Cloning of Gaussian states by linear optics

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We analyze in details a scheme for cloning of Gaussian states based on linear optical components and homodyne detection recently demonstrated by Andersen *et al.* [Phys. Rev. Lett. **94**, 240503 (2005)]. The input-output fidelity is evaluated for a generic (pure or mixed) Gaussian state taking into account the effect of nonunit quantum efficiency and unbalanced mode mixing. In addition, since in most quantum information protocols the covariance matrix of the set of input states is not perfectly known, we evaluate the average cloning fidelity for classes of Gaussian states with the degree of squeezing and the number of thermal photons being only partially known.

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I. INTRODUCTION

The generation of perfect copies of an unknown quantum state is impossible according to the very nature of quantum mechanics. This is succinctly formulated by the no-cloning theorem [1–4]. It is, however, possible to make approximate copies of a quantum state by using a quantum cloning machine [5–7]. Originally, such a machine was proposed for cloning of qubits and was later demonstrated experimentally [8]. Shortly after this development, a continuous variable (CV) [9] analog of the qubit quantum cloner was proposed [10,11], and recently it was shown that a CV optimal Gaussian cloner of coherent states can be implemented using an appropriate combination of beam splitters and a single phase-insensitive parametric amplifier [12,13]. Although this proposal sounds experimentally promising, the implementation of an efficient phase-insensitive amplifier operating at the fundamental limit is a challenging task. This problem was solved by Andersen *et al.* [14], who proposed and experimentally realized a much simpler configuration for optimal cloning of coherent states. The realization relies on simple linear optical components and a feed-forward loop. As a consequence of the simplicity, as well as the high quality of the optical devices used in this experiment, performances close to optimal ones were attained. In turn, the resulting cloning machine represents a highly versatile tool for further investigations on transformation of quantum information from a single system to many systems.

A commonly used figure of merit to quantify the performance of cloning machines is the fidelity, which is a measure of the similarity between the hypothetically perfect clone, i.e., the input state, and the actual clone. If the cloning fidelity is independent of the initial state, the machine is referred to as a *universal* cloner. On the other hand, if the efficiency of the cloning action depends on the input state, then the proper measure in order to assess the performance of the machine is the average fidelity, which weights the fidelities associated to possible input states with the corresponding occurrence probabilities. In other words, for nonuniversal cloners, the alphabet of input states, and the distribution thereof, must be taken into account while evaluating the fidelity. Such an average fidelity has been considered in [15–17]. However, in all these references, it is assumed that the input alphabet is only consisting of coherent states, hereby keeping the covariance matrix of all the possible input states constant. On the other hand, in some experimental realizations, the covariance matrix is not perfectly known due to uncontrollable fluctuations, and therefore it is important to include this uncertainty into the analysis.

The aim of this paper is twofold. At first we present a thorough theoretical description of the cloning machine described in Ref. [14] using a suitable phase-space analysis. In this way the full quantum dynamics of the machine can be taken into account; in particular, we include the effect of losses in the detection scheme, as well as variations in the setups beam splitter (BS) ratios. The second topic of the paper is to investigate the average fidelity of the cloning machine for different ensembles of input states such as sets made of displaced squeezed or displaced thermal states with the squeezing parameter, or the number of thermal photons, distributed according to predefined distributions.

The paper is structured as follows: In Sec. II we review the main components of the cloning machine based on linear optics, and in Sec. III we calculate the input-output fidelities for the case of generic Gaussian states, and for specific classes including coherent, displaced squeezed, and displaced thermal states. Finally, Sec. IV closes the paper with some concluding remarks.

II. THE LINEAR CLONING MACHINE

Optimal Gaussian cloning can be realized using a phase-insensitive amplifier and a beam splitter [12,13]. However, it has been recently shown, theoretically and experimentally,

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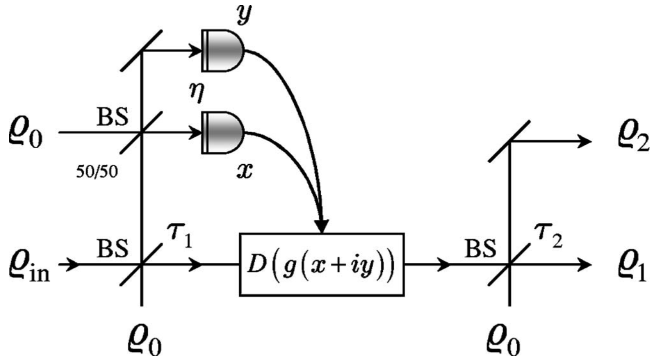

FIG. 1. Cloning of Gaussian states by linear optics: the input state ϱ_{in} is mixed with the vacuum ϱ_0 at a beam splitter (BS) of transmissivity τ_1. The reflected beam is measured by double-homodyne detection and the outcome of the measurement $x+iy$ is forwarded to a modulator, which imposes a displacement $g(x+iy)$ on the transmitted beam, g being a suitable amplification factor. Finally, the displaced state is impinged onto a second beam splitter of transmissivity τ_2. The two outputs, ϱ_1 and ϱ_2, from the beam splitter represent the two clones.

that the parametric amplifier can be replaced by a simpler scheme involving only linear optical components, homodyne detection, and a feed-forward loop [14]. This scheme, which is schematically depicted in Fig. 1, will be referred to as the linear cloning machine throughout the paper.

The input state, denoted by the density operator ϱ_{in}, is mixed with the vacuum at a beam splitter with transmittivity τ_1. On the reflected part, double-homodyne detection is performed using two detectors with equal quantum efficiencies η: this measurement is executed by splitting the state at a balanced beam splitter, and then by measuring the two conjugate quadratures $\hat{x}=\frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger)$ and $\hat{y}=\frac{1}{i\sqrt{2}}(\hat{a}-\hat{a}^\dagger)$, with $\hat{a}$ and $\hat{a}^\dagger$ being the field annihilation and creation operators, respectively. The outcome of the double-homodyne detector gives the complex number $\alpha=x+iy$. According to these outcomes, the transmitted part of the input state undergoes a displacement by an amount $g\alpha$, where g is a suitable electronic amplification factor, and, finally, the two output states, denoted by the density operators ϱ_1 and ϱ_2, are obtained by dividing the displaced state using another beam splitter with transmittivity τ_2. When $\tau_1=\tau_2=1/2$, $g=1$, and $\eta=1$, the scheme reduces to that of Ref. [14], which was shown to be optimal for Gaussian cloning of coherent states on the basis of a description in the Heisenberg picture. Here we apply a different approach, which captures all the essential features of the machine. Towards this aim, in the following we carry out a thorough description of the machine using the characteristic function approach.

