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DYNAMICS OF QUANTUM CORRELATIONS OF TWO INTERFERING BIPARTITE GAUSSIAN STATES

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We address the interference of a pair of two-mode Gaussian states, interacting pairwise through a beam-splitter Hamiltonian. In the framework of a suitable phase-space analysis, the correlations generated through the interaction are studied by considering a quantity proportional to the variance of difference between the detected photocurrents of all the possible couples of modes. We use this quantity to demonstrate the invariance through the interaction and the correlations swapping also in the presence of nonideal photodetection.

Keywords: Quantum interference; quantum correlations; Gaussian states.

1. Introduction

The interference between uncorrelated quantum states may give rise to classical or quantum correlations. Among the possible interference mechanisms, the one that involves mixing of two modes of the radiation field at a beam splitter plays a leading role in the optical implementations of quantum information processing, due to its experimental feasibility and versatility.[1] In particular, the class of Gaussian states, i.e. states with a Gaussian characteristic function, has triggered the attention of the quantum optics community and has been deeply characterized and studied for both quantum information purposes and fundamental issues.[2,3] For example, the interference at a beam splitter of two squeezed states can generate Gaussian entanglement,[4–10] which has been used so far to achieve continuous variable teleportation.[11]

The properties of the correlated states emerging from a beam splitter have been thoroughly investigated in the past years, either to optimize the generation of entanglement[12,13] or to find relations between their entanglement and purities[14] or teleportation fidelity.[15] Furthermore, a recent work[16] has proved that there exists a strict relation between the fidelity (similarity) of the Gaussian states interacting through a beam-splitter Hamiltonian and the birth of entanglement at the output.

On the other hand, the interference at a beam splitter can also lead to the "invariance," that is the outgoing two-mode state is overall unaffected by the interaction and it is left in the same state of the two-mode input.[16–18] This effect may be extremely useful in the case of multi-mode Gaussian states and, more precisely, in the case of bipartite Gaussian states.[19] In Ref. 17, we showed that the correlations lost by a two-mode squeezed vacuum state, whose modes are reflected by two beam splitters with the same transmissivity, can be totally recovered by sending to the other ports of the beam splitters a two-mode squeezed vacuum state with the same characteristics of the input one. This effect has been experimentally applied to restore the two-mode squeezing in the presence of strong optical losses.[20]

In order to better understand the dynamics of the correlations and the limits of the setup addressed in Ref. 17, in this paper we consider as input states more general bipartite Gaussian states than the (pure) two-mode Gaussian states with zero first moments addressed so far. Moreover, to look at the correlations, we focus on an experimentally measurable quantity proportional to the variance of the difference between the detected photocurrents of all the possible couples of modes, also in the presence on nonunit quantum efficiency.

The paper is structured as follows. In Sec. 2, we review the basic elements of the interferometric setup, including the input Gaussian states. The evolution of the system is described in Sec. 3 by means of a suitable phase-space analysis. The characterization of the evolved states is addressed in Sec. 4, where we summarize the results concerning the correlations invariance and swapping for a particular choice of the involved parameters. Finally, Sec. 5 closes the paper with some concluding remarks.

2. The Interferometric Scheme

In this section, we describe the interferometric scheme to study the correlations arising from the interference of two bipartite Gaussian states. The main components of the scheme are sketched in Fig. 1. We consider as initial input states the two bipartite Gaussian states ϱ_{12} and ϱ_{34} generated by the nonlinear crystals NLC_{12} and NLC_{34} (see Fig. 1), which we assume to be a pair of two-mode squeezed thermal states described by the following density matrices:

$$\varrho_{12} = S_2(r)\nu_1(N_{\mathrm{th},1}) \otimes \nu_2(N_{\mathrm{th},2}) S_2^\dagger(r), \tag{1a}$$

$$\varrho_{34} = S_2(s)\nu_3(N_{\mathrm{th},3}) \otimes \nu_4(N_{\mathrm{th},4}) S_2^\dagger(s), \tag{1b}$$

respectively, where $S_2(z) = \exp\{z(a_h^\dagger a_k^\dagger - a_h a_k)\}$ is the two-mode squeezing operator, $r, s, z \in \mathbb{R}$, a_k is the annihilation operator of the kth mode, and:

$$\nu_k(N_{\mathrm{th},k}) = \frac{(N_{\mathrm{th},k})^{a_k^\dagger a_k}}{(1+N_{\mathrm{th},k})^{a_k^\dagger a_k + 1}}, \tag{2}$$

