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Review

Quantum optics in the phase space

A tutorial on Gaussian states

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Abstract. In this tutorial, we introduce the basic concepts and mathematical tools needed for phase-space description of a very common class of states, whose phase properties are described by Gaussian Wigner functions: the Gaussian states. In particular, we address their manipulation, evolution and characterization in view of their application to quantum information.

1 Introduction

A Gaussian state is a state with Gaussian Wigner functions [1]. In recent years, an increasing attention has been devoted to this class of states, as they play a major role in quantum information with continuous variables [2–4]. Besides quantum optics, where they are generated with current quantum optical technology, Gaussian states naturally appear in the description of optomechanical and nanomechanical oscillators, gases of cold atoms and ion traps. Furthermore, Gaussian states exhibit extremality properties: among the continuous variable states, they tend to be extremal if one imposes some constraints on the covariance matrix [5].

The better way to deal theoretically with Gaussian states is to use a suitable phase-space analysis. In fact, if the Gaussian character is preserved during the dynamics, it is natural to think of the evolution of a Gaussian state as a transformation of the covariance matrix and first-moments vector that fully characterize it. As we will see, this is the case if we consider the linear and bilinear interactions used in quantum optics labs to generate and manipulate quantum states. Furthermore, the Gaussian properties of these states may be preserved also during the dissipative evolution through noisy channels, both Markovian and non-Markovian.

The main purpose of this tutorial is to introduce the reader to the phase-space description of Gaussian states in view of their applications to quantum information. After the definition of Gaussian state and its basic properties in relation with the covariance matrix and first-moments vector (Sect. 2), we will focus on the unitary evolution through suitable symplectic transformations (Sect. 3). We will illustrate the phase-space approach to describe the generation and manipulation of Gaussian states with linear and bilinear interaction of modes (Sect. 4). Single-mode Gaussian states

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will be briefly addressed in Sect. 5, while we will discuss two-mode Gaussian states in more details in Sect. 6. In the latter case, the concepts of symplectic eigenvalues as well as of standard form of the covariance matrix and local symplectic invariants will be introduced and applied to calculate the entropies and the mutual information of two-mode Gaussian states and to investigate their separability, entanglement and Gaussian quantum discord (Sect. 6). Dynamics through Markovian noisy channels will be addressed in Sect. 7 and Sect. 8 will consider the effect of a Gaussian measurement performed on a multimode Gaussian state. We will review the main results concerning the fidelity between Gaussian states in Sect. 9. Sect. 10 will close the tutorial with some concluding remarks.

2 Basic notation and definition of Gaussian state

Each mode $k = 1, \ldots, n$ of a system made of n modes or, more in general, n bosons is described by the annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, respectively, with commutation relations $[\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}$. The Hilbert space of the whole system, $\mathcal{H} = \otimes_{k=1}^n \mathcal{F}_k$, is the tensor product of the infinite dimensional Fock spaces $\mathcal{F}_k$ of the n modes, each spanned by the number basis $\{|m\rangle_k\}_{m\in\mathbb{N}}$, i.e., by the eigenstates of the number operator $\hat{a}_k^\dagger \hat{a}_k$. If we address non-interacting modes and use the natural units, the free Hamiltonian of the system may be simply written as $H = \sum_{k=1}^n (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2})$ and the corresponding position- and momentum-like operators for the k-th mode are:

$$\hat{q}_k = \frac{1}{\sqrt{2}} (\hat{a}_k + \hat{a}_k^\dagger), \quad \text{and} \quad \hat{p}_k = \frac{1}{i\sqrt{2}} (\hat{a}_k - \hat{a}_k^\dagger), \tag{1}$$

respectively. The commutation relations $[\hat{q}_k, \hat{p}_l] = i\,\delta_{kl}$ associated with $\hat{q}_k$ and $\hat{p}_l$ can be rewritten in the following compact form, which will turn out to be very useful for the phase-space analysis:

$$[\hat{R}_k, \hat{R}_l] = i\,\Omega_{kl}\ , \tag{2}$$

where $\hat{\boldsymbol{R}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T$ is a vector of operators and $\Omega_{kl} \equiv [\boldsymbol{\Omega}]_{kl}$ are the elements of the symplectic matrix:

$$\boldsymbol{\Omega} = \bigoplus_{k=1}^{n} \boldsymbol{\omega}\,, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Note that $\boldsymbol{\Omega}^T = -\boldsymbol{\Omega} = \boldsymbol{\Omega}^{-1}$.

We can now introduce the leading element of this tutorial. A n-mode state described by the density matrix ϱ is a Gaussian state if its characteristic function:

$$\chi[\varrho](\boldsymbol{\Lambda}) = \mathrm{Tr}[\varrho\, \exp\{-i\boldsymbol{\Lambda}^T \boldsymbol{\Omega} \hat{\boldsymbol{R}}\}] \tag{4}$$

is Gaussian, namely, if $\chi[\varrho](\boldsymbol{\Lambda})$ can be written in the following form:

$$\chi[\varrho](\boldsymbol{\Lambda}) = \exp\{-\tfrac{1}{2}\boldsymbol{\Lambda}^T \boldsymbol{\Omega}\boldsymbol{\sigma}\boldsymbol{\Omega}^T \boldsymbol{\Lambda} - i\boldsymbol{\Lambda}^T \boldsymbol{\Omega} \langle \hat{\boldsymbol{R}} \rangle\}\,, \tag{5}$$

with $\boldsymbol{\Lambda} = (\mathrm{a}_1, \mathrm{b}_1, \ldots, \mathrm{a}_n, \mathrm{b}_n)^T \in \mathbb{R}^{2n}$ and we defined the covariance matrix (CM):

$$\sigma_{kl} \equiv [\boldsymbol{\sigma}]_{kl} = \tfrac{1}{2}\langle \hat{R}_k, \hat{R}_l + \hat{R}_l, \hat{R}_k \rangle - \langle \hat{R}_k \rangle \langle \hat{R}_l \rangle\,, \tag{6}$$

and $\langle \hat{O} \rangle = \mathrm{Tr}[\varrho\, \hat{O}]$ is the expectation value of the operator $\hat{O}$. The vector $\langle \hat{\boldsymbol{R}} \rangle \equiv \mathrm{Tr}[\varrho \hat{\boldsymbol{R}}]$ is usually referred to as first-moments vector. The uncertainty relations among

canonical operators impose a constraint on the CM, corresponding to the inequality [6,7]:

$$\boldsymbol{\sigma} + \frac{i}{2}\,\boldsymbol{\Omega} \geq 0\,, \tag{7}$$

that expresses, in a compact form, the positivity of the density matrix ϱ.

The exponential appearing in Eq. (4) is called displacement operator:

$$\exp\{-i\boldsymbol{\Lambda}^T\boldsymbol{\Omega}\hat{\boldsymbol{R}}\} = D(\boldsymbol{\Lambda}) \equiv D(\boldsymbol{\lambda}) = \bigotimes_{k=1}^{n} D_k(\lambda_k)\,, \tag{8}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{C}^n$, $\lambda_k = \frac{1}{\sqrt{2}}(\mathrm{a}_k + i\mathrm{b}_k)$, and $D_k(\lambda_k) = \exp\{\lambda_k \hat{a}_k^\dagger - \lambda_k^* \hat{a}_k\}$ are single-mode displacement operators acting on the k-th mode. Displacement operator takes its name after its action on the mode operators, namely:

$$D^\dagger(\boldsymbol{\lambda})\,\hat{a}_k\,D(\boldsymbol{\lambda}) = \hat{a}_k + \lambda_k, \quad \text{and} \quad D^\dagger(\boldsymbol{\Lambda})\,\hat{\boldsymbol{R}}\,D(\boldsymbol{\Lambda}) = \hat{\boldsymbol{R}} + \boldsymbol{\Lambda}\,. \tag{9}$$

In other words, in phase space composed of the couples of conjugate variables $\hat{q}_k$ and $\hat{p}_k$, $k = 1, \ldots, n$, it displaces a state by an amount $\boldsymbol{\Lambda}$ [8].

By Fourier transforming the characteristic function (4), we obtain the so-called Wigner function of ϱ [9–11]:

$$W[\varrho](\boldsymbol{X}) = \frac{1}{(2\pi^2)^n}\int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{\Lambda}\,\exp\{i\boldsymbol{\Lambda}^T\boldsymbol{\Omega}\boldsymbol{X}\}\,\chi[\varrho](\boldsymbol{\Lambda})\,, \tag{10}$$

with $\boldsymbol{X} = (x_1, y_1, \ldots, x_n, y_n)^T$. Note that:

$$\frac{1}{\pi^{2n}}\int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{\Lambda}\,\exp\{i\boldsymbol{\Lambda}^T\boldsymbol{\Omega}\boldsymbol{X}\} = 2^n\,\delta^{(2n)}(\boldsymbol{X}), \tag{11}$$

$\delta^{(2n)}(\boldsymbol{X})$ being the $2n$-dimensional δ-function. From the identity:

$$\int_{\mathbb{R}^n} d^{2n}\boldsymbol{\Lambda}\exp\{-\tfrac{1}{2}\boldsymbol{\Lambda}^T\boldsymbol{Q}\boldsymbol{\Lambda} + i\boldsymbol{\Lambda}^T\boldsymbol{X}\} = \frac{(2\pi)^n\exp\{-\frac{1}{2}\boldsymbol{X}^T\boldsymbol{Q}^{-1}\boldsymbol{X}\}}{\sqrt{\det[\boldsymbol{Q}]}}\,, \tag{12}$$

where $\boldsymbol{Q}$ is a real, positive-definite symmetric $2n \times 2n$ matrix, it follows that in the case of the Gaussian state (5) we have:

$$W[\varrho](\boldsymbol{X}) = \frac{\exp\{-\frac{1}{2}(\boldsymbol{X} - \langle\hat{\boldsymbol{R}}\rangle)^T\boldsymbol{\sigma}^{-1}(\boldsymbol{X} - \langle\hat{\boldsymbol{R}}\rangle)\}}{\pi^n\sqrt{\det[\boldsymbol{\sigma}]}}\,, \tag{13}$$

that is still Gaussian. It is possible to show that pure Gaussian states are the only pure states with positive Wigner function [12,13].

