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Non-dissipative decoherence in Rabi oscillation experiments

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Abstract. Decoherence may be due to the fluctuations of some classical variable or parameter of a system, and not only to the entanglement with the environment. Here we derive a model-independent formalism for this non-dissipative decoherence, which is then applied to describe the decoherence observed in some recent Rabi oscillation experiments.

1. Introduction

Decoherence is the rapid transformation of a pure linear superposition state into the corresponding statistical mixture. This process does not preserve the purity of the state, and therefore it has to be described in terms of a non-unitary evolution. The most common approach is the so-called environment-induced decoherence [1] which is based on the consideration that it is extremely difficult to isolate perfectly a system from uncontrollable degrees of freedom (the 'environment'). The non-unitary evolution of the system of interest is obtained by considering the interaction with these uncontrolled degrees of freedom and tracing over them. In this approach, decoherence is caused by the entanglement of the two states of the superposition with two approximately orthogonal states of the environment; the system energy is usually not conserved and the interaction with the environment also accounts for the irreversible thermalization of the system of interest. This approach is inevitably *model-dependent*, because one has to assume a model Hamiltonian for the environment and the interaction between system and environment. This modelization, and therefore any quantitative prediction, becomes problematic whenever the environmental degrees of freedom responsible for decoherence are not easily recognizable.

Decoherence is not always necessarily due to the entanglement with an environment, but it may be due, as well, to the fluctuations of some classical parameter or internal variable of the system. This kind of decoherence is present even in isolated systems, where environment-induced decoherence can be neglected. In these cases the system energy is conserved, and one has a different form of decoherence, which we shall call 'non-dissipative decoherence'. In such cases, every single experimental run is characterized by the usual unitary evolution

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generated by the Hamiltonian system. However, definite statistical prediction is obtained only by repeating the experiment many times and this is when decoherence takes place, because each run corresponds to a different random value or stochastic realization of the fluctuating classical variable. The experimental results correspond therefore to an average over these fluctuations and they will describe in general an effective non-unitary evolution.

In this paper we shall present a quite general theory of non-dissipative decoherence for isolated systems which can be applied for two different kinds of fluctuating variables or parameters: the case of a random evolution time and the case of a fluctuating Rabi frequency yielding a fluctuation of the Hamiltonian. In both cases one has random phases $\exp(-iE_n t/\hbar)$ in the energy eigenstates basis that, once averaged over many experimental runs, lead to the decay of off-diagonal matrix elements of the density operator, while leaving the diagonal ones unchanged.

The outline of the paper is as follows. In section 2 we shall derive the theory under general assumptions, following closely the original derivation presented in [2, 3]. In section 3 we shall apply this theory in order to describe the decoherence effects observed in a Rabi oscillations experiment based on the resonant interaction between a Rydberg atom and a microwave cavity mode [4]. In section 4 we shall apply our approach to a Rabi oscillation experiment performed with a trapped ion [5] and section 5 gives concluding remarks.

2. The general formalism

The formalism describing non-dissipative decoherence of isolated systems has been derived in [2, 3] by considering the case of a system with random evolution time. The evolution time may be random because of the finite time needed to prepare the initial state of the system, because of the randomness of the detection time, as well as many other reasons. For example, in cavity QED experiments, the evolution time is the interaction time, which is determined by the time of flight of the atoms within the cavity and this time can be random due to atomic velocity dispersion.

In these cases, the experimental observations are not described by the usual density matrix of the whole system $\rho(t)$, but by its time averaged counterpart [2, 3]

$$\bar{\rho}(t) = \int_0^\infty \mathrm{d}t' P(t,t')\rho(t'), \tag{1}$$

where $\rho(t') = \exp\{-iLt'\}\rho(0)$ is the usual unitarily evolved density operator from the initial state and $L\ldots = [H,\ldots]/\hbar$. Therefore t' denotes the random evolution time, while t is a parameter describing the usual 'clock' time. Using equation (1), one can write

$$\bar{\rho}(t) = V(t)\rho(0), \tag{2}$$

where

$$V(t) = \int_0^\infty \mathrm{d}t' P(t,t') \exp(-iLt') \tag{3}$$

is the evolution operator for the averaged state of the system. Following [2, 3], we determine the function $P(t,t')$ by imposing the following plausible conditions: (i)

$\bar{\rho}(t)$ must be a density operator, i.e. it must be self-adjoint, positive-definite, and with unit-trace. This leads to the condition that $P(t,t')$ must be non-negative and normalized, i.e. a probability density in t' so that equation (1) is a completely positive mapping. (ii) $V(t)$ satisfies the semigroup property $V(t_1+t_2)=V(t_1)V(t_2)$, with $t_1,t_2\geq 0$.

