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Non-divisibility and non-Markovianity in a Gaussian dissipative dynamics

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ABSTRACT

We study a stochastic Schrödinger equation that generates a family of Gaussian dynamical maps in one dimension permitting a detailed exam of two different definitions of non-Markovianity: one related to the explicit dependence of the generator on the starting time, the other to the non-divisibility of the time-evolution maps. The model shows instances where one has non-Markovianity in both senses and cases when one has Markovianity in the second sense but not in the first one.

Recent theoretical and experimental advances have aroused a lot of interest in non-Markovian effects when quantum systems interact with an environment which cannot be considered at equilibrium [1–15]. More specifically, consider a system S embedded in an environment E, under the hypothesis of an initial factorized state, i.e., a density matrix of the form $\rho \otimes \rho_E$; tracing away the environment degrees of freedom obtains an exact completely positive (CP) reduced dynamics for S that sends an initial state ρ at time $t_0 \geqslant 0$ into a state ρ_{t,t_0} at time $t \geqslant t_0$. This irreversible time-evolution is generated by an integro-differential equation of the form

$$\partial_t \rho_{t,t_0} = \int_{t_0}^{t} \mathrm{d}u\, K_{t,u}[\rho_{u,t_0}], \quad \rho_{t_0,t_0} = \rho, \tag{1}$$

where the operator kernel embodies the dependence on the past history of the system. The previous equation can be cast in the convolution-less form [10]

$$\partial_t \rho_{t,t_0} = \mathbb{L}_{t,t_0}[\rho_{t,t_0}], \tag{2}$$

where the presence of memory effects is now incorporated in the dependence of the generator on the initial time t_0. Because of this, the CP maps which solve (2),

$$\Gamma_{t,t_0} = \mathcal{T} \exp\left(\int_{t_0}^{t_1} \mathrm{d}u\, \mathbb{L}_{u,t_0}\right), \tag{3}$$

with $\mathcal{T}$ time-ordering, violate, in general, the (two-parameter) semigroup composition law, namely

$$\Gamma_{t,t_1} \circ \Gamma_{t_1,t_0} \neq \Gamma_{t,t_0}, \quad 0 \leqslant t_0 \leqslant t_1 \leqslant t. \tag{4}$$

Indeed, if $\mathbb{L}_{u,t_0} = \mathbb{L}_u$ then (3) yields the equality in (4); vice versa, if in (4) the equality holds, by taking the time derivative of both sides with respect to t one obtains $\mathbb{L}_{t,t_1} = \mathbb{L}_{t,t_0}$ for all $t_1 \geqslant t_0 \geqslant 0$. In [10], the dependence of the generator $\mathbb{L}_{t,t_0}$ on t_0 and thus (4) is taken as a criterion of non-Markovianity.

On the other hand, in [12–14] a different approach is considered whereby, given a one-parameter family of CP maps γ_t, $t \geqslant 0$, their non-Markovianity is related to non-divisibility, namely to the fact that no CP map $\Lambda_{t,u}$, $t \geqslant u \geqslant 0$, exists that connects the maps γ_t. In other words, the criterion of non-Markovianity becomes

$$\gamma_t = \Lambda_{t,u} \circ \gamma_u \quad \Longrightarrow \quad \Lambda_{t,t_0} \quad \text{not CP}. \tag{5}$$

If a CP $\Lambda_{t,u}$ existed, it would follow that certain CP monotone like the trace distance, the fidelity or the relative entropy should be decreasing: then, non-Markovianity is identified by the increase in time of such quantities which can also be taken as a measure of non-Markovianity.