The same Wigner function (13) can be calculated also as follows:

$$W[\varrho](\boldsymbol{X}) = \frac{1}{(2\pi^2)^n}\int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{\Lambda}\,\exp\{i\boldsymbol{\Lambda}^T\boldsymbol{X}\}\,\tilde{\chi}[\varrho](\boldsymbol{\Lambda})\,, \tag{14}$$

where $\tilde{\chi}[\varrho](\boldsymbol{\Lambda}) = \chi[\varrho](\boldsymbol{\Omega}\boldsymbol{\Lambda})$, namely:

$$\tilde{\chi}[\varrho](\boldsymbol{\Lambda}) = \exp\{-\tfrac{1}{2}\boldsymbol{\Lambda}^T\boldsymbol{\sigma}\boldsymbol{\Lambda} - i\boldsymbol{\Lambda}^T\langle\hat{\boldsymbol{R}}\rangle\}\,. \tag{15}$$

The equivalence between Eq. (10) and Eq. (14) is due to the fact that an equivalent definition of Eq. (11) is:

$$\frac{1}{\pi^{2n}}\int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{\Lambda}\,\exp\{i\boldsymbol{\Lambda}^T\boldsymbol{X}\} = 2^n\,\delta^{(2n)}(\boldsymbol{X}). \tag{16}$$

Eq. (13) is a particular case of the more general s-ordered Wigner function [9,10,14]:

$$W_s[\varrho](\boldsymbol{X}) = \frac{1}{(2\pi^2)^n} \int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{\Lambda}\ \exp\{\tfrac{1}{2}s|\boldsymbol{\Lambda}|^2 + i\boldsymbol{\Lambda}^T\boldsymbol{\Omega}\boldsymbol{X}\}\ \chi[\varrho](\boldsymbol{\Lambda}). \tag{17}$$

If $s = 0$, then we have the usual Wigner function (13); if $s = -1$ or $s = 1$ we obtain the Husimi Q-function or the Glauber-Sudarshan P-function, respectively [9,10]. Furthermore, by using the relation:

$$W_s[\varrho](\boldsymbol{X}) = \int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{Y}\ \frac{1}{\pi(r-s)} \exp\left\{-\frac{|\boldsymbol{Y}-\boldsymbol{X}|^2}{r-s}\right\} W_r[\varrho](\boldsymbol{Y}), \tag{18}$$

a r-ordered Wigner function can be transformed into a s-ordered one. The s-ordered Wigner function is used to define the nonclassical depth $\mathcal{T}$ of a quantum state [15]:

$$\mathcal{T} = \tfrac{1}{2}(1-\overline{s}), \tag{19}$$

where $\overline{s}$ is the maximum value for which $W_s[\varrho](\boldsymbol{X})$ becomes positive and semidefinite, i.e., a probability distribution. One has $\mathcal{T} = 1$ for number states and $\mathcal{T} = 0$ for coherent states. The nonclassical depth can be interpreted as the minimum number of thermal photons that has to be added to a quantum state in order to erase all the quantum features of state [15].

A Gaussian state is fully characterized by its CM and first-moments vector. For instance, the purity $\mu(\varrho) = \mathrm{Tr}[\varrho^2]$ of the Gaussian state depends only on its CM and reads:

$$\mu(\varrho) = \frac{1}{2^n\sqrt{\mathrm{Det}[\boldsymbol{\sigma}]}}, \tag{20}$$

where we used the trace rule in the phase space:

$$\mathrm{Tr}\left[\hat{O}_1\,\hat{O}_2\right] = \left(\frac{\pi}{2}\right)^n \int_{R^{2n}} d^{2n}\boldsymbol{X}\ W[\hat{O}_1](\boldsymbol{X})\, W[\hat{O}_2](\boldsymbol{X}), \tag{21}$$

which follows from the expansion:

$$\hat{O} = \int_{R^{2n}} d^{2n}\boldsymbol{X}\ W[\hat{O}](\boldsymbol{X})\ D(\boldsymbol{X})\boldsymbol{\Pi}D^{\dagger}(\boldsymbol{X}), \tag{22}$$

where $\boldsymbol{\Pi} = \otimes_{k=1}^{n}(-1)^{\hat{a}^{\dagger}\hat{a}}$ is the parity operator and $D(\boldsymbol{X})\boldsymbol{\Pi}D^{\dagger}(\boldsymbol{X}) = D(2\boldsymbol{X})\boldsymbol{\Pi} = \boldsymbol{\Pi}D^{\dagger}(2\boldsymbol{X})$, or, equivalently, by using the characteristic function formalism:

$$\mathrm{Tr}\left[\hat{O}_1\,\hat{O}_2\right] = \frac{1}{(2\pi)^n} \int_{R^{2n}} d^{2n}\boldsymbol{\Lambda}\ \chi[\hat{O}_1](\boldsymbol{\Lambda})\ \chi[\hat{O}_2](-\boldsymbol{\Lambda}), \tag{23}$$

which follows from:

$$\hat{O} = \frac{1}{(2\pi)^n} \int_{R^{2n}} d^{2n}\boldsymbol{\Lambda}\ \chi[\hat{O}](\boldsymbol{\Lambda})\ D^{\dagger}(\boldsymbol{\Lambda}). \tag{24}$$

We recall also that $\mathrm{Tr}[D(\boldsymbol{\Lambda})] = (2\pi)^n\delta^{(2n)}(\boldsymbol{\Lambda})$ and $\mathrm{Tr}[D(\boldsymbol{X})] = (2\pi)^n\delta^{(2n)}(\boldsymbol{X})$. Starting from Eq. (22) we can also obtain the trace form for the Wigner function:

$$W[\hat{O}] = \left(\frac{2}{\pi}\right)^n \mathrm{Tr}[\hat{O}\ D(\boldsymbol{X})\boldsymbol{\Pi}D^{\dagger}(\boldsymbol{X})]. \tag{25}$$

Note that the identity operator for n modes has a Wigner function given by $W[\mathbb{I}](\boldsymbol{X}) = \pi^{-n}$, thus, form Eq. (21) we have $\mathrm{Tr}[\hat{O}] = 2^{-n} \int_{\mathbb{C}^n} d^{2n}\boldsymbol{X}\, W[\hat{O}](\boldsymbol{X})$, from which follows the normalization of the Wigner function (13). The Wigner function formalism allows to easily calculate the expectations of symmetrically ordered products of field operators [16], namely:

$$\mathrm{Tr}\left[\varrho\left[(\hat{a}_s^\dagger)^h\,\hat{a}_t^k\right]_{\mathrm{s}}\right] = \frac{1}{2^n}\int_{\mathbb{R}^{2n}} d^{2n}\boldsymbol{X}\, W[\varrho](\boldsymbol{X})\,(\alpha_s^*)^h\,\alpha_t^k\,, \tag{26}$$

with, as usual, $\boldsymbol{X} = (x_1, y_1, \ldots, x_n, y_n)^T$, $\alpha_k = \frac{1}{\sqrt{2}}(x_k + i y_k)$, and:

$$\left[(\hat{a}_s^\dagger)^h\,\hat{a}_t^k\right]_{\mathrm{s}} = \frac{\partial^{h+k}}{\partial x^h\,\partial y^k}\frac{(x\,\hat{a}_s^\dagger + y\,\hat{a}_t)^{k+h}}{(k+h)!}\bigg|_{x=y=0}\,. \tag{27}$$

For the sake of completeness, we observe that the expectations in Eq. (26) can be also obtained starting from the characteristic function expressed in complex notation:

$$\mathrm{Tr}\left[\varrho\left[(\hat{a}_s^\dagger)^h\,\hat{a}_t^k\right]_{\mathrm{s}}\right] = (-1)^k\frac{\partial^{h+k}}{\partial\lambda_s^h\,\partial\lambda_t^{*k}}\,\chi[\varrho](\boldsymbol{\lambda})\bigg|_{\boldsymbol{\lambda}=\boldsymbol{0}} \tag{28}$$

where $\chi[\varrho](\boldsymbol{\lambda}) = \mathrm{Tr}[\varrho\, D(\boldsymbol{\lambda})]$ and $D(\boldsymbol{\lambda})$ has been defined in Eq. (8): since its derivatives in the origin of the complex plane generates symmetrically ordered moments of mode operators, the characteristic function is also known as the moment-generating function of the signal ϱ.

In order to become more familiar with the covariance matrix formalism, we consider the multi-mode state at thermal equilibrium at temperature T described by the density matrix $\nu = \bigotimes_{k=1}^n \nu_{\mathrm{th}}(N_k)$ with:

$$\nu_{\mathrm{th}}(N_k) = \frac{e^{-\beta_k\hat{a}_k^\dagger\hat{a}_k}}{\mathrm{Tr}\left[e^{-\beta_k\hat{a}_k^\dagger\hat{a}_k}\right]} = \frac{N_k^{\hat{a}_k^\dagger\hat{a}_k}}{(1+N_k)^{\hat{a}_k^\dagger\hat{a}_k+1}}, \tag{29}$$

$$= \frac{1}{1+N_k}\sum_{m=0}^{\infty}\left(\frac{N_k}{1+N_k}\right)^m\,|m\rangle_{kk}\langle m|\,, \tag{30}$$

where $\beta_k = \hbar\omega_k/(k_{\mathrm{B}}T)$, k_{B} being the Boltzmann constant, and $N_k = (e^{\beta_k}-1)^{-1}$ is the average number of quanta in the k-th mode with frequency ω_k. Its CM $\boldsymbol{\sigma}_\nu$ turns out to be diagonal and reads:

$$\boldsymbol{\sigma}_\nu = \bigoplus_{k=1}^n \boldsymbol{\sigma}_{\mathrm{th}}(N_k), \tag{31}$$

where $\boldsymbol{\sigma}_{\mathrm{th}}(N_k) = \frac{1}{2}(1+2N_k)\mathbb{1}_2$ is the 2×2 CM of the k-th single-mode thermal state with N_k average photons and $\mathbb{1}_m$ is the $m\times m$ identity matrix. Moreover, recalling that $\langle n|D(\lambda)|n\rangle = e^{-\frac{1}{2}|\lambda|^2}L_n(|\lambda|^2)$, $L_n(z)$ being Laguerre polynomials, one can easily calculate the expression of the characteristic function of the thermal state (30), that turns out to be a Gaussian state. Note that if $N_k \to 0\ \forall k$, then $\boldsymbol{\sigma}_\nu \to \frac{1}{2}\mathbb{1}_{2n}$, that is the CM of the vacuum state of n bosons.

