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Phase estimation in the presence of phase diffusion: the qubit case

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Abstract
In this paper we address the estimation of the phase shift imposed on a qubit in the presence of phase diffusion. We evaluate the ultimate quantum limits to precision and determine optimal probes and measurements achieving those bounds. We also analyse in detail the performances of spin measurements and have found that although the corresponding Fisher information depends on the unknown value of the phase shift, we may still achieve the ultimate bound using a two-step adaptive method.

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Let us consider a qubit system subjected to an unknown phase shift ϕ, i.e. a rotation about a given axis, say z, described by the unitary $U_\phi = \exp\{-\mathrm{i}\sigma_z\phi\}$. In ideal conditions, the estimation of the phase shift consists of preparing the qubit in a known pure state and then performing suitable measurements on the (pure) shifted state. In any realistic implementation, however, the propagation of a qubit unavoidably involves some noise, which influences the estimation scheme and usually degrades overall precision. In this paper, we address the estimation in the presence of non-dissipative phase noise. This is the most detrimental kind of noise for a phase gate since it destroys the off-diagonal elements of the density matrix and thus the information on the imposed phase shift. Our goal is to evaluate the ultimate quantum limits to precision in the presence of noise, as established by local quantum estimation theory and to determine both the optimal preparation of the probe qubit and the optimal measurement to be performed at the output. In view of an experimental implementation, we also analyse in detail the performances of spin measurements and have found that they allow for optimal estimation upon using a two-step adaptive method.

In our scheme, a single qubit initially prepared in the pure state $\varrho = |\varphi\rangle\langle\varphi|$ undergoes an unknown phase shift ϕ imposed by the unitary U_ϕ. Before being measured, the shifted state $\varrho_\phi = U_\phi \varrho U_\phi^\dagger$ is degraded by a non-dissipative phase noise occurring during propagation. The effect of this kind of noise on the qubit density matrix can be described by the master equation (ME) $\dot{\varrho}_{\phi,\Delta^2} = \gamma\, \mathcal{L}[\sigma_+\sigma_-]\varrho_{\phi,\Delta^2}$, where $\mathcal{L}[A]\varrho_{\phi,\Delta^2} = \frac{1}{2}\{[A\varrho_{\phi,\Delta^2}, A^\dagger] + [A, \varrho_{\phi,\Delta^2}A^\dagger]\}$ is a Lindblad superoperator and $\Delta^2 = \gamma t/2$ is, as we will see in the following lines, the effective noise factor. Since $\mathcal{L}[\sigma_+\sigma_-]$ and σ_z commute, we can focus on the evolution of ϱ, i.e. $\dot{\varrho}_{\Delta^2} = \gamma\, \mathcal{L}[\sigma_+\sigma_-]\varrho_{\Delta^2}$. Upon writing ϱ_{Δ^2} in the eigenbasis of σ_z, the ME may be written as a set of differential equations $\dot{\varrho}_{nm} = -\frac{1}{2}\gamma\,(n-m)^2\varrho_{nm}$ for the matrix elements $\varrho_{nm} = \langle n|\varrho_{\Delta^2}|m\rangle$. The solution reads as follows:

$$\varrho_{nm}(t) = \mathrm{e}^{-\Delta^2(n-m)^2}\varrho_{nm}(0), \qquad (1)$$

where $\varrho_{nm}(0)$ denotes the initial density matrix elements. From equation (1) it is clear that the diagonal elements are left unchanged, and in turn energy is conserved, whereas the off-diagonal ones are progressively destroyed. Finally, the solution of the ME is $\varrho_{\phi,\Delta^2}(t) = U_\phi \varrho_{\Delta^2} U_\phi^\dagger$. Since we can consider the noise factor Δ^2 as a fixed parameter, in the following we will not write explicitly the dependence on it. It is worth noting that the evolution (1) also corresponds to the application of a random, zero-mean Gaussian-distributed phase shift to a quantum state.

