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INTERFERENCE OF MULTI-MODE GAUSSIAN STATES AND "NON APPEARANCE" OF QUANTUM CORRELATIONS

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We theoretically investigate bilinear, mode-mixing interactions involving two modes of uncorrelated multi-mode Gaussian states. In particular, we introduce the notion of "locally the same states" (LSS) and prove that two uncorrelated LSS modes are invariant under the mode mixing, i.e. the interaction does not lead to the birth of correlations between the outgoing modes. We also study the interference of orthogonally polarized Gaussian states by means of an interferometric scheme based on a beam splitter, rotators of polarization and polarization filters.

Keywords: Quantum interference; quantum correlations; Gaussian states.

1. Introduction

Among the possible mechanisms to generate quantum correlations, the one consisting in the mixing of Gaussian states, that is quantum states with Gaussian characteristic functions, at a beam splitter (BS) is of special interest in view of its feasibility.[1,2] In fact, the Gaussian states are generated and manipulated by means of linear and bilinear Hamiltonians, which are actually implemented in all quantum optical labs. In particular, the interference at a BS of two squeezed states can generate Gaussian entanglement,[3–9] which has been used so far to achieve continuous variable teleportation.[10] The properties of the correlated states emerging from a BS have been thoroughly investigated in the past years, either to optimize the generation of entanglement[11,12] or to find relations between their entanglement and purities[13] or teleportation fidelity.[14] Furthermore, a recent work[15] has proved that there exists a strict relation between the fidelity (similarity) of the Gaussian states entering the BS and the birth of entanglement at the output.

On the other hand, the symmetries exhibited by interfering Gaussian states may lead to the invariance under mode-mixing interactions.[16,17] This effect has been experimentally verified and exploited to restore the nonlocal correlations lost by a two-mode squeezed vacuum state.[18]

Motivated by this result and considering the intrinsic multi-mode nature of the Gaussian states produced in the laboratories,[19,20] in this paper we address the conditions leading to the "non appearance" of the correlations and state the main theorems underlying this effect.

The plan of the paper is as follows. In Sec. 2 we review the formalism to describe multi-mode Gaussian states and introduce the notion of "locally the same states" (LSS). We also state the main theorem and the corollaries concerning the invariance of two uncorrelated Gaussian states through a bilinear interaction. Section 3 addresses the interference of polarized Gaussian states and we close the paper drawing some concluding remarks in Sec. 4.

2. Interference of LSS

Gaussian states are completely characterized by their covariance matrix (CM) and mean values vector.[1] If we define the vector of operators $\boldsymbol{R} = \{q_1, p_1, \ldots, q_N, p_N\}$, where we introduced the quadrature operators $q_k = \frac{1}{\sqrt{2}}(a_k + a_k^\dagger)$ and $p_k = \frac{1}{i\sqrt{2}}(a_k - a_k^\dagger)$, and a_k is the field operator of mode k, the $2N \times 2N$ CM $\boldsymbol{\Sigma}_{\boldsymbol{v}}$ of a N-mode Gaussian state $\varrho_{\boldsymbol{v}}$, $\boldsymbol{v} = \{1, \ldots, N\}$, can be written in the following way:

$$[\boldsymbol{\Sigma}_{\boldsymbol{v}}]_{nm} = \frac{1}{2}\langle R_n R_m + R_m R_n \rangle - \langle R_n \rangle \langle R_m \rangle, \tag{1}$$

where $\langle A \rangle = \mathrm{Tr}[\varrho_{\boldsymbol{v}} A]$ and $R_k = [\boldsymbol{R}]_k$. The first-moments vector is thus given by $\langle \boldsymbol{R} \rangle$. In order to simplify the formalism, we write $\boldsymbol{\Sigma}_{\boldsymbol{v}}$ in the block form:

$$\boldsymbol{\Sigma}_{\boldsymbol{v}} = \begin{pmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{\delta}_{12} & \cdots & \boldsymbol{\delta}_{1N} \\ \boldsymbol{\delta}_{12}^T & \boldsymbol{\sigma}_2 & \cdots & \boldsymbol{\delta}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\delta}_{1N}^T & \boldsymbol{\delta}_{2N}^T & \cdots & \boldsymbol{\sigma}_N \end{pmatrix}, \tag{2}$$

