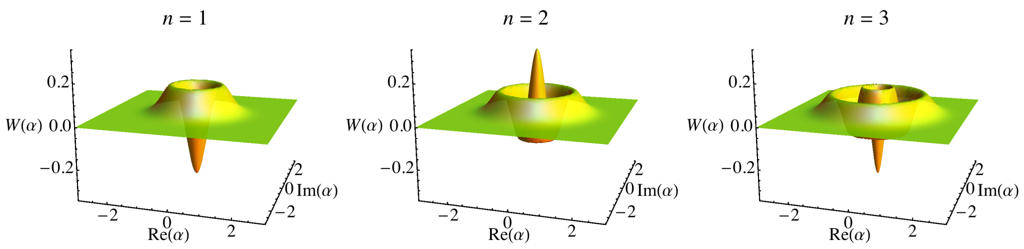

FIGURE 5. Wigner function of the Fock state $|n\rangle$ for three values of n. Note that Wigner functions may have negative values: this is a sufficient (but not necessary!) condition for nonclassicality of a state.

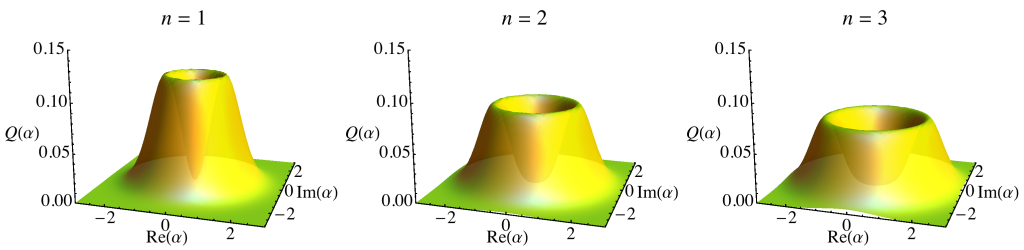

FIGURE 6. Q-function of the Fock state $|n\rangle$ for three values of n. Note that, with respect to the Wigner functions in Fig. 5, the corresponding Q-functions are always positive.

12.3. **The Husimi or Q-function.** The Q-function follows from Eq. (72) when $p = -1$:

$$Q(\alpha) = W(\alpha, 0) = \frac{1}{\pi}\langle\alpha|\hat{\varrho}|\alpha\rangle. \tag{81}$$

Note that in this case we really have a probability distribution, since $Q(\alpha) \geqslant 0$ and normalized.

□ **Example 11.** *If we consider a Fock state $|n\rangle$ we have:*

$$W(\alpha, 1) = P(\alpha) = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{m!} \left(\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right)^m \delta^{(2)}(\alpha),$$

$$W(\alpha, 0) = W(\alpha) = \frac{1}{\pi}(-1)^n e^{-2|\alpha|^2} L_n(4|\alpha|^2),$$

$$W(\alpha, -1) = Q(\alpha) = \frac{1}{\pi} e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}.$$

It is clear that decreasing the value of p from 1 to -1 the corresponding $W(\alpha, p)$ becomes more and more regular. In Figs. 5 and 6 we plot the Wigner function and the Q-function, respectively, of a Fock state $|n\rangle$ for different values of n. ■

As in the case of the characteristic function, given the Wigner function $W(\alpha)$ of the state $\hat{\varrho}$ we have:

$$\hat{D}(\beta)\,\hat{\varrho}\,\hat{D}^\dagger(\beta) \to W(\alpha - \beta), \quad \text{and} \quad \hat{S}(r)\,\hat{\varrho}\,\hat{S}^\dagger(r) \to W(\alpha\cosh r - \alpha^*\sinh r). \tag{82}$$