More in general, the $2n\times 2n$ CM $\boldsymbol{\Sigma}_{\boldsymbol{v}}$ of a n-mode Gaussian state $\varrho_{\boldsymbol{v}}$, $\boldsymbol{v} = \{1,\ldots,n\}$, can be re-written in the following block form:

$$\boldsymbol{\Sigma}_{\boldsymbol{v}} = \begin{pmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{\delta}_{12} & \cdots & \boldsymbol{\delta}_{1n} \\ \boldsymbol{\delta}_{12}^T & \boldsymbol{\sigma}_2 & \cdots & \boldsymbol{\delta}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\delta}_{1n}^T & \boldsymbol{\delta}_{2n}^T & \cdots & \boldsymbol{\sigma}_n \end{pmatrix}, \tag{32}$$

where $\boldsymbol{\sigma}_k$ and $\boldsymbol{\delta}_{hk}$ are 2×2 real matrices. In particular, $\boldsymbol{\sigma}_k$ corresponds to the CM of the state $\varrho_k = \mathrm{Tr}_{\boldsymbol{v}\setminus\{k\}}[\varrho_{\boldsymbol{v}}]$ and $\boldsymbol{\delta}_{hk}$ is related to the (classical or quantum) correlations between the modes h ad k: if $\boldsymbol{\delta}_{hk} = \mathbf{0}$, then $\varrho_{hk} = \mathrm{Tr}_{\boldsymbol{v}\setminus\{h,k\}}[\varrho_{\boldsymbol{v}}] = \varrho_h \otimes \varrho_k$, that is the two modes are uncorrelated, and the CM:

$$\boldsymbol{\Sigma}_{hk} = \begin{pmatrix} \boldsymbol{\sigma}_h & \boldsymbol{\delta}_{hk} \\ \boldsymbol{\delta}_{hk}^T & \boldsymbol{\sigma}_k \end{pmatrix} \tag{33}$$

of the state ϱ_{hk} reduces to the direct sum of the two single-mode CMs, namely, $\boldsymbol{\Sigma}_{hk} = \boldsymbol{\sigma}_h \oplus \boldsymbol{\sigma}_k$.

3 Evolution of Gaussian states

When an evolution preserves the Gaussian character of a state, it can be described with suitable transformations of the position- and momentum-like operators or, equivalently, of $\hat{\boldsymbol{R}}$, that preserve the commutation relations (2). These transformations are called symplectic transformations and are the main tool used to describe the kinematics of Gaussian states in the phase space.

First of all, we recall that the equations of motion of a classical system of n particles described by coordinates $\{q_1,\dots,q_n\}$ and conjugated momenta $\{p_1,\dots,p_n\}$ with Hamiltonian H can be summarized as:

$$\dot{R}_k = \Omega_{kl}\,\frac{\partial H}{\partial R_l}, \tag{34}$$

where $\boldsymbol{R} = (q_1, p_1, \dots, q_n, p_n)^T$ and $\dot{x}$ denotes time derivative and $\boldsymbol{\Omega}$ is the symplectic matrix defined in Eq. (3). Given a transformation of coordinates $\boldsymbol{R} \to \boldsymbol{R'} \equiv \boldsymbol{F}\boldsymbol{R}$, one has:

$$\dot{R}'_k = F_{ks}\Omega_{st}F_{lt}\,\frac{\partial H}{\partial R'_l}\,, \tag{35}$$

and thus the equations of motions remain invarians if and only if:

$$\boldsymbol{F}\,\boldsymbol{\Omega}\boldsymbol{F}^T = \boldsymbol{\Omega}\,, \quad \text{(symplectic condition)} \tag{36}$$

which characterizes the symplectic transformations and, in turn, describes the canonical transformations of coordinates.

Form the quantum mechanical point of view, a mode transformation $\hat{\boldsymbol{R}}' = \boldsymbol{F}\hat{\boldsymbol{R}}$ leaves the kinematics invariant if it preserves canonical commutation relations (2): the $2n\times 2n$ matrix $\boldsymbol{F}$ should satisfy the symplectic condition (36). It is worth noting that if $\boldsymbol{F}$ and $\boldsymbol{G}$ are symplectic transformations, then also $\boldsymbol{F}^T$, $\boldsymbol{F}^{-1} = \boldsymbol{\Omega}\boldsymbol{F}^T\boldsymbol{\Omega}^{-1}$ and $\boldsymbol{F}\boldsymbol{G}$ are symplectic: the set of the $2n\times 2n$ matrices satisfying the condition Eq. (36) form the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$.

An important theorem due to J. Williamson [17] guarantees that every CM can be diagonalized through a symplectic transformation [18]. More in detail, if $\{d_k\}_{k=1}^n$ is the set of the symplectic eigenvalues of the $2n\times 2n$ CM $\boldsymbol{\sigma}$, namely the moduli of the eigenvalues $\{\pm d_k\}_{k=1}^n$ of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$, where $\boldsymbol{\Omega}$ is given in Eq. (3), then:

$$\boldsymbol{\sigma} = \boldsymbol{S}\,\boldsymbol{W}\,\boldsymbol{S}^T, \tag{37}$$

where $\boldsymbol{W} = \bigoplus_{k=1}^n d_k\,\mathbb{1}_2$ is a n-mode thermal state with $N_k = d_k - \frac{1}{2}$ average number of photon in the k-th mode [see Eq. (31)], and $\boldsymbol{S}$ is the matrix which performs the

symplectic diagonalization (as we will see in the following, if $\boldsymbol{\sigma}$ describes a physical state, then $d_k \geq 1/2$, and, thus, $N_k \geq 0$). Now, since the whole set of the symplectic transformations is generated by Hamiltonians which are linear and bilinear in the field modes [19,20], the physical statement implied by decomposition (37) is that every Gaussian state ϱ can be obtained from a thermal state ν by performing the unitary transformation $U_{\boldsymbol{S}}$ associated with the symplectic matrix $\boldsymbol{S}$, which, in turn, can be generated by linear and bilinear interactions. Hence, the density matrix corresponding to the decomposition (37) can be written as:

$$\varrho = U_{\boldsymbol{S}}\, \nu\, U_{\boldsymbol{S}}^{\dagger}\,. \tag{38}$$

By using the uncertainty relation (7), which is invariant under the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$, and the decomposition (37), we have:

$$\boldsymbol{S}\,\boldsymbol{W}\,\boldsymbol{S}^{T} + \frac{i}{2}\boldsymbol{\Omega} \geq 0 \;\Rightarrow\; \boldsymbol{W} \geq -\frac{i}{2}\boldsymbol{\Omega} \;\Rightarrow\; d_k \geq \frac{1}{2},\, \forall k, \tag{39}$$

that is the constraint on the CM by the uncertainty relation leads to the constraints $d_k \geq 1/2$ on its symplectic eigenvalues. From Eq. (20) it is straightforward to see that a Gaussian state is pure if and only if $d_k = 1/2$.

4 Linear and bilinear Hamiltonians

In order to preserve Gaussian states, a Hamiltonian should be linear or bilinear in the fields mode [1]. This kind of Hamiltonian can be experimentally realized by means of parametric processes in quantum optical [21,22], optomechanical [23,24], micromechanical [25] and cold gases [26–31] systems. Though the actual realization of these transformations necessarily involves parametric interactions in nonlinear media, their quantum optical implementation is often referred to as quantum information processing with linear optics, according to the linearity of mode evolution.

The most general Hamiltonian of this kind can be written as:

$$H = \sum_{k=1}^{n} g_k^{(1)}\, \hat{a}_k^{\dagger} + \sum_{k\geq l=1}^{n} g_{kl}^{(2)}\, \hat{a}_k^{\dagger}\hat{a}_l + \sum_{k,l=1}^{n} g_{kl}^{(3)}\, \hat{a}_k^{\dagger}\hat{a}_l^{\dagger} + h.c., \tag{40}$$

and contains three main building blocks, which represent the generators of the corresponding unitary evolutions to be described in the following subsections. The mode transformation imposed by the Hamiltonian (40) and, thus, the evolution of the vector $\hat{\boldsymbol{R}}$ and of the CM $\boldsymbol{\sigma}$ writes:

$$\hat{\boldsymbol{R}} \to \boldsymbol{F}\hat{\boldsymbol{R}} + \boldsymbol{d}, \quad \text{and} \quad \boldsymbol{\sigma} \to \boldsymbol{F}\,\boldsymbol{\sigma}\,\boldsymbol{F}^{T}, \tag{41}$$

where $\boldsymbol{d}$ is a real vector and $\boldsymbol{F}$ a symplectic transformations. Remarkably, the converse is also true, i.e., any symplectic transformation of the form (41) is generated by a unitary transformation induced by Hamiltonians of the form (40) [19,20]. In this context, it is worth noting that a useful decomposition of a generic symplectic transformation $\boldsymbol{F}$ is the following:

$$\boldsymbol{F} = \boldsymbol{O}\begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}^{-1} \end{pmatrix}\boldsymbol{O'}\,, \quad \text{(Euler decomposition)} \tag{42}$$

where $\boldsymbol{O}$ and $\boldsymbol{O'}$ are orthogonal and symplectic matrices, while $\boldsymbol{D}$ is a positive diagonal matrix. The physical implication of the Euler decomposition (42) is that every symplectic transformation may be implemented by means of two passive devices (described by the orthogonal matrices $\boldsymbol{O}$ and $\boldsymbol{O'}$) and by single-mode squeezers (described by $\boldsymbol{D}$) [32] to be addressed in the following subsections.

4.1 Displacement operator and coherent states

The first block of the Hamiltonian in Eq. (40) contains terms of the form $H \propto g^{(1)}\hat{a}^\dagger + h.c.$ and is linear in the field modes. The corresponding unitary transformations are the set of displacement operators we used in Sect. 2 to define the characteristic function. The comparison between Eqs. (9) and (41) shows that the CM is left unchanged by the displacement operator while the first-moments vector is displaced.

The displacement operator is strictly connected with coherent states [8]. For a single mode, coherent states are defined as the eigenstates of the mode operator, i.e., $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C}$ is a complex number. Using Eq. (9), it can be shown that coherent states may be defined also as $|\alpha\rangle = D(\alpha)|0\rangle$, that is the unitary evolution of the vacuum through the displacement operator. Properties of coherent states, such as the overcompleteness and the nonorthogonality directly follow from those of the displacement operator.

Since the CM of coherent states is the same as the vacuum state one [see Eq. (31) with $N_k = 0$], they are minimum uncertainty states, i.e., they fulfill Ineq. (7) with equality sign and, in addition, with uncertainties that are equal for the position- and momentum-like operators (this is directly seen from the CM).