where $\boldsymbol{\sigma}_k$ and $\boldsymbol{\delta}_{hk}$ are 2×2 real matrices. In particular, $\boldsymbol{\sigma}_k$ corresponds to the CM of the state $\varrho_k = \mathrm{Tr}_{\boldsymbol{v}\setminus\{k\}}[\varrho_{\boldsymbol{v}}]$ and $\boldsymbol{\delta}_{hk}$ is related to the (classical or quantum) correlations between the modes h ad k: if $\boldsymbol{\delta}_{hk} = \boldsymbol{0}$, then $\varrho_{hk} = \mathrm{Tr}_{\boldsymbol{v}\setminus\{h,k\}}[\varrho_{\boldsymbol{v}}] = \varrho_h \otimes \varrho_k$, that is the two modes are uncorrelated, and the CM:

$$\boldsymbol{\Sigma}_{hk} = \begin{pmatrix} \boldsymbol{\sigma}_h & \boldsymbol{\delta}_{hk} \\ \boldsymbol{\delta}_{hk}^T & \boldsymbol{\sigma}_k \end{pmatrix} \tag{3}$$

of the state ϱ_{hk} reduces to the direct sum $\mathbf{\Sigma}_{hk} = \boldsymbol{\sigma}_h \oplus \boldsymbol{\sigma}_k$ of the two single-mode CMs. We can now introduce the following:

Definition 1 (Locally the same). Given the N-mode state described by the density matrix $\varrho_{\boldsymbol{v}}$, $\boldsymbol{v} = \{1, \ldots, N\}$, two modes $h, k \in \boldsymbol{v}$ are LSS if $\varrho_h = \varrho_k$, with $\varrho_l = \mathrm{Tr}_{\boldsymbol{v}\setminus\{l\}}[\varrho_{\boldsymbol{v}}]$.

As an example, it is easy to verify that each of the two beams of the two-mode squeezed vacuum state $|\psi\rangle = \sqrt{1-\lambda^2}\sum_n \lambda^n |n\rangle_1 |n\rangle_2$, with $\lambda = \tanh r$, where r is the two-mode squeezing parameter, and a single-mode thermal state with average number of photons $N_{\mathrm{th}} = \sinh^2 r$ are LSS. Indeed, two Gaussian states which are LSS also have the same single-mode CM: for what concerns the example cited above, the CM of the two-mode squeezed vacuum $\mathbf{\Sigma}_{12}$ is obtained from Eq. (3) by setting $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 = \frac{1}{2}(1 + 2\sinh^2 r)\,\mathbb{1}_2$ and $\boldsymbol{\delta}_{12} = \frac{1}{2}\sinh(2r)\boldsymbol{\sigma}_z$, where $\mathbb{1}_k$ is the $k \times k$ identity matrix and $\boldsymbol{\sigma}_z = \mathrm{Diag}(1, -1)$ is the Pauli matrix, whereas the CM of the thermal state reads $\boldsymbol{\sigma} = \frac{1}{2}(1 + 2N_{\mathrm{th}})\,\mathbb{1}_2$.

Let us now turn the attention on the evolution of a Gaussian state. If we act on a Gaussian state by means of linear or bilinear interaction Hamiltonian (actually these are most of the interactions implementable in a quantum optics laboratory), then its Gaussian character is preserved and we can describe its evolution by a suitable symplectic transformation acting on the CM and the mean values vector.[21] By denoting with U the unitary operator associated with the interaction Hamiltonian and with $\boldsymbol{S}$ the corresponding symplectic transformation, we have:

$$\varrho_{\boldsymbol{v}} \to U \varrho_{\boldsymbol{v}} U^\dagger \Rightarrow \begin{cases} \mathbf{\Sigma}_{\boldsymbol{v}} \to \boldsymbol{S}\mathbf{\Sigma}_{\boldsymbol{v}}\boldsymbol{S}^T, \\ \langle \boldsymbol{R} \rangle \to \boldsymbol{S}\langle \boldsymbol{R} \rangle. \end{cases} \tag{4}$$

In this paper, besides all the possible bilinear interactions, we are interested in the properties of the Hamiltonian $H \propto a_h^\dagger a_k + a_h a_k^\dagger$, which describes the mixing of the two modes h and k at a BS. The corresponding unitary operator describing such an evolution writes $U_{hk}(\phi) = \exp\{\phi(a_h^\dagger a_k - a_h a_k^\dagger)\}$, where $T = \cos^2\phi$ is the transmissivity of the BS and, without loss of generality, we take $\phi \in \mathbb{R}$. The 4×4 symplectic matrix associated with $U_{hk}(\phi)$ is given by:

$$\boldsymbol{S}_{hk}(\phi) = \begin{pmatrix} \cos\phi\,\mathbb{1}_2 & \sin\phi\,\mathbb{1}_2 \\ -\sin\phi\,\mathbb{1}_2 & \cos\phi\,\mathbb{1}_2 \end{pmatrix}. \tag{5}$$