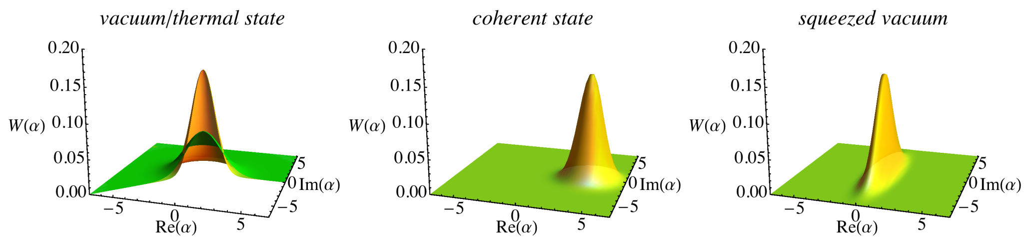

FIGURE 7. Wigner functions of the vacuum state (left plot, yellow) and of a thermal state (left plot, green), of a coherent state (center) and of a squeezed vacuum state (right). Note that, though the squeezed vacuum state is nonclassical, its Wigner function if positive definite (it is a Gaussian function).

In Fig. 7 we plot the Wigner functions of the vacuum, the thermal, the coherent and of the squeezed vacuum states, respectively.

12.4. **Trace rule for the Wigner functions.** Given two operators $\hat{A}$ and $\hat{B}$ and the corresponding Wigner functions $W_{\hat{A}}(\alpha)$ and $W_{\hat{B}}(\alpha)$, we have:

$$\mathrm{Tr}[\hat{A}\hat{B}] = \pi \int_{\mathbb{C}} \mathrm{d}^2\alpha\, W_{\hat{A}}(\alpha)\, W_{\hat{B}}(\alpha). \tag{83}$$

13. WIGNER FUNCTION AND QUADRATURES

If we introduce the quadrature $\hat{x} = \hat{a} + \hat{a}^\dagger$ with eigenvectors $|x\rangle$, namely, $\hat{x}|x\rangle = x|x\rangle$, the generic quadrature $\hat{x}_\theta$ can be written as:

$$\hat{x}_\theta = \hat{a}\,\mathrm{e}^{-i\theta} + \hat{a}^\dagger\,\mathrm{e}^{i\theta} \equiv \hat{U}_\theta^\dagger\,\hat{x}\,\hat{U}_\theta, \tag{84}$$

where $\hat{U}_\theta = \exp(-i\theta\,\hat{a}^\dagger\hat{a})$ is a phase-shift operator. If $\hat{x}_\theta|x\rangle_\theta = x_\theta|x\rangle_\theta$, then we have:

$$\underbrace{\hat{U}_\theta^\dagger\,\hat{x}\,\hat{U}_\theta}_{\hat{x}_\theta}\,|x\rangle_\theta = x_\theta|x\rangle_\theta,$$

that is:

$$\hat{x}\,\underbrace{\hat{U}_\theta|x\rangle_\theta}_{|x_\theta\rangle} = x_\theta\,\underbrace{\hat{U}_\theta|x\rangle_\theta}_{|x_\theta\rangle}, \quad \Rightarrow \quad \hat{x}|x_\theta\rangle = x_\theta|x_\theta\rangle,$$

that is the state $|x_\theta\rangle = \hat{U}_\theta|x\rangle_\theta$ is eigenstate of $\hat{x}$ with eigenvalue x_θ. Therefore, the probability $p(x;\theta)$ to obtain an outcome x measuring the quadrature $\hat{x}_\theta$ given the state $\hat{\varrho}$ can be written as:

$$p(x;\theta) = \langle x|\hat{U}_\theta\,\hat{\varrho}\,\hat{U}_\theta^\dagger|x\rangle, \tag{85}$$

since $\hat{U}_\theta^\dagger|x\rangle$ is eigenstate of $\hat{x}_\theta$ with eigenvalue x.

