Optimal unambiguous comparison of two unknown squeezed vacua

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We propose a scheme for the unambiguous state comparison (USC) of two unknown squeezed vacuum states of the electromagnetic field. Our setup is based on linear optical elements and photon-number detectors, and it achieves optimal USC in an ideal case of unit quantum efficiency. In realistic conditions, i.e., for nonunit quantum efficiency of photodetectors, we evaluate the probability of getting an ambiguous result as well as the reliability of the scheme, thus showing its robustness in comparison to previous proposals.

I. INTRODUCTION

The possibility of creating physical systems with identical properties is crucial to any physical theory that is verifiable by experiments. Comparison of preparators—a procedure of determining whether they prepare the same objects or not—is one of the basic experiments we would like to do when testing a theory, because it allows us to operationally define the equivalence of such devices for their further use. In the framework of classical physics, we can in principle measure and determine the state of the system perfectly without disturbing it. Thus, to compare the states of two systems it suffices to measure each system separately. However, in quantum theory, due to its statistical nature, we cannot make deterministic conclusions or predictions even for the simplest experimental situations. Therefore, the comparison of quantum states is different from the classical situation.

Imagine we are given two independently prepared quantum systems of the same physical nature (e.g., two photons or two electrons). We would like to determine unambiguously whether the (internal) states of these two systems are the same or not. If we have just a single copy of each of the states and we possess no further information about the preparation, then a measurement performed on each system separately cannot determine the states precisely enough to allow an error-free comparison. In this case, all other strategies would also fail, because our knowledge about the states is insufficient [1], e.g., if each of the systems can be in an arbitrary mixed state, then it is impossible to unambiguously test whether the states are equal or not. However, there are often situations in which we have some additional *a priori* information on the states we want to compare. For example, we might know that each system has been prepared in a pure state. This kind of scenario has been considered in Ref. [2] for two qudits and in Ref. [3] for the comparison of a larger number of systems. Thereafter, the comparison of coherent states and its application to quantum cryptography has been addressed in

Ref. [4]. Sedlák *et al.* [5] analyzed the comparison with more copies of the two systems and proposed an optimal comparator for coherent states, which, on this subset, outperforms the optimal universal comparator [2] working for all pure states.

In the present paper we analyze the unambiguous quantum state comparison (USC) of two unknown squeezed vacuum states, that is, we would like to unambiguously determine whether two unknown squeezed-vacuum states are the same or not. The conclusion has to be drawn from a procedure using only a single copy of the states. At the end of the procedure, using only the outcome of the measurement, we have to decide whether the two states given to us have been the same or different or that we don't know which of the former conclusions is true. We strive to find an optimal procedure, i.e., one maximizing the probability of correctly judging the equivalence of the compared squeezed states.

Our proposal relies on the interference of two squeezed states at a beam splitter and on the subsequent measurement of the difference between the number of detected photons at the two output ports. In Ref. [4], the unambiguous comparison of coherent states has been considered in detail and a short remark is devoted to the comparison of squeezed vacua. In the setup of Ref. [4], after interference at a beam splitter, one needs to measure the parity of the detected number of photons: a detection of an odd number of photons indicates the difference between the inputs. As a consequence, the quantum efficiency of the detectors is a critical parameter and plays a crucial role in the robustness of the scheme. As we will show, this problem is less relevant in our case, since our setup requires the measurement of the difference of the detected number of photons. Our configuration also allows us to prove the optimality of our setup.

The plan of the paper is as follows. In Sec. II we introduce our scheme to compare two squeezed vacuum states, whereas the proof of the optimality of the setup is given in Sec. III. The performance of our scheme, also in the presence of imperfections at the detection stage, is investigated in Sec. IV, together with its reliability in the presence of noise. Section V close the paper with some concluding remarks.

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II. COMPARISON OF SQUEEZED VACUUM STATES