It is now straightforward to prove the following:

Theorem 1 (Local invariance). *Given the two-mode Gaussian state described by the density matrix ϱ_{12}, such that the two modes are LSS and $\mathrm{Tr}[R_l\,\varrho_{12}] = 0$, $l = 1, 2$ and given the unitary bilinear transformation $U_{12}(\phi) = \exp\{\phi(a_1^\dagger a_2 - a_1 a_2^\dagger)\}$, then:*

$$U_{12}(\phi)\varrho_{12}U_{12}^\dagger(\phi) = \varrho_{12}, \tag{6}$$

if and only if $\varrho_{12} = \varrho_1 \otimes \varrho_2$, that is if and only if the two modes are uncorrelated.

Proof. Since the CM of the evolved state reads:

$$\boldsymbol{S}_{12}(\phi)\begin{pmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{\delta}_{12} \\ \boldsymbol{\delta}_{12}^T & \boldsymbol{\sigma}_2 \end{pmatrix}\boldsymbol{S}_{12}^T(\phi) = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^T & \boldsymbol{\Sigma}_2 \end{pmatrix}, \tag{7}$$

with:

$$\boldsymbol{\Sigma}_1 = \cos^2(\phi)\boldsymbol{\sigma}_1 + \sin^2(\phi)\boldsymbol{\sigma}_2 - \frac{1}{2}\sin(2\phi)(\boldsymbol{\delta}_{12} + \boldsymbol{\delta}_{12}^T), \tag{8a}$$

$$\boldsymbol{\Sigma}_2 = \sin^2(\phi)\boldsymbol{\sigma}_1 + \cos^2(\phi)\,\boldsymbol{\sigma}_2 + \frac{1}{2}\sin(2\phi)(\boldsymbol{\delta}_{12} + \boldsymbol{\delta}_{12}^T), \tag{8b}$$

$$\boldsymbol{\Sigma}_{12} = \frac{1}{2}\sin(2\phi)(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) + \cos^2(\phi)\boldsymbol{\delta}_{12} - \sin^2(\phi)\boldsymbol{\delta}_{12}^T, \tag{8c}$$

if $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$, i.e. if the input states are LSS, then $\boldsymbol{\Sigma}_k = \boldsymbol{\sigma}_k$ and $\boldsymbol{\Sigma}_{12} = \boldsymbol{\delta}_{12}$, i.e. the two-mode CM is left unchanged, if and only if $\boldsymbol{\delta}_{12} = \boldsymbol{0}$: the two LSS initial states should also be uncorrelated, namely, $\varrho_{12} = \varrho_1 \otimes \varrho_2$. □

If the input states are not LSS, then the two modes emerging from the BS exhibit correlations, whose classical or quantum nature has been thoroughly investigated elsewhere.[11–15] Indeed, under the hypotheses of Theorem 1 the two uncorrelated modes may belong to the same multi-mode Gaussian state, as pointed out in the following:

Corollary 1. *Given the N-mode Gaussian state $\varrho_{\boldsymbol{v}}$, whose two modes h and k are LSS and uncorrelated, i.e. $\varrho_{hk} = \mathrm{Tr}_{\boldsymbol{v}\setminus\{h,k\}}[\varrho_{\boldsymbol{v}}] = \varrho_h \otimes \varrho_k$, and $\mathrm{Tr}[R_l\,\varrho_{hk}] = 0$, $l = h, k$, then:*

$$\mathrm{Tr}_{\boldsymbol{v}\setminus\{h,k\}}[U_{hk}(\phi)\varrho_{\boldsymbol{v}}U_{hk}^\dagger(\phi)] = \varrho_{hk}, \tag{9}$$

where $U_{hk}(\phi) = \exp\{\phi(a_h^\dagger a_k - a_h a_k^\dagger)\}$. Moreover, if $\varrho_{\boldsymbol{v}} = \otimes_{n=1}^N \varrho_n$, then one also has $U_{hk}(\phi)\varrho_{\boldsymbol{v}}U_{hk}^\dagger(\phi) = \varrho_{\boldsymbol{v}}$.

More in general, the Corollary 1 states that if two LSS modes interacting through the BS are not correlated with each other, but may be correlated with other modes, they are still left unchanged by the interaction. Nevertheless, as we will see in the following, the presence of the BS affects the correlations existing between the interacting modes and the other ones.