In this section we will show how the Wigner function $W(\alpha)$ of the state $\hat{\varrho}$ is related to the probability $p(x;\theta)$. First of all we should rewrite the characteristic function $\chi(\lambda)$ of $\hat{\varrho}$ in cartesian notation, that is, we write $\lambda = u + iv$, where, $u, v \in \mathbb{R}$. We have:

$$\chi(\lambda) = \mathrm{Tr}[\hat{\varrho}\,\mathrm{e}^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}] = \mathrm{Tr}[\hat{\varrho}\,\mathrm{e}^{i(v\,\hat{x} - u\,\hat{y})}],$$

where $\hat{y} = i(\hat{a}^\dagger - \hat{a})$. Now, using Theorem 1 and evaluating the trace on the basis $\{|x\rangle\}$ of the eigenstates of $\hat{x}$ we have:

$$\mathrm{Tr}[\hat{\varrho}\, \mathrm{e}^{iv\,\hat{x}}\, \mathrm{e}^{-iu\,\hat{y}}]\, \mathrm{e}^{-iuv} = \int_{\mathbb{R}} \mathrm{d}x\, \langle x|\hat{\varrho}\, \mathrm{e}^{iv\,\hat{x}}\, \mathrm{e}^{-iu\,\hat{y}}|x\rangle\, \mathrm{e}^{-iuv}.$$

Since $[\hat{x}, \hat{y}] = 2i$, we have $\mathrm{e}^{-iu\,\hat{y}}|x\rangle = |x + 2u\rangle$, thus we obtain:

$$\int_{\mathbb{R}} \mathrm{d}x\, \langle x|\hat{\varrho}\, \mathrm{e}^{iv\,\hat{x}}|x + 2u\rangle\, \mathrm{e}^{-iuv} = \int_{\mathbb{R}} \mathrm{d}q\, \langle q - u|\hat{\varrho}|q + u\rangle\, \mathrm{e}^{ivq},$$

where we used the change of variable $x = q - u$. Summarizing, we can write the characteristic function as:

$$\chi(u, v) = \int_{\mathbb{R}} \mathrm{d}q\, \langle q - u|\hat{\varrho}|q + u\rangle\, \mathrm{e}^{ivq}. \tag{86}$$

The corresponding Wigner function is given by Eq. (72). In order to write also $W(\alpha)$ in cartesian notation we should write $\alpha = (x + iy)/2$, where the factor $1/2$ is due to the definition of the quadrature operator. Since $\alpha\lambda^* - \alpha^*\lambda = i(uy - vx)$ we have:

$$\begin{aligned} W(x, y) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}v\, \chi(u, v)\, \mathrm{e}^{i(uy - vx)} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}q\, \langle q - u|\hat{\varrho}|q + u\rangle\, \mathrm{e}^{iuy}\, \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}v\, \mathrm{e}^{iv(q - x)}}_{\delta^{(2)}(q - x)}. \end{aligned} \tag{87}$$

Therefore, after the integration over q, we obtain the original definition of the Wigner function:

$$W(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}u\, \langle x - u|\hat{\varrho}|x + u\rangle\, \mathrm{e}^{iuy}. \tag{88}$$

It is now straightforward to see that:

$$p(x) = \int_{\mathbb{R}} \mathrm{d}y\, W(x, y) \equiv \langle x|\hat{\varrho}|x\rangle, \tag{89}$$

or, more in general, since the Wigner associated with the state $\hat{U}_\theta\, \hat{\varrho}\, \hat{U}_\theta^\dagger$ is:

$$\hat{U}_\theta\, \hat{\varrho}\, \hat{U}_\theta^\dagger \to W(x \cos\theta - y \sin\theta, y \cos\theta + x \sin\theta), \tag{90}$$

we obtain that the probability distribution of the outcomes of the generic quadrature $\hat{x}_\theta$ is given by the following marginal of the suitably transformed Wigner function:

$$p(x; \theta) = \int_{\mathbb{R}} \mathrm{d}y\, W(x \cos\theta - y \sin\theta, y \cos\theta + x \sin\theta) \equiv \langle x|\hat{U}_\theta\, \hat{\varrho}\, \hat{U}_\theta^\dagger|x\rangle. \tag{91}$$

In Fig. 8 we show the Wigner function $W(x, y)$ of a displaced squeezed vacuum state and its marginal $p(x; \theta)$.