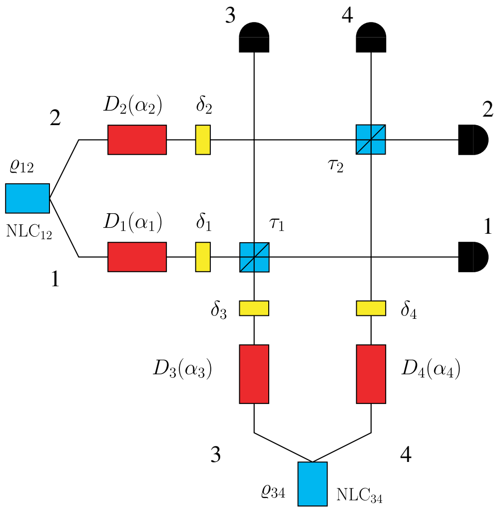

Fig. 1. Scheme to investigate the correlations arising from the pairwise interference of two bipartite Gaussian states. See the text for details.

is a thermal state of mode kth with $N_{\text{th},k}$ average photons. If $N_{\text{th},h}, N_{\text{th},k} \to 0$ in Eqs. (1a) and (1b), then ϱ_{hk} reduces to the maximally entangled two-mode squeezed vacuum state (the twin-beam state) addressed in Ref. 17.

In order to make our analysis more general, we apply a displacement operator $D_k(\alpha_k) = \exp(\alpha_k a_k^\dagger - \alpha_k^* a_k^\dagger)$ to the mode $k = 1, 2, 3, 4$, which are thus displaced by an amount α_k, respectively, and we also add a phase shift δ_k, as depicted in Fig. 1.

After the displacements and the phase shifts (see Fig. 1), the modes 1 and 3 and the modes 2 and 4 are mixed at two beam splitters with transmissivities $\tau_1 = \cos^2\phi_1$ and $\tau_2 = \cos^2\phi_2$, respectively, giving rise to the interference between the involved states. Finally, each outgoing mode undergoes a photodetection process and the following quantity is evaluated for all the possible couples of modes h–k:

$$\Delta_{hk}(\eta) = \frac{\mathrm{Var}[\mathcal{D}_{hk}(\eta)]}{\mathcal{I}_h(\eta) + \mathcal{I}_k(\eta)}, \tag{3}$$

with $\mathcal{D}_{hk}(\eta) = \mathcal{I}_h(\eta) - \mathcal{I}_k(\eta)$, where $\mathcal{I}_k(\eta)$ is the photocurrent measured at the detector $k = 1, 2, 3, 4$ in Fig. 1 and η is the corresponding quantum efficiency (we are assuming that all the photodetectors have the same quantum efficiency).

The next session will be devoted to the description of the evolution of the input states through the interferometer and the calculation of $\Delta_{hk}(\eta)$.

3. System Evolution

Since the states and the operations involved in our scheme are all Gaussian, namely, the states have a Gaussian characteristic function and the operations acting on them preserve their Gaussian character, in this section we address the evolution of the system by using the phase-space approach. In particular, since the Gaussian states are fully characterized by their covariance matrix (CM) and first-moments vector, we can describe the evolution of these quantities through suitable symplectic transformations describing the unitary evolutions.[3] For the sake of simplicity, from now on we set:

$$N_{\text{th},2} = N_{\text{th},1} = N_{\text{th}}, \quad N_{\text{th},4} = N_{\text{th},3} = M_{\text{th}}, \tag{4}$$

$$\alpha_1 = \alpha_2 = \alpha, \quad \alpha_3 = \alpha_4 = \beta, \quad \delta_2 = \delta_4 = 0, \tag{5}$$

with $\alpha, \beta \in R$.