The semigroup condition is satisfied by an exponential dependence on t

$$V(t)=\{V_1\}^{-t/\tau_2}, \tag{4}$$

where τ_2 naturally appears as a scaling time. A solution satisfying all the conditions we have imposed can be found by separating V_1 in its hermitian and antihermitian part $V_1=A+\mathrm{i}B$ and by considering the Gamma function integral identity [6]

$$(V_1)^{-t/\tau_2}=(A+\mathrm{i}B)^{-t/\tau_2}=\frac{1}{\Gamma\left(\dfrac{t}{\tau_2}\right)}\int_0^\infty \mathrm{d}\lambda\lambda^{(t/\tau_2)-1}\exp(-\lambda A)\exp(-\mathrm{i}\lambda B). \tag{5}$$

Now the right-hand side of equation (5) can be identified with the right-hand side of equation (3) if we impose the following conditions: $\lambda=t'/\tau_1$, where τ_1 is another scaling time, generally different from τ_2; $B=L\tau_1$ in order to make the exponential terms identical, and $A=1$ in order to get a normalized probability distribution $P(t,t')$. This choice yields the following expressions for the evolution operator for the averaged density matrix $V(t)$ and for the probability density $P(t,t',\tau_1,\tau_2)$ [2, 3]

$$V(t)=(1+\mathrm{i}L\tau_1)^{-t/\tau_2}, \tag{6}$$

$$P(t,t',\tau_1,\tau_2)=\frac{\exp(-t'/\tau_1)}{\tau_1}\frac{(t'/\tau_1)^{(t/\tau_2)-1}}{\Gamma(t/\tau_2)}. \tag{7}$$

Notice that the ordinary quantum evolution is recovered when $\tau_1=\tau_2=\tau\to 0$; in this limit $P(t,t',\tau_1,\tau_2)\to\delta(t-t')$ so that $\bar{\rho}(t)=\rho(t)$ and $V(t)=\exp\{-\mathrm{i}Lt\}$ is the usual unitary evolution. The semigroup condition leads to the form of the probability distribution $P(t,t',\tau_1,\tau_2)$ we use to perform the average on the fluctuating evolution times. However, notice that this probability distribution depends on both the two scaling times τ_1 and τ_2 only apparently. In fact, if we change variable in the time integral, $t''=(\tau_2/\tau_1)t'$, it is possible to rewrite the integral expression for $V(t)$ in the following way

$$V(t)=(1+\mathrm{i}L\tau_1)^{-t/\tau_2}=\int_0^\infty \mathrm{d}t''P(t,t'',\tau_2)\exp[-\mathrm{i}L(\tau_1/\tau_2)t''], \tag{8}$$

where

$$P(t,t'',\tau_2)=\frac{\exp(-t''/\tau_2)}{\tau_2}\frac{(t''/\tau_2)^{(t/\tau_2)-1}}{\Gamma(t/\tau_2)}. \tag{9}$$

This probability density depends only on τ_2. However equation (8) contains an effective *rescaled* time evolution generator $L_{\mathrm{eff}}=L(\tau_1/\tau_2)$. The physical meaning of the probability distribution of equation (9), of the rescaled evolution operator, and of the two scaling times can be understood if we consider the following simple example. Let us consider a system with Hamiltonian $H(t)=f(t)H_0$, where

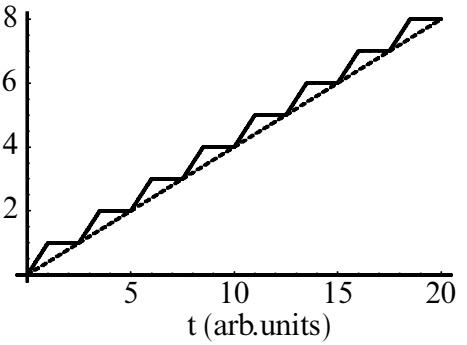

Figure 1. The function $F(t)$ defined in equation (11) (full line) is plotted as a function of time (expressed in arbitrary units) and compared with its 'linear approximation', the rescaled time $t\tau_1/\tau_2$ (dashed line). The relative error between them is given by $(\tau_2 - \tau_1)/t$ and is negligible at large times t.

$$f(t) = \sum_{n=0}^{\infty} \theta(t - n\tau_2)\theta(n\tau_2 + \tau_1 - t) \tag{10}$$

($\theta(t)$ is the Heaviside step function), that is, a system with Hamiltonian H_0 which is periodically applied for a time τ_1, with time period τ_2 ($\tau_2 \geq \tau_1$) and which is 'turned off' otherwise. The unitary evolution operator for this system is $U(t) = \exp[-iF(t)L_0]$, where $L_0 = [H_0, \ldots]$ and

$$F(t) = \begin{cases} t + n(\tau_1 - \tau_2), & n\tau_2 \leq t \leq n\tau_2 + \tau_1, \\ (n+1)\tau_1, & n\tau_2 + \tau_1 \leq t \leq (n+1)\tau_2, \end{cases} \tag{11}$$