The goal of an estimation procedure is not only to retrieve the value of the unknown parameter, but also to obtain this information with minimum uncertainty. The ultimate limit to the precision of any estimation procedure is given by the quantum Cramér–Rao bound [1–4]: $\mathrm{Var}[\phi] = H^{-1}$, where H is the quantum Fisher information (QFI), which, for the estimation of a shift-parameter imposed by a unitary, is independent of ϕ. For a pure probe state the QFI equals four

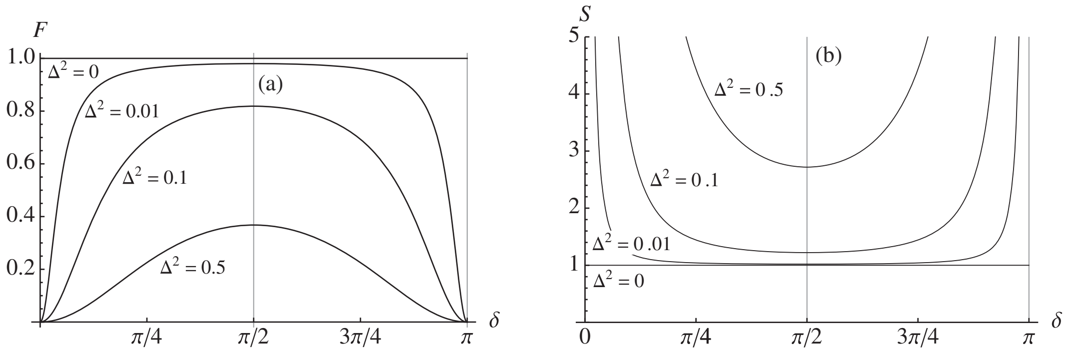

Figure 1. Plot of (a) the Fisher information $F(\phi, \Delta^2)$ and (b) the sensitivity $S(\phi, \Delta^2)$ in the case of equatorial qubit probe states ($\theta = \pi/4$) as a function of $\delta = \alpha - \phi$ and different values of Δ^2.

times the fluctuation of σ_z, i.e. $H = 4(1 - \langle\varphi|\sigma_z|\varphi\rangle^2)$. For mixed states one has [5]

$$H = 2\sum_{n\neq m} \frac{(\lambda_n - \lambda_m)^2}{\lambda_n + \lambda_m} |\langle\psi_m|\partial_\phi\psi_n\rangle|^2 , \qquad (2)$$

where $|\psi_n\rangle \equiv U_\phi|\varphi_n\rangle$ is the eigenvector of the shifted state and λ_n is its (shift independent) eigenvalue, $\varrho_\phi = \sum_n \lambda_n|\psi_n\rangle\langle\psi_n| = \sum_n \lambda_n U_\phi|\varphi_n\rangle\langle\varphi_n|U_\phi^\dagger$. If we decompose $|\psi_n\rangle$ in the standard basis $|\psi_n\rangle = U_\phi \sum_k r_{nk}|k\rangle$ and then plug this into the eigenvalue equation $\varrho_\phi|\psi_n\rangle = \lambda_n|\psi_n\rangle$, we arrive at $\sum_k \varrho_{nk}(0)\,\mathrm{e}^{-\Delta^2(n-k)^2} r_{qk} = \lambda_q r_{qn}$, $\forall n$. Moreover, since $|\partial_\phi\psi_n\rangle = \mathrm{i}\sum_k k\, r_{nk}\, \mathrm{e}^{\mathrm{i}k\phi}|k\rangle$, we have $|\langle\psi_m|\partial_\phi\psi_n\rangle|^2 = |\sum_k k\, r_{mk}\, r_{nk}|^2$ and thus, given λ_n and r_{nk}, we can evaluate the QFI.

The calculation of eigenvalue λ_n and the coefficient r_{nk} of the standard basis decomposition may be a difficult task to be performed by resorting to numerical diagonalization of the perturbed state. However, in the case of qubit systems, calculations can be carried out analytically, as we are going to show in the following. Using the Bloch sphere representation, we write the initial qubit state as $\varrho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ where $\mathbb{1}$ is the 2×2 identity matrix, $\boldsymbol{r} = (r_x, r_y, r_z)$, $|\boldsymbol{r}|^2 \leqslant 1$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrices vector. The evolution under the action of the ME corresponds to the transformation of the Bloch vector $\boldsymbol{r}$ $(r_x, r_y, r_z) \to (r_x \mathrm{e}^{-\Delta^2}, r_y \mathrm{e}^{-\Delta^2}, r_z)$, i.e. the Bloch sphere is deformed in such a way that the z-component is left unchanged, while the x and y ones are scaled by a factor $\mathrm{e}^{-\Delta^2}$. Now, due to symmetry considerations and without lack of generality, we focus on the initial pure state with the Bloch vector: $\boldsymbol{r} = (\sin 2\theta, 0, \cos 2\theta)$, 2θ being the azimuthal angle. In the density matrix representation the evolved state reads as follows:

$$\varrho = \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\Delta^2}\cos\theta\sin\theta \\ \mathrm{e}^{-\Delta^2}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}. \qquad (3)$$