Our goal is the comparison of two squeezed vacuum states $|\xi\rangle \equiv S(\xi)|0\rangle$ and $|\zeta\rangle \equiv S(\zeta)|0\rangle$, where $S(\gamma) = \exp[\frac{1}{2}\gamma(a^\dagger)^2 - \frac{1}{2}\gamma^* a^2]$ is the single-mode squeezing operator, $\xi, \zeta, \gamma \in \mathbb{C}$ [6]. We let $\xi = re^{i\psi}$ and $\zeta = se^{i\varphi}$, where $r = |\xi|$, $\psi = \arg(\xi)$, $s = |\zeta|$, $\varphi = \arg(\zeta)$. We recall that a comparator is a measuring device with two systems at the input and two or more possible outcomes, aimed at determining whether the two systems have been prepared in the same state. The setup we propose for the comparison of the two squeezed vacuum states is composed of a phase shifter, beam splitter, and photon-counting detectors and can be implemented with a current technology. The basic idea is sketched in Fig. 1(a): we start from the two squeezed vacuum states we wish to compare, $S(\xi)|0\rangle$ and $S(\zeta)|0\rangle$. At the first stage of our protocol, one of the two states, say $S(\xi)|0\rangle$, undergoes a phase shift $U(\pi/2)$, i.e., $U(\pi/2)S(\xi)|0\rangle = S(-\xi)|0\rangle$; then we mix the states, having now orthogonal phases, at a balanced beam splitter (BS). If $\xi = \zeta$, i.e., the input states are equal, then the output state is the two-mode squeezed vacuum state of radiation (twin-beam state, TWB) [7], namely,

$$|\Psi_{\mathrm{out}}(\xi,\xi)\rangle\!\rangle = U_{\mathrm{BS}} S(\xi) \otimes S(-\xi)|0\rangle \equiv S_2(\xi)|0\rangle \quad (1)$$

$$= \sqrt{1 - |\lambda(\xi)|^2} \sum_{n=0}^{\infty} \lambda(\xi)^n |n\rangle|n\rangle, \quad (2)$$

where $|n\rangle|n\rangle \equiv |n\rangle \otimes |n\rangle$, U_{BS} is the unitary operator describing the action of the BS, $S_2(\xi) = \exp(\xi a^\dagger b^\dagger - \xi^* ab)$ is the two-mode squeezing operator acting on the two modes a and b, respectively, and $\lambda(\gamma) = e^{i\,\arg(\gamma)}\tanh|\gamma|$. One finds perfect correlations in the photon number of the two beams, which can be detected, e.g., by measuring the difference between the number of photons at the outputs (see Fig. 1), which, in this case, is always equal to zero. On the contrary, if $\xi \neq \zeta$, a different number of photons can be detected in the two beams, as we are going to show in the following.

Though the setup in Fig. 1(a) works for generic ξ and ζ, in the following we address the scenario in which the two squeezing phases are unknown but equal, i.e., $\arg(\xi) = \arg(\zeta)$
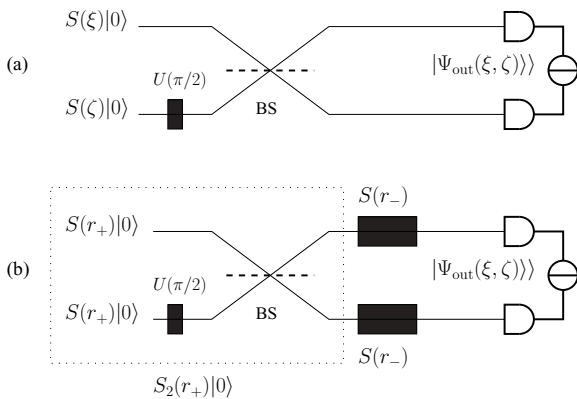

FIG. 1. (a) Schematic diagram of a setup for USC of the squeezed vacuum states $S(\xi)|0\rangle$ and $S(\zeta)|0\rangle$. (b) Scheme leading to the same output state $|\Psi_{\mathrm{out}}(\xi,\zeta)\rangle\!\rangle$ as in (a) when the two squeezing phases are unknown but equal, i.e., $\arg(\xi) = \arg(\zeta)$. We defined $r_\pm = (\xi \pm \zeta)/2$. See the text for comments and details.

[note that in Eq. (2) a two-mode squeezed vacuum state results if and only if the two input squeezed states are the same, no matter the value of their phase]. This allows us to write the output state $|\Psi_{\mathrm{out}}(\xi,\zeta)\rangle\!\rangle$ in a simple form that will turn out to be useful for characterizing our setup. Now, the same result of the evolution as in Fig. 1(a) can be obtained considering the scheme displayed in Fig. 1(b) (see Appendix A). Here the two input states with squeezing parameters ξ and ζ are substituted with two squeezed vacuum states having the same squeezing parameter amplitude $r_+ = (\xi + \zeta)/2$. It is also worth noting that the scheme in Fig. 1(b) cannot be directly used for the comparison, since it requires a prior knowledge about the squeezing parameters to assess $r_\pm$. Now, after the mixing at the BS, the outgoing modes undergo two local squeezing operations with amplitude $r_- = (\xi - \zeta)/2$. In the formula, one has the (formal) equivalence:

$$U_{\mathrm{BS}} S(\xi) \otimes S(-\zeta)|0\rangle = S(r_-) \otimes S(r_-)\, S_2(r_+)|0\rangle. \quad (3)$$