which can be however well approximated by the 'rescaled' evolution operator $U_{\text{eff}}(t) = \exp[-iL_0 t(\tau_1/\tau_2)]$. In fact, the maximum relative error in replacing $F(t)$ with $t(\tau_1/\tau_2)$ is $(\tau_2 - \tau_1)/t$ and becomes negligible with large times (see figure 1). This fact suggests interpreting the time average of equation (8) as an average over unitary evolutions generated by L, taking place randomly in time, with mean time width τ_1, and separated by a mean time interval τ_2. This interpretation is confirmed by the fact that when $t = k\tau_2$, for integer k, the probability distribution $P(t, t'', \tau_2)$ of equation (9) is a known statistical distribution giving the probability density that the waiting time for k independent events is t'' when τ_2 is the mean time interval between two events. A particularly clear example of the random process in time implied by the above equations is provided by the micromaser [7] in which a microwave cavity is crossed by a beam of resonant atoms with mean injection rate $R = 1/\tau_2$, and a mean interaction time within the cavity corresponding to τ_1. In micromaser theory, the non-unitary operator M describing the effective dynamics of the microwave mode during each atomic crossing replaces the evolution operator $\exp(-iL\tau_1)$ [2]. Another example of interrupted evolution is provided by the experimental scheme proposed in [8] for the quantum non-demolition (QND) measurement [9] of the photon number in a high-Q cavity. In this proposal, the photon number is determined by measuring the phase shift induced on a train of Rydberg atoms sent through the microwave cavity with mean

rate $1/\tau_2$, and interacting dispersively with the cavity mode. These two examples show that the two scaling times τ_1 and τ_2 have not to be considered as new universal constants, but as two characteristic times of the system under study.

However, in most cases, one does not have an interrupted evolution as in micromaser-like situations, but a standard, continuous evolution generated by a Hamiltonian H. In this case the 'scaled' effective evolution operator has to coincide with the usual one, L, and this is possible only if $\tau_1 = \tau_2 = \tau$. In this case τ is simply the parameter characterizing the strength of the fluctuations of the random evolution time. This meaning of the parameter τ in the case of equal scaling times is confirmed by the expressions of the mean and the variance of the probability distribution of equation (7)

$$\langle t' \rangle = \frac{\tau_1}{\tau_2} t, \tag{12}$$

$$\sigma^2(t') = \langle t'^2 \rangle - \langle t' \rangle^2 = \frac{\tau_1^2}{\tau_2} t. \tag{13}$$

When $\tau_1 = \tau_2$, the mean evolution time coincides with the 'clock' time t, while the variance of the evolution time becomes $\sigma^2(t') = t\tau$. In the rest of the paper we shall always consider the standard situation of an isolated system with Hamiltonian H, continuously evolving in time, and we shall always assume $\tau_1 = \tau_2 = \tau$.

When $\tau = 0$, $V(t) = \exp\{-\mathrm{i}Lt\}$ is the usual unitary evolution. For finite τ, conversely, the evolution equation (6) describes a decay of the off-diagonal matrix elements in the energy representation, whereas the diagonal matrix elements remain constant, i.e. the energy is still a constant of motion. In fact, in the energy eigenbasis, equations (2) and (6) yield

$$\begin{aligned} \bar{\rho}_{n,m}(t) &= \frac{1}{(1 + \mathrm{i}\omega_{n,m}\tau_1)^{t/\tau_2}} \rho_{n,m}(0) \\ &= \frac{\exp(-\mathrm{i}\nu_{n,m}t)}{(1 + \omega_{n,m}^2\tau_1^2)^{t/2\tau_2}} \\ &= \exp(-\gamma_{n,m}t)\exp(-\mathrm{i}\nu_{n,m}t)\rho_{n,m}(0), \end{aligned} \tag{14}$$

where $\omega_{n,m} = (E_n - E_m)/\hbar$ and

$$\gamma_{n,m} = \frac{1}{2\tau_2} \log\left(1 + \omega_{n,m}^2\tau_1^2\right), \tag{15}$$

$$\nu_{n,m} = \frac{1}{\tau_2} \arctan(\omega_{n,m}\tau_1). \tag{16}$$

This means that, in general, the effect of the average over the fluctuating evolution time yields an exponential decay and a frequency shift $\omega_{n,m} \to \nu_{n,m}$ of every term oscillating in time with frequency $\omega_{n,m}$.