4.2 Free evolution and two-mode mixing

The second block appearing in the Hamiltonian (40), i.e., $\sum_{k\geq l=1}^{n} g_{kl}^{(2)}\,\hat{a}_k^\dagger \hat{a}_l$, represents two different physical processes.

4.2.1 Phase shift

The first process refers to the terms proportional to $g^{(2)}\hat{a}_k^\dagger\hat{a}_k$ and describes the free evolution of the modes: in most cases these terms can be eliminated by choosing a suitable interaction picture. The effect of free evolution is to add an overall phase shift that, for single-mode fields, has no physical meaning, but it is of extreme relevance in the case of interference phenomena involving different beams of light, such as the interferometric scheme used to implement the homodyne detection. The evolution operator may be written as $U(\theta) = \exp\{-i\theta\,\hat{a}_k^\dagger\hat{a}_k\}$ and acts as a phase rotation on the field mode $\hat{a}_k$, i.e., $U^\dagger(\theta)\,\hat{a}_k\,U(\theta) = e^{-i\theta}\,\hat{a}_k$. Hence, the corresponding symplectic matrix reads:

$$\boldsymbol{\mathcal{R}}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{43}$$

and the evolution of the first-moments vector and the single-mode CM follows form Eqs. (41), with $\boldsymbol{d} = \boldsymbol{0}$.

4.2.2 Two-mode mixing

The second process, involving different mode operators, describes a linear mixing of two modes and, in the quantum optics context, the simplest example corresponds to a Hamiltonian of the form $H \propto \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}$, where for the sake of simplicity we consider a system of two modes $\hat{a} \equiv \hat{a}_1$ and $\hat{b} \equiv \hat{a}_2$. This Hamiltonian describes the action of a beam splitter, i.e., the interaction taking place in a linear optical medium such as a dielectric plate. The evolution operator can be recast in the form:

$$U(\zeta) = \exp\{\zeta\hat{a}^\dagger\hat{b} - \zeta^*\hat{a}\hat{b}^\dagger\}, \tag{44}$$

where the coupling $\zeta = \phi\, e^{i\theta} \in \mathbb{C}$ is proportional to the interaction length (time) and to the linear susceptibility of the medium. The two-mode mixer is a "passive" device, i.e., the total number of quanta in the two modes is a constant of motion.

The Heisenberg evolutions of modes $\hat{a}$ and $\hat{b}$ are given by:

$$U^\dagger(\zeta)\, \hat{a}\, U(\zeta) = \cos\phi\, \hat{a} + e^{i\theta} \sin\phi\, \hat{b}\,, \tag{45a}$$

$$U^\dagger(\zeta)\, \hat{b}\, U(\zeta) = \cos\phi\, \hat{b} - e^{-i\theta} \sin\phi\, \hat{a}\,, \tag{45b}$$

respectively, and the corresponding symplectic matrix $\boldsymbol{S}_\zeta$ reads [2]:

$$\boldsymbol{S}_\zeta = \begin{pmatrix} \cos\phi\, \mathbb{1}_2 & \sin\phi\, \boldsymbol{\mathcal{R}}_\theta \\ -\sin\phi\, \boldsymbol{\mathcal{R}}_\theta^T & \cos\phi\, \mathbb{1}_2 \end{pmatrix}, \tag{46}$$

where the 2×2 matrix $\boldsymbol{\mathcal{R}}_\theta$ is defined in Eq. (43). The first-moments vector and two-mode CMs evolve as usual according to Eqs. (41), only with $\boldsymbol{d} = \boldsymbol{0}$.

4.3 Single-mode squeezing

In the particular case of quantum optics, the last block of the Hamiltonian (40) describes $\chi^{(2)}$ nonlinear interactions in which a photon in the input (pump) is converted into two photons, conserving both the energy and the momentum. If the so-called phase-matching conditions are arranged in order to emit the two photons into the same mode $\hat{a}$, we obtain the single-mode squeezing transformations, which, thus, correspond to Hamiltonians of the form $H \propto (\hat{a}^\dagger)^2 + h.c.$ [58]. Squeezing has been firstly introduced for quadrature operators and refers to a phenomenon in which an observable or a set of observables exhibit a second moment below the corresponding vacuum level [33].

The single-mode squeezing operator is usually written as:

$$S(\xi) = \exp\left\{\tfrac{1}{2}\left[\xi(\hat{a}^\dagger)^2 - \xi^* \hat{a}^2\right]\right\}, \tag{47}$$

which corresponds to the following mode evolutions:

$$S^\dagger(\xi)\, \hat{a}\, S(\xi) = \cosh r\, \hat{a} + e^{i\psi} \sinh r\, \hat{a}^\dagger, \tag{48a}$$

$$S^\dagger(\xi)\, \hat{a}^\dagger\, S(\xi) = \cosh r\, \hat{a}^\dagger + e^{-i\psi} \sinh r\, \hat{a}\,, \tag{48b}$$

with $\xi = r e^{i\psi}$. By using the mode transformation in Eqs. (48) and the definition of the quadrature operators (1), it is straightforward to find the symplectic transformation $\boldsymbol{\Sigma}_\xi$ describing the single-mode squeezing, namely:

$$\boldsymbol{\Sigma}_\xi = \cosh r\, \mathbb{1}_2 + \boldsymbol{R}_\xi \quad \text{with} \quad \boldsymbol{R}_\xi = \sinh r \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}, \tag{49}$$

which allows to calculate the evolution of the first-moments vector and CM according to Eqs. (41), but with $\boldsymbol{d} = \boldsymbol{0}$.

4.4 Two-mode squeezing

The two-mode squeezing transformations correspond to Hamiltonians of the form $H \propto \hat{a}^\dagger \hat{b}^\dagger + h.c.$ and describe $\chi^{(2)}$ nonlinear interactions introduced in the previous subsection but with the two photons emitted in different modes. The evolution operator is usually written as:

$$S_2(\xi) = \exp\{\xi \hat{a}^\dagger \hat{b}^\dagger - \xi^* \hat{a}\hat{b}\} \,. \tag{50}$$

The corresponding evolutions of the two modes read:

$$S_2^\dagger(\xi)\, \hat{a}\, S_2(\xi) = \cosh r\, \hat{a} + e^{i\psi} \sinh r\, \hat{b}^\dagger, \tag{51a}$$

$$S_2^\dagger(\xi)\, \hat{b}\, S_2(\xi) = \cosh r\, \hat{b} - e^{i\psi} \sinh r\, \hat{a}^\dagger, \tag{51b}$$

where $\xi = re^{i\psi}$ and the symplectic transformation associated with the two-mode squeezer is represented by the block matrix:

$$\boldsymbol{\Sigma}_{2\xi} = \begin{pmatrix} \cosh r\, \mathbb{1}_2 & \boldsymbol{R}_\xi \\ \boldsymbol{R}_\xi & \cosh r\, \mathbb{1}_2 \end{pmatrix}, \tag{52}$$

where $\boldsymbol{R}_\xi$ is defined as in (49) and, as usual, the CM $\boldsymbol{\sigma}$ and the first-moments vector of a bipartite input state transform according to Eqs. (41) with $\boldsymbol{d} = \boldsymbol{0}$.

5 Single-mode Gaussian states

In the case of a single-mode Gaussian state, Eq. (38) may be written as [34]:

$$\varrho = D(\alpha) S(\xi)\, \nu_{\rm th}(N)\, S^\dagger(\xi) D^\dagger(\alpha) \,, \tag{53}$$

and the corresponding CM and first-moments vector can be easily calculated by applying the phase-space analysis introduced in Sect. 4. In particular the CM can be calculated applying the squeezing transformation (49) to the CM $\boldsymbol{\sigma}_{\rm th}(N)$ of the thermal state $\nu_{\rm th}(N)$, namely $\boldsymbol{\sigma} = \boldsymbol{\Sigma}_\xi\, \boldsymbol{\sigma}_{\rm th}(N)\, \boldsymbol{\Sigma}_\xi^T$. The explicit expressions of its elements are:

$$\sigma_{kk} = \frac{1+2N}{2}\, [\cosh(2r) - (-1)^k \sinh(2r) \cos\psi], \quad (k = 1, 2) \tag{54a}$$

$$\sigma_{12} = \sigma_{21} = \frac{1+2N}{2}\, \sinh(2r) \sin\psi \,, \tag{54b}$$

with $\xi = re^{i\psi}$, while the first-moments vector reads $\langle \hat{\boldsymbol{R}} \rangle = \sqrt{2}(\Re{\rm e}[\alpha], \Im{\rm m}[\alpha])^T$. By using Eq. (20) we can calculate the purity of the Gaussian state (53), that reads $\mu(\varrho) = (1+2N)^{-1}$: the purity of a generic single-mode Gaussian state depends only on the average number of thermal photons, as one should expect since displacement and squeezing are unitary operations and, thus, do not affect the purity of a state. The same observation holds when we address the von Neumann entropy:

$$S_V(\varrho) = -{\rm Tr}[\varrho \ln \varrho] \,. \tag{55}$$

For a single-mode Gaussian state we have:

$$S_V(\varrho) = f(\sqrt{\det[\boldsymbol{\sigma}]}), \tag{56}$$

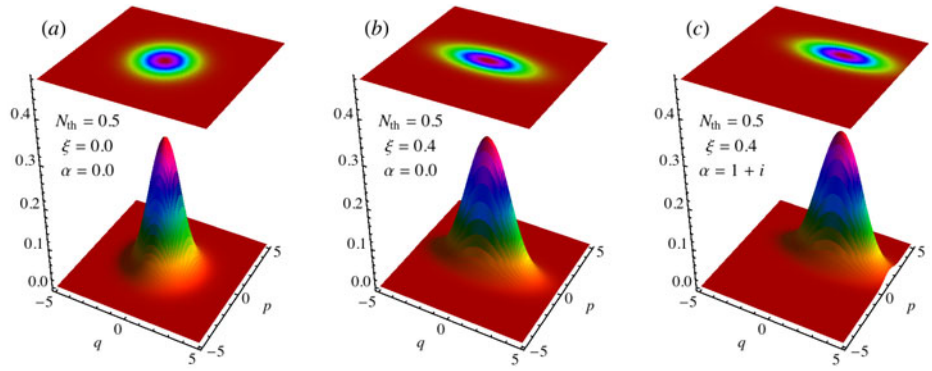

Fig. 1. (Color online) Plots of the Wigner function of the single-mode Gaussian state $\varrho = D(\alpha)\, S(\xi)\, \nu(N_{\rm th})\, S^{\dagger}(\xi)\, D^{\dagger}(\alpha)$, for different values of $N_{\rm th}$, ξ and α: (a) thermal state, (b) squeezed thermal state and (c) displaced squeezed thermal state.

where:

$$f(x) = \left(x + \tfrac{1}{2}\right) \ln\left(x + \tfrac{1}{2}\right) - \left(x - \tfrac{1}{2}\right) \ln\left(x - \tfrac{1}{2}\right) \,, \tag{57}$$

and $\sqrt{\det[\boldsymbol{\sigma}]} = N + \frac{1}{2} = [2\mu(\varrho)]^{-1}$ corresponds to the only symplectic eigenvalue of the 2×2 positive-definite symmetric matrix $\boldsymbol{\sigma}$, since, in this case, the two eigenvalues of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ are $d_{\pm} = \pm\sqrt{\det[\boldsymbol{\sigma}]}$, as follows from the Williamson's theorem applied to a single-mode CM.