Theorem 1 can be extended to address the bilinear interactions between couples of modes belonging to different uncorrelated multi-mode states. We have the:

Corollary 2. (Multi-mode invariance). *Given two uncorrelated N-mode Gaussian states with zero first moments, $\varrho_{\boldsymbol{A}}$ and $\varrho_{\boldsymbol{B}}$, where $\boldsymbol{A} = \{1,\ldots,N\}$ and $\boldsymbol{B} = \{N+1,\ldots,2N\}$, and the transformation $U_{\mathrm{BS},N}(\phi) = \otimes_{k=1}^N \exp\{\phi(a_k^\dagger a_{N+k} - a_k a_{N+k}^\dagger)\}$, which mixes the mode k of the state $\varrho_{\boldsymbol{A}}$ with the mode $N+k$ of the state $\varrho_{\boldsymbol{B}}$, $k = 1,\ldots,N$, then:*

$$U_{\mathrm{BS},N}(\phi)\varrho_{\boldsymbol{A}} \otimes \varrho_{\boldsymbol{B}} U_{\mathrm{BS},N}^\dagger(\phi) = \varrho_{\boldsymbol{A}} \otimes \varrho_{\boldsymbol{B}}, \tag{10}$$

if and only if $\varrho_{\boldsymbol{A}}$ and $\varrho_{\boldsymbol{B}}$ are excited in the same state.

Proof. The proof follows by noting that the $4N \times 4N$ CM of the $2N$-mode state $\varrho_{\boldsymbol{AB}} = \varrho_{\boldsymbol{A}} \otimes \varrho_{\boldsymbol{B}}$ is $\boldsymbol{\Sigma}_{\boldsymbol{AB}} = \boldsymbol{\Sigma}_{\boldsymbol{A}} \oplus \boldsymbol{\Sigma}_{\boldsymbol{B}}$, where $\boldsymbol{\Sigma}_k$ is the $2N \times 2N$ CM of the state ϱ_k, $k = \boldsymbol{A}, \boldsymbol{B}$ and that the symplectic matrix associated with $U_{\mathrm{BS},N}(\phi)$ reads:

$$\boldsymbol{S}_{\mathrm{BS},N}(\phi) = \begin{pmatrix} \cos\phi\, \mathbb{1}_{2N} & \sin\phi\, \mathbb{1}_{2N} \\ -\sin\phi\, \mathbb{1}_{2N} & \cos\phi\, \mathbb{1}_{2N} \end{pmatrix}. \tag{11}$$

$\mathbb{1}_{2N}$ being the $2N \times 2N$ identity matrix. Now, one has:

$$\boldsymbol{S}_{\mathrm{BS},N}(\phi)\boldsymbol{\Sigma}_{\boldsymbol{AB}}\boldsymbol{S}^T_{\mathrm{BS},N}(\phi) = \begin{pmatrix} \cos^2\!\phi\boldsymbol{\Sigma}_{\boldsymbol{A}} + \sin^2\!\phi\boldsymbol{\Sigma}_{\boldsymbol{B}} & \dfrac{1}{2}(\boldsymbol{\Sigma}_{\boldsymbol{B}} - \boldsymbol{\Sigma}_{\boldsymbol{A}})\sin(2\phi) \\ \dfrac{1}{2}(\boldsymbol{\Sigma}_{\boldsymbol{B}} - \boldsymbol{\Sigma}_{\boldsymbol{A}})\sin(2\phi) & \cos^2\!\phi\boldsymbol{\Sigma}_{\boldsymbol{B}} + \sin^2\!\phi\boldsymbol{\Sigma}_{\boldsymbol{A}} \end{pmatrix} \tag{12}$$

and $\boldsymbol{S}_{\mathrm{BS},N}(\phi)\boldsymbol{\Sigma}_{\boldsymbol{AB}}\boldsymbol{S}^T_{\mathrm{BS},N}(\phi) = \boldsymbol{\Sigma}_{\boldsymbol{AB}}$ if and only if $\boldsymbol{\Sigma}_{\boldsymbol{A}} = \boldsymbol{\Sigma}_{\boldsymbol{B}}$, and, in turn, $\varrho_{\boldsymbol{A}}$ and $\varrho_{\boldsymbol{B}}$ describe the same N-mode state. □

The results summarized by the previous theorem and corollaries can be used to design suitable scheme to control decoherence in lossy channels,[16–18] to investigate the birth of nonclassical correlations[15] or to eliminate mode coupling in communication schemes.[22]

