

## ESTIMATION OF PURITY FOR A QUANTUM HARMONIC OSCILLATOR INITIALLY PREPARED IN A DISPLACED THERMAL STATE

MATTEO BRUNELLI

*Dipartimento di Fisica,  
Università degli Studi di Milano, I-20133 Milano, Italy*

STEFANO OLIVARES

*Dipartimento di Fisica,  
Università degli Studi di Milano, I-20133 Milano, Italy  
CNISM UdR Milano Statale, I-20133 Milano, Italy  
[stefano.olivares@fisica.unimi.it](mailto:stefano.olivares@fisica.unimi.it)*

MAURO PATERNOSTRO

*Centre for Theoretical Atomic, Molecular and Optical Physics,  
School of Mathematics and Physics,  
Queen's University Belfast, Belfast BT7 1NN, UK*

MATTEO G. A. PARIS

*Dipartimento di Fisica,  
Università degli Studi di Milano, I-20133 Milano, Italy  
CNISM UdR Milano Statale, I-20133 Milano, Italy  
[matteo.paris@fisica.unimi.it](mailto:matteo.paris@fisica.unimi.it)*

Received 10 October 2012

Revised 19 December 2012

Accepted 2 January 2013

Published 28 February 2013

We address the estimation of purity for a quantum oscillator initially prepared in a displaced thermal state and probed by a suitably prepared qubit interacting with the oscillator via Jaynes–Cummings Hamiltonian without the rotating-wave approximation. We evaluate the quantum Fisher information (QFI) and show that optimal estimation of purity can be achieved by measuring the population of the qubit after a properly chosen interaction time. We also address the estimation of purity at fixed total energy and show that the corresponding precision is independent of the presence of a coherent amplitude.

*Keywords:* Quantum estimation.

## 1. Introduction

The development of quantum technology requires accurate characterization and control of quantum systems and operations. As a matter of fact, however, it is often the case that the signals and devices that we want to design and characterize are not fully or easily addressable. System interrogation can only be performed, in such cases, in an indirect way through the use of probes of an appropriate nature.<sup>1,2</sup>

In many situations, a key parameter is represented by the purity of the state of the system under scrutiny, given that unwanted effects related to decoherence typically spoil the quantum features involved in coherent evolutions and quantum enhancement. In turn, having a precise quantitative estimate of the system purity is crucial not only for prediction purposes but also to design, in the best possible way, a quantum-enforcing protocol that accounts for such undesired effects *ab initio*.

In this paper we consider a paradigmatic example within this framework. We address the estimation of purity for a quantum oscillator probed by a suitably prepared qubit coupled to it via the Jaynes–Cummings Hamiltonian,<sup>3</sup> beyond the rotating-wave approximation. We assume the oscillator initially prepared in a partially coherent state and investigate how well the purity of such initial state may be inferred from measurements performed on the qubit after a suitably tailored interaction time. We first evaluate the quantum Fisher information to obtain the ultimate quantum benchmark on the precision of the purity estimation, and then show that the optimal estimation may be achieved by measuring the population of the qubit. We also address the estimation of purity at fixed total energy and show that the corresponding precision is independent of the presence of a coherent amplitude.

Our analysis may be useful for several system of interest in quantum technology and information processing, e.g. hybrid devices<sup>4,5</sup> consisting of a mechanical mode such as a microscopic cantilever (a nanoscopic beam) coupled to a two-level system embodied by an atom<sup>6</sup> or a quantum dot<sup>7</sup> (a superconducting qubit<sup>8</sup>). In such contexts, it is important to have information on the purity of the mechanical state, modeled as a quantum harmonic oscillator, which undergoes a nontrivial open-system dynamics that typically spoils any enforced mechanical quantumness.<sup>9,10</sup>

The paper is structured as follows. In the next section, we introduce the relevant notation and describe the interaction model, as well as the probe state after the interaction. In Sec. 3, we briefly describe the basic tools of estimation theory, introducing both the Fisher information and its quantum analogue. In Sec. 4, we report the evaluation of the quantum Fisher information and its comparison with the Fisher information of population measurement. We also discuss estimation of purity at fixed total energy and the joint estimation of purity and coherent amplitude. In Sec. 5, we present some concluding remarks.