The characteristic function $\chi_{\text{in}}(\mathbf{\Lambda}_1)\equiv\chi[\varrho_{\text{in}}](\mathbf{\Lambda}_1)$ associated with a generic Gaussian input state ϱ_{in} of mode 1 reads

$$\chi_{\text{in}}(\mathbf{\Lambda}_1)=\exp\left\{-\tfrac{1}{2}\mathbf{\Lambda}_1^T\boldsymbol{\sigma}_{\text{in}}\mathbf{\Lambda}_1-i\mathbf{\Lambda}_1^T\boldsymbol{X}_{\text{in}}\right\},\qquad(1)$$

where $\mathbf{\Lambda}_1=(\mathsf{x}_1,\mathsf{y}_1)^T$, $(\cdots)^T$ denotes the transposition operation, and

$$\boldsymbol{\sigma}_{\text{in}}=\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix},\qquad(2)$$

with $\gamma_{12}=\gamma_{21}$, is the covariance matrix. $\boldsymbol{X}_{\text{in}}=\text{Tr}[\varrho_{\text{in}}(\hat{x},\hat{y})^T]$ is the vector of mean values, $\hat{x}$ and $\hat{y}$ being the quadrature operators defined above. The vacuum state $\varrho_0=|0\rangle\langle 0|$ of mode 2 is described by the (Gaussian) characteristic function

$$\chi_0(\mathbf{\Lambda}_2)\equiv\chi[\varrho_0](\mathbf{\Lambda}_2)=\exp\left\{-\tfrac{1}{2}\mathbf{\Lambda}_2^T\boldsymbol{\sigma}_0\mathbf{\Lambda}_2\right\},\qquad(3)$$

where $\boldsymbol{\sigma}_0=\frac{1}{2}\mathbb{1}_2$, $\mathbb{1}_2$ being the 2×2 identity matrix. In turn, the initial two-mode state $\varrho=\varrho_{\text{in}}\otimes\varrho_0$ is Gaussian and its two-mode characteristic function reads

$$\chi[\varrho](\mathbf{\Lambda})=\exp\left\{-\tfrac{1}{2}\mathbf{\Lambda}^T\tilde{\boldsymbol{\sigma}}\mathbf{\Lambda}-i\mathbf{\Lambda}^T\tilde{\boldsymbol{X}}\right\},\qquad(4)$$

with

$$\tilde{\boldsymbol{\sigma}}=\left(\begin{array}{c|c}\boldsymbol{\sigma}_{\text{in}} & \mathbf{0}\\ \hline \mathbf{0} & \boldsymbol{\sigma}_0\end{array}\right),\quad \tilde{\boldsymbol{X}}=(\boldsymbol{X}_{\text{in}},\mathbf{0})^T,\qquad(5)$$

and $\mathbf{\Lambda}=(\mathbf{\Lambda}_1,\mathbf{\Lambda}_2)^T$. Under the action of the first BS, the state $\chi[\varrho](\mathbf{\Lambda})$ preserves its Gaussian form, namely

$$\chi[\varrho](\mathbf{\Lambda})\rightsquigarrow\chi[\varrho'](\mathbf{\Lambda})=\exp\left\{-\tfrac{1}{2}\mathbf{\Lambda}^T\boldsymbol{\sigma}\mathbf{\Lambda}-i\mathbf{\Lambda}^T\boldsymbol{X}\right\},\qquad(6)$$

where $\varrho'=U_{\text{BS},1}\varrho_{\text{in}}\otimes\varrho_0 U_{\text{BS},1}^\dagger$, while its covariance matrix and mean values transform as [18]:

$$\tilde{\boldsymbol{\sigma}}\rightsquigarrow\boldsymbol{\sigma}\equiv\boldsymbol{S}_{\text{BS},1}^T\tilde{\boldsymbol{\sigma}}\boldsymbol{S}_{\text{BS},1}=\left(\begin{array}{c|c}\boldsymbol{A} & \boldsymbol{C}\\ \hline \boldsymbol{C}^T & \boldsymbol{B}\end{array}\right),\qquad(7)$$

$$\tilde{\boldsymbol{X}}\rightsquigarrow\boldsymbol{X}\equiv\boldsymbol{S}_{\text{BS},1}^T\tilde{\boldsymbol{X}}=(\boldsymbol{X}_1,\boldsymbol{X}_2)^T,\qquad(8)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are 2×2 matrices, and

$$\boldsymbol{S}_{\text{BS},1}=\left(\begin{array}{c|c}\sqrt{\tau_1}\mathbb{1}_2 & \sqrt{1-\tau_1}\mathbb{1}_2\\ \hline -\sqrt{1-\tau_1}\mathbb{1}_2 & \sqrt{\tau_1}\mathbb{1}_2\end{array}\right),\qquad(9)$$

is the symplectic transformation associated with the evolution operator $U_{\text{BS},1}$ of the BS with transmission τ_1. Note that ϱ' is an entangled state if the set of states to be cloned consists of nonclassical states; i.e., states with singular Glauber P-function or negative Wigner function [19,20].