In order to study the two criteria of non-Markovianity, we consider a stochastic Schrödinger equation originally proposed as a non-Markovian mechanism for the wave function collapse [16]. Specifically, we take a particle in one dimension subjected to a time-dependent random Hamiltonian of the form (for sake of simplicity, in the following, vector and matrix multiplication will be understood)

$$\hat{H}_t^{\boldsymbol{w}} = \hat{H} - \boldsymbol{w}^T(t)\hat{\boldsymbol{r}}, \tag{6}$$

where the Hamiltonian $\hat{H}$ is at most quadratic in position and momentum operators $\hat{\boldsymbol{r}}^T = (\hat{r}_1, \hat{r}_2) = (\hat{q}, \hat{p})$, while $\boldsymbol{w}^T(t) = (w_1(t), w_2(t))$ is a Gaussian noise vector with zero mean and 2×2 correlation matrix $\boldsymbol{D}(t,s)$:

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$$\big[\boldsymbol{D}(t,s)\big]_{ij} = \langle\!\langle w_i(t) w_j(s)\rangle\!\rangle, \tag{7}$$

where $\langle\!\langle\cdot\rangle\!\rangle$ denotes the average over the noise. This latter matrix is real symmetric, $D_{ij}(t,s) = D_{ji}(s,t)$, and of positive-definite type, that is

$$\sum_{i,j;t_a,t_b} \xi_i(t_a)\xi_j(t_b) D_{ij}(t_a,t_b) \geqslant 0, \quad \forall \xi(t_a) \in \mathbb{R}^2, \tag{8}$$

for any choice of times $\{t_a\}_{a=1}^n$. For each realization of the noise, the Schrödinger equation ($\hbar = 1$)

$$i\frac{\mathrm{d}|\psi_t^{\boldsymbol{w}}\rangle}{\mathrm{d}t} = \big[\hat{H} - \boldsymbol{w}^T(t)\hat{\boldsymbol{r}}\big]|\psi_t^{\boldsymbol{w}}\rangle, \tag{9}$$

generates unitary maps $\hat{U}_{t,t_0}^{\boldsymbol{w}}$ on the system Hilbert space that send an initial vector state $|\psi\rangle$ at time $t = t_0$ into $|\psi_{t,t_0}^{\boldsymbol{w}}\rangle$ at time t. Averaging the projector $|\psi_{t,t_0}^{\boldsymbol{w}}\rangle\langle\psi_{t,t_0}^{\boldsymbol{w}}|$ over the noise yields a density matrix

$$\rho_{t,t_0} = \langle\!\langle |\psi_{t,t_0}^{\boldsymbol{w}}\rangle\langle\psi_{t,t_0}^{\boldsymbol{w}}| \rangle\!\rangle. \tag{10}$$

In order to find $\hat{U}_{t,t_0}^{\boldsymbol{w}}$, one first goes to the interaction representation and sets:

$$|\widetilde{\psi}_{t,t_0}^{\boldsymbol{w}}\rangle = \hat{U}_{t-t_0}^\dagger |\psi_{t,t_0}^{\boldsymbol{w}}\rangle,$$
$$i\frac{\mathrm{d}|\widetilde{\psi}_{t,t_0}^{\boldsymbol{w}}\rangle}{\mathrm{d}t} = \boldsymbol{w}^T(t)\hat{\boldsymbol{r}}(t-t_0)|\widetilde{\psi}_{t,t_0}^{\boldsymbol{w}}\rangle, \tag{11}$$

where $\hat{U}_t = \exp(-i\hat{H}t)$ and:

$$\hat{\boldsymbol{r}}(t) = \hat{U}_t^\dagger \hat{\boldsymbol{r}} \hat{U}_t \equiv \boldsymbol{S}_t \hat{\boldsymbol{r}}, \tag{12}$$

$\boldsymbol{S}_t$ being a suitable symplectic matrix. For a given realization of the noise $\boldsymbol{w}(t)$, the solution is of the form $|\widetilde{\psi}_{t,t_0}^{\boldsymbol{w}}\rangle = \widetilde{U}_{t,t_0}^{\boldsymbol{w}}|\psi\rangle$ where, a part for a pure phase,

$$\widetilde{U}_{t,t_0}^{\boldsymbol{w}} = \exp\left\{-i\int_{t_0}^t \mathrm{d}u\, \hat{\boldsymbol{w}}^T(u)\hat{\boldsymbol{r}}(u-t_0)\right\}, \tag{13}$$

$$|\psi_{t,t_0}^{\boldsymbol{w}}\rangle = \hat{U}_{t-t_0}\hat{U}_{t,t_0}^{\boldsymbol{w}}|\psi\rangle. \tag{14}$$

