

Improving information/disturbance and estimation/distortion trade-offs with non-universal protocols

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Abstract

We analyse in detail a conditional measurement scheme based on linear optical components, feed-forward loop and homodyne detection. The scheme may be used to achieve two different tasks. On the one hand it allows the extraction of information with minimum disturbance about a set of coherent states. On the other hand, it represents a nondemolitive measurement scheme for the annihilation operator, i.e. an indirect measurement of the Q -function. We investigate the information/disturbance trade-off for state inference and introduce the estimation/distortion trade-off to assess estimation of the Q -function. For coherent states chosen from a Gaussian set, we evaluate both information/disturbance and estimation/distortion trade-offs and found that non-universal protocols may be optimized in order to achieve better performances than universal ones. For Fock number states we prove that universal protocols do not exist and evaluate the estimation/distortion trade-off for a thermal distribution.

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1. Introduction

Any measurement performed on a quantum system alters the state of the system itself. As a consequence, any scheme aimed to extract *information* about the state of a system unavoidably produces a *disturbance*. The same is true if we focus on a specific quantity rather than the state as a whole: any scheme devised for the *estimation* of an observable or generalized observable produces a *distortion* of the probability distribution of the measured quantity.

The trade-off between the amounts of the extracted information about a quantum state and the corresponding added disturbance, from now on the *information/disturbance* trade-off, has received much attention [1–6]. Besides fundamental interest this is motivated by practical applications in quantum communication and quantum cryptography [7–9]. The information/disturbance trade-off crucially depends on the set of input states and may be

quantified in terms of fidelities. For finite-dimensional systems, inequalities on fidelities, which express the bounds on precision imposed by quantum mechanics, have been derived in several cases. These include a single copy of an unknown pure state [6], many copies of identically prepared pure qubits [10], a single copy of a pure state generated by independent phase-shifts [11], an unknown spin coherent state [12], and single copy of an unknown maximally entangled state [13]. Optimal measurement schemes, which saturate the bounds, have been also devised [14–16] and implemented [17].

A relevant concept in quantum estimation is universality. A protocol is said to be universal if the fidelities are independent of the input state, at least within the class of states under investigation. In fact, for systems with infinite-dimensional Hilbert space an information/disturbance trade-off has been derived for an unknown coherent state (i.e. for a set of coherent states with flat distribution of the amplitude) assuming universality and Gaussian operations [18]. An optimal measurement scheme saturating this bound has also been proposed and realized [18]. In addition, it has been shown how to slightly improve the trade-off using non-Gaussian operations [19].

The quantum-mechanical back-action in the measurement of a specific observable has been extensively studied in the context of quantum nondemolition measurements (QND) [20]. In a QND scheme, an observable is measured, without destroying the state carrying the information, with the aim of keeping the distortion (back-action) in the conjugated observable and thus preserving the value of the observable itself [21]. The corresponding *estimation/distortion* trade-off has been mostly analysed in terms of variances.

In this paper, we report a detailed analysis of an optical scheme based on linear optical components and homodyne detection that can be used to achieve two different, though related, tasks. On the one hand, it allows us to implement non-universal estimation protocol for Gaussian sets of coherent states and to improve the trade-off, i.e. the extraction of information with minimum disturbance, in comparison with universal protocols. On the other hand, it represents a nondemolitive measurement scheme of a generalized observable, the annihilation operator, suitable for a generic set of states, i.e. an indirect measurement of the Q -function. We assess its QND performances in terms of fidelities [22], which quantify how much the measured distribution resembles the Q -function of the input states, and how much the distribution of the output states has been distorted by the measurement protocol. Going beyond variances allows us to investigate non-Gaussian states. Indeed, we analyse the *estimation/distortion* trade-off for Gaussian sets of coherent states and thermal sets of Fock number states.

The scheme under investigation is that used in [18] to investigate the universal Gaussian information/disturbance trade-off for an unknown coherent state. The same scheme has also been used to demonstrate $1 \rightarrow 2$ optimal Gaussian cloning of coherent states [23] and suggested for more general cloning task, such as $1 \rightarrow m$ cloning of coherent states [24, 25] or cloning of general Gaussian states [26].