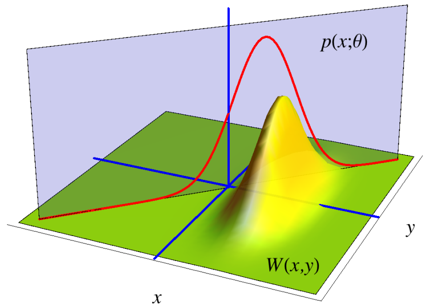

FIGURE 8. Winger function of a displaced squeezed vacuum state (yellow surface) and its marginal $p(x;\theta)$ (red line).

14. MEASURING THE QUADRATURE OPERATOR

In this section we describe the *balanced homodyne detection*, a method to measure the quadrature operator. We have seen that when two modes $\hat{a}$ and $\hat{b}$ interact through a balanced BS their evolution is given by Eqs. (40a) and we can write the outgoing modes $\hat{c}$ and $\hat{d}$ as:

$$\hat{c} = \frac{\hat{a}+\hat{b}}{\sqrt{2}}, \quad \text{and} \quad \hat{d} = \frac{\hat{b}-\hat{a}}{\sqrt{2}}. \tag{92}$$

If we define the sum and difference output photocurrent $\hat{I}_{\pm} = \hat{c}^\dagger\hat{c} \pm \hat{d}^\dagger\hat{d}$ we have that $\hat{I}_+ = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ (energy is conserved) and:

$$\hat{I}_- = \hat{c}^\dagger\hat{c} - \hat{d}^\dagger\hat{d} = \hat{b}^\dagger\hat{a} + \hat{b}\hat{a}^\dagger\,. \tag{93}$$

If we now assume that the input state is the factorized state $\hat{\varrho}\otimes|z\,e^{i\theta}\rangle\langle z\,e^{i\theta}|$, where $|z\,e^{i\theta}\rangle$ is a coherent state, $z\in\mathbb{R}$, we obtain the following expectation:

$$\langle\hat{I}_-\rangle = z\,\langle\hat{x}_\theta\rangle, \quad\Rightarrow\quad \langle\hat{x}_\theta\rangle = \frac{\langle\hat{I}_-\rangle}{z} = \langle\hat{\mathcal{I}}\rangle, \tag{94}$$

where $\hat{x}_\theta = \hat{a}\,e^{-i\theta} + \hat{a}^\dagger\,e^{i\theta}$, $\langle\hat{x}_\theta\rangle = \mathrm{Tr}[\hat{\varrho}\,\hat{x}_\theta]$ and $\hat{\mathcal{I}} = (\hat{b}^\dagger\hat{a} + \hat{b}\hat{a}^\dagger)/z$. Therefore, the mean value of the quadrature $\hat{x}_\theta$ given the state $\hat{\varrho}$ can be measured mixing the state at a balanced BS with a *local oscillator* $|z\,e^{i\theta}\rangle$, measuring the different photocurrent and normalizing the outcome with respect to z. On the other hand, if we evaluate the second moment $\langle\mathcal{I}^2\rangle$, upon the substitution $\hat{b}\to z\,e^{i\theta}$, we find:

$$\hat{\mathcal{I}}^2 = \hat{x}_\theta^2 + \frac{\hat{a}^\dagger\hat{a}}{z^2}, \tag{95}$$

thus one should have $\langle\hat{a}^\dagger\hat{a}\rangle = \mathrm{Tr}[\hat{\varrho}\,\hat{a}^\dagger\hat{a}] \ll z^2$ in order to obtain the actual quadrature moment, namely $\langle\hat{\mathcal{I}}^2\rangle \approx \langle\hat{x}_\theta^2\rangle$.

In Fig. 9 we show the Monte Carlo simulations of homodyne traces in the case of the vacuum state, a coherent state and a squeezed vacuum state.