By averaging over the noise, the corresponding density matrix (10) satisfies:

$$i\partial_t\rho_{t,t_0} = [\hat{H}, \rho_{t,t_0}] - \sum_{j=1}^2 \big[\hat{r}_j, \langle\!\langle w_j(t)|\psi_{t,t_0}^{\boldsymbol{w}}\rangle\langle\psi_{t,t_0}^{\boldsymbol{w}}|\rangle\!\rangle\big].$$

This stochastic Liouville equation can be turned into a standard master equation by means of the Furutsu–Novikov–Donsker relation [17]:

$$\langle\!\langle \boldsymbol{w}(s)\boldsymbol{X}[\boldsymbol{w}]\rangle\!\rangle = \int_{-\infty}^{+\infty} \mathrm{d}u\, \langle\!\langle \boldsymbol{w}(s)\boldsymbol{w}(u)\rangle\!\rangle \left\langle\!\!\left\langle \frac{\delta R[\boldsymbol{w}]}{\delta\boldsymbol{w}(u)}\right\rangle\!\!\right\rangle, \tag{15}$$

where $\boldsymbol{X}[\boldsymbol{w}]$ is a functional of the noise, $\delta/\delta\boldsymbol{w}(u)$ denotes the functional derivative with respect to the noise and $R[\boldsymbol{w}]$ is the density operator of the system. With $R[\boldsymbol{w}] = |\psi_{t,t_0}^{\boldsymbol{w}}\rangle\langle\psi_{t,t_0}^{\boldsymbol{w}}|$, one gets:

$$\partial_t\rho_{t,t_0} = \mathbb{L}_{t,t_0}[\rho_{t,t_0}] = -i[\hat{H}, \rho_{t,t_0}] + \mathbb{N}_{t,t_0}[\rho_{t,t_0}] \tag{16}$$

with:

$$\mathbb{N}_{t,t_0}[\rho] = \sum_{i,j=1}^2 C_{ij}(t,t_0)\left(\hat{r}_i\rho\hat{r}_j - \frac{1}{2}\{\hat{r}_j\hat{r}_i, \rho\}\right), \tag{17}$$

$$\boldsymbol{C}(t,t_0) = \int_{t_0}^t \mathrm{d}u\,\big[\boldsymbol{D}(t,u)\boldsymbol{S}_{u-t} + \boldsymbol{S}_{u-t}^T\boldsymbol{D}^T(t,u)\big]. \tag{18}$$

If $\boldsymbol{D}(t,u) = \delta(t-u)\boldsymbol{D}$ (i.e., white noise) then one reduces to the Markovian Lindblad type dynamics with a time-independent positive Kossakowski matrix, namely $\boldsymbol{C}(t,t_0) = \boldsymbol{D}$ [18,19]. In the time-dependent case, in order that the maps Γ_{t,t_0} generated by $\mathbb{L}_{t,t_0}$ be CP, the Kossakowski matrix $\boldsymbol{C}(t,t_0)$ need not to be positive, as we explicitly show in the following. We shall seek a solution of (16) in the form

$$\rho_{t,t_0} = \Gamma_{t,t_0}[\rho] = \int \frac{\mathrm{d}^2\boldsymbol{r}}{2\pi} G_{t,t_0}(\boldsymbol{r}) R(\boldsymbol{r}) \hat{W}(\boldsymbol{S}_{t-t_0}\boldsymbol{r}), \tag{19}$$

where we have introduced the Weyl operators:

$$\hat{W}(\boldsymbol{r}) = \mathrm{e}^{i\boldsymbol{r}^T\boldsymbol{\Omega}\hat{\boldsymbol{r}}} = \mathrm{e}^{i(q\hat{p}-p\hat{q})}, \tag{20}$$

with $\boldsymbol{r}^T = (q,p) \in \mathbb{R}^2$ and $\boldsymbol{\Omega} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$, and $R(\boldsymbol{r}) = \mathrm{Tr}[\rho\hat{W}(-\boldsymbol{r})]$ is related to the initial condition by:

$$\rho_{t_0,t_0} = \rho = \int \frac{\mathrm{d}^2\boldsymbol{r}}{2\pi} R(\boldsymbol{r})\hat{W}(\boldsymbol{r}).$$

Because the Hamiltonian $\hat{H}$ is at most quadratic and the matrix S_t in (12) is symplectic, one finds:

$$\hat{U}_t\hat{W}(\boldsymbol{r})\hat{U}_t^\dagger = \hat{W}(\boldsymbol{S}_t\boldsymbol{r}).$$

Direct insertion of (19) into (16) yields

$$\partial_t G_{t,t_0}(\boldsymbol{r}) = -\big[\boldsymbol{r}^T\boldsymbol{S}_{t-t_0}^T\boldsymbol{C}(t,t_0)\boldsymbol{S}_{t-t_0}\boldsymbol{r}\big]G_{t,t_0}(\boldsymbol{r}),$$

whence $G_{t,t_0}(\boldsymbol{r}) = \exp[-\frac{1}{2}\boldsymbol{r}^T\boldsymbol{g}(t,t_0)\boldsymbol{r}]$ with

$$\boldsymbol{g}(t,t_0) = 2\int_{t_0}^t \mathrm{d}u\, \boldsymbol{S}_{u-t_0}^T\boldsymbol{C}(u,t_0)\boldsymbol{S}_{u-t_0} \tag{21}$$

$$= \int_{t_0}^t \mathrm{d}u \int_{t_0}^t \mathrm{d}v\, \boldsymbol{S}_{u-t_0}^T\boldsymbol{D}(u,v)\boldsymbol{S}_{v-t_0}. \tag{22}$$

Furthermore, since $\boldsymbol{D}(u,v)$ is of positive type, the matrix $\boldsymbol{g}(t,t_0)$ is positive definite and $G_{t,t_0}(\boldsymbol{r})$ a real Gaussian function; the solution $\Gamma_{t,t_0}[\rho]$ can then be cast in a continuous Kraus–Stinespring decomposition which guarantees the complete positivity of the maps Γ_{t,t_0}. Let $G_{t,t_0}(\boldsymbol{r}) = \int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{x}\,\delta(\boldsymbol{x}-\boldsymbol{r})G_{t,t_0}(\boldsymbol{x})$ with

$$\delta(\boldsymbol{x}-\boldsymbol{r}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}^2\boldsymbol{y}\,\mathrm{e}^{i\boldsymbol{y}^T\boldsymbol{\Omega}(\boldsymbol{x}-\boldsymbol{r})}.$$

By inserting it into (19) and using $\hat{W}(\boldsymbol{x})\hat{W}(\boldsymbol{r})\hat{W}^\dagger(\boldsymbol{x}) = \mathrm{e}^{-i\boldsymbol{x}^T\boldsymbol{\Omega}\boldsymbol{r}} \times \hat{W}(\boldsymbol{r})$, one rewrites

$$\Gamma_{t,t_0}[\rho] = \int_{\mathbb{R}^2}\frac{\mathrm{d}^2\boldsymbol{y}}{2\pi}F_{t,t_0}(\boldsymbol{y})\hat{U}_{t-t_0}\hat{W}(\boldsymbol{x})\rho\hat{W}^\dagger(\boldsymbol{x})\hat{U}_{t-t_0}^\dagger \tag{23}$$

with the Fourier transform

$$F_{t,t_0}(\boldsymbol{y}) = \int_{\mathbb{R}^2}\frac{\mathrm{d}^2\boldsymbol{x}}{2\pi}\mathrm{e}^{i\boldsymbol{y}^T\boldsymbol{\Omega}\boldsymbol{x}}G_{t,t_0}(\boldsymbol{x}), \tag{24}$$

also a real Gaussian, hence a positive function.