It is worth noting that two LSS modes really interfere at the BS, though the final state is the same as the input one. The interaction can be demonstrated considering the scheme in Fig. 1, where the two modes 1 and 2, in the mixed states $\varrho_1 = \varrho_2 = \varrho$, are LSS and uncorrelated, but the mode 2 is correlated with mode 3, i.e. $\varrho_{23} \neq \varrho_2 \otimes \varrho_3$. It is worth noting that since modes 2 and 3 are correlated, then ϱ_2 and ϱ_3 should be mixed states. Thanks to the Corollary 1, after the interference modes 1 and 2 are still left unchanged and uncorrelated, however, because of the interaction, part of the correlations shared between modes 2 and 3 are now shared between modes 1 and 3. This can be seen by looking at the evolved CM of the whole state of the three modes in the presence of the BS. The CM of the initial state

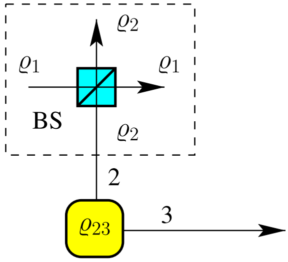

Fig. 1. In this scheme, the modes 1 and 2 are LSS and interfere at a BS, but the mode 2 is correlated with the mode 3: because of the interaction mode 1 becomes correlated with mode 3, though it is still uncorrelated with mode 1. See the text for details.

$\varrho_{123} = \varrho_1 \otimes \varrho_{23}$ reads:

$$\boldsymbol{\Sigma}_{123} = \begin{pmatrix} \boldsymbol{\sigma}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_2 & \boldsymbol{\delta}_{23} \\ \mathbf{0} & \boldsymbol{\delta}_{23}^T & \boldsymbol{\sigma}_3 \end{pmatrix}, \tag{13}$$

where $\boldsymbol{\sigma}_k$ is the 2×2 single-mode CM of mode $k = 1, 2, 3$, $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}$ and the matrix $\boldsymbol{\delta}_{23} \neq \mathbf{0}$ contains the correlations between modes 2 and 3. Since the BS acts only on modes 1 and 2, the 6×6 symplectic matrix associated with the unitary evolution $U_{12}(\phi) \otimes \mathbb{I}$ can be written as the direct sum $\boldsymbol{S}_{12}(\phi) \oplus \mathbb{1}_2$, where $\boldsymbol{S}_{12}(\phi)$ is given in Eq. (5). We have:

$$\boldsymbol{\Sigma}_{123}^{(\text{out})} = \boldsymbol{S}_{12}(\phi) \oplus \mathbb{1}_2 \boldsymbol{\Sigma}_{123} \boldsymbol{S}_{12}^T(\phi) \oplus \mathbb{1}_2 = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} & \sin\phi\, \boldsymbol{\delta}_{23} \\ \mathbf{0} & \boldsymbol{\sigma} & \cos\phi\, \boldsymbol{\delta}_{23} \\ \sin\phi\, \boldsymbol{\delta}_{23} & \cos\phi\, \boldsymbol{\delta}_{23} & \boldsymbol{\sigma}_3 \end{pmatrix}. \tag{14}$$

The comparison between Eqs. (13) and (14) shows that while, according to the Corollary 1, the states of modes 1 and 2 are (locally) left unchanged (and uncorrelated), now both modes 1 and 2 are correlated with mode 3. In fact we have the following CMs of the three reduced two-mode subsystems:

$$\boldsymbol{\Sigma}_{12}^{(\text{out})} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}, \tag{15a}$$

$$\boldsymbol{\Sigma}_{13}^{(\text{out})} = \begin{pmatrix} \boldsymbol{\sigma} & \sin\phi\, \boldsymbol{\delta}_{23} \\ \sin\phi\, \boldsymbol{\delta}_{23}^T & \boldsymbol{\sigma}_3 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{23}^{(\text{out})} = \begin{pmatrix} \boldsymbol{\sigma} & \cos\phi\, \boldsymbol{\delta}_{23} \\ \cos\phi\, \boldsymbol{\delta}_{23}^T & \boldsymbol{\sigma}_3 \end{pmatrix}, \tag{15b}$$

respectively. Furthermore, the degree of correlations between the modes 2 and 3 is decreased ($\boldsymbol{\delta}_{23} \to \cos\phi\, \boldsymbol{\delta}_{23}$) for the benefit of the birth of correlations between the previously uncorrelated modes 1 and 3 ($\mathbf{0} \to \sin\phi\, \boldsymbol{\delta}_{23}$). We conclude that the modes 1 and 2 are actually mixed at the BS, since mode 1 becomes correlated with mode 3 at the expenses of the initial correlations between modes 1 and 2. Nevertheless, this happens in such a way that no overall correlations arise between the interacting modes. This effect has been very recently experimentally demonstrated addressing pseudo-thermal (Gaussian) states.[23]

3. Interference of Polarized Gaussian States

In this section we address the effect of the polarization on the interference of Gaussian states. Without loss of generality, we still consider Gaussian states with zero first moments.