The 4×4 CM of the the two-mode state $\varrho_{hk} = S_2(r)\nu_h(N) \otimes \nu_k(N) S_2^\dagger(r)$ reads[3]:

$$\boldsymbol{\sigma}_{hk}(r, N) = \frac{1+2N}{2} \begin{pmatrix} \cosh(2r)\,\mathbb{1} & \sinh(2r)\,\sigma_3 \\ \sinh(2r)\,\sigma_3 & \cosh(2r)\,\mathbb{1} \end{pmatrix}, \tag{6}$$

where $\mathbb{1}$ is the 2×2 identity matrix and $\sigma_3 = \text{diag}(1, -1)$ is the Pauli matrix, while the first-moments vector $\boldsymbol{X}_{hk} = \text{Tr}[\varrho_{hk}\,(q_h, p_h, q_k, p_k)^T]$ reduces to $\boldsymbol{X}_{hk} = (0, 0, 0, 0)^T$, where $(\cdots)^T$ is the transposition operation and $q_k = \frac{1}{\sqrt{2}}(a_k^\dagger + a_k)$ and $p_k = \frac{i}{\sqrt{2}}(a_k^\dagger - a_k)$ are the quadrature operators of mode k. Thus, the 8×8 CM of the four-mode input state $\varrho_{12} \otimes \varrho_{34}$ with the assumptions in Eq. (4) reads:

$$\boldsymbol{\Sigma}_{\text{in}} \equiv \boldsymbol{\Sigma}_{\text{in}}(r, N_{\text{th}}; s, M_{\text{th}}) = \begin{pmatrix} \boldsymbol{\sigma}_{12}(r, N_{\text{th}}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_{34}(s, M_{\text{th}}) \end{pmatrix}, \tag{7}$$

while the first-moments vector is $\boldsymbol{X}_{\text{in}} = (\boldsymbol{X}_{12}, \boldsymbol{X}_{34})^T$. The displacement operations leave the CM unchanged, but "displace" the vector $\boldsymbol{X}_{\text{in}}$, i.e. (here we use the same symbol $\boldsymbol{X}_{\text{in}}$ for both the initial and the displaced first-moments vector)[3]:

$$\boldsymbol{X}_{\text{in}} \to \boldsymbol{X}_{\text{in}} = \sqrt{2}(\alpha, 0, \alpha, 0, \beta, 0, \beta, 0)^T, \tag{8}$$

where we used the assumptions (4).

The symplectic transformation associated with the single-mode phase shift of an amount δ acting on mode k is given by the following 2×2 matrix:

$$\boldsymbol{S}_{\text{ps},k}(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}; \tag{9}$$

in turn, the symplectic transformation acting on the four modes (see also Fig. 1) is just the direct sum:

$$\boldsymbol{S}_{\text{ps}}(\boldsymbol{\delta}) = \bigotimes_{k=1}^{4} \boldsymbol{S}_{\text{ps},k}(\delta_k), \tag{10}$$

with $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4) = (\delta_1, 0, \delta_3, 0)$. Now, both $\boldsymbol{\Sigma}(r, N_{\text{th}}; s, M_{\text{th}})$ and the displaced $\boldsymbol{X}_{\text{in}}$ are transformed as follows:

$$\boldsymbol{\Sigma}_{\text{in}} \to \boldsymbol{S}_{\text{ps}}(\boldsymbol{\delta})\boldsymbol{\Sigma}_{\text{in}}\boldsymbol{S}_{\text{ps}}^T(\boldsymbol{\delta}), \quad \boldsymbol{X}_{\text{in}} \to \boldsymbol{S}_{\text{ps}}(\boldsymbol{\delta})\boldsymbol{X}_{\text{in}}. \tag{11}$$