The phase diffusion aspects of the present approach can also be seen if the evolution equation of the averaged density matrix $\bar{\rho}(t)$ is considered. In fact, by differentiating with respect to time equation (2) and using (6), one gets the following master equation for $\bar{\rho}(t)$ (we consider the case $\tau_1 = \tau_2 = \tau$)

$$\dot{\bar{\rho}}(t) = -\frac{1}{\tau}\log(1+\mathrm{i}L\tau)\bar{\rho}(t); \tag{17}$$

expanding the logarithm at second order in $L\tau$, one obtains

$$\dot{\bar{\rho}}(t) = -\frac{\mathrm{i}}{\hbar}[H,\bar{\rho}(t)] - \frac{\tau}{2\hbar^2}[H,[H,\bar{\rho}(t)]], \tag{18}$$

which is the well-known phase-destroying master equation [10]. Hence equation (17) appears as a *generalized* phase-destroying master equation taking into account higher order terms in τ. Notice, however, that the present approach is different from the usual master equation approach in the sense that it is model-independent and no perturbative and specific statistical assumptions are made. The solution of equation (18) gives an expression for $\bar{\rho}_{n,m}(t)$ similar to that of equation (14), but with [10]

$$\gamma_{n,m} = \frac{\omega_{n,m}^2\tau}{2}, \tag{19}$$

$$\nu_{n,m} = \omega_{n,m}, \tag{20}$$

which are nonetheless obtained also as a first-order expansion in $\tau_1 = \tau_2 = \tau$ of equations (15) and (16). The opposite limit $\omega_{m,n}\tau \gg 1$ has been discussed in detail in [2].

Finally a comment concerning the form of the evolution operator for the averaged density matrix $V(t)$ of equation (6). At first sight it seems that $V(t)$ is in general a multivalued function of the Liouvillian L, and that $V(t)$ is uniquely defined only when $t/\tau_2 = k$, k integer. However, this form for $V(t)$ is a consequence of the time average over $P(t,t',\tau_1,\tau_2)$ of equation (7), which is a properly defined, non-negative probability distribution only if the algebraic definition of the power law function $(t'/\tau_1)^{(t/\tau_2)-1}$ is assumed. This means that in equation (6) one has to take the first determination of the power-law function and in this way $V(t)$ is univocally defined.

3. Rabi oscillations in a high-Q cavity

A first experimental situation in which the above formalism can be applied is the Rabi oscillation experiment of [4], in which the resonant interaction between a quantized mode in a high-Q microwave cavity (with annihilation operator a) and two circular Rydberg states ($|e\rangle$ and $|g\rangle$) of a Rb atom has been studied. This interaction is well described by the usual Jaynes–Cummings [11] model, which in the interaction picture reads

$$H = \hbar\Omega_{\mathrm{R}}\left(|e\rangle\langle g|a + |g\rangle\langle e|a^{\dagger}\right), \tag{21}$$

where Ω_{R} is the Rabi frequency.

The Rabi oscillations describing the exchange of excitations between atom and cavity mode are studied by injecting the velocity-selected Rydberg atom, prepared in the excited state $|e\rangle$, in the high-Q cavity and measuring the population of the lower atomic level g, $P_{eg}(t)$, as a function of the interaction time t, which is varied by changing the Rydberg atom velocity. Different initial states of the cavity mode have been considered in [4]. We shall restrict ourselves only to the case of vacuum state induced Rabi oscillations, where the decoherence effect is particularly

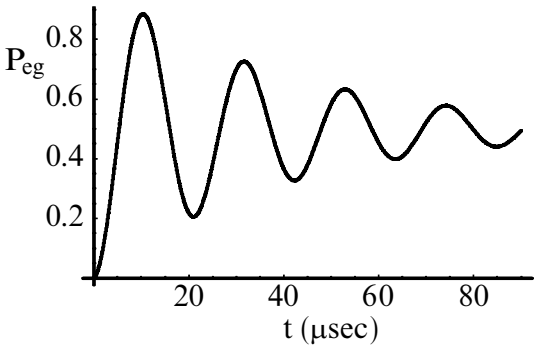

Figure 2. The Rabi oscillations of the transition probability $P_{eg}(t)$ as a function of time, according to the fitting function of equation (23).

evident. The Hamiltonian evolution according to equation (21) predicts in this case Rabi oscillations of the form

$$P_{eg}(t) = \tfrac{1}{2}(1 - \cos(2\Omega_R t)). \tag{22}$$

Experimentally instead, damped oscillations are observed, which are well fitted by

$$P_{eg}^{\exp}(t) = \tfrac{1}{2}(1 - \exp(-\gamma t)\cos(2\Omega_R t)), \tag{23}$$

where the decay time fitting the experimental data is $\gamma^{-1} = 40\,\mu s$ [12] and the corresponding Rabi frequency is $\Omega_R/2\pi = 25\,kHz$ (see figure 2). This decay of quantum coherence cannot be associated with photon leakage out of the cavity because the cavity relaxation time is larger ($220\,\mu s$) and also because in this case one would have an asymptotic limit $P_{eg}^{\exp}(\infty) = 1$. Therefore decoherence in this case has certainly a non-dissipative origin, and dark counts of the atomic detectors, dephasing collisions with background gas or stray magnetic fields within the cavity have been suggested as possible sources of the damped oscillations [4, 12].