Using (19) one can study the composition properties of the maps Γ_{t,t_0}; since:

$$\Gamma_{t_2,t_1}\circ\Gamma_{t_1,t_0}[\rho]$$
$$= \int\frac{\mathrm{d}^2\boldsymbol{r}}{2\pi}G_{t_2,t_1}(\boldsymbol{S}_{t_1-t_0}\boldsymbol{r})G_{t_1,t_0}(\boldsymbol{r})R(\boldsymbol{r})\hat{W}(\boldsymbol{S}_{t_2-t_0}\boldsymbol{r}),$$

in order to satisfy the semigroup composition law $\Gamma_{t_2,t_1} \circ \Gamma_{t_1,t_0} = \Gamma_{t_1,t_0}$ one should have

$$G_{t_2,t_1}(\boldsymbol{S}_{t_1-t_0}\boldsymbol{r})G_{t_1,t_0}(\boldsymbol{r}) = G_{t_2,t_0}(\boldsymbol{r}).$$

Using (22), one instead finds that

$$\left(\int_{t_1}^{t_2}\int_{t_1}^{t_2} + \int_{t_0}^{t_1}\int_{t_0}^{t_1} \mathrm{d}u\,\mathrm{d}v\right)\left(\boldsymbol{S}^T_{u-t_0}\boldsymbol{D}(u,v)\boldsymbol{S}_{v-t_0}\right)$$
$$\neq \int_{t_0}^{t_2}\mathrm{d}u\int_{t_0}^{t_2}\mathrm{d}v\,\boldsymbol{S}^T_{u-t_0}\boldsymbol{D}(u,v)\boldsymbol{S}_{v-t_0}. \tag{25}$$

This fact remains true even when $\boldsymbol{D}(s,u) = \boldsymbol{D}(|s-u|)$ in which case from (22) we have

$$\boldsymbol{g}(t,t_0) = \int_0^{t-t_0}\mathrm{d}u\int_0^{t-t_0}\mathrm{d}v\,\boldsymbol{S}^T_u\boldsymbol{D}(u,v)\boldsymbol{S}_v$$

and $\Gamma_{t,t_0} = \Gamma_{t-t_0,0}$.

Consider the master equation (16); if $t_0 = 0$ its solutions $\rho_{t,0} = \Gamma_{t,0}[\rho]$ propagate the initial state ρ from $t_0 = 0$ to $t \geqslant 0$. Because of the above result, $\Gamma_{t,0} \neq \Gamma_{t,t_0} \circ \Gamma_{t_0,0}$. However, setting $t_0 = 0$ in (16) and searching a solution $\Lambda_{t,t_0}[\rho]$ in the form (19), one gets

$$\Lambda_{t,t_0}[\rho] = \int \frac{\mathrm{d}^2\boldsymbol{r}}{2\pi} L_{t,t_0}(\boldsymbol{r})R(\boldsymbol{r})\hat{W}(\boldsymbol{S}_{t-t_0}\boldsymbol{r}) \tag{26}$$

where $L_{t,t_0}(\boldsymbol{r}) = \exp\{-\frac{1}{2}\boldsymbol{r}^T\boldsymbol{\ell}(t,t_0)\boldsymbol{r}\}$ with:

$$\boldsymbol{\ell}(t,t_0) = \int_{t_0}^{t}\mathrm{d}u\,\boldsymbol{S}^T_{u-t_0}\boldsymbol{C}(u,0)\boldsymbol{S}_{u-t_0} \tag{27}$$
$$= \int_{t_0}^{t}\mathrm{d}u\int_0^{u}\mathrm{d}v\,\boldsymbol{S}^T_{u-t_0}\boldsymbol{D}(u,v)\boldsymbol{S}_{v-t_0}. \tag{28}$$