Finally, the mode-mixing at the two beam splitters is described by the symplectic transformation:

$$\boldsymbol{S}_{13,24}(\phi_1,\phi_2) = \begin{pmatrix} \cos\phi_1\,\mathbb{1} & \mathbf{0} & -\sin\phi_1\,\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \cos\phi_2\,\mathbb{1} & \mathbf{0} & -\sin\phi_2\,\mathbb{1} \\ \sin\phi_1\,\mathbb{1} & \mathbf{0} & \cos\phi_1\,\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \sin\phi_2\,\mathbb{1} & \mathbf{0} & \cos\phi_2\,\mathbb{1} \end{pmatrix}. \tag{12}$$

The CM $\boldsymbol{\Sigma}_{\text{out}} \equiv \boldsymbol{\Sigma}_{\text{out}}(r, N_{\text{th}}, \delta_1, \phi_1; s, M_{\text{th}}, \delta_3, \phi_2)$ and the first-moments vector $\boldsymbol{X}_{\text{out}} \equiv \boldsymbol{X}_{\text{out}}(\alpha, \delta_1, \phi_1; \beta, \delta_3, \phi_2)$ of the final four-mode Gaussian state are given by

$$\boldsymbol{\Sigma}_{\text{out}} = \boldsymbol{S}_{13,24}(\phi_1,\phi_2)\boldsymbol{S}_{\text{ps}}(\boldsymbol{\delta})\boldsymbol{\Sigma}_{\text{in}}\boldsymbol{S}^T_{\text{ps}}(\boldsymbol{\delta})\boldsymbol{S}^T_{13,24}(\phi_1,\phi_2), \tag{13}$$

$$\boldsymbol{X}_{\text{out}} = \boldsymbol{S}_{13,24}(\phi_1,\phi_2)\boldsymbol{S}_{\text{ps}}(\boldsymbol{\delta})\boldsymbol{X}_{\text{in}}, \tag{14}$$

respectively. In order to calculate the quantity $\Delta_{hk}(\eta)$ defined in Eq. (3) and which we will study in the following section, we will use the results reported in the Appendix A.

4. Output State Characterization

We cannot report explicitly the analytic expression of $\Delta_{hk}(\eta)$, since it is clumsy. However, in this section we report some relevant result obtained focusing on the couple of modes 1, 2 and 1, 4 (analogous results are obtained considering the other couples of modes). In particular, we consider two interesting effects due to the interference: the correlations invariance (through the interaction) and the correlations swapping. In the first case, the correlations between the input modes 1 and 2 are overall not affected by the presence of the beam splitters[17,20]: the same values of $\Delta_{hk}(\eta)$ for $\delta_1 = 0, 2\pi$ are obtained in the absence of the beam splitters. In the second case, the beam splitters cause the swap of the correlations between the modes[17]: the correlations exhibited by the modes 1 and 2 are swapped to the modes 1 and 4 as one can verify by looking at the values of $\Delta_{hk}(\eta)$ at $\delta_1 = \pi$ (it is worth noting that the modes 1 and 4 do not directly interact).

In Figs. 2 and 3, we plot $\Delta_{12}(\eta)$ and $\Delta_{14}(\eta)$: in both the cases it is clear that the effect of the correlations invariance and swapping is according to the particular choice of the transmissivities of the beam splitters.

The presence of the displacement operators does not affect the amount of correlations at the output, nevertheless it changes the actual value of $\Delta_{hk}(\eta)$. In Fig. 4, we show how it is possible to remove the contribution to $\Delta_{12}(\eta)$ of the displacement operations, for a suitable choice of the parameters and in the special case of equal two-mode input states. In fact, by setting, e.g. $\phi_1 = \pi/4$, $\phi_2 = \phi_1 + \pi/2$, and $\delta_1 = \pi$ (with $\delta_2 = 0$) one has

$$\sqrt{2}(\alpha,0,\alpha,0,\alpha,0,\alpha,0) \to (0,0,0,0,-2\alpha,0,-2\alpha,0), \tag{15}$$

that is, the displacements are moved to modes 3 and 4 due to the interference at the BSs and, in turn, $\Delta_{12}(\eta)$ decreases. Though we are working in the correlation