The eigenvalues are given by $\lambda_\pm = \frac{1}{2}[1 \pm \frac{1}{\sqrt{2}}(1 + f(\theta, \Delta^2))]$ with $f(\theta, \Delta^2) = \sqrt{\mathrm{e}^{-2\Delta^2} + (1 - \mathrm{e}^{-2\Delta^2})\cos 4\theta}$. The corresponding eigenvectors read are given by $|\psi_\pm\rangle = \frac{1}{Z_\pm}(g_\pm(\theta, \Delta^2)|-\rangle + |+\rangle)$ with $\langle\psi_+|\psi_-\rangle = 0$ and

$$g_\pm(\theta, \Delta^2) = \cos 2\theta \pm f(\theta, \Delta^2)/(\sqrt{2}\sin 2\theta), \qquad (4a)$$

$$Z_\pm = \sqrt{1 + [g_\pm(\theta, \Delta^2)]^2}. \qquad (4b)$$

Substituting the previous equations into equation (2), we obtain

$$H(\theta, \Delta^2) = \mathrm{e}^{-2\Delta^2}\sin^2 2\theta , \qquad (5)$$

which reaches the maximum for $\theta = \pi/4$: the best states for phase estimation in the presence of phase noise are the equatorial ones, i.e. the states lying in the x–y plane of the Bloch sphere. Since the Bures metrics, and then the Bures distance [6–12] between states, is proportional to the QFI [5, 13], this result can be easily understood under a geometrical point of view. If we choose two states, one of them equatorial and the other not, and shift them by the same amount ϕ, then the *distance* on the Bloch sphere between the initial states and the shifted counterparts is larger for the equatorial states: the equatorial states allow better estimation.

The optimal quantum estimator (the observable to be measured) can be written as [5] $O_\phi = \phi\mathbb{I} + L_\phi/H(\theta, \Delta^2)$, where we introduced the symmetric logarithmic derivative $\partial_\phi\varrho(\phi, \Delta^2) = \frac{1}{2}(L_\phi\varrho(\phi, \Delta^2) + \varrho(\phi, \Delta^2)L_\phi)$. One finds

$$L_\phi = \mathrm{i}\,\frac{2g_+g_- (g_+ - g_-)(\lambda_- - \lambda_+)}{Z_+^2 Z_-^2(\lambda_+ + \lambda_-)}\left(\sigma_+\,\mathrm{e}^{\mathrm{i}\phi} - \sigma_-\mathrm{e}^{-\mathrm{i}\phi}\right), \qquad (6)$$

where $\lambda_\pm$, $g_\pm \equiv g_\pm(\theta, \Delta^2)$ and $Z_\pm$ are as given above. If we choose $\theta = \pi/4$, equation (6) reduces to $L_\phi = \mathrm{i}\mathrm{e}^{-\Delta^2}(\sigma_+\mathrm{e}^{\mathrm{i}\phi} - \sigma_-\mathrm{e}^{-\mathrm{i}\phi})$.

Let us now consider a realistic scenario where, in order to estimate ϕ, we measure the spin in a generic direction in the plane, i.e. the observable $\Theta_\alpha = \sigma_x\cos\alpha + \sigma_y\sin\alpha$. The probabilities to obtain the outcomes ± 1 given the phase shift ϕ read

$$P_{\Delta^2}(\pm 1|\phi) = \tfrac{1}{2}[1 \pm \mathrm{e}^{-\Delta^2}\cos(\alpha - \phi)\sin 2\theta], \qquad (7)$$

and the expectation value is $\langle\Theta_\alpha\rangle = \mathrm{Tr}[\Theta_\alpha\,\varrho(\phi, \Delta^2)] = \mathrm{e}^{-\Delta^2}\cos(\alpha - \phi)\sin 2\theta$. The corresponding Fisher information turns out to be

$$F(\phi, \Delta^2) = \sum_{k=\pm 1} P_{\Delta^2}(k|\phi)\left[\partial_\phi \ln P_{\Delta^2}(k|\phi)\right]^2 \qquad (8)$$

$$= \frac{\mathrm{e}^{-2\Delta^2}\sin^2(\alpha - \phi)\sin^2 2\theta}{1 - \mathrm{e}^{-2\Delta^2}\cos^2(\alpha - \phi)\sin^2 2\theta}, \qquad (9)$$

which is plotted in figure 1(a) for equatorial probe states ($\theta = \pi/4$) as a function of $\delta = \alpha - \phi$ and different values of