Since $S_2(r_+)|0\rangle = \sqrt{1 - |\lambda(r_+)|^2} \sum_n \lambda(r_+)^n |n\rangle|n\rangle$, we obtain

$$|\Psi_{\mathrm{out}}(\xi,\zeta)\rangle\!\rangle = \sqrt{1 - |\lambda(r_+)|^2} \sum_{n=0}^{\infty} \lambda(r_+)^n |\psi_n\rangle|\psi_n\rangle, \quad (4)$$

where we defined the new basis $|\psi_n\rangle = S(r_-)|n\rangle$. Finally, the probability of measuring h and k photons in the two beams, respectively, is given by

$$p(h,k) = |\langle h|\langle k|\Psi_{\mathrm{out}}(\xi,\zeta)\rangle\!\rangle|^2, \quad (5)$$

with

$$\langle h|\langle k|\Psi_{\mathrm{out}}(\xi,\zeta)\rangle\!\rangle$$
$$= \sqrt{1 - |\lambda(r_+)|^2} \sum_n \lambda(r_+)^n [S(r_-)]_{hn}\, [S(r_-)]_{kn}, \quad (6)$$

where $[S(r_-)]_{lm} = \langle l|S(r_-)|m\rangle$ are the matrix elements of the squeezing operator, whose analytical expressions are given, e.g., in Ref. [8]. If $\xi = \zeta$ and $h \neq k$, then $\langle h|\langle k|\Psi_{\mathrm{out}}(r,r)\rangle\!\rangle = 0$ and $p(h,k) = 0$, as one can see from Eq. (2). Thus, the probability $p(h,k)$ for $h \neq k$ can be nonzero only if $\xi \neq \zeta$, that is, only if the input states are different.

In the ideal case (unit quantum efficiency of the detectors) the measurement apparatus we want to use gives two possible outcomes: zero or nonzero photon-counting difference. Thus, the positive operator-valued measure (POVM) describing the measurement is defined by the effects E_0 and E_D, corresponding to the "zero" and "nonzero" photon-counting events, respectively, given by

$$E_0 = \sum_{n=0}^{\infty} |n\rangle\langle n| \otimes |n\rangle\langle n|, \quad E_D = \mathbb{I} - E_0. \quad (7)$$

The occurrence of the "D" event implies that the incident squeezed-vacuum states could not have been identical [see Eqs. (4) and (6)]. The occurrence of the "0" event, on the other hand, implies nothing, as each possible pair of squeezed-vacuum states leads to a nonzero overlap with any of the states $|n\rangle|n\rangle$. Thus, event "D" unambiguously indicates the difference of the compared squeezed states, whereas "0" is an inconclusive outcome.

III. PROOF OF THE OPTIMALITY OF THE SETUP

In this section we prove the optimality of the proposed setup for two situations: (i) the restricted scenario in which the squeezing phases of the compared states are unknown but equal for both states and (ii) for the general situation, when no assumption on the squeezing phases is taken. We first tackle the former scenario, which is considered in most of the paper, and at the end of the proof we comment on the differences in proving scenario (ii).

Let us denote by $\mathcal{S}^\varphi \equiv \{S(re^{i\varphi})|0\rangle; r \in \mathbb{R}\}$ the set of squeezed states from which we randomly chose the states to be compared. We also define the sets $\mathcal{S}_S^\varphi \equiv \{S(re^{i\varphi})|0\rangle \otimes S(re^{i\varphi})|0\rangle; r \in \mathbb{R}\}$ and $\mathcal{S}_D^\varphi \equiv \mathcal{S}^\varphi \otimes \mathcal{S}^\varphi \setminus \mathcal{S}_S^\varphi$, composed by pairs of identical and different squeezed vacuum states, respectively. We assume a generic measurement with three outcomes ("same", "different", and "don't know") described by the POVM $\Pi_S + \Pi_D + \Pi_0 = \mathbb{I}$, and we optimize the overall probability

$$P = z_S \int_{\mathcal{S}_S^\varphi} d\Phi \; p_S(\Phi) \langle \Phi | \Pi_S | \Phi \rangle$$
$$+ z_D \int_{\mathcal{S}_D^\varphi} d\Phi \; p_D(\Phi) \langle \Phi | \Pi_D | \Phi \rangle, \qquad (8)$$