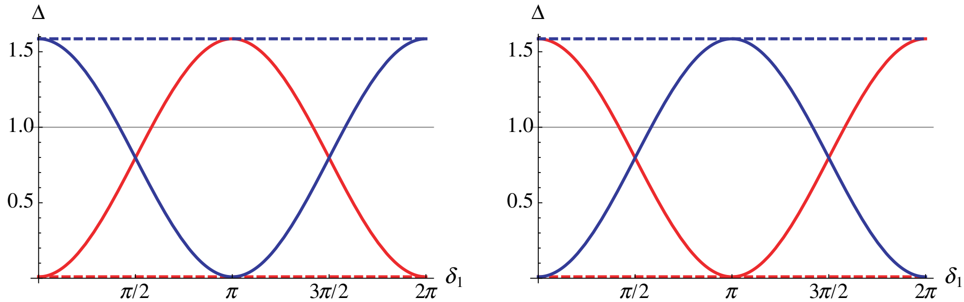

Fig. 2. (Color online.) Plot of $\Delta_{12}(\eta)$, red lines, and $\Delta_{14}(\eta)$, blue lines, with (solid lines) and without (dashed lines) the interaction at the beam splitters as functions of δ_1 (without lack of generality we put $\delta_2 = 0.0$), in the case of equal input states. We set: $\eta = 1.0$, $r = s = 0.7$, $N_{\text{th}} = M_{\text{th}} = 0.0$, $\alpha = \beta = 0.0$. Left: correlations invariance configuration ($\phi_1 = \phi_2 = \pi/4$); right: correlations swapping configuration ($\phi_1 = \pi/4$ and $\phi_2 = \phi_1 + \pi/2$).

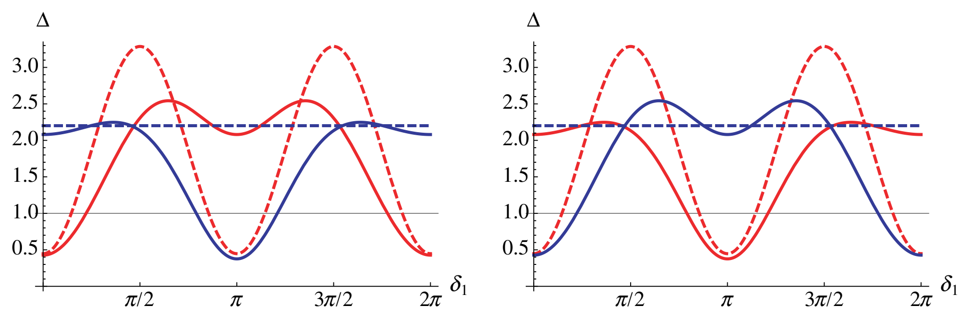

Fig. 3. (Color online.) Same plot as in Fig. 2, but for different input states, we set: $\eta = 0.8$, $r = s = 0.7$, $N_{\text{th}} = 0.2$, $M_{\text{th}} = 0.1$, $\alpha = 2.0$, $\beta = 1.0$. Left: correlations invariance configuration ($\phi_1 = \phi_2 = \pi/4$); right: correlations swapping configuration ($\phi_1 = \pi/4$ and $\phi_2 = \phi_1 + \pi/2$).

swapping configuration (i.e. $\phi_2 = \phi_1 + \pi/2$), since we set $\delta_1 = \pi$, we have the following output CM:

$$\boldsymbol{\Sigma}_{\text{out}} = \begin{pmatrix} \boldsymbol{\sigma}_{12}(-r, N_{\text{th}}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_{34}(r, N_{\text{th}}), \end{pmatrix}, \tag{16}$$

that is, the input states ϱ_{12} and ϱ_{34} are left unchanged (invariance), up to a "swapping" of the modes 1 and 2, which corresponds to the transformation $r \to -r$ [see the upper left 2×2 block matrix in Eq. (16) and the corresponding one in Eq. (7)]. This effect survives also if the input states are not perfectly equal.