The function $L_{t,t_0}(r)$ plays the role of $G_{t,t_0}(\boldsymbol{r})$ in (19) to which it reduces when $t_0 = 0$; that is $\Lambda_{t,0} = \Gamma_{t,0}$. Note however that, in contrast to $\boldsymbol{g}(t,t_0)$ in (21), in $\boldsymbol{\ell}(t,t_0)$ one integrates $\boldsymbol{C}(u,0)$, not $\boldsymbol{C}(u,t_0)$, from t_0 to t. As a consequence, $\Gamma_{t,0} = \Lambda_{t,t_0} \circ \Gamma_{t_0,0}$; indeed,

$$\Lambda_{t,t_0} \circ \Gamma_{t_0,0}[\rho] = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2\boldsymbol{r}}{2\pi} L_{t,t_0}(\boldsymbol{S}_{t_0}\boldsymbol{r})L_{t_0,0}(\boldsymbol{r})R(\boldsymbol{r})\hat{W}(\boldsymbol{S}_t\boldsymbol{r}),$$

where now, unlike in (25),

$$\boldsymbol{S}^T_{t_0}\ell(t,t_0)\boldsymbol{S}_{t_0} + \boldsymbol{\ell}(t_0,0) = \left(\int_{t_0}^{t}\int_0^{u} + \int_0^{t_0}\int_0^{u}\right)\mathrm{d}u\,\mathrm{d}v\,\boldsymbol{S}^T_u\boldsymbol{D}(u,v)\boldsymbol{S}_v$$
$$= \int_0^{t}\mathrm{d}u\int_0^{u}\mathrm{d}v\,\boldsymbol{S}^T_u\boldsymbol{D}(u,v)\boldsymbol{S}_v = \boldsymbol{\ell}(t,0).$$

However, contrary to the maps $\Lambda_{t,0} = \Gamma_{t,0}$ which, as we have seen, are CP, the maps Λ_{t,t_0} cannot be CP as this would imply [9] the positive definiteness of the matrix $\boldsymbol{C}(t,t_0)$ in (17). In fact, the maps Λ_{t,t_0} are in general not even positive.

All these various possibilities can be seen in a concrete example; consider a free particle of unit mass, $\hat{H} = \hat{p}^2/2$, so that $\boldsymbol{S}_t = \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$, and a diagonal noise with correlation matrix given by

$$\boldsymbol{D}(t,u) = \frac{\gamma\,\mathrm{e}^{-\gamma|t-u|}}{2}\begin{pmatrix} d_q & 0\\ 0 & d_p\end{pmatrix}. \tag{29}$$

First suppose the noise couples only to the position operator: $d_q = 1$, $d_p = 0$; then, from (18),

$$\boldsymbol{C}(t,t_0) = \begin{pmatrix} 1-\mathrm{e}^{-\gamma(t-t_0)} & \frac{\mathrm{e}^{-\gamma(t-t_0)}[1+\gamma(t-t_0)]-1}{2\gamma}\\ \frac{\mathrm{e}^{-\gamma(t-t_0)}[1+\gamma(t-t_0)]-1}{2\gamma} & 0\end{pmatrix} \tag{30}$$

has a negative eigenvalue for all $t > t_0 \geqslant 0$. In spite of the non-positivity of the Kossakowski matrix in (18), the maps Γ_{t,t_0} in (23) are nevertheless CP for all $0 \leqslant t_0 \leqslant t$.