The paper is structured as follows: in section 2 we describe the measurement scheme as well its statistics and dynamics. In section 3 we introduce the inference rules and the fidelities, whereas in sections 4 and 5 we explicitly evaluate the trade-offs for Gaussian sets of coherent states and thermal sets of Fock number states, respectively. Section 6 closes the paper with some concluding remarks.

2. The measurement scheme

The measurement scheme we are going to analyse is schematically depicted in figure 1, where we show the configuration used for state inference (left) as well as that used for nondemolitive measurement of the Q -function (right). In our scheme, the signal $q^{(\text{in})}$, i.e. mode 1, is mixed

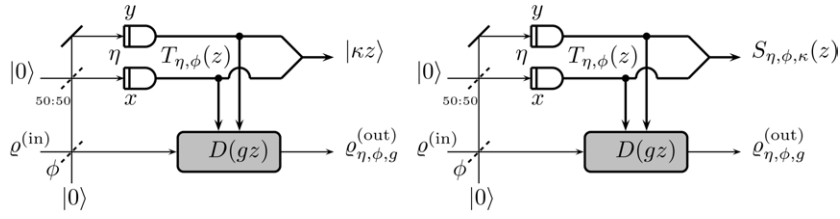


Figure 1. Linear optical schemes for state inference (left) and indirect measurement of the Q -function (right). In both schemes the input signal $\varrho^{(in)}$ impinges onto a beam splitter with transmissivity $\tau = \cos^2 \phi$: the reflected part is measured by double homodyne detection, and the complex measurement outcome $z = x + iy$ is used to displace the transmitted beam by an amount gz , g being a suitable gain factor. In the state inference scheme (left), the measurement outcome is used to infer the input state according to the rule $z \rightsquigarrow |\kappa z\rangle$, κ being a real number and $|z\rangle$ a coherent state. In the estimation scheme (right), the measurement outcomes are collected to form the distribution $S_{\eta, \phi, \kappa}(z)$, which is used as an estimate of the Q -function of the input signal. See text for more details.

with the vacuum at a beam splitter with transmissivity $\tau = \cos^2 \phi$. The reflected beam is then measured by a double homodyne detector with quantum efficiency η . The outcomes of the measurement are complex numbers $z = x + iy$, x and y being the outcomes from the two homodyne detectors, which are used either to infer the state at the input, or collected to build an estimate of the input Q -function. In both cases, a suitable real rescaling factor κ may be used to optimize the fidelities. The outcome of the measurement is also sent to the transmitted beam, which is displaced by an amount gz , g being a suitable additional gain. The positive operator-valued measure (POVM) of the double homodyne is given by

$$\Pi_{\eta}(z) = \int_{\mathbb{C}} \frac{d^2 \mu}{\pi \Delta_{\eta}^2} \exp \left\{ -\frac{|\mu - z|^2}{\Delta_{\eta}^2} \right\} \frac{|\mu\rangle\langle\mu|}{\pi}, \quad (1)$$

with $\Delta_{\eta}^2 = (1 - \eta)/\eta$, η being the quantum efficiency of each detector (here we assume that both the detectors have the same efficiency). The probability distribution of the raw outcomes z is given by

$$T_{\eta, \phi}(z) = \text{Tr}_{12} [U_{\phi} \varrho^{(in)} \otimes |0\rangle\langle 0| U_{\phi}^{\dagger} \mathbb{1} \otimes \Pi_{\eta}(z)], \quad (2)$$

where $U_{\phi} = \exp \{ \phi (a_1^{\dagger} a_2 - a_1 a_2^{\dagger}) \}$ is the evolution operator of the beam splitter. The conditional state of mode 1, after the outcome z is given by

$$\varrho_{\eta, \phi}(z) = \frac{\text{Tr}_2 [U_{\phi} \varrho^{(in)} \otimes |0\rangle\langle 0| U_{\phi}^{\dagger} \mathbb{1} \otimes \Pi_{\eta}(z)]}{T_{\eta, \phi}(z)}, \quad (3)$$

