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Quantum estimation of states and operations from incomplete data

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Abstract. We review minimum Kullback entropy principle for estimation of quantum states and operations and discuss its application to qubit and harmonic oscillator systems. In particular, we address the estimation of displacement and squeezing operations from incomplete data and show how to estimate the displacement or squeezing amplitude starting from photon-number resolving or on/off photodetection.

1 Introduction

A quantum system may be characterized by measuring an observable, or a set of observables, on repeated preparations of the system. As a matter of fact, the set of observables is generally not complete, i.e., it is not sufficient to give a complete quantum information on the system [1,2]. In these cases, the question is not that of finding the actual state of the system, but rather that of estimating the state that best represents the knowledge we have acquired about the system from the measured data [3–5]. The problem is thus to find the most appropriate density matrix ϱ describing a quantum state satisfying some given constraints, which in turn express the results of the measurements performed on the system itself. If there is no a priori information, then the optimal choice is given by the density matrix ϱ which maximizes the von Neumann entropy (MaxEnt principle) [3,4]

$$S(\varrho) = -\mathrm{Tr}[\varrho \log \varrho], \tag{1}$$

and satisfies the constraints, i.e., reproduces the observed data. This principle directly generalizes the classical Jaynes maximum-entropy principle (MaxEnt) [7,8], which formalizes the idea that we have to include only the information obtained by measurements, while not allowing any conclusions not warranted by the data themselves.

On the other hand, when some a priori information about the system under investigation is accessible, then the state to be estimated has a bias toward a prior

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one τ. This is the case, for example, of a quantum system evolved according to an unknown but weak Hamiltonian starting from a known initial state. An estimation scheme for these situations may effectively constructed starting from the quantum relative (Kullback) entropy. Quantum Kullback entropy is defined as follows [9–12]:

$$K(\varrho|\tau) = \mathrm{Tr}\left[\varrho(\log\varrho - \log\tau)\right], \tag{2}$$

and quantifies the relative quantum entropy of the density matrix ϱ with respect to τ. This leads to formulate a minimum Kullback-entropy principle (mKE) [13], which states that an estimate of ϱ may be found by minimizing $K(\varrho|\tau)$ with the constraint of reproducing the data. The intuition behind mKE is that the quantum way to incorporate some a priori information is that of bias of the state to be estimated toward a given quantum state and, with the Kullback entropy effectively quantifying this bias. Quantum mKE principle generalizes classical mKE principle [14–21], which found applications in several branches of science [22–26]. Indeed, minimizing the relative entropy has all the important attributes of the maximum entropy approach with the advantage that prior information may be easily included.

If we assume that the mean values of N different observables A_k, $k = 1, \ldots, N$ are experimentally accessible, then we have N constraints $\langle A_k \rangle$, to be considered. In this case the best estimate according to mKE is given by [27,28]:

$$\varrho = \frac{1}{Z} e^{-\frac{1}{2}\sum_k A_k \lambda_k}\, \tau\, e^{-\frac{1}{2}\sum_k A_k \lambda_k} \tag{3}$$

where the partition function is given by $Z = \mathrm{Tr}[\tau\, e^{-\sum_k A_k \lambda_k}]$, and the values of the Lagrange multipliers λ_k, coming from the minimization [13,27], are obtained by solving the system of equations

$$\mathrm{Tr}[\varrho\, A_k] = \langle A_k \rangle \qquad k = 1, \ldots, N\,. \tag{4}$$

Let us now consider a scenario in which the measurement of the full distributions of a single observable is achievable. In this case the set of observables to be taken into account are the orthogonal (commuting) eigenprojectors $A_k = |\varphi_k\rangle\langle\varphi_k|$, $\langle\varphi_k|\varphi_s\rangle = \delta_{ks}$ of the measured observables. The constraints $\mathrm{Tr}[\varrho\, A_k] = p_k$, correspond to the measured distribution. The partition function and the probabilities rewrites as

$$Z = \sum_k e^{-\lambda_k}\, \langle\varphi_k|\tau|\varphi_k\rangle, \quad p_k = \frac{1}{Z} e^{-\lambda_k} \langle\varphi_k|\tau|\varphi_k\rangle\,. \tag{5}$$

Finally, taking the matrix elements of (3), back-substituting the Lagrange multipliers, and using (5) it is possible to reconstruct the posterior state, given the initial density matrix τ and the measured probabilities p_k:

$$\varrho = \sum_{n,m} \frac{\langle\varphi_m|\tau|\varphi_n\rangle}{\sqrt{\langle\varphi_m|\tau|\varphi_m\rangle\langle\varphi_n|\tau|\varphi_n\rangle}}\, \sqrt{p_m\, p_n}\, |\varphi_m\rangle\langle\varphi_n|. \tag{6}$$