We consider, as initial condition at t_0, a Gaussian state ρ_σ with covariance matrix (CM) $\boldsymbol{\sigma}$ and zero first moments, $\mathrm{Tr}[\rho_\sigma\hat{W}(-\boldsymbol{r})] = \exp\{-\frac{1}{2}\boldsymbol{r}^T(\boldsymbol{\Omega}\boldsymbol{\sigma}\boldsymbol{\Omega}^T)\boldsymbol{r}\}$. Using (19), Γ_{t,t_0} maps ρ_σ to the Gaussian state $\mathrm{Tr}[\Gamma_{t,t_0}[\rho_\sigma]\hat{W}(\boldsymbol{r})] = \exp\{-\frac{1}{2}\boldsymbol{r}^T\boldsymbol{\Omega}^T\boldsymbol{\sigma}_{t,t_0}\boldsymbol{\Omega}\boldsymbol{r}\}$, where $\boldsymbol{\sigma}_{t,t_0} = \boldsymbol{S}_{t-t_0}\boldsymbol{\sigma}\boldsymbol{S}^T_{t-t_0} + \widetilde{\boldsymbol{g}}(t,t_0)$ with

$$\widetilde{\boldsymbol{g}}(t,t_0) = \int_{t_0}^{t}\mathrm{d}u\,\boldsymbol{\Omega}^T\boldsymbol{S}^T_{u-t}\boldsymbol{C}(u,t_0)\boldsymbol{S}_{u-t}\boldsymbol{\Omega}. \tag{31}$$

Instead, if the same initial condition is taken for the maps Λ_{t,t_0}, the matrix $\widetilde{\boldsymbol{g}}(t,t_0)$ is to be substituted by

$$\widetilde{\boldsymbol{\ell}}(t,t_0) = \int_{t_0}^{t}\mathrm{d}u\,\boldsymbol{\Omega}^T\boldsymbol{S}^T_{u-t}\boldsymbol{C}(u,0)\boldsymbol{S}_{u-t}\boldsymbol{\Omega}. \tag{32}$$

If we choose $\boldsymbol{\sigma} = \boldsymbol{S}_{t_0-t}\boldsymbol{\sigma}_0\boldsymbol{S}^T_{t_0-t}$ and expand $\boldsymbol{\sigma}_{t,t_0} = \boldsymbol{\sigma}_0 + \widetilde{\boldsymbol{\ell}}(t,t_0)$ to first order about t_0, we have:

$$\boldsymbol{\sigma}_{t,t_0} \simeq \boldsymbol{\sigma}_0 + (t-t_0)\boldsymbol{\Omega}^T\boldsymbol{C}(t_0,0)\boldsymbol{\Omega}, \tag{33}$$

where $\boldsymbol{C}(t_0,0)$ is calculated from Eq. (30). Now, the second matrix at the l.h.s. is real symmetric and has one positive and one negative eigenvalue, $\lambda \geqslant 0$ and $-\mu < 0$; let $\boldsymbol{V}$ be the symplectic, orthogonal matrix which diagonalizes it. Then, choosing an initial state with CM diagonal in the same basis, i.e., $\boldsymbol{\sigma}_0 = \mathrm{Diag}[\sigma_{qq},\sigma_{pp}]$, such that $\boldsymbol{\sigma}_0 + \frac{i}{2}\boldsymbol{\Omega} \geq 0$ (positivity of the initial state), one gets:

$$\boldsymbol{\sigma}_{t,t_0} \simeq \boldsymbol{V}^T\begin{pmatrix}\sigma_{qq}+\lambda(t-t_0) & 0\\ 0 & \sigma_{pp}-\mu(t-t_0)\end{pmatrix}\boldsymbol{V},$$

and a sufficiently small σ_{pp} would yield a non-positive-definite CM σ_{t,t_0}, thus exhibiting the non-positivity of the map Λ_{t,t_0}. The non-positive preserving character of Λ_{t,t_0} is exposed by very specific states; on other states as, for instance, on all those of the form $\Gamma_{t_0,0}[\rho]$ it acts perfectly well for $\Lambda_{t,t_0} \circ \Gamma_{t_0,0} = \Gamma_{t,0}$. In addition, starting from $t_0 = 0$, $\Lambda_{t,0} = \Gamma_{t,0}$ is CP.