and the overall output state $\varrho_{\eta, \phi, g}^{(out)}$ is obtained averaging over all the possible outcomes

$$\varrho_{\eta, \phi, g}^{(out)} = \int_{\mathbb{C}} d^2 z T_{\eta, \phi}(z) D(gz) \varrho_{\eta, \phi}(z) D^{\dagger}(gz), \quad (4)$$

with $D(gz) = \exp \{ gza^{\dagger} - gz^* a \}$.

Using the Glauber–Sudarshan P -function representation, we can write the input state as

$$\varrho^{(in)} = \int_{\mathbb{C}} d^2 \xi P^{(in)}(\xi) |\xi\rangle\langle\xi|. \quad (5)$$

In turn, the probability distribution of the outcomes may be written as

$$T_{\eta, \phi}(z) = \frac{1}{\sin^2 \phi} W_{s_1}[\varrho^{(in)}] \left(\frac{z}{\sin \phi} \right), \quad (6)$$

with

$$s_1 = 1 - \frac{2}{\eta \sin^2 \phi}, \quad (7)$$

where $W_s[\varrho](\alpha)$ denotes the s -ordered Wigner function of the state ϱ [27]. We also made use of the relation, valid for $r > s$,

$$W_s[\varrho](\zeta) = \int_{\mathbb{C}} d^2\xi \frac{2}{\pi(r-s)} \exp\left\{-\frac{2|\xi-\zeta|^2}{r-s}\right\} W_r[\varrho](\xi). \quad (8)$$

In addition, it is straightforward to prove that

$$\varrho_{\eta,\phi,g}^{(\text{out})} = \int_{\mathbb{C}} d^2\xi P_{\eta,\phi,g}^{(\text{out})}(\xi) |\xi\rangle\langle\xi|, \quad (9)$$

with

$$P_{\eta,\phi,g}^{(\text{out})}(\xi) = \frac{1}{(\cos\phi + g \sin\phi)^2} W_{s_2}[\varrho^{(\text{in})}]\left(\frac{\xi}{\cos\phi + g \sin\phi}\right), \quad (10)$$

and

$$s_2 = 1 - \frac{2g^2}{\eta(\cos\phi + g \sin\phi)^2}. \quad (11)$$

Finally, thanks to equation (8), the Q -function at the output, corresponding to the state (9), can be written as

$$Q_{\eta,\phi,g}^{(\text{out})}(z) = \frac{1}{(\cos\phi + g \sin\phi)^2} W_{s_3}[\varrho^{(\text{in})}]\left(\frac{1}{\cos\phi + g \sin\phi}\right), \quad (12)$$

with

$$s_3 = 1 - \frac{2(\eta - g^2)}{\eta(\cos\phi + g \sin\phi)^2}. \quad (13)$$

3. Inferences and fidelities

In this section, we introduce inference rules and fidelities to quantify the information/disturbance trade-off for state inference and the estimation/distortion trade-off for measurement of the Q -function.

3.1. State inference: information fidelity and disturbance fidelity

If the input signal belongs to a Gaussian set of coherent states then the reflected beam is still a coherent state and a natural inference rule [28] after having observed the outcome z is the following: $z \rightsquigarrow |\kappa z\rangle$, with $\kappa \geq 0$. In order to assess our inference, assuming a set of pure states at the input, we use the state overlap between the inferred state and the input one. By averaging over the possible outcomes z , we arrive at the *information fidelity*:

$$G_{\eta,\kappa}(\phi) = \int_{\mathbb{C}} d^2z T_{\eta,\phi}(z) \langle \kappa z | \varrho^{(\text{in})} | \kappa z \rangle, \quad (14)$$

which can be optimized, i.e., maximized, with respect to the parameter κ . Similarly, the amount of disturbance can be evaluated by the overlap between the input state and the conditional one. By averaging over the possible outcomes z we have the *disturbance fidelity*:

$$F_{\eta,g}(\phi) = \int_{\mathbb{C}} d^2z T_{\eta,\phi}(z) \text{Tr}[\varrho^{(\text{in})} D(gz) \varrho_{\eta,\phi}(z) D^\dagger(z)] = \text{Tr}[\varrho^{(\text{in})} \varrho_{\eta,\phi,g}^{(\text{out})}], \quad (15)$$

where, again, we assumed pure states at the input. The disturbance fidelity can be optimized, i.e., maximized, with respect to the parameter g .