The subsequent step is to describe double-homodyne detection with quantum efficiency η on the reflected beam. This action can be described by the following positive operator-valued measure:

$$\Pi_\eta(\alpha)=\int_{\mathbb{C}}d^2\xi\frac{1}{\pi\sigma_\eta}\exp\left\{-\frac{|\alpha-\xi|^2}{\sigma_\eta}\right\}\frac{|\xi\rangle\langle\xi|}{\pi},\qquad(10)$$

where $\sigma_\eta=(1-\eta)/\eta$ and $|\xi\rangle$ is a coherent state. Equation (10) describes a Gaussian measurement, the characteristic function associated with $\Pi_\eta(\alpha)$ has the form

$$\chi[\Pi_\eta(\alpha)](\mathbf{\Lambda}_2)=\frac{1}{\pi}\exp\left\{-\tfrac{1}{2}\mathbf{\Lambda}_2^T\boldsymbol{\sigma}_{\text{M}}\mathbf{\Lambda}_2-i\mathbf{\Lambda}_2^T\boldsymbol{X}_{\text{M}}\right\},\qquad(11)$$

with $\boldsymbol{X}_{\text{M}}=(\text{Re}[\alpha],\text{Im}[\alpha])^T$ and

$$\boldsymbol{\sigma}_{\mathrm{M}}=\Delta^2\,\mathbb{1}_2,\quad \Delta^2=\frac{1}{2}+\sigma_\eta=\frac{2-\eta}{2\eta}. \tag{12}$$

The probability of obtaining the outcome α is given by

$$p_\eta(\alpha)=\mathrm{Tr}_{12}[\varrho'\mathbb{1}\otimes\Pi_\eta(\alpha)] \tag{13}$$

$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}d^4\boldsymbol{\Lambda}\chi[\varrho'](\boldsymbol{\Lambda})\chi[\mathbb{1}\otimes\Pi_\eta(\alpha)](-\boldsymbol{\Lambda}) \tag{14}$$

$$=\frac{\exp\left\{-\frac{1}{2}(\boldsymbol{X}_{\mathrm{M}}-\boldsymbol{X}_2)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_{\mathrm{M}}-\boldsymbol{X}_2)\right\}}{\pi\sqrt{\mathrm{Det}[\boldsymbol{\Sigma}]}}, \tag{15}$$

where $\chi[\mathbb{1}\otimes\Pi_\eta(\alpha)](\boldsymbol{\Lambda})\equiv\chi[\mathbb{1}](\boldsymbol{\Lambda}_1)\chi[\Pi_\eta(\alpha)](\boldsymbol{\Lambda}_2)$, $\chi[\mathbb{1}](\boldsymbol{\Lambda}_1)=2\pi\delta^{(2)}(\boldsymbol{\Lambda}_1)$ and $\delta^{(2)}(\zeta)$ is the complex Dirac's delta function. We also introduced the 2×2 matrix $\boldsymbol{\Sigma}=\boldsymbol{B}+\boldsymbol{\sigma}_{\mathrm{M}}$.

The conditional state ϱ_{c} of the transmitted beam, obtained when the outcome of the measurement is α, i.e.,

$$\varrho_{\mathrm{c}}=\frac{\mathrm{Tr}_2[\varrho'\Pi_\eta(\alpha)]}{p_\eta(\alpha)}, \tag{16}$$

has the following characteristic function (for the sake of clarity we explicitly write the dependence on $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$)

$$\chi[\varrho_{\mathrm{c}}](\boldsymbol{\Lambda}_1)=\int_{\mathbb{R}^2}d^2\boldsymbol{\Lambda}_2\frac{\chi[\varrho'](\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2)\chi[\Pi_\eta(\alpha)](-\boldsymbol{\Lambda}_2)}{p_\eta(\alpha)} \tag{17}$$

$$\begin{aligned}&=\exp\left\{-\tfrac{1}{2}\boldsymbol{\Lambda}_1^T[\boldsymbol{A}-\boldsymbol{C}\boldsymbol{\Sigma}^{-1}\boldsymbol{C}^T]\boldsymbol{\Lambda}_1-\tfrac{1}{2}\boldsymbol{X}_2^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_2\right.\\&\quad\left.+i\boldsymbol{\Lambda}_1^T[\boldsymbol{C}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_2-\boldsymbol{X}_1]\right\}\exp\left\{-\tfrac{1}{2}\boldsymbol{X}_{\mathrm{M}}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\mathrm{M}}\right.\\&\quad\left.+i\boldsymbol{X}_{\mathrm{M}}^T[i\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_2+\boldsymbol{\Sigma}^{-1}\boldsymbol{C}^T\boldsymbol{\Lambda}_1]\right\}.\end{aligned} \tag{18}$$

Now, the conditional state ϱ_{c} is displaced by the amount $g\alpha$ resulting from the measurement amplified by a factor g. By averaging over all possible outcomes of the double-homodyne detection, we obtain the following output state:

$$\varrho_{\mathrm{d}}=\int_{\mathbb{C}}d^2\alpha p_\eta(\alpha)D(g\alpha)\varrho_c D^\dagger(g\alpha), \tag{19}$$

with $D(z)$ being the displacement operator. In turn, the characteristic function reads as follows:

$$\chi[\varrho_{\mathrm{d}}](\boldsymbol{\Lambda}_1)=\exp\left\{-\tfrac{1}{2}\boldsymbol{\Lambda}_1^T\boldsymbol{\sigma}_{\mathrm{d}}\boldsymbol{\Lambda}_1-i\boldsymbol{\Lambda}_1^T\boldsymbol{X}_{\mathrm{d}}\right\}, \tag{20}$$

with $\boldsymbol{\sigma}_{\mathrm{d}}=\boldsymbol{A}+g(\boldsymbol{\Sigma}+2\boldsymbol{C}^T)$ and $\boldsymbol{X}_{\mathrm{d}}=\boldsymbol{X}_1+g\boldsymbol{X}_2$. The conditioned state (19) is then sent to a second beam splitter with transmission τ_2 (see Fig. 1), where it is mixed with the vacuum ϱ_0, and finally the two clones are generated. Note that, in practice, the average over all the possible outcomes α in Eq. (19) should be performed at this stage; that is, after the second beam splitter. On the other hand, because of the linearity of the integration, the results are identical, but performing the averaging just before the beam splitter simplifies the calculations. Since ϱ_{d} is still Gaussian, the two-mode state $\varrho_{\mathrm{f}}=\varrho_{\mathrm{d}}\otimes\varrho_0$ is a Gaussian with covariance matrix and mean given by

$$\boldsymbol{\sigma}_{\mathrm{f}}=\left(\begin{array}{c|c}\boldsymbol{\sigma}_{\mathrm{d}}&\boldsymbol{0}\\\hline\boldsymbol{0}&\boldsymbol{\sigma}_0\end{array}\right),\quad \boldsymbol{X}_{\mathrm{f}}=(\boldsymbol{X}_{\mathrm{d}},\boldsymbol{0})^T, \tag{21}$$