Remarkably, mKE principle may be effectively applied also to the estimation of quantum operations, at least when they are weak, i.e., not too different from the identity. In these cases the evolved state is not so different from the initial one and, then, there is a natural bias toward the unperturbed state. Therefore the estimation of a weak Hamiltonian H may be pursued by means of the mKE and suitable measurements onto the evolved states. This allows one to estimate the parameters (matrix elements) of the operations from data obtained by an incomplete set (i.e. not tomographically complete [2,29]) of measurements on the evolved state, i.e., to use mKE principle as a effective tool for process estimation. In particular, for the sake of simplicity, we will focus on weak Hamiltonian processes.

The paper is structured as follows. In Sect. 2 we apply the mKE principle to estimate states and operations for a qubit system, whereas Sect. 3 is devoted to continuous variable systems (harmonic oscillators). Section 4 closes the paper with some concluding remarks.

2 mKE estimation for qubit systems

In this section we exploit mKE to estimate the state of a qubit and the (coupling) parameters of a weak qubit Hamiltonian. Let us start with the estimation of a qubit state starting from the measurement of a *single* observable [30]. In order to apply the mKE principle we assume to have a bias toward the state $\tau = \frac{1}{2}(\mathbb{1} + \boldsymbol{\tau}\cdot\boldsymbol{\sigma}), \quad |\boldsymbol{\tau}| \leq 1$, where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is the Bloch vector associated to τ and we have defined $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, σ_k, $k = 1, 2, 3$, being the Pauli matrices. The measured quantity is the spin along direction $\boldsymbol{n}$, which is described by the operator $A = \boldsymbol{n}\cdot\boldsymbol{\sigma}$. Eq. (3) rewrites as

$$\varrho = \frac{1}{2}(\mathbb{1} + \boldsymbol{v}\cdot\boldsymbol{\sigma}), \quad \boldsymbol{v} = \frac{\boldsymbol{\tau} + 2\sinh^2(\lambda/2)(\boldsymbol{n}\cdot\boldsymbol{\tau})\boldsymbol{n} - \sinh\lambda\,\boldsymbol{n}}{\cosh\lambda - \boldsymbol{\tau}\cdot\boldsymbol{n}\,\sinh\lambda}, \tag{7}$$

Now, the constraint $\mathrm{Tr}[\varrho\,\boldsymbol{n}\cdot\boldsymbol{\sigma}] = \langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle$, leads to the following estimate of the vector $\boldsymbol{v}$ [13]:

$$\boldsymbol{v}\cdot\boldsymbol{n}_1 = \langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle \tag{8}$$

$$\boldsymbol{v}\cdot\boldsymbol{n}_k = \boldsymbol{\tau}\cdot\boldsymbol{n}_k\sqrt{\frac{1-\langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle^2}{1-(\boldsymbol{\tau}\cdot\boldsymbol{n})^2}} \quad (k = 2, 3), \tag{9}$$

where we considered an operator basis composed by spin operators along three orthogonal directions $\boldsymbol{n}_1 \perp \boldsymbol{n}_2 \perp \boldsymbol{n}_3$, with $\boldsymbol{n}_1 \equiv \boldsymbol{n}$. Eqs. (8) and (9) say that the estimated Bloch component in the direction of the measured observable is equal to the measured mean value, whereas the two other orthogonal components are obtained from the prior one by a common shrinking factor.

Let us now exploit the above results for the estimation of "weak" interactions, i.e. Hamiltonians that can be written as $H = \sum_{\nu=0}^{3} \mathrm{h}_\nu\,\boldsymbol{\sigma}_\nu$ where $\mathbf{h}$ is unknown and small, i.e. $|\mathbf{h}| \ll 1$. Under this assumption we may expand the evolution equation for initial state τ $\varrho_t = e^{-iHt}\tau\,e^{iHt}$ at the first order in $\mathbf{h}$, thus obtaining

$$\varrho_t = \tau + it[\tau, H] + o(|\mathbf{h}|^2), \qquad [\tau, H] = i\sum_{k,s=1}^{3} \tau_s\,\mathrm{h}_k\,\varepsilon_{ksl}\,\boldsymbol{\sigma}_l, \tag{10}$$

where ε_{ksl} is the totally antisymmetric tensor, $\varepsilon_{123} = 1$. Being the Hamiltonian weak, the evolved state ϱ_t has a natural bias toward the initial one and it can be written as $\varrho_t = \frac{1}{2}(\mathbb{1} + \boldsymbol{w}\cdot\boldsymbol{\sigma})$, where the Bloch vector is now given by :

$$\boldsymbol{w} \equiv \boldsymbol{\tau} + 2\,\mathbf{h}\times\boldsymbol{\tau}. \tag{11}$$