3.2. Measurement of the Q -function: estimation fidelity and distortion fidelity

Performing double-homodyne detection on the reflected beam provides an estimate of the input Q -function upon a suitable rescaling of raw outcomes. As an estimate we adopt the distribution $S_{\eta,\phi,\kappa}(z)$, defined as follows

$$S_{\eta,\phi,\kappa}(z) = \frac{1}{\kappa^2} T_{\eta,\phi}\left(\frac{z}{\kappa}\right), \tag{16}$$

with $\kappa > 0$ (see the right panel of figure 1). In order to evaluate the similarity of the inferred Q -function to the input one $Q^{(\text{in})}(z) = \frac{1}{\pi} \langle z | \rho^{(\text{in})} | z \rangle$ we introduce the *estimation fidelity*:

$$H_{\eta,\kappa}(\phi) = \int_{\mathbb{C}} d^2z \sqrt{Q^{(\text{in})}(z) S_{\eta,\phi,\kappa}(z)}. \tag{17}$$

$H_{\eta,\kappa}(\phi)$ is a proper fidelity, i.e. $0 \leq H_{\eta,\kappa}(\phi) \leq 1$, with $H_{\eta,\kappa}(\phi) = 1$ iff the inferred distribution is equal to the actual Q -function. The protocol can be optimized, by maximizing $H_{\eta,\kappa}(\phi)$ with respect to the parameter κ . Since the output state is altered by the measurement, the corresponding Q -function, $Q_{\eta,\phi,g}^{(\text{out})}(z)$, is a distorted version of input one. The degree of this modification can be evaluated by means of the *distortion fidelity*

$$K_{\eta,g}(\phi) = \int_{\mathbb{C}} d^2z \sqrt{Q^{(\text{in})}(z) Q_{\eta,\phi,g}^{(\text{out})}(z)}, \tag{18}$$

which can be optimized, i.e., maximized, with respect to the parameter g .

4. Coherent states

In this section, we evaluate explicitly the information/disturbance and the estimation/distortion trade-offs for a set of coherent states $\rho^{(\text{in})} = |\beta\rangle\langle\beta|$ with complex amplitudes distributed according to the Gaussian:

$$\mathcal{P}(\beta) = \frac{1}{\pi \Omega^2} \exp\left\{-\frac{|\beta|^2}{\Omega^2}\right\}. \tag{19}$$

Such a distribution of coherent states can be obtained, e.g., starting from the single output states of a continuous variable teleportation protocol as well as at the output of a Gaussian noise channel with vacuum input.

4.1. Information/disturbance trade-off

Since the Glauber P -function of a coherent state $P^{(\text{in})}(\xi) = \delta^{(2)}(\xi - \beta)$ is a delta function in the complex plane the information fidelity is given by

$$G_{\eta,\kappa}(\phi, \beta) = \frac{\eta}{\eta + \kappa^2} \exp\left\{-\frac{\eta(1 - \kappa \sin \phi)^2}{\eta + \kappa^2} |\beta|^2\right\}. \tag{20}$$

The protocol is universal (i.e. G does not depend on β) if $\kappa = 1/\sin \phi$; the corresponding fidelity is given by

$$G_{\eta}(\phi) = \frac{\eta \sin^2 \phi}{1 + \eta \sin^2 \phi}. \tag{21}$$

For the disturbance fidelity we have

$$F_{\eta,g}(\phi, \beta) = \frac{\eta}{\eta + g^2} \exp\left\{-\frac{\eta(1 - \cos \phi - g \sin \phi)^2}{\eta + g^2} |\beta|^2\right\}. \tag{22}$$