respectively, which, as in the case of Eqs. (7) and (8), under the action of the BS transform as follows:

$$\boldsymbol{\sigma}_{\mathrm{f}}\rightsquigarrow\boldsymbol{\sigma}_{\mathrm{out}}\equiv\boldsymbol{S}_{\mathrm{BS},2}^T\boldsymbol{\sigma}_f\boldsymbol{S}_{\mathrm{BS},2}=\left(\begin{array}{c|c}\mathcal{A}_1&\mathcal{C}\\\hline\mathcal{C}^T&\mathcal{A}_2\end{array}\right), \tag{22}$$

$$\boldsymbol{X}_{\mathrm{f}}\rightsquigarrow\boldsymbol{X}_{\mathrm{out}}\equiv\boldsymbol{S}_{\mathrm{BS},2}^T\boldsymbol{X}_{\mathrm{f}}=(\mathcal{X}_1,\mathcal{X}_2)^T, \tag{23}$$

where $\mathcal{A}_k$ and $\mathcal{C}$ are 2×2 matrices, and $\boldsymbol{S}_{\mathrm{BS},2}$ is the symplectic matrix given by Eq. (9) with τ_1 replaced by τ_2. Finally, the (Gaussian) characteristic function of the clone ϱ_k, $k=1,2$, is obtained by integrating over $\boldsymbol{\Lambda}_h$, $h\neq k$, the two-mode characteristic function $\chi[\varrho_{\mathrm{out}}](\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2)$, where $\varrho_{\mathrm{out}}=U_{\mathrm{BS},2}\varrho_{\mathrm{f}}\otimes\varrho_0U_{\mathrm{BS},2}^\dagger$, i.e.,

$$\chi[\varrho_k](\boldsymbol{\Lambda}_k)=\frac{1}{2\pi}\int_{\mathbb{R}^2}d^2\boldsymbol{\Lambda}_h\chi[\varrho_{\mathrm{out}}](\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2) \tag{24}$$

$$=\exp\left\{-\tfrac{1}{2}\boldsymbol{\Lambda}_k^T\mathcal{A}_k\boldsymbol{\Lambda}_k-i\boldsymbol{\Lambda}_k^T\mathcal{X}_k\right\}. \tag{25}$$

Let us now focus our attention on $\boldsymbol{X}_{\mathrm{out}}$: the explicit expressions of $\mathcal{X}_1$ and $\mathcal{X}_2$ are

$$\mathcal{X}_1=\sqrt{\tau_2}(\sqrt{\tau_1}+g\sqrt{1-\tau_1})\boldsymbol{X}_{\mathrm{in}}, \tag{26}$$

$$\mathcal{X}_2=\sqrt{1-\tau_2}(\sqrt{\tau_1}+g\sqrt{1-\tau_1})\boldsymbol{X}_{\mathrm{in}}. \tag{27}$$

As a matter of fact, in order to have two output Gaussian states with the same means $\mathcal{X}_1=\mathcal{X}_2$, one should set $\tau_2=1/2$; furthermore, if one also sets

$$g=g_{\mathrm{s}}\equiv\sqrt{\frac{2}{1-\tau_1}}-\sqrt{\frac{\tau_1}{1-\tau_1}}, \tag{28}$$

then $\mathcal{X}_1=\mathcal{X}_2=\boldsymbol{X}_{\mathrm{in}}$, corresponding to unity gain cloning. On the other hand, $\mathcal{A}_k$ can be written in a compact form as follows:

$$\mathcal{A}_1=\tfrac{1}{2}(1-\tau_2)\mathbb{1}+\tau_2\boldsymbol{\Gamma}(\boldsymbol{\sigma}_{\mathrm{in}}), \tag{29a}$$

$$\mathcal{A}_2=\tfrac{1}{2}\tau_2\mathbb{1}+(1-\tau_2)\boldsymbol{\Gamma}(\boldsymbol{\sigma}_{\mathrm{in}}), \tag{29b}$$

where

$$\boldsymbol{\Gamma}(\boldsymbol{\sigma}_{\mathrm{in}})=\begin{pmatrix}\mathcal{F}(\gamma_{11})&\mathcal{G}(\gamma_{12})\\\mathcal{G}(\gamma_{21})&\mathcal{F}(\gamma_{22})\end{pmatrix}, \tag{30}$$

with

$$\mathcal{F}(\gamma)=1-\tau_1+g[\tau_1-2\sqrt{(1-\tau_1)\tau_1}+\Delta^2]+\mathcal{G}(\gamma), \tag{31}$$

$$\mathcal{G}(\gamma)=[\tau_1+g(1-\tau_1+2\sqrt{(1-\tau_1)\tau_1})]\gamma. \tag{32}$$

Now, if $\tau_2=1/2$ and $g=g_{\mathrm{s}}$, one has $\mathcal{A}_1=\mathcal{A}_2$ and $\mathcal{X}_1=\mathcal{X}_2$, as we have seen above; i.e., the cloning becomes *symmetric*. Furthermore, when also $\tau_1=1/2$, thanks to Eqs. (25) and (29) we have that the cloning map for the scheme in Fig. 1 is given by the following Gaussian map:

$$\mathcal{G}_{\sigma_{\rm GN}}(\varrho_{\rm in}) = \int_{\mathbb{C}} \frac{d^2\gamma}{\pi\sigma_{\rm GN}^2} \exp\left\{-\frac{|\gamma|}{\sigma_{\rm GN}^2}\right\} D(\gamma)\varrho_{\rm in}D^\dagger(\gamma), \tag{33}$$

where $\sigma_{\rm GN}^2=\frac{1}{2}+\Delta^2$. Finally, although $\mathcal{C}$ does not appear in Eq. (25), for the sake of completeness, we give its analytic expression:

$$\mathcal{C} = \sqrt{(1-\tau_2)\tau_2}\left[\tfrac{1}{2}\mathbb{1} - \boldsymbol{\Gamma}(\boldsymbol{\sigma}_{\rm in})\right]. \tag{34}$$

In the following we will analyze the input-output fidelities for a generic (pure or mixed) Gaussian state. In particular, we will consider three classes of Gaussian states; i.e., coherent, displaced squeezed, and displaced thermal states.