Starting from Eqs. (53) and (54) one can obtain the CMs of the coherent state (by setting $N_{\rm th} = \xi = 0$) or of the squeezed vacuum state (with $\alpha = \xi = 0$). In Fig. 1 the Wigner function of the Gaussian state (53) is plotted for different values of the involved parameters.

6 Two-mode Gaussian states

We can identify different classes of two-mode or, more in general, bipartite Gaussian states. However, since these states are the simplest scenario where to investigate the fundamental issue of entanglement in quantum information, it is useful to introduce equivalence classes of Gaussian states with the same amount of entanglement, i.e., Gaussian states whose CMs are connected by local symplectic transformations and, thus, are locally equivalent. If we write the CM of a two-mode Gaussian state as:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^{T} & \boldsymbol{B} \end{pmatrix} \,, \tag{58}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are 2×2 matrices, then we can define four local symplectic invariants, i.e., quantities that are left unchanged by local symplectic transformations:

$$I_1 = \det[\boldsymbol{A}], \quad I_2 = \det[\boldsymbol{B}], \quad I_3 = \det[\boldsymbol{C}], \quad I_4 = \det[\boldsymbol{\sigma}] \,. \tag{59}$$

The CMs of locally equivalent states can be reduced to the following standard or normal form [35,36]:

$$\boldsymbol{\sigma} = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix} \,, \tag{60}$$

where the values of a, b, c_1, and c_2 are determined by the local symplectic invariants (59), namely $a^2 = I_1$, $b^2 = I_2$, $c_1 c_2 = I_3$ and $(ab - c_1^2)(ab - c_2^2) = I_4$.

The two symplectic eigenvalues of the CM of a generic two-mode Gaussian state can be computed in terms of the symplectic invariants [37]:

$$d_{\pm} = \sqrt{\frac{\Delta(\boldsymbol{\sigma}) \pm \sqrt{\Delta(\boldsymbol{\sigma})^2 - 4I_4}}{2}}\,, \tag{61}$$

with $\Delta(\boldsymbol{\sigma}) = I_1 + I_2 + 2I_3$ and, in turn, the uncertainty relation (39) reduces to:

$$d_- \geq 1/2\,. \tag{62}$$

Note that for a pure two-mode Gaussian state we have $I_4 = 1/16$ and $\Delta(\boldsymbol{\sigma}) = 1/2$, i.e., a pure Gaussian state has minimum uncertainty. Moreover, bipartite pure states necessarily have a symmetric normal form, i.e., $a = b$ in Eq. (60), as can be seen by equating the entropies of the subsystems.

A relevant subclass of Gaussian states is formed by the two-mode squeezed thermal states (for a general parameterization of an arbitrary bipartite Gaussian state, by means of a proper symplectic diagonalization, see Ref. [37]), i.e., states generated by applying the two-mode squeezing operator to a two-mode thermal state, namely:

$$\varrho = S_2(\xi)\, \nu_{\rm th}(N_1) \otimes \nu_{\rm th}(N_2)\, S_2^{\dagger}(\xi)\,. \tag{63}$$

We can calculate the CM of the state (63) as $\boldsymbol{\sigma} = \boldsymbol{\Sigma}_{2\xi} \boldsymbol{\sigma}_{\nu} \boldsymbol{\Sigma}_{2\xi}^{T}$, where $\boldsymbol{\Sigma}_{2\xi}$ is the symplectic two-mode squeezing matrix (52) and $\boldsymbol{\sigma}_{\nu}$ is the CM of the thermal state ν given in Eq. (31) with $n = 2$. In formula,

$$\boldsymbol{\sigma} = \frac{1}{2} \begin{pmatrix} A\, \mathbb{1}_2 & C\, \boldsymbol{R}_{\xi} \\ C\, \boldsymbol{R}_{\xi} & B\, \mathbb{1}_2 \end{pmatrix}\,, \tag{64}$$

$\boldsymbol{R}_{\xi}$ being defined in Eq. (52) and, if we assume $\xi = r \in \mathbb{R}$:

$$A \equiv A(r, N_1, N_2) = (1 + N_1 + N_2) \cosh(2r) + (N_1 - N_2)\,, \tag{65a}$$

$$B \equiv B(r, N_1, N_2) = (1 + N_1 + N_2) \cosh(2r) - (N_1 - N_2)\,, \tag{65b}$$

$$C \equiv C(r, N_1, N_2) = (1 + N_1 + N_2)\, \sinh(2r)\,. \tag{65c}$$

In particular, if $N_1 = N_2 = 0$ we have so-called twin-beam state (TWB) or two-mode squeezed vacuum, that plays a leading role in quantum information with continuous variable. The first name, TWB, refers to the fact that it shows perfect correlation in the photon number, i.e., it is an eigenstate of the photon number difference $\hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}$, which is a constant of motion as the reader can verify. The second name is instead connected with a duality under the action of a balanced beam splitter, since one has:

$$U^{\dagger}(\tfrac{\pi}{4}\, e^{i\theta})\, S_2(\xi)\, U(\tfrac{\pi}{4}\, e^{i\theta}) = S(\xi e^{i\theta}) \otimes S(-\xi e^{-i\theta})\,, \tag{66}$$

where $U(\zeta = \frac{\pi}{4}\, e^{i\theta})$ is the evolution operator of Eq. (44) for a balanced beam splitter, $S_2(\xi)$ is the two-mode squeezing operator (50), and $S(\xi)$ is the single-mode squeezing operator of Eq. (47) acting on the evolved mode out of the mixer. In other words, a TWB entering a balanced beam splitter evolves into a factorized state composed of two squeezed vacua with opposite squeezing phases [38] and, *viceversa*, a TWB may be generated by using single-mode squeezers and a linear mixer as in the first continuous variable teleportation experiment [39].

6.1 Entropies and mutual information

6.1.1 Von Neumann entropy

As we have seen in Sect. 3, a consequence of Williamson's theorem is that every Guassian state can be generated acting on a thermal state with unitary transformations. Thus, the von Neumann entropy of a generic Gaussian state reduces to that of the thermal state obtained from it by symplectic diagonalization, since unitary operations do not affect the entropy of the whole state. In the case of a two-mode Gaussian state ϱ_{AB} with CM $\boldsymbol{\sigma}$, using Eq. (56) and the additivity of von Neumann entropy (55) for tensor product states, i.e., $S_V(\varrho_A \otimes \varrho_B) = S_V(\varrho_A) + S_V(\varrho_B)$, we obtain:

$$S_V(\varrho_{AB}) = f(d_+) + f(d_-)\,, \tag{67}$$

where $f(x)$ has been defined in Eq. (57) and $d_\pm$ are the symplectic eigenvalues of $\boldsymbol{\sigma}$ written explicitly in Eq. (61).

6.1.2 Mutual information and conditional entropies

Starting from the von Neumann entropies of the state ϱ_{AB} and of the two subsystems $\varrho_A = \mathrm{Tr}_B[\varrho_{AB}]$ and $\varrho_B = \mathrm{Tr}_A[\varrho_{AB}]$ it is possible to assess how much information about ϱ_{AB} one can obtain by addressing the single parties. This is of course related to the correlations between the two modes and can be quantified by means of the quantum mutual information or the conditional entropies [40].

The quantum mutual information is defined as:

$$I_M\left(\varrho_{AB}\right) = S_V\left(\varrho_A\right) + S_V\left(\varrho_B\right) - S_V\left(\varrho_{AB}\right), \tag{68}$$

and can be easily expressed in terms of the symplectic invariants (59) of $\boldsymbol{\sigma}$ and its symplectic eigenvalues (61) as follows:

$$I_M(\varrho_{AB}) = f(\sqrt{I_1}) + f(\sqrt{I_2}) - f(d_+) - f(d_-). \tag{69}$$

Note that $f(\sqrt{I_1}) = S_V(\varrho_A)$ and $f(\sqrt{I_2}) = S_V(\varrho_B)$, since ϱ_A and ϱ_b are a single-mode Gaussian states (see Sect. 5). It is also worth noting that, in the case of pure states, the entropies $S_V(\varrho_A) = S_V(\varrho_B)$ correspond to the unique measure of entanglement for pure bipartite states [41].

The conditional entropies are defined as:

$$\begin{aligned} S_{A|B}(\varrho_{AB}) &= S_V(\varrho_{AB}) - S_V(\varrho_B)\,, \\ &= f(d_+) + f(d_-) - f(\sqrt{I_2})\,, \end{aligned} \tag{70a}$$

$$\begin{aligned} S_{B|A}(\varrho_{AB}) &= S_V(\varrho_{AB}) - S_V(\varrho_A) \\ &= f(d_+) + f(d_-) - f(\sqrt{I_1})\,, \end{aligned} \tag{70b}$$

and can also assume negative values. If $S_{A|B}(\varrho_{AB}) \geq 0$, the conditional entropy gives the amount of information that the party A should send to the party B in order to allow for the full knowledge of the overall state ϱ_{AB}. If $S_{A|B}(\varrho_{AB}) < 0$, the party A does not need to send any information to the other and, in addition, they gain $-S_{A|B}(\varrho_{AB})$ bits of entanglement, respectively [analogous considerations hold for $S_{B|A}(\varrho_{AB})$]. This has been proved for the case of discrete variable quantum systems [42] and conjectured [43] for infinite dimensional ones.