Eq. (11) represent a system of equation for the unknowns $\mathbf{h}$. The transfer matrix is singular, but the system may be anyway inverted using the Moore-Penrose generalized inverse [45,46], thus leading to the following estimate for $\mathbf{h}$ [13]:

$$\mathbf{h} = \frac{1}{2|\boldsymbol{\tau}|^2}\frac{(1-\sqrt{1-\kappa^2})\,\boldsymbol{\tau}\cdot\boldsymbol{n} - \kappa}{1 - \boldsymbol{\tau}\cdot\boldsymbol{n}\,\kappa}\,\boldsymbol{\tau}\times\boldsymbol{n}, \qquad \kappa = \frac{\boldsymbol{\tau}\cdot\boldsymbol{n} - \langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle}{1 - \langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle(\boldsymbol{\tau}\cdot\boldsymbol{n})}. \tag{12}$$

By a repeated randomized choice of the measurement, an effective reconstruction may be achieved for any, but weak, qubit Hamiltonian.

3 mKE estimation for harmonic systems

In this section we exploit mKE to estimate the state of a single-mode field (harmonic oscillator) and the (coupling) parameters of a weak oscillator Hamiltonian.

Let us start with the problem of estimating the state of a harmonic oscillator with a bias toward a coherent state $\tau = |\alpha\rangle\langle\alpha|$, with $\alpha \in \mathbb{C}$. Let us consider a photon number measurement, i.e., if a denotes the annihilation operator, $[a, a^\dagger] = 1$, then the observable is expressed by $A \equiv a^\dagger a = \sum_n n|n\rangle\langle n|$, $\{|n\rangle\}$ being the photon number basis. Since $\langle n|\tau|m\rangle = (n!m!)^{-1/2}\alpha^n\bar{\alpha}^m\, e^{-|\alpha^2|}$, we have that the mKE estimated state is [13]:

$$\varrho = e^{-N}\sum_{nm}\left(N/|\alpha|^2\right)^{(n+m)/2}\frac{\alpha^n\bar{\alpha}^m}{\sqrt{n!m!}} \equiv |\sqrt{N}e^{i\phi}\rangle\langle\sqrt{N}e^{i\phi}|\,, \tag{13}$$

with $N = \mathrm{Tr}[\varrho\, a^\dagger a]$ and $\phi = \arg\alpha$: the best estimate according to mKE is a coherent state with average number of photons equal to the measured one and phase equal to that of the prior coherent state. Notice that the best estimate obtained using the MaxEnt principle with the same constraint on the average number of photons, but without the bias, would have been a thermal state with N thermal photons [13].

If by some means the complete photon distribution p_n is available, the reconstructed state, given by Eq. (6), reads as follows (we assumed that the bias is still toward $\tau = |\alpha\rangle\langle\alpha|$, $\alpha \in \mathbb{C}$):

$$\varrho = \sum_{n,m}\sqrt{p_n\, p_m}\, e^{i\phi(n-m)}|n\rangle\langle m|\,. \tag{14}$$

Remarkably, Eq. (14) no longer depends on the amplitude of the prior state, which enters only trough the phase ϕ. This makes the above scheme quite promising though measuring the photon distribution is, in general, a challenging task. On the other hand, in the optical case it is possible to reconstruct the p_n by means of on/off photodetection and maximum likelihood algorithm [31–34], a method that has been recently verified in laboratory [35,36]. Multichannel fiber loop detectors [37–39], and hybrid photodetectors [40–42] may be also used.

Finally, we notice that a special case of bias is that toward a Gaussian state. In fact the Kullback relative entropy of a state ϱ with respect to a Gaussian state τ, *with the same covariance matrix* reduces to the difference of the Von Neumann entropies $K(\varrho|\tau) = S(\tau) - S(\varrho)$ and thus the mKE principle reduces to MaxEnt and it is equivalent to minimizing the nonGaussianity of the estimated state [43,44]. Notice, however, that this is not in contrast with the results above, since in that case the results of the measurement do not impose the equality of the covariance matrices.