III. CLONING OF GAUSSIAN STATES

A. Fidelity

Usually, the performance of cloning machines are quantified by the fidelity, which is a measure of the similarity between the hypothetically perfect clone and the actual clone. In its most general form, the fidelity is given by Uhlmann's transition probability [21]

$$F(\varrho_{\rm in},\varrho_k) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\varrho_{\rm in}}\varrho_k\sqrt{\varrho_{\rm in}}}\right]\right)^2, \tag{35}$$

and satisfies the natural axioms

(i) $F(\varrho_{\rm in},\varrho_k)\leq 1$ and $F(\varrho_{\rm in},\varrho_k)=1$ if and only if $\varrho_{\rm in}=\varrho_k$;

(ii) $F(\varrho_{\rm in},\varrho_k)=F(\varrho_k,\varrho_{\rm in})$;

(iii) if $\varrho_{\rm in}$ is a pure state $\varrho_{\rm in}=|\psi_{\rm in}\rangle\langle\psi_{\rm in}|$, then we have $F(\varrho_{\rm in},\varrho_k)=\langle\psi_{\rm in}|\varrho_k|\psi_{\rm in}\rangle$;

(iv) $F(\varrho_{\rm in},\varrho_k)$ is invariant under unitary transformations on the state space.

Furthermore, when $\varrho_{\rm in}$ and ϱ_k are Gaussian states of the form (1) and (25), the fidelity (35) becomes [22,23]

$$F_\eta \equiv F(\varrho_{\rm in},\varrho_k) = \frac{1}{\sqrt{\mathrm{Det}[\boldsymbol{\sigma}_{\rm in}+\mathcal{A}_k]+\delta}-\sqrt{\delta}} \times\exp\left\{-\tfrac{1}{2}(\boldsymbol{X}_{\rm in}-\mathcal{X}_k)^T(\boldsymbol{\sigma}_{\rm in}+\mathcal{A}_k)^{-1}(\boldsymbol{X}_{\rm in}-\mathcal{X}_k)\right\}, \tag{36}$$

where $\delta=4\left(\mathrm{Det}[\boldsymbol{\sigma}_{\rm in}]-\frac{1}{4}\right)\left(\mathrm{Det}[\mathcal{A}_k]-\frac{1}{4}\right)$. Note that for pure Gaussian states $\mathrm{Det}[\boldsymbol{\sigma}_{\rm in}]=\frac{1}{4}$, and in turn $\delta=0$.

For *symmetric* cloning, i.e., for $\tau_2=1/2$ and $g=g_{\rm s}$ in Eq. (28), then Eq. (36) reduces to

$$F_\eta = \frac{1}{\sqrt{\mathrm{Det}[\boldsymbol{\sigma}_{\rm in}+\mathcal{A}_k]+\delta}-\sqrt{\delta}}. \tag{37}$$

Notice that for CV systems, the lowest cloning fidelity, corresponding to clones containing the minimum information about the original state, is equal to $F=0$. This is the CV analog of the $F=1/d$ lower bound for the cloning of a d-dimensional quantum system [7].

In general, the cloning fidelity in (36) is state dependent, and therefore the figure of merit to be considered is the mean cloning fidelity, averaged over the ensemble of possible input states. In order to evaluate this quantity, we parametrize the input ensemble (class) $\{\varrho_{\rm in}(\boldsymbol{\lambda})\}$ of different Gaussian states, by $\boldsymbol{\lambda}\in\Omega$ and consider each of them occurring with the *a priori* probability $p(\boldsymbol{\lambda})$. The average fidelity then reads

$$\bar{F}_\eta = \int_\Omega d\boldsymbol{\lambda}\, p(\boldsymbol{\lambda})F_\eta(\boldsymbol{\lambda}). \tag{38}$$

Within the set of possible states, both the mean values as well as the covariance matrices may vary. Assuming that the probability distribution $p(\boldsymbol{\lambda})$ is factorizable into a distribution for the mean values $p(\alpha)$ and a distribution for the covariance matrix, $p(\boldsymbol{\sigma}_{\rm in})$, we may write $p(\boldsymbol{\lambda})=p(\alpha)p(\boldsymbol{\sigma}_{\rm in})$, and the average fidelity reads

$$\bar{F}_\eta = \int_\Omega d\boldsymbol{\sigma}_{\rm in}\, d^2\alpha\, p(\alpha)p(\boldsymbol{\sigma}_{\rm in})F_\eta(\boldsymbol{\sigma}_{\rm in},\alpha). \tag{39}$$

In the extreme case where both $\boldsymbol{\sigma}_{\rm in}$ and α are fixed, the input state is completely known, and perfect cloning with unit fidelity is, of course, possible. A more interesting scenario is when the covariance matrix is fixed, as, for example, the case in which the set is made up by coherent states, while the displacement (that is, the mean value) is random. In this case the average fidelity reduces to

$$\bar{F}_\eta = \int_{\mathbb{C}} d^2\alpha\, p(\alpha)F_\eta(\alpha). \tag{40}$$

If $\tau_1=1/2$ and $g=g_s$, the map (33) is covariant with respect to displacements, meaning that if two input states are identical up to a displacement, their respective clones should be identical up to the same displacement [24]. Indeed, if the input state is of the form $\varrho_{\rm in}(\alpha)=D(\alpha)\varrho_{\rm s}D^\dagger(\alpha)$, $\varrho_{\rm s}$ being a seed state, then the fidelity $F_\eta(\alpha)$ actually does not depend on complex parameter α, and, as a consequence, $\bar{F}_\eta=F_\eta$. Therefore, in this case the noise added by the cloning process (33) is the minimum amount of noise allowed by quantum mechanics for a joint measurement of conjugated quadratures [25]. Notice also that this is the same noise added in the cloning of coherent states; i.e., the cloning is *optimal*. The corresponding optimal fidelity $\bar{F}$, however, is not necessarily equal to 2/3 [see Eq. (37)].