## 2. The Qubit-Oscillator Model

Let us consider a single-mode bosonic field prepared in the displaced thermal state  $\varrho = D(\alpha)\nu D^\dagger(\alpha)$ , where  $D(\alpha) = \exp\{\alpha a^\dagger - \bar{\alpha}a\}$  is the displacement operator,  $\alpha$  is a

complex amplitude and

$$\nu = \frac{2\mu}{1+\mu} \left( \frac{1-\mu}{1+\mu} \right)^{a^\dagger a}, \quad (1)$$

is a thermal equilibrium state. The quantity  $\mu = \text{Tr}[\varrho^2] = (2N+1)^{-1}$  is the purity of the state and is related to the effective temperature of the oscillator via the average number of thermal excitations  $N = (e^\beta - 1)^{-1}$ ,  $\beta = \frac{\Omega}{k_B T}$  being the inverse temperature, which embodies the oscillator's frequency  $\Omega$  (throughout the paper we use units such that  $\hbar = 1$ ).

Purity is a quadratic functional of the density operator and thus cannot be associated with any observable. We thus address the problem of its estimation by exploiting the interaction with a qubit and investigate whether the partial coherence of the initial state, i.e. the presence of an initial coherent amplitude, has any influence on the estimation procedure.

We assume that the oscillator is coupled to an ancillary two level system, initially prepared in a generic pure state  $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ . The two systems are assumed to be initially uncorrelated and undergo a unitary evolution governed by the interaction Hamiltonian<sup>11</sup>

$$H_I = g\hat{X} \otimes \hat{\sigma}_1, \quad (2)$$

where  $\hat{X} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$  is the in-phase quadrature operator of the harmonic oscillator,  $g$  is the coupling strength, and  $\hat{\sigma}_1$  is the  $x$ -Pauli spin operator. This Hamiltonian embodies the Jaynes–Cummings coupling beyond the so-called rotating wave approximation.<sup>3</sup>

After the interaction, quantum-limited measurements of the excited-state population are performed on the probe qubit, which is the qubit state after the evolution, obtained as the partial trace of the evolved state of the system over the degrees of freedom of the field. We have

$$\varrho = \text{Tr}_F[U|\psi\rangle\langle\psi| \otimes D(\alpha)\nu D^\dagger(\alpha)U^\dagger] \quad (3)$$

$$= e^{-i\tau a\sigma_1} \varrho_\nu e^{i\tau a\sigma_1}, \quad (4)$$

where  $a = \sqrt{2}\Re(\alpha)$ ,  $\tau = gt$ ,  $\hat{X}|x\rangle = x|x\rangle$ , and

$$\varrho_\nu = \int dx \langle x|\nu|x\rangle e^{-i\tau x\sigma_1} |\psi\rangle\langle\psi| e^{i\tau x\sigma_1}.$$

By explicitly computing the matrix elements of the probe state one finds

$$\varrho_{00} = \frac{1}{2} \left\{ 1 + [\cos\theta \cos(2a\tau) + \sin\theta \sin\varphi \sin(2a\tau)] \exp\left(-\frac{\tau^2}{\mu}\right) \right\}, \quad (5)$$

$$\varrho_{01} = \frac{1}{2} \sin\theta \cos\varphi - \frac{i}{2} [\sin\theta \sin\varphi \cos(2a\tau) - \cos\theta \sin(2a\tau)] \exp\left(-\frac{\tau^2}{\mu}\right), \quad (6)$$

$$\varrho_{10} = \varrho_{01}^*, \quad \text{and} \quad \varrho_{11} = 1 - \varrho_{00}. \quad (7)$$

Purity may be thus estimated by exploiting its influence on the matrix element of the qubit. This may be pursued by measuring the population of the qubit, or by other measurements involving the off-diagonal elements, and then by processing data either inverting the above equations or by a Bayesian analysis.<sup>12,13</sup> In the next section, we introduce the tools of estimation theory, which allows one to look for the optimal measurement and thus assess the different estimation schemes that may be implemented experimentally.