The damped behaviour of equation (23) can be easily obtained if one applies the formalism described above. In fact, from the linearity of equation (1), one has that the time averaging procedure is also valid for mean values and matrix elements of each subsystem. Therefore one has

$$\bar{P}_{eg}(t) = \int_0^\infty dt' P(t,t') P_{eg}(t'). \tag{24}$$

Using equations (2), (6), (7) and (22), equation (24) can be rewritten in the same form of equation (23)

$$\bar{P}_{eg}(t) = \tfrac{1}{2}(1 - \exp(-\gamma t)\cos(\nu t)), \tag{25}$$

where, using equations (15) and (16),

$$\gamma = \frac{1}{2\tau}\log(1 + 4\Omega_R^2\tau^2), \tag{26}$$

$$\nu = \frac{1}{\tau}\arctan(2\Omega_R\tau). \tag{27}$$

If the characteristic time τ is sufficiently small, i.e. $\Omega_R\tau \ll 1$, there is no phase shift, $\nu \simeq 2\Omega_R$, and

$$\gamma = 2\Omega_R^2\tau \tag{28}$$

(see also equations (19) and (20)). The fact that in [4] the Rabi oscillation frequency essentially coincides with the theoretically expected one, suggests that the time τ characterizing the fluctuations of the interaction time is sufficiently small so that it is reasonable to use equation (28). Using the above values for γ and Ω_R, one can derive an estimate for τ, thus obtaining $\tau \simeq 0.5\,\mu s$. This estimate is consistent with the assumption $\Omega_R\tau \ll 1$ we have made, but, more importantly, it turns out to be comparable to the experimental value of the uncertainty in the interaction time. In fact, the fluctuations of the interaction time are mainly due to the experimental uncertainty of the atomic velocity v, that is $\delta t/t \simeq \delta v/v = 1\%$ (see [4]), and taking an average interaction time $\bar{t} \simeq 50\,\mu s$, one obtains $\tau \simeq \delta t = \bar{t}\delta v/v = 0.5\,\mu s$, which is just the estimate we have derived from the experimental values. This simple argument supports the interpretation that the decoherence observed in [4] is essentially due to the randomness of the interaction time. In fact, in our opinion, the other effects proposed as possible sources of decoherence, such as dark counts of the atomic detectors, dephasing collisions with background gas or stray magnetic fields within the cavity, would give an overall, time-independent, contrast reduction of the Rabi oscillations, different from the observed exponential decay.

Results similar to that of [4] have been very recently obtained by H. Walther's group at the Max Planck Institut für Quantenoptik, in a Rabi oscillation experiment involving a high-Q microwave cavity mode resonantly interacting with Rydberg atoms [13]. In this case, three different initial Fock states $|n\rangle$ of the cavity mode, $n = 0,1,2$, have been studied, and preliminary results show a good quantitative agreement of the experimental data with our theoretical approach based on the dispersion of the interaction times.

4. Rabi oscillation experiments in trapped ions

Another interesting Rabi oscillation experiment, performed on a different system, that is, a trapped ion [5], has recently observed a decoherence effect which cannot be attributed to dissipation. In the trapped ion experiment of [5], the interaction between two internal states ($|\uparrow\rangle$ and $|\downarrow\rangle$) of a Be ion and the centre-of-mass vibrations in the z direction, induced by two driving Raman lasers is studied. In the interaction picture with respect to the free vibrational and internal Hamiltonian, this interaction is described by the following Hamiltonian [14]

$$H = \hbar\Omega|\uparrow\rangle\langle\downarrow| \exp\{i[\eta(a \exp(-i\omega_z t) + a^\dagger \exp(i\omega_z t)) - \delta t + \phi\} + H.C., \tag{29}$$

where a denotes the annihiliation operator for the vibrations along the z direction, ω_z is the corresponding frequency and δ is the detuning between the internal transition and the frequency difference between the two Raman lasers. The Rabi frequency Ω is proportional to the two Raman laser intensities and η is the Lamb–Dicke parameter [5, 14]. When the two Raman lasers are tuned to the first blue sideband, i.e. $\delta = \omega_z$, Hamiltonian (29) predicts Rabi oscillations between $|\downarrow,n\rangle$ and $|\uparrow,n+1\rangle$ ($|n\rangle$ is a vibrational Fock state) with a frequency [14]

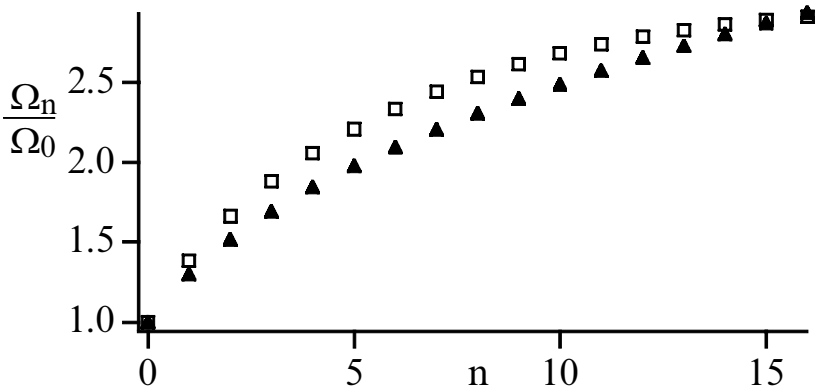

Figure 3. The ratio between Rabi frequencies Ω_n/Ω_0 experimentally measured in [5], and well fitted by the theoretical prediction of equation (30), is plotted as a function of the initial vibrational number n (□), and compared with the power-law approximation of equation (33), $(n+1)^{0.35}$ (▲).