Let us now assume that the expansion $\varrho(t) = \tau + it[\tau, H] + o(|H|^2)$, describe the state of a harmonic oscillator evolving under the action of a weak Hamiltonian H that we want to estimate. We also assume that the full distribution of a single observable of the evolved state can be measured. Therefore, the evolved density matrix ϱ_t may be reconstructed by mKE starting from the observation level $A_k = |\varphi_k\rangle\langle\varphi_k|$, and thus obtaining the state (6). Using the basis $\{|\varphi_k\rangle\}$ we can write the matix element of the evolved state as follows:

$$\varrho_{mn}(t) = \tau_{mn} + it\sum_s\left(H_{ms}\tau_{sn} + \tau_{ms}H_{sn}\right), \tag{15}$$

with $H_{mn} \equiv \langle\varphi_m|H|\varphi_n\rangle$. Using the mKE estimate (6) for the evolved density matrix we obtain the following hierarchy of equations

$$\sum_s\left(H_{ms}\tau_{sn} + \tau_{ms}H_{sn}\right) = \frac{i}{t}\tau_{mn}\left(1 - \sqrt{\frac{p_n p_m}{\langle\varphi_n|\tau|\varphi_n\rangle\langle\varphi_m|\tau|\varphi_m\rangle}}\right) \tag{16}$$

where p_n are the measured probabilities (the constraints used for the mKE) and the matrix elements H_{nm} are the unknowns.

A relevant example, in which the Hamiltonian can be effectively estimated using mKE, is that corresponding to $H = (ga + \text{h.c.})$, a being the annihilation operator starting from the sole measurement of the photon distributions. The evolution imposed by the Hamiltonian H corresponds to the unitary displacement operator $D(\beta) = \exp(\beta a^\dagger - \beta^* a)$, $\beta = gt$. The problem is then to estimate the displacement amplitude β from the measured photon distribution. We assume that the initial state is a coherent state $\varrho(\alpha) = |\alpha\rangle\langle\alpha|$. For the sake of simplicity we take α and β as real. Using the photon number basis the evolved state may be written as $\varrho(\beta) = e^{(\alpha+\beta)^2} \sum_{n,m} (n!\,m!)^{-1/2} (\alpha+\beta)^{n+m} |n\rangle\langle m|$. Now, upon assuming that the measurement of the photon number is made on the evolved state, and that mKE principle is used to estimate the density matrix, we equate the above expression to that given in Eq. (14) for $\phi = 0$ (recall that α and β are taken as real). We thus obtain the following set of equations:

$$-(\alpha+\beta)^2 + (n+m)\,\ln(\alpha+\beta) = \ln\sqrt{n!\,m!\,p_n\,p_m}, \tag{17}$$

to be solved for β. It is worth noting that in order to estimate β one can choose to measure a *finite* number of p_k, i.e., $k = 0, \ldots, N-1$. As a matter of fact, this choice also select a subspace of the Hilbert space where the reconstructed state is defined. In turn, (17) corresponds to N^2 determinations of the same parameter β. Notice that without using the mKE principle the only way to exploit the information at disposal, i.e., the elements of the probability distribution $p_n = e^{-(\alpha+\beta)^2}(\alpha+\beta)^{2n}/n!$, $n = 0, \ldots, N-1$, is to invert those relations. In order to estimate β one should solve the set of equations

$$-(\alpha+\beta)^2 + 2n\,\ln(\alpha+\beta) = \ln(\,n!\,p_n), \tag{18}$$

which provide only N determinations of β. In Fig. 1 we plot the estimated displacement β_{est} as a function of the actual one β, taken as real, with and without the use of the mKE principle in the case of a simulated experiment. The photon distribution p_n of the coherent state has been obtained by using on/off photodetection and maximum likelihood algorithm [31–34]. Even if the estimation actually depends on the reconstructed p_n, we can see that mKE achieves better results (see the plots on the left in Figs. 1). It is worth noting that in this particular scenario, i.e., displacement onto coherent states, we have good results also when the Hamiltonian is not so weak.

Let us consider the estimation of a weak squeezing amplitude starting from the sole photon distribution. The squeezing Hamiltonian is given by $H = (g' a^2 + \text{h.c.})$ and the evolution operator corresponds to the unitary squeezing operator $S(r) = \exp(\frac{1}{2} r a^{\dagger 2} - \frac{1}{2} r a^2)$, $r = 2g't \in \mathbb{R}$. The problem is to estimate the squeezing amplitude r from the measured photon distribution assuming that r is small. To this aim we assume that the initial state of the harmonic oscillator is the vacuum $|0\rangle$, then it is evolves according to the squeezing Hamiltonian, and finally we measure the photon distribution on the output state. Since the squeezing is small the measured state has a bias toward the vacuum and therefore the estimated mKE state in that Eq. (14). The squeezed vacuum state $S(r)|0\rangle$ may be written as

$$\varrho_r = \sum_{nm} f_{2m}(r) f_{2n}(r) |2n\rangle\langle 2m|\,, \tag{19}$$

where

$$f_{2n}(r) = \left(\frac{\tanh r}{2}\right)^n \sqrt{\binom{2n}{n} \frac{1}{\cosh r}}.$$