### 3. Basic Tools of Estimation Theory

Our analysis relies on the application of tools from (local) quantum estimation theory (QET) to the coupled qubit-oscillator system. In any estimation procedure, the information about the quantity of interest is inferred from some suitable measurement performed on the system. Once the measurement has been chosen, an estimator is needed, i.e. a function from the data sample to the quantity of interest,  $\mu$  in our case.

As a matter of fact, the variance  $\text{Var}(\mu)$  of any unbiased estimator is lower-bounded, as stated by the Cramér–Rao inequality

$$\text{Var}(\mu) \geq \frac{1}{MF(\mu)} \quad (8)$$

with  $M$  the number of measurements employed in the estimation and  $F(\mu)$  the Fisher information relative to the purity  $\mu$ , which is defined as

$$F(\mu) = \sum_j p_j (\partial_\mu \ln p_j)^2 = \sum_j \frac{|\partial_\mu p_j|^2}{p_j}, \quad (9)$$

where  $p_j$  represents the probability to get outcome  $j$  from a measurement performed over the qubit probe state  $\varrho(\mu)$ . Quantum mechanically, such probabilities are calculated via the Born rule assuming the oscillator to have purity  $\mu$ , i.e.  $p_j = \text{Tr}_q[\varrho(\mu)\hat{\Pi}_j]$ , and the observable to be measured is generally described by the positive operator valued measurement (POVM)  $\{\hat{\Pi}_j : \hat{\Pi}_j \geq 0, \sum_j \hat{\Pi}_j = \mathbb{I}\}$ . In particular, for the case of population measurements, the POVM reduces to the projective measure  $\{\hat{\Pi}_j\} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ . Once the observable is fixed, a maximization of the FI over the qubit parameters  $(\vartheta, \varphi)$ , as well as over the interaction time  $\tau$  and the coherent amplitude  $\alpha$ , leads to the minimum variance, and hence to the maximum precision attainable for that particular measurement.

On the other hand, one can even maximize the FI over all possible quantum measurements, obtaining the quantum mechanical counterpart of the Fisher Information

$$H(\mu) = \text{Tr}[\varrho \hat{L}^2(\mu)] \quad (10)$$

with  $\hat{L}(\mu)$  the symmetric logarithmic derivative operator (SLD), satisfying the equation

$$\partial_\mu \varrho = [\hat{L}(\mu)\varrho + \varrho\hat{L}(\mu)]/2. \quad (11)$$

The quantum Fisher Information (QFI) is an upper bound for  $F(\mu)$  as it embodies the optimization of the Fisher Information over *any* possible measurement performed on the probing qubit states. The QFI is thus independent of the specific measurement strategy and is an intrinsic feature of the family of probing states. This leads to the extension of the Cramér–Rao bound to the quantum domain

$$\text{Var}(\mu) \geq \frac{1}{MH(\mu)}, \quad (12)$$

which embodies the ultimate limit to the precision of the estimate of  $\mu$ . A measurement is optimal when the corresponding Fisher information  $F(\mu)$  equals the quantum Fisher Information  $H(\mu)$ . Although various instances of optimal measurement may be found, depending on the model at hand, the observable embodied by the spectral measure of  $\hat{L}(\mu)$  is certainly optimal. Upon diagonalization of the probe state  $\varrho = \varrho_+|\psi_+\rangle\langle\psi_+| + \varrho_-|\psi_-\rangle\langle\psi_-|$ , the QFI can be computed explicitly as

$$H(\mu) = \sum_{k=\pm} \frac{(\partial_\mu \varrho_k)^2}{\varrho_k} + 2\gamma \sum_{k \neq l = \pm} \left| \sum_{j=0,1} (\partial_\mu \langle j|\psi_k\rangle) \langle \psi_l|j \rangle \right|^2 \quad (13)$$

with  $\gamma = (1 - 2\varrho_+)^2$  (see Ref. 11).