6.2 Separability of Gaussian states

A bipartite state $\varrho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is separable if it can be written as a convex combination of product states [44], namely, $\varrho_{AB} = \sum_k p_k \varrho_k^{(A)} \otimes \varrho_k^{(B)}$ where $p_k \geq 0$, $\sum_k p_k = 1$, and $\varrho_k^{(h)} \in \mathcal{H}_h$, $h = A, B$. Finding the convex combination of a separable state is a challenging task; nevertheless the separability can be revealed with the aid of positive but not completely positive maps. In particular, positivity under partial transposition (`ppt`), that is the transposition applied only to a part of a system, has been introduced in entanglement theory by A. Peres [45] as a necessary condition for separability. In fact, if we apply, for instance, transposition only to elements of the first subsystem A of a separable state ϱ_{AB}, we have $\varrho_{AB}^{T_A} = \sum_k p_k \left(\varrho_k^{(A)}\right)^T \otimes \varrho_k^{(B)}$. Now, since $\left[\varrho_k^{(A)}\right]^T = \left[\varrho_k^{(A)}\right]^*$, transposition corresponds to complex conjugation and the transposed matrix is a legitimate density matrix itself, being non-negative, self-adjoint and with unit trace. Then none of the eigenvalues of ϱ^{T_A} is negative if ϱ is separable. The `ppt` criterion is usually only necessary and entangled states with positive partial transposed density matrix are known to exist and are called bound-entangled states [46]. R. Simon, however, has proved that for two-mode Gaussian states it represents also a sufficient condition for separability [47].

Since complex conjugation corresponds to time reversal of the Schrödinger equation, in terms of continuous variables transposition corresponds to a sign change of the momentum variables, i.e., a mirror reflection. For a two-mode system described by the density matrix ϱ_{AB}, partial transposition with respect to system A will be performed on the phase space through the action of the matrix $\mathbf{\Delta}_A = \mathrm{Diag}(1, -1) \oplus \mathbb{1}_2$, where the first factor of the direct sum, representing the mirror reflection, refers to subsystem A and the second one to subsystem B (partial transposition with respect to subsystem B is obtained in a similar way). Hence, the positivity of the partial transposed operator leads to the following uncertainty relation:

$$\widetilde{\boldsymbol{\sigma}} + \frac{i}{2}\,\mathbf{\Omega} \geq 0\,, \quad \text{or} \quad \boldsymbol{\sigma} \geq -\frac{i}{2}\,\widetilde{\mathbf{\Omega}}_A\,, \tag{71}$$

where $\widetilde{\boldsymbol{\sigma}} = \mathbf{\Delta}_A\,\boldsymbol{\sigma}\,\mathbf{\Delta}_A$ and $\widetilde{\mathbf{\Omega}}_A = \mathbf{\Delta}_A\,\mathbf{\Omega}\,\mathbf{\Delta}_A$. Furthermore, recalling the definition (59) of the four local symplectic invariants, now we have:

$$\tilde{I}_1 = I_1\,, \qquad \tilde{I}_2 = I_2\,, \qquad \tilde{I}_3 = -I_3\,, \qquad \tilde{I}_4 = I_4\,, \tag{72}$$

where $\tilde{I}_k$ are the symplectic invariants referred to $\widetilde{\boldsymbol{\sigma}}$. Thus, in terms of the symplectic eigenvalues $\tilde{d}_\pm$ of the partially transposed CM the `ppt` criterion reduces to:

$$\tilde{d}_- \geq 1/2\,. \tag{73}$$

with:

$$\tilde{d}_\pm = \sqrt{\frac{\widetilde{\Delta}(\boldsymbol{\sigma}) \pm \sqrt{\widetilde{\Delta}(\boldsymbol{\sigma})^2 - 4I_4}}{2}}\,, \tag{74}$$

where $\widetilde{\Delta}(\boldsymbol{\sigma}) = I_1 + I_2 - 2I_3$. Here we have shown that the `ppt` criterion is necessary for separability. As for its sufficiency we refer to the original paper [47].

An equivalent necessary and sufficient criterion is based on the evaluation of the sum of the variances associated with a pair of EPR-like operators, defined on the two different subsystems [48]. The insight underlying this criterion is that for an entangled state it is possible to gain information on one of the subsystems suitably measuring

the other one. This criterion leads to an inequality that can be expressed in terms of elements of the CM expressed in the standard form (60), namely:

$$\tilde{a}\gamma^2 + \frac{\tilde{b}}{\gamma^2} - |\tilde{c}_1| - |\tilde{c}_2| - \left(\gamma^2 + \frac{1}{\gamma^2}\right) < 0, \tag{75}$$

where we introduced the quantities: $\gamma^2 = \sqrt{(\tilde{b} - 1/2)/(\tilde{a} - 1/2)}$, $\tilde{a} = 2a\cosh 2r_1$, $\tilde{b} = 2b\cosh 2r_2$, $\tilde{c}_1 = 2c_1 \exp(r_1 + r_2)$, $\tilde{c}_2 = 2c_2 \exp[-(r_1 + r_2)]$, and r_1 and r_2 are suitable squeezing parameters to transform the CM (60) into the so-called Duan canonical form (see Ref. [48] for details). A separable state, whether Gaussian or not, will not satisfy the above inequality.

It is worth noting that also the negativity of the conditional entropies (70) is a sufficient condition for entanglement [49].

6.3 On the quantification of Gaussian entanglement

For a two-mode state, a quantitative measure of entanglement can be given on the observation that the larger is the violation $\tilde{d}_- < 1/2$ the stronger is the entanglement, or more properly, the stronger the resilience of entanglement to noise [50–53]. The logarithmic negativity for a two-mode Gaussian state, is given by [54]:

$$E(\boldsymbol{\sigma}) = \max\{0, -\log 2\tilde{d}_-\}, \tag{76}$$

and it is a simple increasing monotone function of the minimum symplectic eigenvalue $\tilde{d}_-$ (for $0 < \tilde{d}_- < 1/2$). Thus, it represents a good candidate for evaluating entanglement in a quantitative way.

Another convenient and useful way of looking at the entanglement evolution in continuous variable systems is by means of the entanglement of formation (EoF), which corresponds to the minimal amount of entanglement of any ensemble of pure bipartite states realizing the given state [55,56]. In general the derivation of an expression of the EoF for arbitrary states is not a simple task.

In the case of a symmetric bipartite Gaussian state with CM given by (60) with $a = b$, the EoF reads [56]:

$$E_F = f(x_m), \tag{77}$$

where $f(x)$ is defined in Eq. (57), $x_m = (\tilde{d}_-^2 + 1/4)/(2\tilde{d}_-)$, and $\tilde{d}_-$ is the minimum symplectic eigenvalue of the partially transposed CM given in Eq. (61).

For the two-mode squeezed thermal state (63), in which the standard form of the CM is obtained from Eq. (60) with $a \geq b$ and $c_1 = -c_2 = c \geq 0$, the EoF is still given by Eq. (77) but with [57]:

$$x_m = \frac{(a+b)(ab - c^2 + \frac{1}{4}) - 2c\sqrt{\det(\boldsymbol{\sigma} + \frac{i}{2}\boldsymbol{\Omega})}}{(a+b)^2 - 4c^2}. \tag{78}$$

The EoF of other classes of two-mode Gaussian states can be evaluated by following the general prescription proposed in Ref. [57].

Quantitative estimation of entanglement can be also obtained by means of entropy functionals [58]. In particular, the degree of entanglement of an ideal bipartite system can be assessed following the analysis presented in [59].

6.4 Gaussian quantum discord

The correlations of a bipartite quantum system ϱ_{AB}, quantified by the mutual information (68), can be divided in a quantum part, known as quantum discord, and a classical part [60]. The classical correlations are defined as the maximum amount of information we can gain on one part of the system by locally measuring the other subsystem, and, thus, can be written as a function of the von Neumann entropies of the two subsystems as follows [61]:

$$\mathcal{C}_{A|B}(\varrho_{AB}) = \max_{\Pi_k}\{S_V(\varrho_A) - \sum_k p_k S_V(\varrho_{A|B}^{\Pi_k})\}, \tag{79}$$

where the set $\{\Pi_k\}$, $\Pi_k \geq 0$ and $\sum_k \Pi_k = \mathbb{I}$, represents a positive operator-valued measure (POVM), $\varrho_{A|B}^{\Pi_k} = \mathrm{Tr}_B[\varrho_{AB}\, \mathbb{I}\otimes\Pi_k]/p_k$ is the conditional state of subsystem A when the k-th outcome occurs in a measurement of subsystem B and $p_k = \mathrm{Tr}_{AB}[\varrho_{AB}\, \mathbb{I}\otimes\Pi_k]$. The maximum is taken over all the POVMs performable on one subsystem. Classical correlations are thus obtained in correspondence of the POVM that minimizes the conditional entropy $\sum_k p_k S_V(\varrho_{A|B}^{\Pi_k})$, allowing one to obtain the highest amount of information on the state of system A. As a matter of fact, the above definition is in general non symmetric with respect to the interchange of the subsystems. The quantum discord is then defined as:

$$\mathcal{D}_{A|B}(\varrho_{AB}) = I_M(\varrho_{AB}) - \mathcal{C}_{A|B}(\varrho_{AB})\,, \tag{80}$$

$I_M(\varrho_{AB})$ being the mutual information (68).

In the particular case of a two-mode Gaussian state, the Gaussian quantum discord is evaluated addressing only Gaussian measurements performed on the subsystems and can be written as (for conditional Gaussian measurements on Gaussian states, see Sect. 8) [62,63]:

$$\mathcal{D}_{A|B}(\varrho_{AB}) = S_V(\varrho_B) - S_V(\varrho_{AB}) + f\left(\sqrt{E_{A|B}^{\min}}\right)\,, \tag{81a}$$

$$= f\left(\sqrt{E_{A|B}^{\min}}\right) - S_{A|B}(\varrho_{AB})\,, \tag{81b}$$

where $f(x)$ has been defined in Eq. (57), $S_{A|B}(\varrho_{AB})$ is the conditional entropy (70a) and, in terms of the symplectic invariants (59), $E_{A|B}^{\min}$ writes [63]:

$$E_{A|B}^{\min} = \begin{cases} \left[\frac{2|I_3| + \sqrt{4I_3^2 + (4I_2-1)(4I_4-I_1)}}{(4I_2-1)}\right]^2 & \text{if } \frac{4(I_1I_2-I_4)^2}{(I_1+4I_4)(1+4I_2)I_3^2} \leq 1\,, \\ \frac{I_1I_2+I_4-I_3^2-\sqrt{\left(I_1I_2+I_4-I_3^2\right)^2-4I_1I_2I_4}}{2I_2} & \text{otherwise.} \end{cases} \tag{82}$$

In the case of the squeezed thermal state (63), one has (we set $\xi = r$):

$$\sqrt{E_{A|B}^{\min}} = \frac{1}{2} + \frac{2N_1(1+N_2)}{1-N_1+N_2+(1+N_1+N_2)\cosh(2r)}\,, \tag{83}$$

and the explicit expression of the quantum discord (81) can be easily evaluated.