15. HOMODYNE TOMOGRAPHY

In this section we show how we can obtain the expectation of an observable $\hat{A}$ given a state $\hat{\varrho}$ and its homodyne data sample $\{(\theta_k, x_k)\}$, $k = 1,\ldots,M$, x_k being the outcome

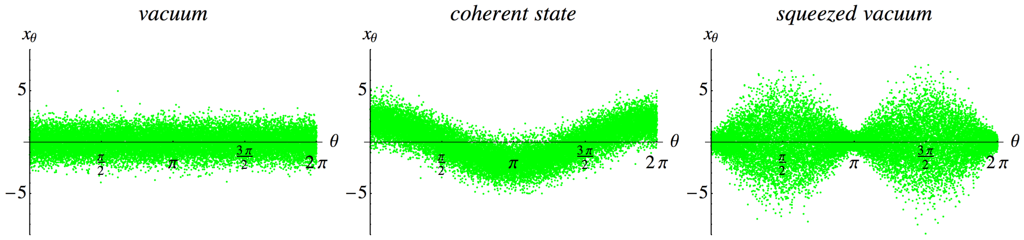

FIGURE 9. Monte Carlo simulations of homodyne traces for the vacuum state, a coherent state $|\alpha\rangle$, $\alpha = 1$, and a squeezed vacuum $\hat{S}(r)|0\rangle$ with $\langle\hat{n}\rangle = 1$ and $r < 0$, corresponding to 7.7 dB of squeezing.

from the observation of the quadrature $\hat{x}_{\theta_k}$. Exploiting the Glauber formula, we can write the operator $\hat{A}$ as:

$$\hat{A} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha\, \mathrm{Tr}\left[\hat{A}\,\hat{D}(\alpha)\right]\hat{D}^\dagger(\alpha)$$

or, writing $\alpha = iy\,\mathrm{e}^{-i\theta}$, as:

$$\hat{A} = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}y\,|y|\,\mathrm{Tr}\left[\hat{A}\,\mathrm{e}^{iy\hat{x}_\theta}\right]\mathrm{e}^{-iy\hat{x}_\theta}. \tag{96}$$

The expectation $\langle\hat{A}\rangle = \mathrm{Tr}[\hat{\varrho}\,\hat{A}]$ is thus given by:

$$\langle\hat{A}\rangle = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}y\,|y|\,\mathrm{Tr}\left[\hat{A}\,\mathrm{e}^{iy\hat{x}_\theta}\right]\mathrm{Tr}\left[\hat{\varrho}\,\mathrm{e}^{-iy\hat{x}_\theta}\right], \tag{97}$$

and, evaluating the last trace using the basis $\{|x\rangle_\theta\}$ of the eigenvectors of the quadrature $\hat{x}_\theta$, $\hat{x}_\theta|x\rangle_\theta = x|x\rangle_\theta$, namely:

$$\mathrm{Tr}\left[\hat{\varrho}\,\mathrm{e}^{-iy\hat{x}_\theta}\right] = \int_{-\infty}^{+\infty} \mathrm{d}x\, \underbrace{{}_\theta\langle x|\hat{\varrho}|x\rangle_\theta}_{p(x;\theta)}\,\mathrm{e}^{-iyx}, \tag{98}$$

we have:

$$\langle\hat{A}\rangle = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x;\theta)\,\mathcal{R}[\hat{A}](x,\theta) \equiv \overline{\mathcal{R}[\hat{A}](x,\theta)}\,, \tag{99}$$

where we introduced the estimator of the operator ensemble average $\langle\hat{A}\rangle$ given by:

$$\mathcal{R}[\hat{A}](x,\theta) = \int_{-\infty}^{+\infty} \mathrm{d}y\,|y|\,\mathrm{Tr}\left[\hat{A}\,\mathrm{e}^{iy(\hat{x}_\theta - x)}\right]. \tag{100}$$

Equation (99) is at the basis of quantum homodyne tomography. It is worth noting that since the tomographic measurement is given in terms of the average $\overline{\mathcal{R}[\hat{A}](x,\theta)}$ the precision of the measurement is given by the quantity (here we consider only the case of $\overline{\mathcal{R}[\hat{A}](x,\theta)} \in \mathbb{R}$):

$$\delta^2 = \overline{\mathcal{R}^2[\hat{A}](x,\theta)} - \left\{\overline{\mathcal{R}[\hat{A}](x,\theta)}\right\}^2, \tag{101}$$

where:

$$\overline{\mathcal{R}^2[\hat{A}](x,\theta)} = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x;\theta)\,\mathcal{R}^2[\hat{A}](x,\theta). \tag{102}$$

TABLE 1. Estimator $\mathcal{R}_\eta[\hat{A}](x,\theta)$ for some operators $\hat{A}$.