Finally, exploiting the fact that  $\text{Tr}[\varrho L(\mu)] = 0$ , one can explicitly build the optimal quantum estimator

$$O_\mu = \mu\mathbb{I} + \frac{L(\mu)}{H(\mu)}, \quad (14)$$

as the one having  $\text{Tr}[\varrho O_\mu] = \mu$  and  $\text{Tr}[\varrho O_\mu^2] = \mu^2 + \frac{\text{Tr}[\varrho L(\mu)^2]}{H(\mu)^2}$ , thus  $\langle O_\mu \rangle = \mu$  and  $\langle \Delta O_\mu^2 \rangle = 1/H(\mu)$ .

#### 4. Estimation of Purity

The FI associated to a population measurement of the qubit state is given by

$$F(\mu) = \frac{(\tau/\mu)^4 [\cos\theta \cos(2a\tau) + \sin\theta \sin\varphi \sin(2a\tau)]^2}{e^{\frac{2\pi^2}{\mu}} - \cos^2\theta \cos^2(2a\tau) - \frac{1}{2} \sin(2\theta) \sin(4a\tau) \sin\varphi - \sin^2\theta \sin^2\varphi \sin^2(2a\tau)}. \quad (15)$$

Note that, although analytic, the FI is quite an involved expression as it depends on both the qubit angles  $(\theta, \varphi)$  and the complex amplitude  $\alpha = |\alpha|e^{i\text{Arg}(\alpha)}$ . However, as

these parameters only appear in the arguments of periodic functions, one may argue that they will simply affect the position of the global maximum of FI, not its value. We first verified that, apart from some cases for which the FI identically vanishes, e.g.  $(\theta, \varphi) = (\pi/2, k\pi)$  with  $k = 0, 1, 2$ , setting  $\varphi$  to some fixed value always guarantee the achievability of the global maximum. Therefore, from now on we set  $\varphi = \pi/2$ . In the same manner, we can set the phase of the coherent amplitude to zero, so that  $\alpha \equiv \Re(\alpha)$ , and for the moment we work at fixed value of the intensity, say  $\alpha = 1$ . The FI consequently becomes

$$F(\mu) = \left(\frac{\tau}{\mu}\right)^4 \frac{\cos^2(2\sqrt{2}\tau - \theta)}{[e^{\frac{2\tau^2}{\mu}} - \cos^2(2\sqrt{2}\tau - \theta)]}. \quad (16)$$

The most challenging conditions to test the precision of the population measurement are those reached very close to the oscillator's ground state, where temperatures are vanishing and the corresponding purities approach unity.

If we inspect the FI for a fixed value of the purity  $\mu = 0.9$ , which corresponds to an effective temperature of  $\beta = 10$ , we can see that, compared to the case  $\alpha = 0$  where no displacement acts on the thermal state, the FI is no longer symmetric with respect to the qubit angle  $\theta$ , as is apparent from the equation. Moreover the FI is no longer maximized by preparing the qubit in one of the poles but the optimal  $\theta$  has to be found numerically. We have thus lost both a simple and global optimal preparation of the qubit. Actually, different choices of  $(\theta, \varphi)$  can lead to the maximum [see full expression of  $F(\mu)$ ], but we maximize over  $\theta$  only as we work at fixed  $\varphi$ .

If we then take into account the effects of variable  $\alpha$  we find that, as one can see in Fig. 1, the displacement modifies the optimal angle  $\theta_{\text{opt}}$  which yields to the maximum of the FI, but does not affect the magnitude of the latter. This is a first evidence of the uselessness of a coherent kick for the purposes of an enhancement in precision.