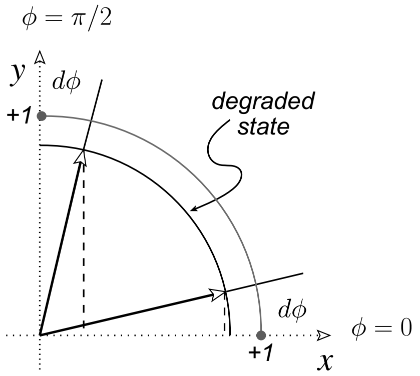

Figure 2. Sensitivity of σ_x measurement to a small change dϕ in the phase shift. The projections onto the x-axis are the expectations $\langle\Theta_0\rangle$, with $\Theta_0 = \sigma_x$. For a fixed change dϕ, the change of $\langle\Theta_0\rangle$ at $\phi = 0$ is smaller than the one at $\phi = \pi/2$. For the sake of simplicity we sketch only the upper right quarter of the Bloch sphere x–y equatorial plane.

Δ^2. As it is apparent from the plot, without noise ($\Delta^2 = 0$) one has $F = 1$, $\forall\alpha$, ϕ, i.e. the Fisher information of spin measurements is equal to the QFI H and thus one may easily achieve optimal estimation. When noise affects the propagation, the maximum of F, which equals the QFI H, is achieved for $\delta = \alpha - \phi = \pi/2$, while it goes to zero as $\alpha - \phi = k\pi$, $k \in \mathbb{N}$. These results are also understood upon considering the sensitivity of the measurement (actually this is the square of the sensitivity):

$$\mathcal{S} = \frac{\mathrm{Var}[\Theta_\alpha]}{(\partial_\phi\langle\Theta_\alpha\rangle)^2} = \frac{1 - \langle\Theta_\alpha\rangle^2}{(\partial_\phi\langle\Theta_\alpha\rangle)^2}, \qquad (10)$$

which is the ratio between the fluctuations of $\langle\Theta_\alpha\rangle$ and how $\langle\Theta_\alpha\rangle$ varies with respect to ϕ. The quantity $\sqrt{\mathcal{S}}$ represents the smallest change of ϕ that can be detected with our measurement (up to the statistical scaling). In figure 1(b) we plot the sensitivity (10) for equatorial probe states as a function of $\delta = \alpha - \phi$ and different values of Δ^2. In the present case, $\mathcal{S}(\phi, \Delta^2)$ is just the inverse of equation (9): the maximum of F (maximum information) corresponds to the case of maximum sensitivity (minimum of $\mathcal{S}$). If $\Delta^2 = 0$, one finds that $\mathrm{Var}[\Theta_\alpha]$ and $(\partial_\phi\langle\Theta_\alpha\rangle)^2$ are always equal, irrespective of the values of α and ϕ. When noise is acting, the maximum of F at $\delta = \pi/2$ corresponds to the minimum of $\mathcal{S}(\phi, \Delta^2)$; this fact can be also understood by geometrical means addressing the special case of $\alpha = 0$ ($\Theta_0 = \sigma_x$). In this case the result of the measurement carried out onto the probe is just the projection onto the x-axis: for a fixed change dϕ, the change of $\langle\Theta_0\rangle$ at $\phi = 0$ is smaller than the one at $\phi = \pi/2$ (see figure 2).

Remarkably, although the Fisher information depends on the unknown value of ϕ, we may still achieve the QFI bound for any value of ϕ using a two-step adaptive method. During the first step, we use a small amount of data to obtain a rough estimate $\tilde{\phi}$ of the phase shift; then, at the second step, we *tune* Θ_α according to the transformation $\alpha \to \tilde{\phi} + \pi/2$ and put the setup in the optimal configuration. The scheme may be iterated, but we found numerically [14] that two steps are enough to achieve the QFI limit. The same goal may be obtained by fixing the measurement at a chosen α and then tuning the probe state by applying a suitable rotation. Overall, spin measurements are good candidates to provide optimal estimation upon the choice of a suitable estimator, e.g. Bayesian ones [15].

In conclusion, we have evaluated the ultimate quantum limits to precision of phase-shift estimation for qubit in the presence of phase diffusion. The ultimate precision does not depend on the value of the phase shift, whereas the optimal measurement does. We have determined the optimal probe and analysed in detail the performance of spin measurements. We found that a simple two-step adaptive permits one to achieve ultimate bounds to precision, thus allowing for experimental implementation with current technology [16].

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