Finally, for what concerns the nonlocal correlations, we observe that the entanglement of formation $\mathscr{E}_{hk}$ of the evolved states of modes 1, 2 and 3, 4 with CM (16) is the same as that of the states (1a) and (1b) and reads[21]:

$$\mathscr{E}_{hk}(r, N_{\text{th}}) = \left(x + \frac{1}{2}\right) \ln\left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) \ln\left(x - \frac{1}{2}\right), \tag{17}$$

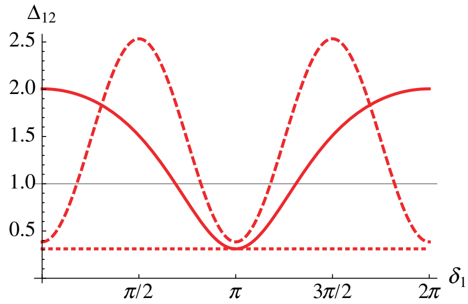

Fig. 4. (Color online.) Plot of $\Delta_{12}(\eta)$ in the correlations swapping configuration ($\phi_1 = \pi/4$ and $\phi_2 = \phi_1 + \pi/2$) as a function of δ_1 (with $\delta_2 = 0$). We set: $\eta = 0.8$, $r = s = 0.7$, $N_{\rm th} = M_{\rm th} = 0.1$, $\alpha = \beta = 1.5$. The solid line refers to $\Delta_{12}(\eta)$ after the mode mixing, the dashed one without the mode mixing and the dotted one without mode mixing and by putting $\alpha = 0$. Actually, due to the interference at the BSs, in this configuration and for $\delta_1 = \pi$, it is possible to remove the contribution due to the displacement operations and, thus, to reduce the value of $\Delta_{12}(\eta)$. See the text for details.

where

$$x = \frac{\left[\left(\frac{1}{2} + N_{\rm th}\right)^2 + \frac{1}{4}\right]\cosh(2r) - N_{\rm th}(1 + N_{\rm th})\sinh(2r)}{1 + 2N_{\rm th}}. \tag{18}$$

Analogously one can evaluate the entanglement of formation of the different couples of modes also for the other configurations.[22]

4.1. *Robustness of the scheme*

Due to the large number of the involved parameters, it is quite complicated to obtain general results about the robustness of the proposed interferometric scheme with respect to the presence of the thermal contributions at the inputs or to the amplitudes of the displacement operators. Here, we focus on the relevant case of equal input states and we comment on the effect of small amount of the thermal contribution and of the displacements ($\alpha = \beta \ll 1,\ \alpha, \beta \in \mathbb{R}$) on the quantities $\Delta_{12}(\eta)$ and $\Delta_{14}(\eta)$. Also in this case, the analytic expressions of the expansions are lengthy and they cannot be explicitly reported here, thus we summarize our main results.

The series expansions of $\Delta_{12}(\eta)$ and $\Delta_{14}(\eta)$ for $N_{\rm th} = M_{\rm th} \ll 1$ show that in both the cases we have a contribution at the first order in $N_{\rm th}$: the quantity we have chosen to characterize the system is quite sensible to the presence of thermal noise at the input.

For what concerns the amplitudes of the displacements, the series expansions for $\alpha = \beta \ll 1$ show that for both $\Delta_{12}(\eta)$ and $\Delta_{14}(\eta)$ we have a contribution at the second order in α^2: this corresponds to a linear dependence on the energy added with the displacement operators.

5. Concluding Remarks

In this paper, we have addressed the dynamics of the correlations of two bipartite Gaussian states, whose beams are mixed at two beam splitters. We have described the evolution of the states by means of the phase-space analysis, focusing on the transformation of the corresponding CM and first-moments vector. The effects of the correlations invariance and correlations swapping between the modes h and k have been proved by means of the quantity $\Delta_{hk}(\eta)$, where η is the quantum efficiency of the detectors. Our analysis has shown that the results of Ref. 17, which have been demonstrated for two-mode squeezed vacuum states, i.e. pure bipartite Gaussian states, can be extended and experimentally verified also in the presence of mixed Gaussian states by measuring a quantity proportional to the variance of the difference between two photocurrents.