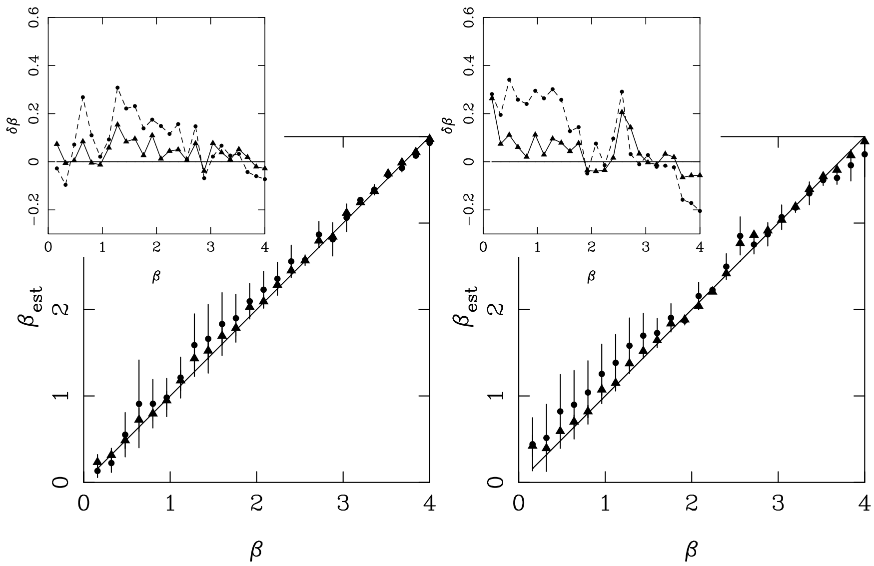

Fig. 1. Displacement amplitude β_{est} estimated using the mKE principle as a function of the actual one β: triangles and disks refer to the estimation with and without mKE, respectively. We set the amplitude of the initial coherent state to $\alpha = 0.5$ (left) or $\alpha = 1.5$ (right). The insets show the absolute deviations $\delta\beta = \beta_{\mathrm{est}} - \beta$ (symbols have the same meaning).

Now, upon denoting the measured photon distribution by p_n, the above estimate, with the condition $r \ll 1$, may be exploited to obtain a set of equations for r. We have

$$r \simeq 2 \left[\frac{1}{p_{2k}\, p_{2s}} \binom{2k}{k} \binom{2s}{s} \right]^{-\frac{1}{2(k+s)}} \tag{20}$$

where $k, s = 0, \ldots, N-1$, N being a suitable truncation of the Hilbert space, e.g. corresponding to the maximum discrimination power of the detectors employed to measure the photon distribution. As in the previous examples, this choice also select a subspace of the Hilbert space where the reconstructed state is defined. Eq. (20) exploits the estimation of the off-diagonal elements by the mKE principle to provide N^2 determinations of the same parameter r, whereas without using the mKE principle the only way to employ the information at disposal, i.e., the elements of the probability distribution, would have been that of inverting the relations $p_n = f_n(n)$ (for even n), leading to only N determinations of r.

4 Conclusions

We have considered quantum estimation of states and weak Hamiltonian operations in situations where one has at disposal data from the measurement of an incomplete set of observables and, at the same time, some *a priori* information on the state itself. By expressing the *a priori* information in terms of a bias toward a given state the best estimate is obtained using the principle of minimum Kullback entropy, i.e., by taking the state that reproduces the data while minimizing relative entropy with respect to the bias. The mKE principle has been used to estimate the quantum state from the measurement of a single observable. In particular, we have analyzed qubit and harmonic systems with some details. We have also considered the problem of estimating a *weak* Hamiltonian processes. In this case there is natural bias of the evolved state toward the initial state and the mKE principle can be used as a tool to estimate the Hamiltonian from a incomplete data. In particular, we have applied mKE principle to estimate the amplitude of a displacement imposed to a single-mode radiation field. We found that mKE principle improves estimation and, in the case of a coherent input signal, may be applied also when the Hamiltonian is not so weak. Overall the minimum Kullback entropy principle appears to be a convenient approach for quantum estimation in realistic situations and a useful tool for the estimation of weak Hamiltonian processes.

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