where z_D and $z_S = 1 - z_D$ are the *a priori* probability of being different or the same, and $p_S(\Phi)$, $p_D(\Phi)$ are the probability densities of choosing $|\Phi\rangle$ from $\mathcal{S}_S^\varphi$, $\mathcal{S}_D^\varphi$, respectively. We also impose the no-error constraints

$$\mathrm{Tr}(\Pi_S |\Phi\rangle\langle\Phi|) = 0, \quad \forall |\Phi\rangle \in \mathcal{S}_D^\varphi, \qquad (9a)$$

$$\mathrm{Tr}(\Pi_D |\Phi\rangle\langle\Phi|) = 0, \quad \forall |\Phi\rangle \in \mathcal{S}_S^\varphi, \qquad (9b)$$

which guarantee the unambiguity of the results. From the mathematical point of view, the constraints (III) restrict the support of the operators Π_S and Π_D. The fact that the possible states in $\mathcal{S}^\varphi$ form a continuous subset of pure states is responsible for the impossibility to unambiguously confirm that the compared states are identical. The proof of this statement can be found in Appendix B and essentially states that due to the no-error conditions (III), we must have $\Pi_S = 0$. Thus, the measurement actually has only two outcomes, the effective POVM is given by $\Pi_D, \Pi_0 = \mathbb{I} - \Pi_D$, and it is clear that increasing the eigenvalues of Π_D without changing its support increases the figure of merit and leaves the no-error conditions satisfied. This is true independently of the distribution p_D, and thus the optimal measurement is formed by Π_D being a projector onto the biggest support allowed by the no-error condition (III) and Π_0 being a projector onto the orthocomplement. Moreover, the quantity that completely characterizes the behavior of the squeezed-states comparator is $p(D|r,s) = \langle \Phi | \Pi_D | \Phi \rangle$, i.e., the conditional probability of obtaining the outcome Π_D if different squeezed states $|\Phi\rangle = S(re^{i\varphi})|0\rangle \otimes S(se^{i\varphi})|0\rangle$ $(r \neq s)$ are sent to the comparator. It is worth noting that in what follows one does not need to know the actual value of φ. Summarizing, in order to find an optimal comparator of squeezed states from $\mathcal{S}^\varphi$ we need to refine the definition of the largest allowed support of Π_D hidden in the no-error condition (9b). To do this we equivalently rewrite Eq. (9b) as

$$\mathrm{Tr}(W\Pi_D W^\dagger W |\Phi\rangle\langle\Phi| W^\dagger) = 0 \quad \forall |\Phi\rangle \in \mathcal{S}_S^\varphi, \qquad (10)$$

which, by denoting $E_D \equiv W\Pi_D W^\dagger$ and choosing W to be the unitary transformation performed by the proposed setup from Fig. 1(a), becomes

$$\mathrm{Tr}[E_D |\Psi_{\mathrm{out}}(r,r)\rangle\rangle \langle\langle \Psi_{\mathrm{out}}(r,r)|] = 0 \quad \forall r \in \mathbb{R}. \qquad (11)$$

The optimality of the proposed setup is proved by showing that the biggest support allowed by the previous condition coincides with the support of the projective measurement E_D we use, see Eq. (7).

From the expression of $|\Psi_{\mathrm{out}}(r,r)\rangle\rangle$, Eq. (2), it is clear that for any operator E_D with the support orthogonal to the span of $|n\rangle|n\rangle$, with $n \in \mathbb{N}$, the unambiguous no-error condition (11) holds. Hence, if any such operator E_D is a part of a POVM, then the emergence of the outcome related to it unambiguously indicates the difference of the squeezing parameters. We now proceed to show that the support of such E_D cannot be further enlarged. Now let us assume that a vector $|v\rangle\rangle = \sum_{h,k=0}^\infty d_{hk} |h\rangle |k\rangle$ with at least one nonzero coefficient d_{ii} is in the support of E_D. As a consequence of the required no-error condition (11), the overlap

$$\langle\langle v | \Psi_{\mathrm{out}}(r,r)\rangle\rangle = \sqrt{1 - |\lambda(r)|^2} \sum_{n=0}^\infty d_{nn}^* \lambda(r)^n \qquad (12)$$

has to be vanishing for all values of r. Equation (12) is vanishing if and only if

$$\frac{\langle\langle v | \Psi_{\mathrm{out}}(r,r)\rangle\rangle}{\sqrt{1 - |\lambda(r)|^2}} = \sum_{n=0}^\infty d_{nn}^* \lambda(r)^n \qquad (13)$$