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Appendix A. Difference Photocurrent of a Two-Mode Gaussian State and Its Variance

In this appendix, we explicitly write the difference between the detected photocurrents of the modes of a bipartite Gaussian state and its variance as functions of the corresponding CM and first moments vector elements. The difference between the photocurrents of the modes 1 and 2 of the state ϱ_{12} is given by (we assume that the two modes are detected with the same quantum efficiency η):

$$\mathcal{D}_{12}(\eta) = \eta\langle a_1^\dagger a_1 - a_2^\dagger a_2\rangle = \eta \mathrm{Tr}[\varrho_{12}(a_1^\dagger a_1 - a_2^\dagger a_2)] = \eta\langle [a_1^\dagger a_1]_\mathrm{s} - [a_2^\dagger a_2]_\mathrm{s}\rangle, \tag{A.1}$$

where $[\cdots]_\mathrm{s}$ denotes (Weyl) symmetric ordering and a_k is the annihilation operator of mode $k = 1, 2$. The variance $\mathrm{Var}[\mathcal{D}_{12}(\eta)]$ of the quantity $\mathcal{D}_{12}(\eta)$ reads[23]:

$$\mathrm{Var}[\mathcal{D}_{12}(\eta)] = \eta^2 \mathrm{Var}[\mathcal{D}_{12}] + \eta(1-\eta)N_\mathrm{tot}, \tag{A.2}$$

where $N_\mathrm{tot} = \langle a_1^\dagger a_1 + a_2^\dagger a_2\rangle = \langle [a_1^\dagger a_1]_\mathrm{s} + [a_2^\dagger a_2]_\mathrm{s}\rangle - 1$, and:

$$\begin{aligned}\mathrm{Var}[\mathcal{D}_{12}] = {} & \langle [(a_1^\dagger)^2 a_1^2]_\mathrm{s}\rangle + \langle [(a_2^\dagger)^2 a_2^2]_\mathrm{s}\rangle - \langle [a_1^\dagger a_1]_\mathrm{s}\rangle^2 - \langle [a_2^\dagger a_2]_\mathrm{s}\rangle^2 \\ & - 2\langle [a_1^\dagger a_1]_\mathrm{s}[a_2^\dagger a_2]_\mathrm{s}\rangle + 2\langle [a_1^\dagger a_1]_\mathrm{s}\rangle\langle [a_2^\dagger a_2]_\mathrm{s}\rangle - \frac{1}{2}.\end{aligned} \tag{A.3}$$

The expectations values of the symmetrically ordered operators involved in the previous quantities can be calculated by using the property[3]:

$$\langle [(a_1^\dagger)^n a_1^m]_\mathrm{s}[(a_2^\dagger)^h a_2^k]_\mathrm{s}\rangle = (-1)^{n+h}\partial^n_{\lambda_1^*}\partial^m_{\lambda_1}\partial^h_{\lambda_2^*}\partial^k_{\lambda_2}\chi(\lambda_1,\lambda_2)|_{\lambda_1=\lambda_2=0}, \tag{A.4}$$

where $\chi(\lambda_1, \lambda_2) = \mathrm{Tr}[\varrho_{12} D_1(\lambda_1) D_2(\lambda_2)]$ is characteristic function of the state ϱ_{12}, $D_k(\alpha)$ is the displacement operator acting on mode $k = 1, 2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. The characteristic function $\chi(\lambda_1, \lambda_2)$ can also be written in the following Cartesian form, which put in evidence the dependence on the 4×4 CM $\boldsymbol{\sigma}$ and on the first-moments vector $\boldsymbol{X}$:

$$\chi(\boldsymbol{\Lambda}) = \exp\left\{ -\frac{1}{2} \boldsymbol{\Lambda}^T \boldsymbol{\sigma} \boldsymbol{\Lambda} + i \boldsymbol{\Lambda}^T \boldsymbol{X} \right\}, \tag{A.5}$$