Universality is obtained for $g = (1 - \cos \phi) / \sin \phi$, i.e.,

$$F_\eta(\phi) = \frac{\eta \sin^2 \phi}{\eta \sin^2 \phi + (1 - \cos \phi)^2}. \quad (23)$$

The (universal) information/disturbance trade-off reads as follows:

$$F_\eta = G \left\{ G + (1 - G) \left[1 - \sqrt{1 - \frac{G}{\eta(1 - G)}} \right]^2 \right\}^{-1}. \quad (24)$$

For $\eta \rightarrow 1$ we recover the optimal trade-off obtained in [18]. Note that also for $\eta \neq 1$ the trade-off (24) is optimal, i.e., the noise added is the minimum allowed by quantum mechanics in a joint measurement of conjugated quadratures [26]. As expected we have $F_\eta < F_1, \forall G, \eta$, i.e., a non-unit value of the quantum efficiency degrades performances.

In the general case, i.e., releasing the request of universality, the average information and disturbance fidelities are given by

$$\overline{G}_{\eta,\kappa}(\phi) = \int_{\mathbb{C}} d^2\beta \mathcal{P}(\beta) G_{\eta,\kappa}(\phi, \beta) = \frac{\eta}{\eta + \kappa^2 + \eta\Omega^2(1 - \kappa \sin \phi)^2}, \quad (25)$$

$$\overline{F}_{\eta,g}(\phi) = \int_{\mathbb{C}} d^2\beta \mathcal{P}(\beta) F_{\eta,g}(\phi, \beta) = \frac{\eta}{\eta + g^2 + \eta\Omega^2(1 - \cos \phi - g \sin \phi)^2}, \quad (26)$$

respectively. The fidelity may be maximized with respect to the parameters κ and g , whose optimal values are given by

$$\kappa = \frac{\eta\Omega^2 \sin \phi}{1 + \eta\Omega^2 \sin^2 \phi}, \quad g = \frac{\eta\Omega^2 \sin^2 \phi (1 - \cos \phi)}{1 + \eta\Omega^2 \sin^2 \phi}, \quad (27)$$

corresponding to

$$\begin{aligned} \overline{G}_{\eta,\Omega}(\phi) &= \frac{1 + \eta\Omega^2 \sin^2 \phi}{1 + \Omega^2 + \eta\Omega^2 \sin^2 \phi}, \\ \overline{F}_{\eta,\Omega}(\phi) &= \frac{1 + \eta\Omega^2 \sin^2 \phi}{1 + \Omega^2[\eta \sin^2 \phi + (1 - \cos \phi)^2]}, \end{aligned} \quad (28)$$

and to the trade-off

$$\overline{F}_{\eta,\Omega} = \frac{(1 + \Omega^2)\overline{G}}{\Omega^2} \left\{ \overline{G} + (1 - \overline{G}) \left[1 - \sqrt{1 - \frac{(1 + \Omega^2)\overline{G} - 1}{\eta\Omega^2(1 - \overline{G})}} \right]^2 \right\}^{-1}. \quad (29)$$

For ‘large’ set of signals, i.e., for $\Omega \rightarrow \infty$, we recover the ‘universal’ trade-off (24), whereas for finite values of Ω we have $\overline{F}_{\eta,\Omega} > F_\eta$. In other words, for finite Ω non-universal protocols may be optimized and achieve superior performances compared to universal one.

In figure 2 we plot the information/disturbance trade-off, which is obtained by tuning ϕ in the interval $[0, \pi/2]$ and, in turn, the transmissivity $\tau = \cos^2 \phi$ of the beam splitter ranges from 1 to 0. When $\tau = 1$, the input states are completely transmitted and only the vacuum is left for the double homodyne detection: in this case the disturbance fidelity $\overline{F}_{\eta,\Omega} = 1$ while the information fidelity $\overline{G}_{\eta,\Omega}$ reaches its minimum. When $\tau = 0$, the input state is completely reflected and nothing is transmitted: now $\overline{F}_{\eta,\Omega}$ is minimum and $\overline{G}_{\eta,\Omega}$ reaches its maximum. The universal protocol has been recently experimentally demonstrated in [18], where the quantum efficiency was approximately $\eta \approx 94\%$.