vanishes for all r. The sum on the right-hand side of Eq. (13) can be seen as a polynomial in $\lambda(r)$ and should vanish for all possible values of $\lambda(r)$, i.e., for all $|\lambda(r)| < 1$. Polynomials of this type on a finite interval form a vector space with linearly independent basis vectors $\lambda(r)^k$, with $k \in \mathbb{N}$. Thus the sum in Eq. (13) vanishes $\forall r \in \langle 0, \infty)$ only if $d_{nn} = 0$, $\forall n \in \mathbb{N}$. This contradicts our assumption about the vector $|v\rangle\rangle$, and therefore the largest support an operator E_D, unambiguously indicating the difference of the squeezing parameters, can have is the orthocomplement of the span of vectors $|n\rangle|n\rangle$, with $n \in \mathbb{N}$. This concludes the proof.

In the case (ii) of compared states with completely arbitrary phases of the complex squeezing parameters, the proof can be done in the same way as before, up to defining accordingly the set of pairs of the same states.

IV. PERFORMANCE OF THE SETUP

In this section we give a thorough analysis of the statistics of our setup also in the presence of nonunit quantum efficiency at the detection stage in order to assess its reliability in Sec. IV C.

A. Probability of revealing the difference

The conditional probability of revealing the difference of compared states with $\xi \neq \zeta$ [but $\arg(\xi) = \arg(\zeta) = \varphi$, though unknown], which is the probability to obtain an E_D outcome, reads

$$p(D|\xi,\zeta) = 1 - p(0|\xi,\zeta), \qquad (14)$$

with

$$p(0|\xi,\zeta) = \langle\langle\Psi_{\text{out}}(\xi,\zeta)|E_0|\Psi_{\text{out}}(\xi,\zeta)\rangle\rangle$$

$$= [1 - |\lambda(r_+)|^2] \sum_{n,m=0}^{\infty} [\lambda(r_+)]^n[\lambda^*(r_+)]^m$$

$$\times \sum_{k=0}^{\infty}\{[S(r_-)]_{kn}\}^2\{[S^\dagger(r_-)]_{mk}\}^2, \quad (15)$$

where $|\Psi_{\text{out}}(\xi,\zeta)\rangle\rangle$ is given in Eq. (4). For $\xi \to \zeta$ we correctly obtain $p(0|\xi,\xi) = 1$. By noting that [8]

$$[S(\gamma)]_{hk} \propto \begin{cases} \exp\left\{i\left(\frac{h-k}{2}\right)\theta\right\} & \text{for } h,k \text{ odd or even,} \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

where $\gamma = |\gamma|e^{i\theta}$, it is straightforward to see that Eq. (15) does not depend on the (equal) phase φ of ξ and ζ. Thus, in order to investigate the performances of the optimal squeezed-states comparator, we may set $\varphi = 0$ and let $\xi = r$ and $\zeta = s$, with $r,s \in \mathbb{R}$, without loss of generality. Furthermore, it is possible to show by numerical means that the probability $p(D|r,s)$ does not depend on the sum of the squeezing parameter $\delta_+ = r + s$, but only on the difference $\delta_- = |r - s|$. In Fig. 2 we plot the probability $p(D|r,s)$ given in Eq. (15) as a function of $\delta_- = |r - s|$, and we compare it with the possible use of the universal comparator [2], which works unambiguously for all pure states leading to

$$p_{\text{UC}}(D|\omega) = \tfrac{1}{2}(1 - \omega^2), \quad (17)$$

where $\omega = |\langle\psi_1|\psi_2\rangle| = (\cosh \delta_-)^{-1/2}$ is the overlap between the two squeezed vacuum states.

B. Influence of nonideal detectors

In a realistic scenario, in which the photon-number resolving detectors have nonunit quantum efficiency η, we should modify the POVM by replacing the projectors $|n\rangle\langle n|$ in Eq. (7) by the following operators [9,10]:

$$\Pi_n(\eta) = \eta^n \sum_{k=n}^{\infty}(1 - \eta)^{k-n}\binom{k}{n}|k\rangle\langle k|, \quad (18)$$
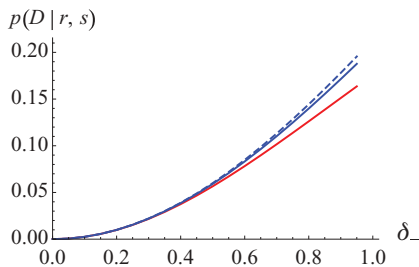