In order to include the polarization in our analysis, each field operator introduced in the previous section should be doubled, according to two orthogonal polarizations (for the sake of simplicity, we use the basis $\{|H\rangle, |V\rangle\}$ for the polarization degree of freedom), namely:

$$a_k \to a_{k,H}, a_{k,V} \quad \text{with } [a_{k,s}, a_{l,t}] = \delta_{h,l}\delta_{s,t}, \tag{16}$$

where $a_{k,s}$ is the annihilation operator of the mode kth with polarization s. In turn, the 2×2 CM matrix characterizing a single-mode Gaussian state is now replaced with a 4×4 CM $\boldsymbol{\sigma}$, which may be written in the block form:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}^{(H)} & \boldsymbol{\sigma}^{(HV)} \\ \boldsymbol{\sigma}^{(VH)} & \boldsymbol{\sigma}^{(V)} \end{pmatrix}, \tag{17}$$

where all the blocks are 2×2 real matrices. Indeed, the CM (17) fulfills all the properties to be a CM.[1,24] The block $\boldsymbol{\sigma}^{(s)}$, $s = H, V$, refers to the CM of the state the reduced s-polarized Gaussian state $\varrho^{(s)} = \langle s|\varrho|s\rangle$, while the block $\boldsymbol{\sigma}^{(HV)}$ represents the eventual correlations between the two orthogonally polarized modes.

In order to manipulate the polarization of the states, we introduce the operator $R_k(\theta)$, which describes the transformation performed by a rotator of polarization and, thus, acts on mode k by rotating its polarization by an amount θ (with respect to the horizontal axis):

$$R_k(\theta) = \exp\Big\{\theta\Big(a^\dagger_{k,H}\, a_{k,V} - a_{k,H}\, a^\dagger_{k,V}\Big)\Big\}. \tag{18}$$

We recall that, by using the Schwinger two-mode representation of the SU(2) algebra,[25] i.e. $J_{k,+} = a^\dagger_{k,H}\, a_{k,V}$, $J_{k,-} = a_{k,H}\, a^\dagger_{k,V}$ and $J_{k,3} = \frac{1}{2}[J_{k,+}, J_{k,-}] = \frac{1}{2}(a^\dagger_{k,H}\, a_{k,H} - a^\dagger_{k,V}\, a_{k,V})$ the operator $R_k(\theta)$ can be also written as:

$$R_k(\theta) = \exp\{\theta(J_{k,+} - J_{k,-})\} \tag{19a}$$

$$= \exp\{-\tan\theta\, J_{k,+}\}(\cos\theta)^{2J_{k,3}}\exp\{\tan\theta\, J_{k,-}\} \tag{19b}$$

$$= \exp\{\tan\theta\, J_{k,-}\}(\cos\theta)^{-2J_{k,3}}\exp\{-\tan\theta\, J_{k,+}\} \tag{19c}$$

and, by using the Hausdorff recursion formula[26]:

$$e^{\alpha A}Be^{-\alpha A} = B + \alpha[A, B] + \frac{\alpha^2}{2!}[A, [A, B]] + \frac{\alpha^3}{3!}[A, [A, [A, B]]] + \cdots,$$

it is straightforward to verify that:

$$R^\dagger_k(\theta)\; a_{k,H}\, R_k(\theta) = \cos\theta\, a_{k,H} + \sin\theta\, a_{k,V}, \tag{20a}$$

$$R^\dagger_k(\theta)\; a_{k,V}\, R_k(\theta) = \cos\theta\, a_{k,V} - \sin\theta\, a_{k,H}. \tag{20b}$$

The symplectic transformation corresponding to $R_k(\theta)$ reads:

$$\mathcal{R}_k(\theta) = \begin{pmatrix} \cos\theta\, \mathbb{1}_2 & \sin\theta\, \mathbb{1}_2 \\ -\sin\theta\, \mathbb{1}_2 & \cos\theta\, \mathbb{1}_2 \end{pmatrix}. \tag{21}$$

Note that the symplectic $\mathcal{R}_k(\theta)$ is similar to the symplectic of the BS given in Eq. (5).