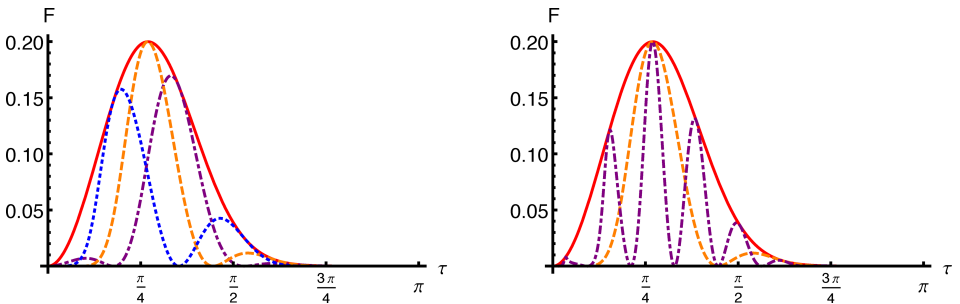


Fig. 1. (Color online) Left panel: temporal evolution of the FI with a fixed purity  $\mu = 0.9$  for different  $\theta$  values; the solid red curve corresponds to the optimal QFI  $H(\mu)_{\text{opt}}$ , while the others are the FI for  $\theta = 0$  (purple),  $\theta = \pi/2$  (blue) and  $\theta = \theta_{\text{opt}} \approx 2.39$  (orange). Right panel: temporal evolution of the FI for different intensities of the displacement  $\alpha$  and for a fixed purity, say  $\mu = 0.9$ ; the red curve corresponds to  $\alpha = 0$ ,  $\theta_{\text{opt}} = 0$  (i.e. the optimal QFI), the orange one to  $\alpha = 1$ ,  $\theta_{\text{opt}} \approx 2.39$  while the purple one is for  $\alpha = 3$  and  $\theta_{\text{opt}} \approx 0.9$ .

Moreover, one can see that the red curve ( $\alpha = 0$ ) is an envelop from the results associated with  $\alpha \neq 0$  and that the higher  $\alpha$  the more oscillating the behavior of the FI. We finally perform a numerical maximization over  $\tau$  and  $\theta$ , i.e. we consider  $\max_{(\tau,\theta)} F(\mu)$ . As shown in Fig. 2 the maximum of the FI and the optimal time  $\tau_{\text{opt}}$  at which it occurs do not depend on  $\alpha$ , while  $\theta_{\text{opt}}$  does. Moreover  $\tau_{\text{opt}}$  does not reach any steady optimal value.

We conclude our analysis by noticing that, thanks to the one-to-one map between  $\beta$  and  $\mu$ , the corresponding FI functions are connected by the following transformation:

$$F(\mu) = \sum_{j=0,1} \frac{[\partial_{\mu} p_j(\mu)]^2}{p_j(\mu)} = F(\beta) \left( \frac{\partial \beta}{\partial \mu} \right)_{\beta=\beta(\mu)}^2 \quad (17)$$

$$= F(\beta(\mu)) \frac{4}{(1 - \mu^2)^2}. \quad (18)$$

In order to understand whether the population measurement, which physically implements the  $\hat{\sigma}_3$  operator, is the best conceivable one, one has to compute the QFI. The QFI relative to the family of displaced thermal states is

$$H(\mu) = \frac{1}{8} \left( \frac{\tau}{\mu} \right)^4 \left[ \coth^2 \left( \frac{\tau^2}{\mu} \right) - 1 \right] [3 + \cos 2\theta - 2 \sin^2 \theta \cos 2\varphi]. \quad (19)$$

At a first glance, one can see that the QFI does not depend on  $\alpha$ . This definitely proves the irrelevance of adding any displacement to the thermal state in order to improve the precision of estimation. The QFI can be either maximized by choosing  $\varphi = \frac{\pi}{2}$ , regardless of  $\theta$ , or by preparing the qubit in one of the poles, i.e.  $\theta = 0, \pi$ . For both the optimal preparations, the QFI reduces to  $H(\mu)_{\text{opt}} = \frac{1}{2} \left( \frac{\tau}{\mu} \right)^4 \left[ \coth^2 \left( \frac{\tau^2}{\mu} \right) - 1 \right]$ , which is equal to the FI of the population measurement without any displacement and with the qubit in one of the poles, namely the solid red curve in Fig. 1. In this case the ultimate answer is given by inspecting the SLD, whose spectral measure gives the best observable, i.e. the one with FI equal to QFI. For  $\varphi = \frac{\pi}{2}$  or for  $\theta = 0, \pi$  the SLD is