$p(D | r, s)$

FIG. 2. (Color online) Conditional probability of revealing the difference of two squeezed vacuum states $|\psi_1\rangle = S(r)|0\rangle$ and $|\psi_2\rangle = S(s)|0\rangle$ in the ideal case ($\eta = 1$) as a function of $\delta_- = |r - s|$. Solid lines, from top to bottom, correspond to the optimal squeezed-states comparator (blue) and the universal comparator (red line). The dashed line is the upper bound on the probability in the case of only two possible squeezings. See Sec. IV C for details.

namely (we assume that the two detectors have the same quantum efficiency),

$$E_0(\eta) = \sum_{n=0}^{\infty} \Pi_n(\eta) \otimes \Pi_n(\eta), \quad (19a)$$

$$E_D(\eta) = \mathbb{I} - E_0(\eta). \quad (19b)$$

The performance of this kind of detector and its reliability to resolve up to tens of photons have been recently investigated in Ref. [11]. Equation (18) shows that the single projector is turned into a (infinite) sum of projectors. This could be a relevant issue for protocols that rely on the discrimination between even and odd number of photons [4], as we mentioned in Sec. I, since it becomes challenging to detect the actual parity of the number of incoming photons, $\Pi_n(\eta)$ being a sum over both even and odd number of photons. For what concerns our setup, as one may expect, the presence of nonunit quantum efficiency no longer guarantees the unambiguous operation.

In principle, the effect of an imperfect detection could be taken into account while designing the comparison procedure. However, this would be mathematically challenging and most probably would not provide an unambiguous procedure anyway because of the form of the noise in realistic detectors. Alternatively, one could try to maximize the reliability (confidence) of the outcomes (for maximum confidence in state discrimination see [12]), nevertheless this would require one to make some particular choice of the prior probabilities z_S and z_D and of the probability distributions $p_S(\Phi)$ and $p_D(\Phi)$ (see Sec. III). By following this type of approach, an optimal comparison of coherent states in realistic conditions can be improved by employing a linear amplifier [13]. On the other hand, as we are going to show in the next section, the reliability of the difference detection of our proposal is quite close to unambiguity if the detector efficiency is high enough.

The conditional probability $p_\eta(D|\xi,\zeta)$ for the detectors with nonunit quantum efficiency η reads

$$p_\eta(D|\xi,\zeta) = 1 - p_\eta(0|\xi,\zeta), \quad (20)$$

with

$$p_\eta(0|\xi,\zeta) = \langle\langle\Psi_{\text{out}}(\xi,\zeta)|E_0(\eta)|\Psi_{\text{out}}(\xi,\zeta)\rangle\rangle$$

$$= [1 - |\lambda(r_+)|^2] \sum_{n,l,m=0}^{\infty} \eta^{2n}[\lambda(r_+)]^l[\lambda^*(r_+)]^m$$

$$\times \sum_{h,k=n}^{\infty}(1-\eta)^{h+k-2n}\binom{h}{n}\binom{k}{n}$$

$$\times [S(r_-)]_{kl}[S(r_-)]_{hl}[S^\dagger(r_-)]_{mk}[S^\dagger(r_-)]_{mh}, \quad (21)$$

which, in the case of $\xi = \zeta$, reduces to

$$p_\eta(0|\xi,\xi) = \langle\langle\Psi_{\text{out}}(\xi,\xi)|E_0(\eta)|\Psi_{\text{out}}(\xi,\xi)\rangle\rangle$$

$$= [1 - |\lambda(\xi)|^2] \sum_{n=0}^{\infty} \eta^{2n}|\lambda(\xi)|^{2n}$$

$$\times {}_2F_1[1 + n, 1 + n, 1, (1-\eta)^2|\lambda(\xi)|^2], \quad (22)$$

where ${}_2F_1$ are hypergeometric functions and $[S(r_-)]_{lm}$ are the matrix elements of the squeezing operator as in Eq. (6).
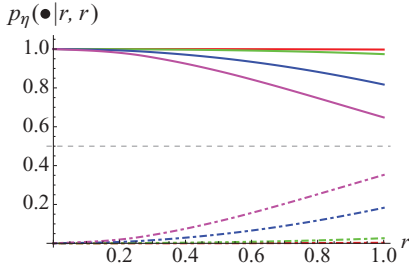
FIG. 3. (Color online) $p_\eta(0|r,r)$ (solid lines) and $p_\eta(D|r,r)$ (dot-dashed lines) as functions of r for different values of the efficiency η; from top to bottom (solid) and from bottom to top (dot-dashed): $\eta = 0.999$ (red), 0.99 (green), 0.90 (blue), 0.50 (magenta).