B. Coherent states

Before addressing the general case, let us reconsider cloning of (pure) coherent states. For this set of states, our linear machine provides universal cloning, i.e., state independent fidelity. In Fig. 2 we plot the fidelity as given by Eq. (37), as a function of τ_1, for different values of η and for $\tau_2=1/2$, $g=g_{\rm s}$. In this case, corresponding to symmetric cloning, the machine yields the optimal fidelity $F=2/3$ predicted for universal Gaussian cloning of coherent states. Notice that the optimal fidelity is achieved with $\tau_1=1/2$ and $\eta=1$; by expanding the fidelity up to the second order around $\tau_1=1/2$, we obtain

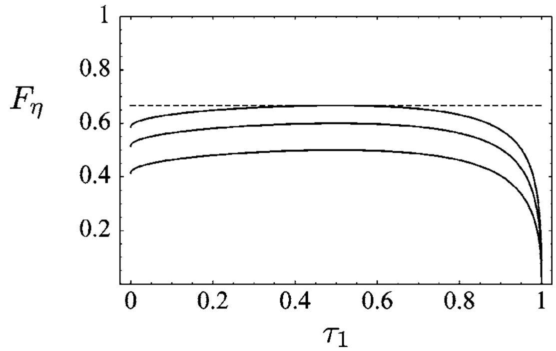

FIG. 2. Linear cloning fidelity F_η for a coherent state as a function of the BS transmittivity τ_1 for different values of the quantum efficiency η: from top to bottom η=1.0, 0.75, and 0.5. We set $\tau_2 = 1/2$ and $g = g_s$ (symmetric cloning). The dashed line is the value 2/3. The fidelity does not depend on the coherent state amplitude.

$$\bar{F}_\eta \equiv F_\eta \simeq \frac{2\eta}{1+2\eta}\left[1 - \frac{\left(\tau_1 - \frac{1}{2}\right)^2}{1+2\eta}\right].$$

From this expression we clearly see that the cloning machine proposed in [14] is robust against fluctuations of the BS ratio. This conclusion can be also directly drawn from Fig. 2.

Let us, however, note that the fidelity value $F=2/3$ is the optimal one only if the input distribution of coherent states is flat; that is, if we have no *a priori* information about the amplitudes. If the set of coherent states is restricted such that the distribution of amplitudes is the Gaussian

$$p_{\mathrm{a}}(\alpha) = \frac{1}{\pi\sigma_{\mathrm{a}}^2} \exp\left\{-\frac{|\alpha|^2}{\sigma_{\mathrm{a}}^2}\right\}, \tag{41}$$

the average fidelity can be increased by choosing a different gain [15]. However, in this scenario the cloning action becomes state dependent, and the integration in (40) should be explicitly performed. By optimizing the gain, we find [15]

$$\bar{F} = \frac{2(1+\sigma_{\mathrm{a}}^2)}{1+3\sigma_{\mathrm{a}}^2}, \quad \text{if } \sigma_{\mathrm{a}}^2 \geq 1+\sqrt{2}, \tag{42}$$

$$\bar{F} = \frac{2}{2+(3-2\sqrt{2})\sigma_{\mathrm{a}}^2}, \quad \text{if } \sigma_{\mathrm{a}}^2 < 1+\sqrt{2}. \tag{43}$$

We have now seen that by fixing the covariance matrix of the input states to coherent states, the fidelity is a function of the distribution (being delta, flat, or Gaussian) of these states. This aspect has been investigated in the literature [15]. In contrast, the case where the covariance matrix may fluctuate has not received much attention heretofore. In the following sections we therefore discuss the average cloning fidelity for classes of states with covariance matrix randomly distributed according to a predetermined distribution. We assume that the displacement of the input state is completely unknown and that the cloner is set to unity gain (that is, invariant with respect to the displacement corresponding to $g=g_s$). In this case the average over the mean value is trivial and the average fidelity can be written as

$$\bar{F}_\eta = \int_\Sigma d\boldsymbol{\sigma}_{\mathrm{in}} p(\boldsymbol{\sigma}_{\mathrm{in}}) F_\eta(\boldsymbol{\sigma}_{\mathrm{in}}). \tag{44}$$

C. Squeezed states

When the input Gaussian state is the squeezed state $|\alpha,\xi\rangle = D(\alpha)S(\xi)|0\rangle$, where $D(\alpha)=\exp\{\alpha a^\dagger - \alpha^* a\}$ and $S(\xi) = \exp\{\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)\}$ are the displacement and squeezing operator, respectively, the entries of the input covariance matrix (2) are

$$\gamma_{11} = \tfrac{1}{2}(\cosh 2|\xi| + \sinh 2|\xi| \cos\varphi), \tag{45a}$$

$$\gamma_{22} = \tfrac{1}{2}(\cosh 2|\xi| - \sinh 2|\xi| \cos\varphi), \tag{45b}$$

$$\gamma_{12} = \gamma_{21} = -\tfrac{1}{2}\sinh 2|\xi| \sin\varphi, \tag{45c}$$

where we put $\xi = |\xi| e^{i\varphi}$; obviously, when $\xi=0$ the squeezed state $|\alpha,\xi\rangle$ reduces to the coherent state $|\alpha\rangle$ and $\boldsymbol{\sigma}_{\mathrm{in}} = \frac{1}{2}\mathbb{1}_2$. In this section we are addressing the case of an *unknown* squeezing parameter ξ (randomly distributed according to a given probability density). If the squeezing parameter is known, the optimal strategy (in the Gaussian regime) is to perform the unsqueezing operation $S^{-1}(\xi)$ just before the cloning machine, proceed as in the case of coherent states, and, at the output stage, apply the squeezing operation $S(\xi)$ to both the clones which yields a fidelity of 2/3 (independent on the amount of fixed squeezing) as in the coherent state case [12].