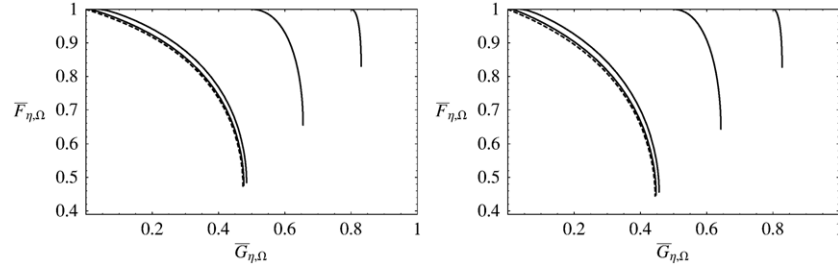


Figure 2. Information/disturbance trade-off for universal and non-universal protocols. We show the disturbance fidelity as a function of the information fidelity for different values of the width Ω and for two different values of the quantum efficiency (see text for details). (Left): $\eta = 0.9$. (Right): $\eta = 0.8$. The dashed line refers to universal protocol whereas, in both the plots, the solid lines are for (from right to left): $\Omega = 0.5, 1.0, 5.0$ and 10.0 .

4.2. Estimation/distortion trade-off

The estimation and distortion fidelities for a given coherent state read as follows:

$$H_{\eta, \kappa}(\phi, \beta) = \frac{2\kappa\sqrt{\eta}}{\eta + \kappa^2} \exp\left\{-\frac{\eta(1 - \kappa \sin \phi)^2}{2(\eta + \kappa^2)} |\beta|^2\right\} \quad (30)$$

$$K_{\eta, g}(\phi, \beta) = \frac{2\sqrt{\eta(\eta + g^2)}}{2\eta + g^2} \exp\left\{\frac{\eta(1 - \cos \phi - g \sin \phi)^2}{2(2\eta + g^2)} |\beta|^2\right\}. \quad (31)$$

Universality conditions are given by $\kappa = 1/\sin \phi$ and $g = (1 - \cos \phi)/\sin \phi$, and the corresponding universal fidelities are

$$H_{\eta}(\phi) = \frac{2\sqrt{\eta} \sin \phi}{1 + \eta \sin^2 \phi} \quad (32)$$

$$K_{\eta}(\phi) = \frac{2 \sin \phi \sqrt{\eta[\eta \sin^2 \phi + (1 - \cos \phi)^2]}}{2\eta \sin^2 \phi + (1 - \cos \phi)^2}. \quad (33)$$

For non-universal protocols we have

$$\bar{H}_{\eta, \kappa}(\phi) = \int_{\mathcal{C}} d^2\beta \mathcal{P}(\beta) H_{\eta, \kappa}(\phi, \beta) = \frac{4\kappa\sqrt{\eta}}{2(\eta + \kappa^2) + \eta\Omega^2(1 + \kappa \sin \phi)^2}, \quad (34)$$

$$\bar{K}_{\eta, g}(\phi) = \int_{\mathcal{C}} d^2\beta \mathcal{P}(\beta) K_{\eta, g}(\phi, \beta) = \frac{4\sqrt{\eta(\eta + g^2)}}{2(2\eta + g^2) + \eta\Omega^2(1 - \cos \phi - g \sin \phi)^2}. \quad (35)$$

The estimation fidelity is maximized for

$$\kappa = \sqrt{\frac{\eta(2 + \Omega^2)}{2 + \eta\Omega^2 \sin^2 \phi}}, \quad (36)$$

whereas the distortion fidelity is maximized when g is equal to the real root of the following cubic equation:

$$g^3(2 + \eta\Omega^2 \sin^2 \phi) + g\eta\Omega^2[2\eta \sin^2(\phi) - (1 - \cos \phi)^2] = \eta^2\Omega^2(2 \sin \phi - \sin 2\phi). \quad (37)$$