$\hat{A}$	$\mathcal{R}_\eta[\hat{A}](x,\theta)$
$\hat{a}$	$e^{i\theta}x$
$\hat{a}^2$	$e^{2i\theta}\left(x^2-\frac{1}{\eta}\right)$
$\hat{x}_\phi$	$2x\cos(\theta-\phi)$
$\hat{x}_\phi^2$	$\left(x^2-\frac{1}{\eta}\right)\{1+2\cos[2(\theta-\phi)]\}+1$
$\hat{a}^\dagger\hat{a}$	$\frac{1}{2}\left(x^2-\frac{1}{\eta}\right)$
$(\hat{a}^\dagger\hat{a})^2$	$\frac{x^4}{6}-\left(\frac{2-\eta}{2\eta}\right)x^2+\frac{1-\eta}{2\eta^2}$

In the case of the homodyne data sample $\{(\theta_k, x_k)\}$, $k = 1, \ldots, M$, where θ_k uniformly spans the interval $[0, \pi]$ and $M \gg 1$, Eq. (99) can be written as:

$$\langle\hat{A}\rangle = \lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\mathcal{R}[\hat{A}](x_k,\theta_k). \tag{103}$$

Indeed, the finite sum:

$$\langle\hat{A}\rangle = \frac{1}{M}\sum_{k=1}^{M}\mathcal{R}[\hat{A}](x_k,\theta_k) \tag{104}$$

gives an approximation of the actual value of $\langle\hat{A}\rangle$. Furthermore, using the central limit theorem, one finds that the tomographic estimation converges with a statistical error that decreases as $1/\sqrt{M}$.

In the case of optical systems, one can reduce the estimation of the expectation of any operator to the estimation of normally ordered products of annihilation and creation operators $(\hat{a}^\dagger)^n\hat{a}^m$, whose estimator is:

$$\mathcal{R}[(\hat{a}^\dagger)^n\hat{a}^m](x,\theta) = e^{i(n-m)\theta}\frac{H_{n+m}(x/\sqrt{2})}{\sqrt{2^{n+m}}\binom{n+m}{n}}, \tag{105}$$

where $H_n(x)$ are Hermite polynomials.

Since quantum tomography has a practical utility, we note that in the presence of homodyne detection with non-unit quantum efficiency η, the homodyne probability is given by the convolution:

$$p_\eta(x;\theta) = \frac{1}{\sqrt{2\pi\delta_\eta^2}}\int_{-\infty}^{+\infty}dx'\,p(x';\theta)\,\exp\left[-\frac{(x'-x)^2}{2\delta_\eta^2}\right] \tag{106}$$

with $\delta_\eta^2 = (1-\eta)/\eta$. In turn, the function in Eq. (105) should be replaced by:

$$\mathcal{R}_\eta[(\hat{a}^\dagger)^n \hat{a}^m](x,\theta) = \mathrm{e}^{i(n-m)\theta} \frac{H_{n+m}(\sqrt{\eta}x/\sqrt{2})}{\sqrt{(2\eta)^{n+m}\binom{n+m}{n}}}\,. \tag{107}$$

In Table 1 we report the analytical expression of the estimator $\mathcal{R}_\eta[\hat{A}](x,\theta)$ for some relevant operators $\hat{A}$.

16. Conclusions

In these pages I have summarized the main theoretical tools to deal with generation, manipulation and characterization of optical quantum states. Indeed, they didn't cover all the aspects of this interesting topics: this is far beyond the scope of this work (the interested reader can found further information in the essential list of references). Nevertheless, I hope that this introduction has been a useful tool both for the student that was looking for an "advanced summary" of quantum optics as well as for the researcher that applies it to describe and investigate our world.

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