As an example of the dynamics of a polarized Gaussian state under the action of $R_k(\theta)$, let us assume that the Gaussian input state $\varrho_{\rm in}$ is initially H-polarized. The 4×4 CM of ϱ then reads (we stress that the diagonal blocks refer to different polarizations):

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_{\rm in} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_0 \end{pmatrix}, \tag{22}$$

where $\boldsymbol{\sigma}_{\rm in}$ and $\boldsymbol{\sigma}_0 = \frac{1}{2}\,\mathbb{1}_2$ are the CMs of the reduced H-polarized and V-polarized modes, respectively (since the input state is H-polarized, the V-polarized mode is in the vacuum state). The evolved state $\varrho_\theta = R(\theta)\varrho R^\dagger(\theta)$ is still a Gaussian state with CM given by:

$$\boldsymbol{\sigma}_\theta = \begin{pmatrix} \cos^2\theta\,\boldsymbol{\sigma}_{\rm in} + \sin^2\theta\,\boldsymbol{\sigma}_0 & \dfrac{1}{2}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{\rm in})\sin(2\theta) \\ \dfrac{1}{2}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{\rm in})\sin(2\theta) & \cos^2\theta\,\boldsymbol{\sigma}_0 + \sin^2\theta\,\boldsymbol{\sigma}_{\rm in} \end{pmatrix}. \tag{23}$$

As one may expect, correlations can arise between the two orthogonally polarized modes, which can be spatially separated by means of a polarizing BS and, thus, exploited for quantum information purposes.

In Fig. 2, we sketched an interferometric scheme consisting of two polarized Gaussian states which interfere at a BS. As inputs, we take two uncorrelated Gaussian states $\varrho_{1,H}$ and $\varrho_{2,V}$ with orthogonal polarizations, H and V, respectively and the CM of the resulting Gaussian state $\varrho_{12} = \varrho_{1,H}\otimes\varrho_{2,V}$ is:

$$\boldsymbol{\Sigma}_{12} = \left(\begin{array}{cc|cc} \boldsymbol{\sigma}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_0 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{\sigma}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\sigma}_2 \end{array}\right), \tag{24}$$

where the upper-left 4×4 block describes the modes $a_{1,p}$ and the lower-right one refers to the modes $a_{2,p}$, $p = H, V$. Indeed, at the BS only the modes with the same

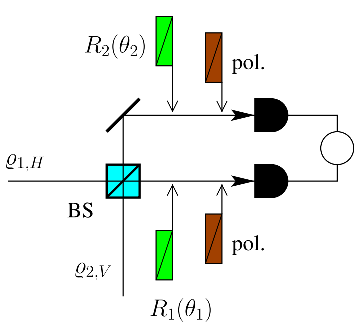

Fig. 2. Interference of polarized Gaussian states (see the text for details).

polarization interfere, thus the symplectic matrix describing the interaction can be written as:

$$\mathcal{S}_{12}(\phi) = \left(\begin{array}{cc|cc} \cos\phi \mathbb{1}_2 & \mathbf{0} & \sin\phi \mathbb{1}_2 & \mathbf{0} \\ \mathbf{0} & \cos\phi \mathbb{1}_2 & \mathbf{0} & \sin\phi \mathbb{1}_2 \\ \hline -\sin\phi \mathbb{1}_2 & \mathbf{0} & \cos\phi \mathbb{1}_2 & \mathbf{0} \\ \mathbf{0} & -\sin\phi \mathbb{1}_2 & \mathbf{0} & \cos\phi \mathbb{1}_2 \end{array}\right). \tag{25}$$

After the BS we can insert two rotators of polarization, $R_k(\theta_k)$, $k = 1, 2$, as depicted in Fig. 2, and the corresponding symplectic transformation $\mathcal{R}_{12}(\theta_1, \theta_2) = \mathcal{R}_1(\theta_1) \oplus \mathcal{R}_2(\theta_2)$ reads:

$$\mathcal{R}_{12}(\theta_1, \theta_2) = \left(\begin{array}{cc|cc} \cos\theta_1 \mathbb{1}_2 & \sin\theta_1 \mathbb{1}_2 & \mathbf{0} & \mathbf{0} \\ -\sin\theta_1 \mathbb{1}_2 & \cos\theta_1 \mathbb{1}_2 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \cos\theta_2 \mathbb{1}_2 & \sin\theta_2 \mathbb{1}_2 \\ \mathbf{0} & \mathbf{0} & -\sin\theta_2 \mathbb{1}_2 & \cos\theta_2 \mathbb{1}_2 \end{array}\right). \tag{26}$$