Therefore, in this case the master equation (16) generates a non-Markovian dynamics both according to the criterion (4), since the generator $\mathbb{L}_{t,t_0}$ depends on the initial time t_0 and also according to the other criterion (5). In fact, the family of maps $\Gamma_{t,0}$ is non-divisible for Λ_{t,t_0} is uniquely defined and non-positive.

Since Λ_{t,t_0} is not (completely) positive, certain quantities that exhibit monotonic behavior under CP maps fail to do so when evolving the system from time t_0 to time t. One of such quantities is the fidelity [15] $\mathcal{F}(t) = \mathcal{F}(\Gamma_{t,0}[\rho_1],\Gamma_{t,0}[\rho_2])$ of two states ρ_1 and ρ_2 evolving in time according to $\Gamma_{t,0}$. While $\mathcal{F}(t) \geqslant \mathcal{F}(0)$ for all $t \geqslant 0$, $\mathcal{F}(t_0+t)$ may become smaller than $\mathcal{F}(t_0)$ for some $t, t_0 > 0$. This is showed in Fig. 1 for two Gaussian states with zero first moments and "squeezed" CM. As one may expect, the effect disappears when γ increases towards the Markovian limit.

On the other hand, if in (9), the noise affects the particle momentum only, namely if $d_q = 0$, $d_p = 1$, then, from (29),

$$C(t,t_0) = \left(1-\mathrm{e}^{-\gamma(t-t_0)}\right)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{34}$$

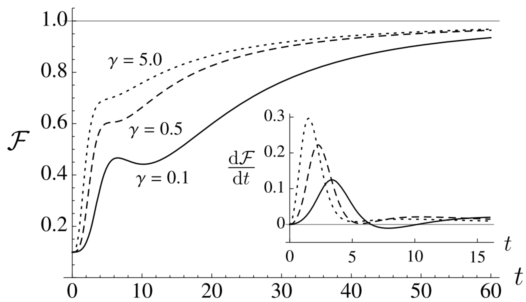

Fig. 1. Plot of the time evolution of the fidelity $\mathcal{F}$ between two Gaussian states ρ_k, $k = 1, 2$, with zero first moments and CMs $\boldsymbol{\sigma}_k = \frac{1}{2}\,\mathrm{Diag}[\exp(2r_k), \exp(-2r_k)]$, with $r_1 = -r_2 = 1.5$, evolving under the map $\Gamma_{t,0}$ for different values of γ. The inset refers to the time derivative of the fidelity. The non-monotonic behavior denotes non-Markovian evolution [15]; note that as γ increases, $\mathcal{F}$ becomes monotonic.

is positive definite. It follows that the intertwining map Λ_{t,t_0} is CP, whence the family of maps $\Gamma_{t,0}$ is divisible and Markovian according to the criterion (5). However, it is non-Markovian according to the other criterion (4). Indeed, the generator resulting from (9) depends on the starting time t_0.

In conclusion, the analysis of above examples indicates that the criterion identifying non-Markovianity with the explicit dependence of the generator $\mathbb{L}_{t,t_0}$ on the starting time t_0 appears stronger than the criterion based on the non-divisibility of the maps $\Gamma_{t,0}$. Indeed, on one hand, we have provided a case where the map $\Gamma_{t,0}$ is divisible, yet the generator of Γ_{t,t_0} explicitly depends on the initial time t_0; on the other hand, a Markovian evolution according to the first criterion readily implies the semigroup composition law, i.e., (4) with the equality sign, hence divisibility of $\Gamma_{t,0}$. Nevertheless, the non-divisibility criterion is the only one at disposal when one is presented just with the family of maps $\Gamma_{t,0}$: in such a case, one may reconstruct the generator $\mathbb{L}_{t,0}$ starting from $t_0 = 0$, but, in general, no information is available on the full generator $\mathbb{L}_{t,t_0}$ at $t_0 > 0$.

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