where $\boldsymbol{\Lambda}^T = (x_1, y_1, x_2, y_2)$ and $\boldsymbol{X} = \mathrm{Tr}[\varrho(q_1, p_1, q_2, p_2)^T]$, with $q_k = \frac{1}{\sqrt{2}}(a_k^\dagger + a_k)$ and $p_k = \frac{i}{\sqrt{2}}(a_k^\dagger - a_k)$. If we write the 4×4 CM and the first-moments vector as follows:

$$\boldsymbol{\sigma} = \begin{pmatrix} a & c & e & f \\ c & b & g & h \\ e & g & A & C \\ f & h & C & B \end{pmatrix}, \quad \boldsymbol{X} = (X_1, Y_1, X_2, Y_2)^T, \tag{A.6}$$

then the characteristic function in Eq. (A.5) can be expressed in the complex notation as[24]:

$$\begin{aligned} \chi(\lambda_1, \lambda_2) = {} & \exp\{-\mathscr{A}|\lambda_1|^2 - \mathscr{B}|\lambda_2|^2 - \mathscr{C}\lambda_1^2 - \mathscr{C}^*\lambda_1^{*2} - \mathscr{D}\lambda_2^2 - \mathscr{D}^*\lambda_2^{*2} \\ & - \mathscr{E}\lambda_1\lambda_2 - \mathscr{E}^*\lambda_1^*\lambda_2^* - \mathscr{F}\lambda_1\lambda_2^* - \mathscr{F}^*\lambda_1^*\lambda_2 \\ & + i[\mathscr{U}^*\lambda_1 + \mathscr{U}\lambda_1^* + \mathscr{V}^*\lambda_2 + \mathscr{V}\lambda_2^*]\}, \end{aligned} \tag{A.7}$$

where

$$\mathscr{A} = \frac{1}{2}(a + b), \quad \mathscr{B} = \frac{1}{2}(A + B), \tag{A.8}$$

$$\mathscr{C} = \frac{1}{4}(a - b - 2ic), \quad \mathscr{D} = \frac{1}{4}(A - B - 2iC), \tag{A.9}$$

$$\mathscr{E} = \frac{1}{2}[e - h - i(f + g)], \quad \mathscr{F} = \frac{1}{2}[e + h + i(f - g)], \tag{A.10}$$

$$\mathscr{U} = \frac{1}{\sqrt{2}}(X_1 + iY_1), \quad \mathscr{V} = \frac{1}{\sqrt{2}}(X_2 + iY_2). \tag{A.11}$$

Finally, we have:

$$\langle [a_1^\dagger a_1]_s \rangle = \mathscr{A} + |\mathscr{U}|^2, \quad \langle [a_2^\dagger a_2]_s \rangle = \mathscr{B} + |\mathscr{V}|^2, \tag{A.12a}$$

$$\begin{aligned} \langle [a_1^\dagger a_1]_s [a_2^\dagger a_2]_s \rangle = {} & |\mathscr{E}|^2 + \mathscr{A}|\mathscr{V}|^2 + \mathscr{B}|\mathscr{U}|^2 + |\mathscr{U}|^2|\mathscr{V}|^2 + |\mathscr{F}|^2 \\ & + \mathscr{A}\mathscr{B} + \mathscr{U}^*\mathscr{V}^*\mathscr{E} + \mathscr{U}\mathscr{V}\mathscr{E}^* + \mathscr{V}^*\mathscr{U}\mathscr{F} + \mathscr{F}^*\mathscr{U}^*\mathscr{V}, \end{aligned} \tag{A.12b}$$

$$\langle [a_1^{\dagger 2} a_1^2]_s \rangle = 2\mathscr{A}^2 + 4\mathscr{A}|\mathscr{U}|^2 + |2\mathscr{C} + \mathscr{U}^2|^2, \tag{A.12c}$$

$$\langle [a_2^{\dagger 2} a_2^2]_s \rangle = 2\mathscr{B}^2 + 4\mathscr{B}|\mathscr{V}|^2 + |2\mathscr{D} + \mathscr{V}^2|^2, \tag{A.12d}$$

which can be directly used to calculate the difference photocurrent (A.1) and its variance (A.2) as functions of the CM and first-moments vector elements given in Eq. (A.6).

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