$$\Omega_n = \Omega \frac{\exp(-\eta^2/2)}{(n+1)^{1/2}} \eta L_n^1(\eta^2), \tag{30}$$

where L_n^1 is the generalized Laguerre polynomial. These Rabi oscillations have been experimentally verified by preparing the initial state $|\downarrow,n\rangle$, (with n ranging from 0 to 16) and measuring the probability $P_\downarrow(t)$ as a function of the interaction time t, which is varied by changing the duration of the Raman laser pulses. Again, as in the cavity QED experiment of [4], the experimental Rabi oscillations are damped and well fitted by [5, 14]

$$P_\downarrow(n,t) = \tfrac{1}{2}(1 + \exp(-\gamma_n t)\cos(2\Omega_n t)), \tag{31}$$

where the measured oscillation frequencies Ω_n are in very good agreement with the theoretical prediction (30) corresponding to the measured Lamb–Dicke parameter $\eta = 0.202$ [5]. As concerns the decay rates γ_n, the experimental values are fitted in [5] by

$$\gamma_n = \gamma_0 (n+1)^{0.7}, \tag{32}$$

where $\gamma_0 = 11.9\,\text{kHz}$. This power-law scaling has attracted the interest of a number of authors and it has been investigated in [15, 16], even if a clear explanation of this behaviour of the decay rates is still lacking. Conversely, the scaling law (32) can be simply accounted for in the previous formalism if we consider the small τ limit of equation (28), which is again suggested by the fact that the experimental and theoretical predictions for the frequencies Ω_n agree. In fact, the n-dependence of the theoretical prediction of equation (30) for $\eta = 0.202$ is well approximated, within 10%, by the power-law dependence (see figure 3)

$$\Omega_n \simeq \Omega_0 (n+1)^{0.35}, \tag{33}$$

so that, using equation (28), one has immediately the power-law dependence $(n+1)^{0.7}$ of equation (32). The value of the parameter τ can be obtained by matching the values corresponding to $n = 0$, and using equation (28), that is

$\tau = \gamma_0/2\Omega_0^2 \simeq 1.5 \times 10^{-8}\,\mathrm{s}$, where we have used the experimental value $\Omega_0/2\pi = 94\,\mathrm{kHz}$.

However, this value of the parameter τ cannot be explained in terms of some interaction time uncertainty, such as the time jitter of the Raman laser pulses, which is experimentally found to be much smaller [17]. In this case, instead, the observed decoherence can be attributed, as already suggested in [14–16], to the fluctuation of the Raman laser intensities, yielding a fluctuating Rabi frequency parameter $\Omega(t)$ of the Hamiltonian (29). In this case, the evolution is driven by a fluctuating Hamiltonian $H(t) = \hbar\Omega(t)\tilde{H}$, where $\tilde{H} = H/\Omega$ in equation (29), so that

$$\rho(t) = \exp\left\{-\mathrm{i}\tilde{L}\int_0^t \mathrm{d}\xi\Omega(\xi)\right\}\rho(0) = \exp\left[-\mathrm{i}\tilde{L}A(t)\right]\rho(0), \tag{34}$$

where $\tilde{L} = [\tilde{H},\ldots]/\hbar$, and we have defined the positive dimensionless random variable $A(t) = \int_0^t \mathrm{d}\xi\Omega(\xi)$, which is proportional to the pulse area. It is now easy to understand that the physical situation is analogous to that characterized by a random interaction time considered in the preceding sections, with L replaced by $\tilde{L}$ and t' by $A(t)$. It is therefore straightforward to adapt the formalism developed in section 2 to this case, in which the fluctuating quantity is the pulse area A, yielding again random phases in the energy basis representation. In analogy with equation (1), one considers an averaged density matrix

$$\bar{\rho}(t) = \int_0^\infty \mathrm{d}A P(t,A)\,\exp\left(-\mathrm{i}\tilde{L}A\right)\rho(0). \tag{35}$$

Imposing again that $\bar{\rho}(t)$ must be a density operator and the semigroup property, one finds results analogous to equations (6) and (7)

$$V(t) = \left(1 + \mathrm{i}\tilde{L}\Omega\tau\right)^{-t/\tau}, \tag{36}$$

$$P(t,A) = \frac{\exp\left(-A/\Omega\tau\right)}{\Omega\tau}\frac{\left(A/\Omega\tau\right)^{(t/\tau)-1}}{\Gamma(t/\tau)}. \tag{37}$$