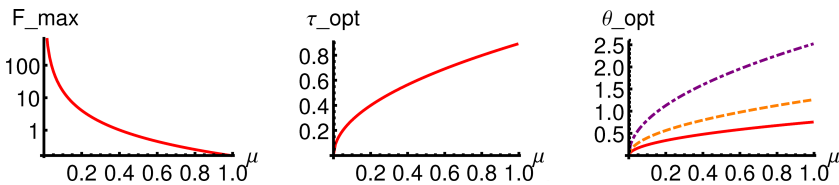


Fig. 2. (Color online) Numerical maximization  $\max_{(\tau,\theta)} F(\mu)$  of the FI with respect to  $\tau$  and  $\theta$ . Left: logarithmic plot of the maximum of the FI as a function of the purity. Center: plot of the optimal time  $\tau_{\text{opt}}$  as a function of  $\mu$ . Right: plot of the optimal angle  $\theta_{\text{opt}}$  for different values of the displacement,  $\alpha = 0.3$  (red),  $\alpha = 0.5$  (orange),  $\alpha = 1$  (purple).

diagonal and reads

$$L(\mu) = \left(\frac{\tau}{\mu}\right) \left[1 - \coth\left(\frac{\tau^2}{\mu}\right)\right] \left(\mathbb{I} - e^{\frac{\tau^2}{\mu}} \sigma_3\right),$$

which provides additional evidence of the optimality of population measurements.

#### 4.1. Estimation of purity at a fixed total energy

Up to now, we have looked for maxima of the FI treating  $\mu$  and  $\alpha$  as independent parameters. Actually one may wonder how things go if we only have a fixed total amount of energy  $\epsilon$  available. Such value will be determined by both the thermal excitations  $N = \frac{1-\mu}{2\mu}$  and the intensity of the initial displacement, so that we can write

$$\epsilon = |\alpha|^2 + \frac{1-\mu}{2\mu}. \quad (20)$$

For  $\alpha = 0$  the only contribution is the thermal one. In Fig. 3 (left) we have randomly generated different purities  $\mu$  and, for each value, performed a numerical optimization over  $\tau$  and  $\theta$ . This has been done for two different values of energy [ $\epsilon = 0.5$  (purple) and  $\epsilon = 2$  (orange)]. We can see that, by increasing the energy  $\epsilon$ , a larger part of the purity close to 0 becomes accessible but it does not affect the value of the FI. In the right panel we compare, for each randomly generated value of the purity,  $\max_{(\tau,\theta)} F(\mu)$  with  $F(\epsilon = \frac{1-\mu}{2\mu})$ , i.e. the FI relative to only the thermal contribution. Since the two plots overlap we conclude that a coherent kick cannot guarantee any improvement in precision.

#### 4.2. Joint estimation of purity and complex amplitude

Let us now consider the case where, by measuring the qubit, we want to estimate both the purity and the coherent amplitude. In this case the relevant tool in assessing the estimation precision is the quantum Fisher information matrix  $\mathbf{H}$  which, thanks to the unitary nature of the displacement operator, turns out to be diagonal. That is  $\mathbf{H}_{\mu\alpha} = \mathbf{H}_{\alpha\mu} = 0$ . The ultimate precision achievable in such a joint estimation

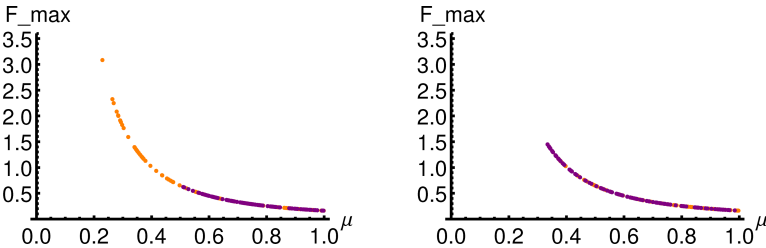


Fig. 3. (Color online) Optimized FI at set value of the total energy. In both panels, we take  $\epsilon = 0.5$  (purple),  $\epsilon = 2$  (orange). The right panel compares  $\max_{(\tau,\theta)} F(\mu)$  to the “thermal” part  $F(\epsilon = \frac{1-\mu}{2\mu})$ .