However, in the case of an unknown squeezing parameter, the squeezing action $S(\xi)$ is not known. Therefore in the following, we investigate the cloning of unknown squeezed states using the cloning machine outlined in this paper. First we note that since the linear elements involved in the cloning machine do not affect the phase of the input state, the fidelity $F_\eta(\xi)$ depends only on $|\xi|$ and, without loss of generality, we may take ξ as real. The fidelity for Gaussian squeezed input states, using the coherent state cloning machine, is given by

$$F_\eta(\xi) = \frac{4}{\sqrt{(5+2\Delta^2)^2 + 16(1+2\Delta^2)\sinh^2|\xi|}}. \tag{46}$$

This fidelity is plotted in Fig. 3 as a function of the squeezing parameter for different values of η. We clearly see that for coherent states (corresponding to $\xi=0$), the fidelity is 2/3 while decreasing with the degree of squeezing, eventually reaching zero for highly squeezed input states.

In order to calculate the average fidelity, we assume that the squeezed state $|\alpha,\xi\rangle$ is drawn from an ensemble of states with *a priori* probability $p(\alpha,\xi) = p(\alpha)p(\xi)$. Above, we mentioned that the cloning action with unity gain is independent on the distribution $p(\alpha)$, which can then be left undefined. The distribution of the squeezing factor is, however, quite important: it is clear that for completely unknown input squeezing (corresponding to a flat distribution) the average fidelity goes to zero. We therefore must restrict the set of input squeezed states to, say, a Gaussian distribution given by

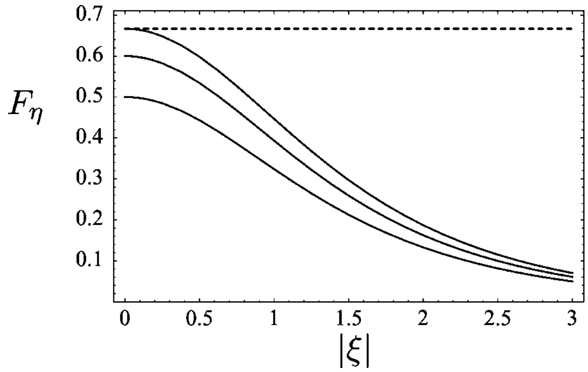

FIG. 3. Plot of the fidelity $F_\eta(\xi)$ of the squeezed state $|\alpha,\xi\rangle$ as a function of $|\xi|$ for different values of η: from top to bottom $\eta=1.0$, 0.75, and 0.5. We set $\tau_1=\tau_2=1/2$ and $g=g_s$ (symmetric cloning). The dashed line is the value 2/3. The fidelity does not depend on the displacement amplitude α.

$$p_s(\xi)=\frac{1}{\pi\sigma_s^2}\exp\left\{-\frac{|\xi|^2}{\sigma_s^2}\right\}. \tag{47}$$

As evident from this expression, we assume the distribution to be centered at $\xi=0$, which corresponds to a coherent state. This means that the coherent state is the most likely member in the set of input states, and we therefore conjecture that our machine is optimal in the Gaussian scenario. If, however, the distribution is centered at a known squeezing amplitude, say $\xi=\xi_0$, we then believe that the optimal machine is the one mentioned above where the input states are unsqueezed $[S^{-1}(\xi_0)]$ before the cloning machine and squeezed $[S(\xi_0)]$ again after the cloning action.

Using the polar coordinates, $d^2\xi=\rho\, d\rho\, d\phi$, $\xi=\rho e^{i\phi}$, and $F_\eta(\xi)=F_\eta(|\xi|)$, the average fidelity now reads

$$\bar{F}_\eta=\int_{\mathbb{C}} d^2\xi\, p(\xi)F_\eta(\xi) \tag{48}$$

$$=2\int_0^{+\infty} d\rho\frac{\rho}{\sigma_s^2}\exp\left\{-\frac{\rho^2}{2\sigma_s^2}\right\}F_\eta(\rho) \tag{49}$$

This function is depicted in Fig. 4 as a function of σ_s for different values of η. If the standard deviation $\sigma_s=0$, the distribution in (47) is a delta function and the input alphabet contains only coherent states. In this case it reduces to the case discussed in the previous section and the expected fidelity is 2/3 (for ideal detection efficiency) as seen in the figure. We also see that the fidelity degrades as the width of the distribution of the squeezing parameter increases, and eventually reaches zero when the *a priori* information is poor. At this point we should note that if one allows for non-Gaussian output clones the fidelity can be improved. For example, it is known that the optimal cloner of coherent states and the optimal universal cloner employ non-Gaussian operations and they yield fidelities of 68.3% [27] and 50% [26], respectively.

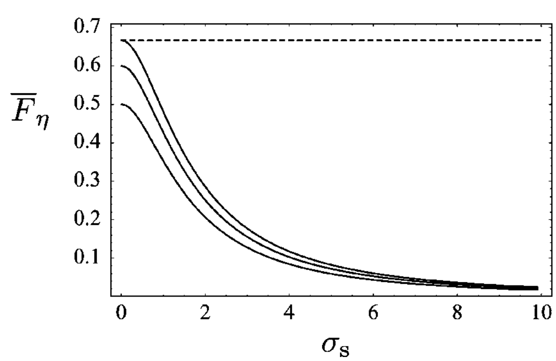

FIG. 4. Plot of the average fidelity $\bar{F}_\eta$ of a set of squeezed states as a function of σ_s (see text for details) and different values of the efficiency η: from top to bottom $\eta=1.0$, 0.75, and 0.5. The dashed line corresponds to 2/3; i.e., the optimal cloning fidelity of coherent states. We put $\tau_1=\tau_2=1/2$ and $g=g_s$.

D. Thermal states

Another interesting class of Gaussian states is the set of displaced thermal states $\varrho_{\mathrm{th},\alpha}=D(\alpha)\nu_{\mathrm{th}}D^\dagger(\alpha)$, which arise, for example, from the propagation of coherent states in a noisy environment [28]. The thermal state ν_{th} is given by

$$\nu_{\mathrm{th}}=\frac{1}{1+N}\sum_{m=0}^{\infty}\left(\frac{N}{1+N}\right)^m|m\rangle\langle m|, \tag{50}$$

where N is the average number of thermal photons. Its covariance matrix is given by $\boldsymbol{\sigma}_{\mathrm{in}}=(N+\frac{1}{2})\mathbb{1}_2$. Since ν_{th} and, in turn, $D(\alpha)\nu_{\mathrm{th}}D^\dagger(\alpha)$ are not pure states, the cloning fidelity $F_\eta(N)$ should be calculated using the full expression of Eq. (36), and the result is plotted in Fig. 5 as a function of N and different values of η. For the unity gain cloner and assuming the detection efficiency to be ideal $(\eta=1)$, we derive the expression

$$F_{\eta=1}(N)=\left(\frac{3}{2}+N(3+2N)-\sqrt{N(2N+1)(2N^2+5N+3)}\right)^{-1}. \tag{51}$$