Here, the parameters Ω and τ are introduced as scaling parameters, but they have a clear meaning, as can be easily seen by considering the mean and the variance of the probability distribution of equation (37),

$$\langle A\rangle = \Omega t, \tag{38}$$

$$\sigma^2(A) = \langle A^2\rangle - \langle A\rangle^2 = \Omega^2 t\tau, \tag{39}$$

implying that Ω has now to be considered as a *mean* Rabi frequency and that τ quantifies the strength of A fluctuations. It is interesting to note that these first two moments of $P(t,A)$ determine the properties of the fluctuating Rabi frequency $\Omega(t)$, which can be written as

$$\Omega(t) = \Omega + \xi(t), \tag{40}$$

$$\langle\xi(t)\rangle = 0, \quad \langle\xi(t)\xi(t')\rangle = \Omega^2\tau\delta(t-t'), \tag{41}$$

that is, the Rabi frequency $\Omega(t)$ is a white, non-gaussian (due to the non-gaussian form of $P(t,A)$) stochastic process. In fact, the semigroup assumption we have made implies a Markovian treatment in which the spectrum of the laser intensity fluctuations is flat in the relevant frequency range. This in particular implies that

we are neglecting the dynamics at small times, of the order of the correlation time of the laser intensity fluctuations.

The estimated value of τ gives a reasonable estimate of the pulse area fluctuations, since it corresponds to a fractional error of the pulse area $[\sigma^2(A)]^{1/2}/\langle A\rangle = (\tau/t)^{1/2}$ of 10% for a pulse duration of $t = 1\,\mu s$, and which is decreasing for increasing pulse durations.

The present analysis shows many similarities with that of [15] which also tries to explain the decay of the Rabi oscillations in the ion trap experiments of [5] in terms of laser intensity fluctuations. The authors of [15] in fact use a phase destroying master equation coinciding with the second-order expansion (18) of our generalized master equation of equation (17) (see equation (16) of [15] with the identifications $G \leftrightarrow H/\hbar$ and $\Gamma \leftrightarrow \tau$) and moreover derive the same numerical estimate for the pulse area fluctuation strength $\Gamma \leftrightarrow \tau$. Despite these similarities, they do not recover the scaling (32) of the decay rates γ_n only because they do not use the general expression of the Rabi frequency (30) (and which is well approximated by the power law (33)), but its Lamb–Dicke limit $\Omega_n = \Omega_0(n+1)^{0.5}$, which is valid only when $\eta \ll 1$. There is however another, more fundamental, difference between our approach and that of [15]. They assume from the beginning that the laser intensity fluctuations have a white and gaussian character, while we make no *a priori* assumption on the statistical properties of the pulse area A. We derive these properties, i.e. the probability distribution (37), only from the semigroup condition, and it is interesting to note that this condition yields a gaussian probability distribution for the pulse area only as a limiting case. In fact, from equation (37) one can see that $P(t,A)$ tends to become a gaussian with the same mean value Ωt and the same width $\Omega^2\tau t$ only in the large time limit $t/\tau \gg 1$

$$P(t,A)_{t\gg\tau} \simeq \frac{1}{(2\pi\Omega^2 t\tau)^{1/2}} \exp\left\{-\frac{(A-\Omega t)^2}{2\Omega^2 t\tau}\right\}. \tag{42}$$

The non-gaussian character of $P(t,A)$ can be traced back to the fact that $P(t,A)$ must be definite and normalized in the interval $0 < A < \infty$ and not in $-\infty < A < +\infty$. Notice that at $t = \tau$, equation (37) assumes the exponential form $P(t,A) = \exp(-A/\Omega\tau)/\Omega\tau$. Only at large times t does the random variable A become the sum of many independent contributions and assume the gaussian form.

Due to the non-gaussian nature of the random variable A, we find that the more generally valid phase-destroying master equation is given by equation (17) (with L replaced by $\Omega\tilde{L}$). The predictions of equation (17) significantly depart from its second-order expansion in $L\tau$, equation (18), corresponding to the gaussian limit, as soon as τ becomes comparable with the typical time scale of the system under study, which, in the present case, is the inverse of the Rabi frequency.

The present analysis of the Rabi oscillation experiment of [5] can be repeated for the very recent experiment with trapped ions performed in Innsbruck [18], in which Rabi oscillations involving the vibrational levels and an optical quadrupole transition of a single $^{40}Ca^{+}$ ion have been observed. Damped oscillations corresponding to initial vibrational numbers $n = 0$ and $n = 1$ are reported. From the data with $n = 0$, $\Omega_0/2\pi = 21\,\text{kHz}$ and $\gamma_0 = 1\,\text{kHz}$, we get $\tau \simeq \gamma_0/2\Omega_0^2 \simeq 3 \times 10^{-8}\,\text{s}$ and this estimate is consistent with attributing again the decoherence to the fluctuations of the Rabi frequency caused by laser intensity fluctuations.