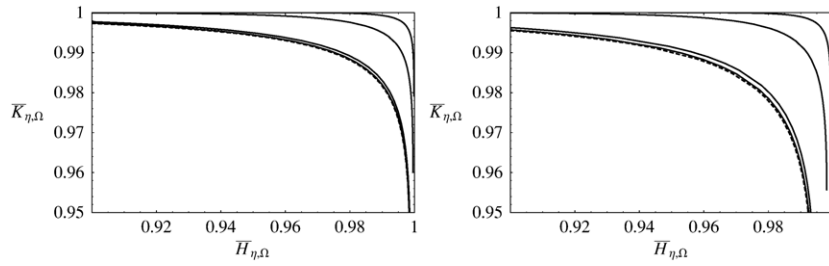


Figure 3. Average estimation/distortion trade-off for different values of Ω and for two different values of the quantum efficiency (see text for details). (Left): $\eta = 0.9$. (Right): $\eta = 0.8$. The dashed line is the universal trade-off whereas, in both the plots, the solid lines are for (from right to left): $\Omega = 0.5, 1.0, 5.0$ and 10.0 .

The optimized estimation fidelity is given by

$$\bar{H}_\eta(\phi) = \frac{2\sqrt{(2 + \Omega^2)(2 + \eta\Omega^2 \sin^2 \phi)}}{(2 + \Omega^2)(2 + \eta\Omega^2 \sin^2 \phi) - \Omega^2 \sin \phi \sqrt{\eta(2 + \Omega^2)(2 + \eta\Omega^2 \sin^2 \phi)}}, \quad (38)$$

whereas we do not report the analytic expression for the optimized distortion fidelity $\bar{K}_\eta(\phi)$ which is quite cumbersome. In figure 3, we show the estimation/distortion trade-off for different values of the quantum efficiency η and the width Ω of the distribution. The universal trade-off is recovered for $\Omega \rightarrow \infty$.

5. Fock number states

Since the s -ordered Wigner function of the Fock state $|n\rangle$ is given by

$$W_s^{(n)}(\xi) = (-1)^n \frac{2}{\pi(1-s)} \left(\frac{1+s}{1-s}\right)^n \exp\left\{-\frac{2|\xi|^2}{1-s}\right\} L_n\left(\frac{4|\xi|^2}{1-s^2}\right), \quad (39)$$

where $L_n(z)$ are Laguerre polynomials, we can evaluate the estimation and distortion fidelities $H_{\eta,k}^{(n)}(\phi)$, $K_{\eta,g}^{(n)}(\phi)$, respectively, as described in sections 2 and 3. For Fock states universal protocols, i.e., protocols independent of n , do not exist. To prove this let us consider the simple case of $|0\rangle$ and $|1\rangle$ as input states. Universality would require that $\forall \eta, \phi$ there exist $\bar{k} = \bar{k}(\eta, \phi)$ and $\bar{g} = \bar{g}(\eta, \phi)$ such that $H_{\eta,\bar{k}}^{(0)}(\phi) = H_{\eta,\bar{k}}^{(1)}(\phi)$ and $K_{\eta,\bar{g}}^{(0)}(\phi) = K_{\eta,\bar{g}}^{(1)}(\phi)$. On the other hand, if we set, for example, $\phi = \pi/3$ and $\eta = 1$, the two conditions are never satisfied, as shown in figure 4. In figure 5, we show the optimized estimation/distortion trade-off for some value of n and the quantum efficiency.