Because of Eq. (16), the probabilities (20) and (21) are still independent of the unknown value of φ, thus, from now on, we set $\varphi = 0$ and put $\xi = r$ and $\zeta = s$, with $r, s \in \mathbb{R}$, without loss of generality. In Fig. 3 we plot $p_\eta(0|r,r)$ and $p_\eta(D|r,r)$ for different values of η. If $r \ll 1$, then Eq. (22) can be expanded up to the second order in r, obtaining

$$p_\eta(0|\xi,\xi) \simeq 1 - 2\eta(1-\eta)r^2. \tag{23}$$

C. Reliability of the setup

To assess the reliability of our setup, we address the scenario in which only two squeezing parameters for each of the squeezed vacua are possible. In such case one knows that the two squeezing parameters are either $\{(r,r),(s,s)\}$ or $\{(r,s),(s,r)\}$ with the same prior probability. Our squeezed-states comparator may not be optimal in this case. However, as one can see in Fig. 2, the performance of our setup is nearly as good as if it were optimized also for this restricted scenario. In particular, the dashed line in Fig. 2 refers to the optimal measurement, unambiguously detecting the difference in the case of only two possible squeezing parameters, in the formula [14]

$$p_{\max}(D|\omega) = \frac{1 - \omega^2}{1 + \omega^2}. \tag{24}$$

We define the reliability R_D of the scheme in revealing the difference of the squeezing parameters r and s as the conditional probability of the two squeezed vacuum states being different if the outcome E_D is found, i.e. (we assume equal prior probabilities),

$$R_D(\eta; r, s) = \frac{p_\eta(D|r,s) + p_\eta(D|s,r)}{\sum_{u,v=r,s} p_\eta(D|u,v)}. \tag{25}$$

In the ideal case, i.e., $\eta = 1$, we have $p_\eta(D|r,r) = 0$ and, thus, $R_D(\eta; r, s) = 1$, which is guaranteed by the construction of the setup. On the other hand, if $\eta < 1$, then $p_\eta(D|r,r) \neq 0$ and, consequently, the conclusion based on the outcome D is not unambiguous anymore. The actual value of R_D can be numerically calculated starting from Eqs. (20) and (21). The reliability $R_D(\eta; r, s)$ is plotted in the upper panel of Fig. 4 as a function of $\delta_- = |r - s|$. Note that differently from the case $\eta = 1$, for $\eta < 1$ the probability $p_\eta(D|r,s)$ depends not only on the difference $\delta_- = |r - s|$ but also on the sum $\delta_+ = r + s$. The dependence on δ_+ is shown in the the lower panel of
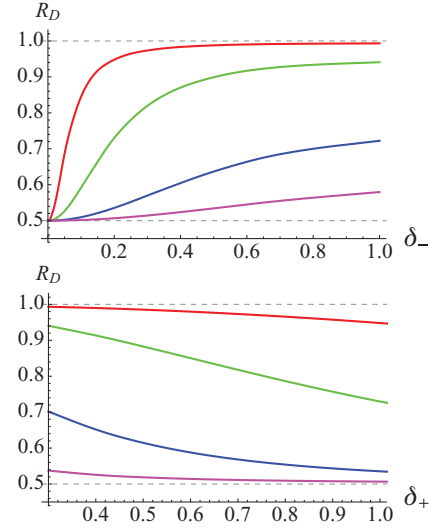

FIG. 4. (Color online) Top: Reliability $R_D(\eta; r, s)$ as a function of δ_- for fixed $\delta_+ = 1.0$ and different values of the efficiency. Bottom: Reliability $R_D(\eta; r, s)$ as a function of δ_+ for difference $\delta_- = 0.2$ and different values of the efficiency. In both plots, from top to bottom: $\eta = 0.999$ (red), 0.99 (green), 0.90 (blue), 0.50 (magenta).

Fig. 4, where we plot $R_D(\eta; r, s)$ as a function of δ_+ for fixed difference $\delta_- = 0.2$.

V. CONCLUDING REMARKS

In this paper we have addressed the comparison of two squeezed vacuum states of which we have a single copy available. We have suggested an optical setup based on a beam splitter, a phase shifter, and two photodetectors which is feasible with the current technology. Even though we analyzed the scenario with an equal, though unknown, phase of the compared states, our setup is able to operate unambiguously with ideal detectors irrespective of the squeezing phases, and without the knowledge of the relative phases of the squeezed states. We have proved the optimality of our scheme for arbitrary phases and ideal detectors, and we analyzed its performance and reliability also in the presence of nonunit quantum efficiency at the detection stage in the case of equal phases. As one may expect, the detection efficiency strongly affects the reliability; nevertheless we have shown that for small energies and not too low quantum efficiency, the setup is still robust.