procedure is ruled by the diagonal elements of the QFI matrix, which read as follow

$$\mathbf{H}_{\mu\mu} = H(\mu), \quad (21)$$

$$\mathbf{H}_{\mu\alpha} = 2\tau^2 e^{-\frac{2\tau^2}{\mu}} [3 + \cos 2\theta - 2 \sin^2 \theta \cos 2\varphi]. \quad (22)$$

Therefore, the appropriate bound for  $\mathbf{H}$  reduces to the case of single parameter estimation, e.g.  $\text{Var}(\mu) \geq \frac{1}{M} (\mathbf{H}^{-1})_{\mu\mu} = \frac{1}{M\mathbf{H}_{\mu\mu}}$ .

## 5. Conclusion

We have addressed the problem of estimating the purity of the state of a quantum harmonic oscillator initially prepared in a displaced thermal state. Our probing system is a qubit, coupled to the oscillator via a Jaynes–Cummings model that also includes the counter-rotating terms, in line with some of experimental systems considering hybrid qubit-oscillator devices. Using the tools provided by quantum estimation theory, we have been able to ascertain the irrelevance of the coherent displacement for the estimation of the purity of the oscillator state. The latter can be optimally inferred by means of population measurements of the qubit state. These results have been extended to the case of a constrained problem, where only a fixed and given amount of total energy of the system is available.

Our results extend previous analyses where temperature estimation was addressed for micromechanical resonators coupled to a superconducting qubit via Jaynes–Cummings interaction in the rotating-wave approximation.<sup>14</sup> As mentioned, our findings are relevant to the increasingly important scenario where direct access to a system is impossible or inconvenient (in light, for instance, of the fragility of the system to the back-action of a direct measurement) and only an indirect inference of the system’s properties is in order. Such a situation is frequently faced in setups involving quantum mesoscopic devices such as those involving mechanical modes operating at the quantum level or quantum many-body systems. The analysis of the simple situation addressed here should be seen as benchmarking more sophisticated approaches to parameter estimation.

## Acknowledgments

MP thanks the UK EPSRC for a Career Acceleration Fellowship and grant under the “New Directions for EPSRC Research Leaders” initiative (EP/G004759/1). SO acknowledges financial support by MIUR (FIRB “LICHIS” — RBFR10YQ3H).

## References

1. M. G. A. Paris, *Int. J. Quantum. Inform.* **7** (2009) 125.
2. S. Campbell, M. Paternostro, S. Bose and M. S. Kim, *Phys. Rev. A* **81** (2010) 050301.
3. B. W. Shore and P. L. Knight, *J. Mod. Opt.* **40** (1993) 1195.
4. D. Hunger *et al.*, *Comptes Rendus Phys.* **12** (2011) 871.

5. A. Schliesser and T. J. Kippenberg, *Physics* **4** (2011) 97.
6. G. Vacanti, R. Fazio, M. S. Kim, G. M. Palma, M. Paternostro and V. Vedral, *Phys. Rev. A* **85** (2012) 022129.
7. I. Wilson-Rae, P. Zoller and A. Imamoglu, *Phys. Rev. Lett.* **92** (2004) 075507.
8. P. Rabl, A. Shirman and P. Zoller, *Phys. Rev. B* **70** (2004) 205304.
9. M. Aspelmeyer et al., *J. Am. Opt. B* **27** (2010) 189.
10. F. Marquardt and S. M. Girvin, *Physics* **2** (2009) 40.
11. M. Brunelli, S. Olivares, M. Paternostro and M. G. A. Paris, *Phys. Rev. A* **86** (2012) 012125.
12. B. Teklu, S. Olivares and M. G. A. Paris, *J. Phys. B* **42** (2009) 035502.
13. D. Brivio, S. Cialdi, S. Vezzoli, B. Teklu, M. G. Genoni, S. Olivares and M. G. A. Paris, *Phys. Rev. A* **81** (2010) 012315.
14. M. Brunelli, S. Olivares and M. G. A. Paris, *Phys. Rev. A* **84** (2011) 032105.