If the different number states are sent to the input according to a thermal probability distribution

$$p_n = \frac{1}{1+N} \left(\frac{N}{1+N}\right)^n, \quad n \geq 0, \quad (40)$$

the average fidelities are given by

$$\bar{H}_\eta(\phi) = \sum_{n=0}^{\infty} p_n H_{\eta,k}^{(n)}(\phi), \quad \bar{K}_{\eta,g}(\phi) = \sum_{n=0}^{\infty} p_n K_{\eta,g}^{(n)}(\phi). \quad (41)$$

We have not been able to find a closed analytical form for the corresponding trade-off. In figure 6, we show the trade-off, as obtained by numerical evaluation of equations (41), upon a suitable truncation of the Hilbert space, for different values of N and η .

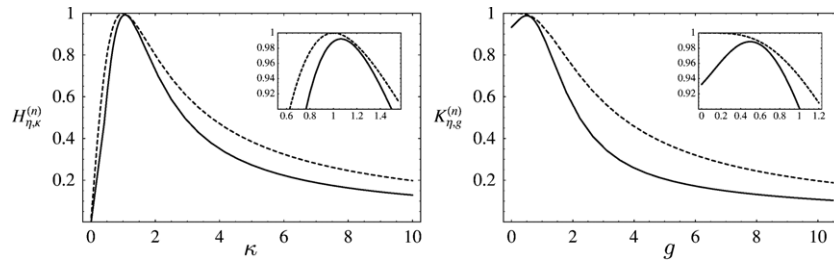


Figure 4. (Right) Plots of $H_{\eta,\kappa}^{(n)}(\phi)$ as a function of the parameter κ ; (left) plots of $K_{\eta,g}^{(n)}(\phi)$ as a function of the parameter g . In both the plots we set $\phi = \pi/3, \eta = 1$ and we have chosen $n = 0$ (dashed lines) and $n = 1$ (solid lines). The insets are magnification of the regions nearby the maxima. Note that there are not intersections.

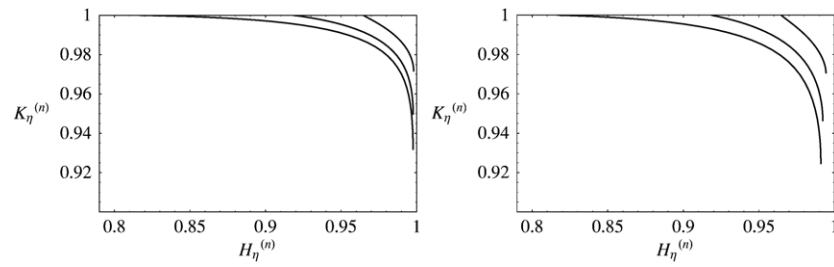


Figure 5. Optimized estimation/distortion trade-off for Fock number states. (Left): $\eta = 0.9$. (Right): $\eta = 0.8$. In both plots, we show the trade-off for (from right to left) $n = 1, 2, 5$.

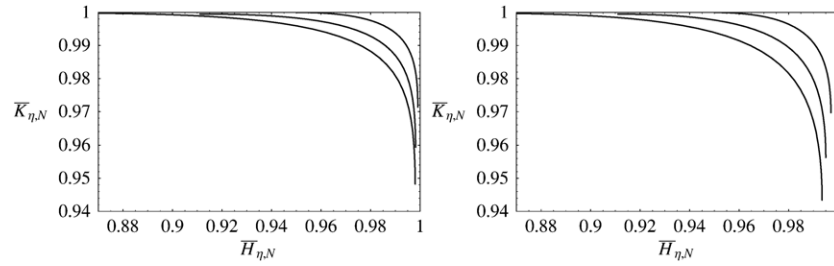


Figure 6. Optimized estimation/distortion trade-off for thermal sets of Fock number states. (Left): $\eta = 0.9$. (Right): $\eta = 0.8$. In both plots, we show the trade-off for (from right to left) $N = 0.5, 1, 2$.

6. Conclusions

We have analysed a linear optical scheme for the extraction of information about a set of coherent states with minimum disturbance. We have shown that non-universal protocols improve the trade-off compared to universal one. We have also introduced the estimation/distortion trade-off to assess the indirect measurement of the Q -function of a generic set of states and explicitly evaluated it for coherent and Fock number states.

Acknowledgments

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