5. Concluding remarks

Decoherence is not always due to the entanglement with an environment, but it may be due, as well, to the fluctuations of some classical parameter or internal variable of a system. This is a different form of decoherence, which is present even in isolated systems, and that we have called non-dissipative decoherence. In this paper we have presented a model-independent theory for non-dissipative decoherence, which can be applied in the case of a random evolution time or in the case of a fluctuating Hamiltonian. This approach proves to be a flexible tool, able to give a quantitative understanding of the decoherence caused by the fluctuations of classical quantities. In fact, in this paper we have given a simple and *unified* description of the decoherence phenomenon observed in recent Rabi oscillation experiments performed in a cavity QED configuration [4] and on a trapped ion [5]. In particular, this approach has allowed us to explain for the first time in simple terms, the power-law scaling of the coherence decay rates of equation (32), observed in the trapped ion experiment.

The relevant aspect of the approach applied here, and introduced in [2], is its model independence. The formalism is in fact derived starting from few, very general assumptions: (i) the average density matrix $\bar{\rho}(t)$ has all the usual properties of a density matrix; (ii) the semigroup property for the time evolution generator $V(t)$ for $\bar{\rho}(t)$. With this respect, this approach seems to provide a very general description of non-dissipative decoherence, in which the random properties of the fluctuating classical variables are characterized by the two, system-dependent, time parameters τ_1 and τ_2. As we have seen in section 2, in the cases where one has a standard, continuous evolution, the two times coincide $\tau_1 = \tau_2 = \tau$. Under ideal conditions of no fluctuating classical variable or parameter, one would have $\tau = 0$, and the usual unitary evolution of an isolated system in quantum mechanics would be recovered. However, the generality of the approach suggests in some way the possibility that the parameter τ, even though system dependent, might have a lower *non-zero* limit, which would be reached just in the case of no fluctuations of experimental origin. This would mean a completely new description of time in quantum mechanics. In fact, the evolution time of a system t' (and not the 'clock' time t) would become an intrinsically random variable with a well-defined probability distribution, *without* the difficulty of introducing an evolution time operator. In [2] a relation of the non-zero limit for τ with the 'energy time' $\hbar/2\Delta E$, where ΔE is the uncertainty in energy, is suggested. This would give a precise meaning to the time-energy uncertainty relation because now τ rules the width of the time distribution function. However, this 'intrinsic assumption' is not necessarily implied by the formalism developed in [2] and applied, with a more pragmatic attitude, in the present paper.

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References

[1] Zurek, W. H., 1991, *Phys. Today*, **44,** 36, and references therein.

[2] Bonifacio, R., 1999, *Nuovo Cimento*, **114B,** 473; LANL e-print archive quant-ph/9901063.
[3] Bonifacio, R., 1999, *Mysteries, Puzzles and Paradoxes in Quantum Mechanics*, edited by R. Bonifacio (Woodbury: AIP), p. 122.
[4] Brune, M., *et al.*, 1996, *Phys. Rev. Lett.*, **76,** 1800.
[5] Meekhof, D. M., *et al.*, 1996, *Phys. Rev. Lett.*, **76,** 1796.
[6] Gradshteyn, I. S., and Ryzhik, I. M., 1980, *Table of Integrals and Series* (Orlando: Academic), p. 317.
[7] Raithel, G., Wagner, C., Walther, H., Narducci, L. M., and Scully, M. O., 1994, *Advances in Atomic, Molecular and Optical Physics*, edited by P. Berman (New York: Academic Press), Suppl. 2, p. 57.
[8] Brune, M., Haroche, S., Raimond, J. M., Davidovich, L., and Zagury, N., 1992, *Phys. Rev.* A, **45,** 5193.
[9] Caves, C. M., Thorne, K. S., Drever, R. W. P., Sandberg, V. D., and Zimmermann, M., 1980, *Rev. mod. Phys.*, **52,** 341.
[10] Walls, D. F., and Milburn, G. J., 1996, *Quantum Optics* (Berlin: Springer).
[11] Jaynes, E. T., and Cummings, F. W., 1963, *Proc. IEEE*, **51,** 89.
[12] Raimond, J. M., private communication.
[13] Varcoe, B. T. H., Brattke, S., Weidinger, M., and Walther, H., 2000, *Nature*, **403,** 743.
[14] Wineland, D. J., Monroe, C., Itano, W. M., Leibfried, D., King, B. E., and Meekhof, D. M., 1998, *J. Res. Nat. Inst. Stand. Technol.*, **103,** 259.
[15] Schneider, S., and Milburn, G. J., 1998, *Phys. Rev.* A, **57,** 3748.
[16] Murao, M., and Knight, P. L., 1998, *Phys. Rev.* A, **58,** 663.
[17] Wineland, D., private communication.
[18] Roos, Ch., Zeiger, Th., Rohde, H., Nägerl, H. C., Eschner, J., Leibfried, D., Schmidt-Kaler, F., and Blott, R., 1999, *Phys. Rev. Lett.*, **83,** 4713.