Finally, two polarization filters ("pol." in Fig. 2) are placed in front of the two photodetectors in order to select the H or V polarization: this last operation corresponds to delete the unwanted rows and columns from CM $\mathbf{\Sigma}_{12}^{(\text{out})}$ of the evolved state, which writes:

$$\mathbf{\Sigma}_{12}^{(\text{out})} = \mathcal{R}_{12}(\theta_1, \theta_2)\mathcal{S}_{12}(\phi)\mathbf{\Sigma}_{12}\mathcal{S}_{12}^T(\phi)\mathcal{R}_{12}^T(\theta_1, \theta_2). \tag{27}$$

In the following we assume that the two inputs are in the same Gaussian state (but with orthogonal polarizations), i.e. $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}$ in Eq. (24). At first we remove the rotators of polarizations, i.e. $\theta_1 = \theta_2 = 0$, and the CM $\mathbf{\Sigma}_{12}^{(\text{out})}$ of the state arriving at the detectors becomes (for the sake of simplicity we set $\phi = \pi/2$):

$$\mathbf{\Sigma}_{12}^{(\text{out})} = \frac{1}{2}\left(\begin{array}{cc|cc} \boldsymbol{\sigma}_+ & \mathbf{0} & -\boldsymbol{\sigma}_- & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_+ & \mathbf{0} & \boldsymbol{\sigma}_- \\ \hline -\boldsymbol{\sigma}_- & \mathbf{0} & \boldsymbol{\sigma}_+ & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_- & \mathbf{0} & \boldsymbol{\sigma}_+ \end{array}\right). \tag{28}$$

If we now place the polarization filters H or V before the detectors, we obtain the following reduced CM:

$$\mathbf{\Sigma}_{12}^{(\text{out})}\Big|_H = \frac{1}{2}\left(\begin{array}{c|c} \boldsymbol{\sigma}_+ & -\boldsymbol{\sigma}_- \\ \hline -\boldsymbol{\sigma}_- & \boldsymbol{\sigma}_+ \end{array}\right), \qquad \mathbf{\Sigma}_{12}^{(\text{out})}\Big|_V = \frac{1}{2}\left(\begin{array}{c|c} \boldsymbol{\sigma}_+ & \boldsymbol{\sigma}_- \\ \hline \boldsymbol{\sigma}_- & \boldsymbol{\sigma}_+ \end{array}\right), \tag{29}$$

respectively, which exhibit correlations between the two detected modes in both the cases. As a matter of fact, the same CMs (29) and, thus, the same output states are obtained by sending separately the two input states, since they do not interfere at the BS.

Now we insert the two rotators of polarization after the BS (see Fig. 2) and set $\theta_1 = \theta_2 = \pi/4$: the resulting Gaussian state has the following CM:

$$\tilde{\mathbf{\Sigma}}_{12}^{(\text{out})} = \frac{1}{2}\left(\begin{array}{cc|cc} \boldsymbol{\sigma}_+ & \mathbf{0} & \mathbf{0} & \boldsymbol{\sigma}_- \\ \mathbf{0} & \boldsymbol{\sigma}_+ & \boldsymbol{\sigma}_- & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\sigma}_- & \boldsymbol{\sigma}_+ & \mathbf{0} \\ \boldsymbol{\sigma}_- & \mathbf{0} & \mathbf{0} & \boldsymbol{\sigma}_+ \end{array}\right), \tag{30}$$

and, after filtering the modes by means of the polarization filters H or V we obtain:

$$\tilde{\mathbf{\Sigma}}_{12}^{(\text{out})}\Big|_H = \tilde{\mathbf{\Sigma}}_{12}^{(\text{out})}\Big|_V = \frac{1}{2}\left(\begin{array}{c|c} \boldsymbol{\sigma}_+ & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\sigma}_+ \end{array}\right), \tag{31}$$

i.e. in both the cases the two modes are no longer correlated. This effect is due to the rotators that let the modes with orthogonal polarization interfere and "erase" the information on the input state carried by its polarization, analogously to what happened in the first optical implementation of the quantum eraser.[27] This kind of interference has been also exploited to fully reconstruct the CM of a two-mode entangled Gaussian state.[28–30]

4. Concluding Remarks

In this paper we have introduced the notion of "LSS" and we have proved that two uncorrelated LSS modes are invariant under the evolution through a BS. The theorem and the corollaries written in Sec. 2 summarize, generalize and formalize the results of previous works, opening the way to the experimental investigation of the invariance in the multi-mode regime. We have included in our study the interference of orthogonally polarized Gaussian states and, in particular, we have proposed an interferometric scheme based on a BS rotators of polarization and polarization filters aimed to explore the dynamics of correlation in polarized continuous-variable systems.

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