The quantity $f\left(\sqrt{E_{A|B}^{\min}}\right)$ corresponds to the average von Neumann entropy of the conditional single-mode Gaussian state in which is left the subsystem A after the Gaussian measurement on subsystem B minimizing the conditional entropy in

Eq. (79) (see Ref. [63] for details of the calculation). $\mathcal{D}_{B|A}(\varrho_{AB})$ can be obtained by exchanging the roles of the two subsystems.

It is worth noting that quantum discord can be nonzero even if the state is separable, which indicates that entanglement is not the only source of quantum correlations. For instance, there are examples of quantum computational algorithms showing a speedup with respect to the classical counterparts, even in the absence of entanglement [64,65]. States with zero discord represent essentially a classical probability distribution embedded in a quantum system, while a positive discord, even on separable (mixed) states, is an indicator of quantumness [66,67], and may operationally be associated with the impossibility of local broadcasting [68].

In the case of pure two-mode Gaussian states, since $\Delta(\boldsymbol{\sigma}) = I_1+I_2+2I_3 = 1/2$ and $I_4 = 1/16$, one has $S_V(\varrho_{AB}) = f\left(\sqrt{E^{\min}_{A|B}}\right) = 0$, and hence the Gaussian quantum discord (81) reduces to the entropy of entanglement, i.e., $\mathcal{D}_{A|B}(\varrho_{AB}) = \mathcal{D}_{B|A}(\varrho_{AB}) = S_V(\varrho_A) = S_V(\varrho_B)$.

7 Gaussian states in noisy channels

As one may expect, the dissipative dynamics of a Gaussian states in a Gaussian environment, or channel, can be reduced to a suitable transformation of its CM and first-moments vector. In this tutorial we focus on Markovian environments, however, it is possible to extend the analysis to non-Markovian ones, as described, for instance, in Refs. [69,70].

The dynamics of a single-mode quantum state ϱ_t through a (Markovian) noisy environment is governed by the following Master equation:

$$\dot{\varrho}_t = \frac{\Gamma}{2}\{(N+1)\mathcal{L}[\hat{a}] + N\mathcal{L}[\hat{a}^\dagger] - M^*\mathcal{D}[\hat{a}] - M\mathcal{D}[\hat{a}^\dagger]\}\,\varrho_t\,, \tag{84}$$

where $\mathcal{L}[\hat{O}]\varrho_t = 2\hat{O}\varrho_t\hat{O}^\dagger - \hat{O}^\dagger\hat{O}\varrho_t - \varrho_t\hat{O}^\dagger\hat{O}$ and $\mathcal{D}[\hat{O}]\varrho_t = 2\hat{O}\varrho_t\hat{O} - \hat{O}\hat{O}\varrho_t - \varrho_t\hat{O}\hat{O}$ are Lindblad superoperators, Γ is the overall damping rate, while $N \in \mathbb{R}$ and $M \in \mathbb{C}$ represent the effective number of photons and the squeezing parameter of the bath, respectively [2]. The terms proportional to $\mathcal{L}[\hat{a}]$ and to $\mathcal{L}[\hat{a}^\dagger]$ describe losses and linear, phase-insensitive, amplification processes, respectively, while the terms proportional to $\mathcal{D}[\hat{a}]$ and $\mathcal{D}[\hat{a}^\dagger]$ describe phase dependent fluctuations. The positivity of the density matrix imposes the constraint $|M|^2 \leq N(N+1)$. At thermal equilibrium, i.e., for $M = 0$, N coincides with the average number of thermal photons in the bath.

In order to explicitly derive the evolution of the CM and first-moments vector, we transform the Master equation (84) into the following Fokker-Planck equation for the Wigner function $W[\varrho_t](\boldsymbol{X})$ associated with ϱ_t [2]:

$$\partial_t W[\varrho_t](\boldsymbol{X}) = \frac{\Gamma}{2}(\partial^T_{\boldsymbol{X}} X + \partial^T_{\boldsymbol{X}}\boldsymbol{\sigma}_\infty\partial_{\boldsymbol{X}})W[\varrho_t](\boldsymbol{X})\,, \tag{85}$$

where $\boldsymbol{X} \equiv (x, y)^T$, $\partial_{\boldsymbol{X}} \equiv (\partial_x, \partial_y)^T$, and we introduced the diffusion matrix $\boldsymbol{\sigma}_\infty$:

$$\boldsymbol{\sigma}_\infty = \begin{pmatrix} \left(\frac{1}{2}+N\right) + \Re\mathrm{e}[M] & \Im\mathrm{m}[M] \\ \Im\mathrm{m}[M] & \left(\frac{1}{2}+N\right) - \Re\mathrm{e}[M] \end{pmatrix}. \tag{86}$$

If the initial state ϱ_0 is a Gaussian state with CM $\boldsymbol{\sigma}_0$ and first-moments vector $\overline{\boldsymbol{X}}_0 \equiv \mathrm{Tr}[\varrho_0\hat{\boldsymbol{R}}]$, respectively, the Wigner function $W(\boldsymbol{X})$ of the evolved state under

the action of the Eq. (86) is still Gaussian, but with CM and first-moments vector given by (see Ref. [2] for the explicit calculation):

$$\boldsymbol{\sigma}_t = e^{-\Gamma t}\boldsymbol{\sigma}_0 + (1 - e^{-\Gamma t})\boldsymbol{\sigma}_\infty, \quad \text{and} \quad \overline{\boldsymbol{X}}_t = e^{-\Gamma t/2}\,\overline{\boldsymbol{X}}_0, \tag{87}$$

respectively, which show that $\boldsymbol{\sigma}_\infty$ is the asymptotic CM when the initial state is Gaussian, while $\overline{\boldsymbol{X}}_t \equiv \mathrm{Tr}[\varrho_t \hat{\boldsymbol{R}}]$ is damped to zero.

The extension to two-mode or, more in general, to n-mode states interacting with uncorrelated environments, each described by a Master equation of the form (84), is straightforward. In this case, if $\boldsymbol{\sigma}_0$ and $\overline{\boldsymbol{X}}_0$ refer to the CM and first-moments vector of the initial state n-mode state and Γ_k, N_k and M_k are the parameter characterizing the environment interacting with the k-th mode, then we have:

$$\boldsymbol{\sigma}_t = \mathbb{G}_t^{1/2}\boldsymbol{\sigma}_0\mathbb{G}_t^{1/2} + (\mathbb{1}_{2n} - \mathbb{G}_t)\,\boldsymbol{\sigma}_\infty, \quad \text{and} \quad \overline{\boldsymbol{X}}_t = \mathbb{G}_t^{1/2}\overline{\boldsymbol{X}}_0, \tag{88}$$

where $\mathbb{G}_t = \bigoplus_{h=1}^{n} e^{-\Gamma_h t}\mathbb{1}_2$ and $\boldsymbol{\sigma}_\infty = \bigoplus_{h=1}^{n}\boldsymbol{\sigma}_{h,\infty}$ with:

$$\boldsymbol{\sigma}_{h,\infty} = \begin{pmatrix} \left(\frac{1}{2} + N_h\right) + \Re\mathrm{e}[M_h] & \Im\mathrm{m}[M_h] \\ \Im\mathrm{m}[M_h] & \left(\frac{1}{2} + N_h\right) - \Re\mathrm{e}[M_h] \end{pmatrix}. \tag{89}$$

Starting form $\boldsymbol{\sigma}_t$ and $\overline{\boldsymbol{X}}_t$, one can easily evaluate the evolution of all the quantities addressed in the previous sections, such as purity and, for two-mode states, the separability thresholds and entropies (the interested reader can find the explicit calculations and a thorough analysis, e.g., in Ref. [2] and references therein).

8 Gaussian measurements onto a Gaussian state

In the previous sections we have reviewed how a Gaussian state can be generated and characterized. We have also addressed its kinematics and evolution through noisy channels. In order to make this tutorial as complete as possible, we now focus on conditional Gaussian measurements [71–74], such as homodyne detection and double homodyne detection [2], performed on Gaussian states (for a PhD tutorial on the manipulation of Gaussian states at the photon level see, e.g., Ref. [75]). Conditional measurements are extremely important in quantum information processing, since they are at the basis of quantum teleportation and telecloning protocols and allow to generate and manipulate new classes of states [76]. Furthermore, a single homodyne detector has been recently used to fully characterize a two-mode squeezed thermal state [77,78].

In order to show how a typical calculation involving Gaussian states and operations is carried out, we explicitly derive the characteristic function of a conditional Gaussian state obtained by performing a Gaussian measurement on one of its n modes.

Let us consider the following Gaussian characteristic function with zero first-moments vector (extension to non-zero first-moments states is straightforward) associated with a n-mode state ϱ [for the sake of simplicity we use the characteristic function as defined in Eq. (15) and drop the explicit dependence on the operators]:

$$\chi(\boldsymbol{\Lambda}) = \exp\{-\tfrac{1}{2}\boldsymbol{\Lambda}^T\boldsymbol{\Sigma}\,\boldsymbol{\Lambda}\}, \tag{90}$$

where $\boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_n)^T \in \mathbb{R}^{2n}$ is a column vector and $\boldsymbol{\Sigma}$ is the $2n \times 2n$ CM. Now we assume to perform a Gaussian measurement on one of the modes, that is a measurement described by a POVM with Gaussian characteristic function. Without

lack of generality, we can assume that the measurement involves mode 1, and, thus, the corresponding characteristic function may be written as:

$$\chi_{\rm M}(\mathbf{\Lambda}_1) = \pi^{-1}\, \exp\{-\tfrac{1}{2}\mathbf{\Lambda}_1^T \boldsymbol{\sigma}_{\rm M}\, \mathbf{\Lambda}_1 - i\mathbf{\Lambda}_1^T \boldsymbol{X}\}, \tag{91}$$

where $\boldsymbol{\sigma}_{\rm M}$ and $\boldsymbol{X}$ are the CM the first-moments vector or, more precisely, the outcome of the measurement, respectively. For the sake of simplicity, we write $\mathbf{\Lambda}$ and the CM $\mathbf{\Sigma}$ in the following block form:

$$\mathbf{\Lambda} = (\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \ldots, \mathbf{\Lambda}_n)^T = (\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}})^T\,, \quad \text{and} \quad \mathbf{\Sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^T & \boldsymbol{B} \end{pmatrix}, \tag{92}$$

where $\boldsymbol{A} \in \mathbb{R}^{\,2} \times \mathbb{R}^{\,2}$ and $\boldsymbol{B} \in \mathbb{R}^{\,2(n-1)} \times \mathbb{R}^{\,2(n-1)}$ are symmetric, and $\boldsymbol{C} \in \mathbb{R}^{\,2} \times \mathbb{R}^{\,2(n-1)}$, making evident the mode undergoing the measurement. The conditional characteristic function of the system after the measurement with outcome $\boldsymbol{X}$ is:

$$\chi'(\tilde{\mathbf{\Lambda}}) = \frac{1}{p(\boldsymbol{X})} \int_{\mathbb{R}^{\,2}} \frac{d^2\mathbf{\Lambda}_1}{2\pi} \chi(\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}})\, \chi_{\rm M}(-\mathbf{\Lambda}_1)\,, \tag{93}$$

where we used the trace rule (23) and $p(\boldsymbol{X})$ is the probability of the outcome $\boldsymbol{X}$:

$$p(\boldsymbol{X}) = \int_{\mathbb{R}^{\,2n}} \frac{d^2\mathbf{\Lambda}_1\, d^{2(n-1)}\tilde{\mathbf{\Lambda}}}{(2\pi)^n}\, \chi(\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}})\, \chi_{\rm M}(-\mathbf{\Lambda}_1)\, (2\pi)^{(n-1)} \delta(-\tilde{\mathbf{\Lambda}}) \tag{94}$$

$$= \frac{\exp\{-\frac{1}{2}\boldsymbol{X}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1}\, \boldsymbol{X}\}}{\pi\sqrt{\det[\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M}]}}\,, \tag{95}$$

where $\delta(-\tilde{\mathbf{\Lambda}}) = \prod_{k=2}^{n} \delta^{(2)}(-\mathbf{\Lambda}_k)$ is the product of Kronecker deltas in $\mathbb{R}^{\,2}$. Note that:

$$\chi(\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}})\, \chi_{\rm M}(-\mathbf{\Lambda}_1) = \pi^{-1}\, \exp\{-\tfrac{1}{2}(\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}})^T \boldsymbol{\sigma}\, (\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}}) + i\mathbf{\Lambda}_1^T \boldsymbol{X}\}\,, \tag{96}$$

with:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} + \boldsymbol{\sigma}_{\rm M} & \boldsymbol{C} \\ \boldsymbol{C}^T & \boldsymbol{B} \end{pmatrix}. \tag{97}$$

In order to perform the integral (93) we observe that $\boldsymbol{\sigma}$ can be rewritten as follows:

$$\boldsymbol{\sigma} = \boldsymbol{M}^T \begin{pmatrix} \boldsymbol{A} + \boldsymbol{\sigma}_{\rm M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} - \boldsymbol{C}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{C} \end{pmatrix} \boldsymbol{M}\,, \quad \boldsymbol{M} = \begin{pmatrix} \mathbb{1}_2 & (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{C} \\ \boldsymbol{0} & \mathbb{1}_{2(n-1)} \end{pmatrix}. \tag{98}$$

The matrix $\boldsymbol{B} - \boldsymbol{C}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{C}$ is the Schur complement of the matrix $\boldsymbol{\sigma}$ with respect to $\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M}$. Now, since:

$$\boldsymbol{M}(\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}}) = \left(\mathbf{\Lambda}_1 + (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{C} \tilde{\mathbf{\Lambda}}, \tilde{\mathbf{\Lambda}}\right), \tag{99}$$

Eq. (93) reduces to:

$$\chi'(\tilde{\mathbf{\Lambda}}) = \frac{1}{p(\boldsymbol{X})}\, \exp\{-\tfrac{1}{2}\tilde{\mathbf{\Lambda}}^T [\boldsymbol{B} - \boldsymbol{C}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{C}]\, \tilde{\mathbf{\Lambda}} - i\tilde{\mathbf{\Lambda}}^T \boldsymbol{C}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{X}\}$$
$$\times \int_{\mathbb{R}^{\,2}} \frac{d^2\mathbf{\Lambda}'}{2\pi^2}\, \exp\{-\tfrac{1}{2}(\mathbf{\Lambda}')^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})\, (\mathbf{\Lambda}') - i(\mathbf{\Lambda}')^T \boldsymbol{X}\}\,, \tag{100}$$

$$= \exp\{-\tfrac{1}{2}\tilde{\mathbf{\Lambda}}^T [\boldsymbol{B} - \boldsymbol{C}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{C}]\, \tilde{\mathbf{\Lambda}} - i\tilde{\mathbf{\Lambda}}^T \boldsymbol{C}^T (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1} \boldsymbol{X}\}\,, \tag{101}$$

where we performed the change of variables $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}_1 + (\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1}\boldsymbol{C}\tilde{\boldsymbol{\Lambda}}$. The conditional state $\chi'(\tilde{\boldsymbol{\Lambda}})$ is a $(n-1)$-mode Gaussian state with CM $\boldsymbol{B} - \boldsymbol{C}^T(\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1}\boldsymbol{C}$ and first-moments vector $\boldsymbol{C}^T(\boldsymbol{A} + \boldsymbol{\sigma}_{\rm M})^{-1}\boldsymbol{X}$. Analogously, if we carry out the measurement on the mode n (actually, the last one), we obtain that the conditional state is still Gaussian but with CM $\boldsymbol{A} - \boldsymbol{C}(\boldsymbol{B} + \boldsymbol{\sigma}_{\rm M})^{-1}\boldsymbol{C}^T$ and first-moments vector $\boldsymbol{X}^T(\boldsymbol{B} + \boldsymbol{\sigma}_{\rm M})^{-1}\boldsymbol{C}$, where, now, $\boldsymbol{A} \in \mathbb{R}^{\,2(n-1)} \times \mathbb{R}^{\,2(n-1)}$, $\boldsymbol{B} \in \mathbb{R}^{\,2} \times \mathbb{R}^{\,2}$ and $\boldsymbol{C} \in \mathbb{R}^{\,2(n-1)} \times \mathbb{R}^{\,2}$.

9 Fidelity between Gaussian states

The fidelity is one of the most important figure of merit in quantum information and quantifies the similarity between two states ϱ_1 and ϱ_2. The Uhlmann's fidelity is defined as [79]:

$$\mathcal{F}(\varrho_1, \varrho_2) = \{\mathrm{Tr}[(\sqrt{\varrho_1}\varrho_2\sqrt{\varrho_1})^{1/2}]\}^2, \tag{102}$$

and corresponds to the maximal transition probability between all purifications of the two states.

In the case of single-mode Gaussian states with CMs $\boldsymbol{\sigma}_k$ and first-moments vectors $\overline{\boldsymbol{X}}_k$, $k = 1, 2$, Eq. (102) leads to [80]:

$$\mathcal{F}(\varrho_1, \varrho_2) = \frac{\exp\left\{-\frac{1}{2}\,(\overline{\boldsymbol{X}}_1 - \overline{\boldsymbol{X}}_2)^T(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^{-1}(\overline{\boldsymbol{X}}_1 - \overline{\boldsymbol{X}}_2)\right\}}{\sqrt{\Delta + \delta} - \sqrt{\delta}}, \tag{103}$$

with $\Delta = \det[\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2]$ and $\delta = 4\prod_{k=1}^{2}(\det[\boldsymbol{\sigma}_k] - \frac{1}{4})$.

The problem of finding an analytical formula for the fidelity between n-mode Gaussian states has been very recently solved in an elegant way [81]. In particular, for two-mode Gaussian states one obtains:

$$\mathcal{F}(\varrho_1, \varrho_2) = \mathrm{Tr}[\varrho_1\varrho_2]\,(\sqrt{\mathcal{X}} + \sqrt{\mathcal{X} - 1})^2, \tag{104}$$

where $\mathcal{X} = 2\sqrt{\mathcal{A}} + 2\sqrt{\mathcal{B}} + \frac{1}{2}$ and:

$$\mathcal{A} = \frac{\det[\boldsymbol{\Omega}\,\boldsymbol{\sigma}_1\,\boldsymbol{\Omega}\,\boldsymbol{\sigma}_2 - \frac{1}{4}\mathbb{1}_4]}{\det[\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2]}\,, \quad \mathcal{B} = \frac{\det[\boldsymbol{\sigma}_1 + \frac{i}{2}\boldsymbol{\Omega}]\,\det[\boldsymbol{\sigma}_2 + \frac{i}{2}\boldsymbol{\Omega}]}{\det[\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2]} \geq 0\,, \tag{105}$$

are symplectic invariants, $\boldsymbol{\Omega} = \boldsymbol{\omega} \oplus \boldsymbol{\omega}$ is the symplectic matrix (3) and, using Eq. (21) or Eq. (23), we have:

$$\mathrm{Tr}[\varrho_1\varrho_2] = \frac{\exp\left\{-\frac{1}{2}\,(\overline{\boldsymbol{X}}_1 - \overline{\boldsymbol{X}}_2)^T(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^{-1}(\overline{\boldsymbol{X}}_1 - \overline{\boldsymbol{X}}_2)\right\}}{\sqrt{\det[\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2]}}. \tag{106}$$

The reader can find the full analysis leading to Eq. (104) and the extension to n-mode Gaussian states in the original paper [81].

10 Conclusions

In this tutorial we have presented the basic tools and results in order to deal with Gaussian states in phase space. We have seen how their generation, manipulation and propagation through noisy channels can be described by means of suitable transformations of the covariance matrix and first-moments vector. Focusing on two-mode

Gaussian states, we have addressed their characterization by means of mutual information and conditional entropies, their separability and entanglement properties. As a pedagogical example, we have explicitly showed how to calculate the conditional state of a multimode Gaussian state that has gone through a Gaussian measurement on one of its modes. We have also presented the latest results about the fidelity between Gaussian states, that is the most important figure of merit in quantum information.

After the last meeting I had with prof. Federico Casagrande, I promised him we would have discussed my research on Gaussian states and my latest results. Unfortunately, we had not enough time. Of course, these pages cannot substitute an afternoon spent with him, his curiosity and his enthusiasm... Nevertheless, I believe he would have appreciated this tutorial and I hope it could be a useful tool for students and scientists interested in quantum optics, the main topic investigated by Federico during his activity at the University of Milano.

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