We see that the fidelity increases with the average number of thermal photons, that is, using the fidelity as a measure, the

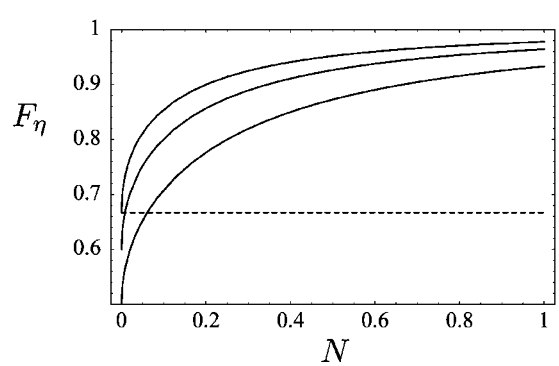

FIG. 5. Plot of the input-output fidelity $F_\eta(N)$ of the displaced thermal state $\varrho_{\mathrm{th},\alpha}$ as a function of the average number of thermal photons N and different values of the efficiency η: from top to bottom $\eta=1.0$, 0.75, and 0.5. The dashed line corresponds to 2/3; i.e., the optimal cloning of coherent states. We put $\tau_1=\tau_2=1/2$ and $g=g_s$.

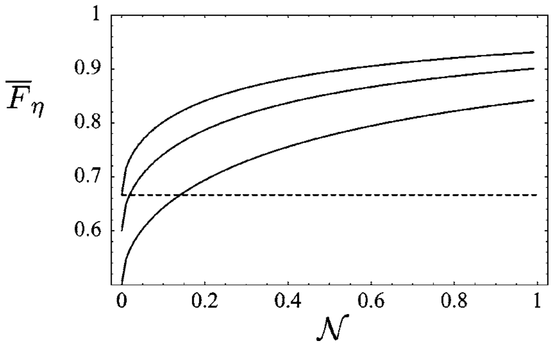

FIG. 6. Plot of the average fidelity $\bar{F}_\eta$ of the set of thermal states distributed according to the top-hat distribution (53) as a function of the threshold value $\mathcal{N}$ and different values of the efficiency η: from top to bottom η=1.0, 0.75, and 0.5. The dashed line corresponds to 2/3. We put $\tau_1=\tau_2=1/2$ and $g=g_s$.

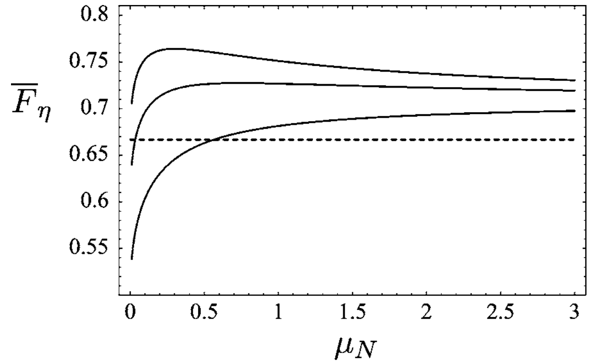

FIG. 7. Plot of the average fidelity $\bar{F}_\eta$ of the set of thermal states distributed according to a "half-Gaussian" distribution (54) as a function of μ_N and different values of the efficiency η: from top to bottom η=1.0, 0.75, and 0.5. The dashed line corresponds to 2/3. We put $\tau_1=\tau_2=1/2$ and $g=g_s$.

quality of the cloning action increases with the mixedness of the input states.

Let us now consider a different ensemble of displaced thermal states, with random displacement and average number of thermal photons N distributed around zero either as a bounded flat, top-hat, distribution or as a "half-Gaussian" distribution. The average fidelity is

$$\bar{F}_\eta = \int_0^{+\infty} dN\, p(N) F_\eta(N), \tag{52}$$

where

$$p(N) = \begin{cases} \mathcal{N}^{-1}, & \text{if } N \in [0,\mathcal{N}] \\ 0, & \text{otherwise} \end{cases} \tag{53}$$

for a top-hat distribution, and

$$p(N) = \frac{2}{\sqrt{2\pi\mu_N^2}} \exp\left\{-\frac{N^2}{2\mu_N^2}\right\}, \quad (N \geq 0) \tag{54}$$

for a (renormalized) "half-Gaussian" distribution. In Figs. 6 and 7 we show the corresponding average fidelities, as functions of $\mathcal{N}$ and μ_N, respectively, for different values of η. For the top-hat distribution the average fidelity monotonically increases as the threshold value $\mathcal{N}$ increases, whereas for the half-Gaussian one the average fidelity shows a maximum value depending on the value of η, as far as $\eta \gtrsim 0.7$.

IV. CONCLUSIONS

We have analyzed in details a recently demonstrated scheme for cloning of Gaussian states [14]. Using a suitable phase-space analysis, the input-output fidelity has been evaluated for a generic (pure or mixed) Gaussian state taking into account the effect of nonunit quantum efficiency of homodyne detection and fluctuations in the beam splitters transmittivity. Our results indicate that the linear cloning machine suggested in [14] is robust against fluctuations of transmissivity and nonunit quantum efficiency.

We have explicitly evaluated the cloning fidelity for specific classes of noncoherent displaced states. We found that a fixed (unknown) squeezing of the input states degrades the fidelity with respect to the coherent level, as one may expect for cloning of highly nonclassical states, while, on the contrary, cloning of displaced thermal states may be achieved with larger fidelity. Using the above results we have evaluated the average cloning fidelity for classes of Gaussian states with fluctuating covariance matrix, as, for example, displaced squeezed or displaced thermal states with the degree of squeezing or the number of thermal photons randomly distributed according to a Gaussian or a uniform distribution. Results indicate that the average fidelity monotonically decreases as the squeezing dispersion increases, whereas the behavior with respect to dispersion of thermal photons is not monotone.

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