Our scheme may be employed not only for the comparison of two squeezed vacua, but also for a more general scenario in which the input states $|\xi\rangle$ and $|\zeta\rangle$ are known to be transformed by two *fixed known* local unitaries U and V, respectively (namely, $U|\xi\rangle \otimes V|\zeta\rangle$) or by any *fixed known* global unitary transformation W (namely, $W|\xi\rangle \otimes |\zeta\rangle$): now it is enough to apply the inverse of the transformation before processing the state with our setup.

APPENDIX A: PROOF OF THE EQUIVALENCE OF THE TWO SCHEMES

In this Appendix we show the equivalence between the schemes in Figs. 1(a) and 1(b). Since the squeezed states are Gaussian states and all operations involved in the schemes (phase shift and beam splitter mixing) preserve the Gaussian character, we use the phase-space description of the system evolution [15]. For the sake of simplicity we focus on the case of real squeezing parameters, i.e., $\xi = r$ and $\zeta = s$, with $r, s \in \mathbb{R}$. The symplectic transformation associated with the squeezing operator $S(r)$ is

$$S(r) = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}, \tag{A1}$$

while the symplectic transformation associated with the balanced beam splitter operator U_{BS} is

$$S_{\mathrm{BS}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}, \tag{A2}$$

where $\mathbb{1}_2$ is a 2×2 identity matrix. The covariance matrix of the outgoing Gaussian state in the scheme in Fig. 1(a) [for the sake of simplicity we used $U(\pi/2)S(s) = S(-s)$ and we do not write explicitly the symplectic transformation of the phase shift]

$$|\Psi_{\mathrm{out}}(r,s)\rangle\rangle = U_{\mathrm{BS}} S(r) \otimes S(-s)|0\rangle \tag{A3}$$

is, thus, given by

$$\boldsymbol{\sigma}_{\mathrm{out}} = S_{\mathrm{BS}} \boldsymbol{LS}(r,-s)\boldsymbol{\sigma}_0 \boldsymbol{LS}(r,-s)^T S_{\mathrm{BS}}^T, \tag{A4}$$

where $\boldsymbol{\sigma}_0 = \frac{1}{2}\mathbb{1}_4$, and

$$\boldsymbol{LS}(r,-s) = \begin{pmatrix} S(r) & \mathbf{0} \\ \mathbf{0} & S(-s) \end{pmatrix} \tag{A5}$$

represents the two local squeezing operations $\boldsymbol{LS}(r,-s) = S(r) \otimes S(-s)$. The explicit form of (A4) reads

$$\boldsymbol{\sigma}_{\mathrm{out}} = \frac{1}{2}\begin{pmatrix} f(r,-s) & 0 & g(r,-s) & 0 \\ 0 & f(-r,s) & 0 & g(-r,s) \\ g(r,-s) & 0 & f(r,-s) & 0 \\ 0 & g(-r,s) & 0 & f(-r,s) \end{pmatrix}, \tag{A6}$$

where

$$f(x,y) = \frac{e^{2x} + e^{2y}}{2} \quad \text{and} \quad g(x,y) = \frac{e^{2x} - e^{2y}}{2}. \tag{A7}$$

Note that by setting $s = r$ one obtains the covariance matrix of the TWB in Eq. (1).

It is now straightforward to verify that the same result of the evolution as in Fig. 1(a), corresponding to the covariance matrix in Eq. (A6), may be obtained considering the setup displayed in Fig. 1(b). Here two input states with same squeezing parameter amplitude $r_+ = (r + s)/2$ are mixed after a phase shift at the BS, and the outgoing modes undergo two local squeezing operations with amplitude $r_- = (r - s)/2$; in formula:

$$\boldsymbol{\sigma}' = \boldsymbol{LS}(r_-,r_-)\, \boldsymbol{S}_2(r_+)\, \boldsymbol{\sigma}_0\, \boldsymbol{S}_2(r_+)^T\, \boldsymbol{LS}(r_-,r_-)^T, \tag{A8}$$

where $\boldsymbol{S}_2(r_+) = \boldsymbol{S}_{\mathrm{BS}}\, \boldsymbol{LS}(r_+, -r_+)$ is the symplectic transformation associated with $S_2(r)$ defined in Eq. (1). By performing the calculation, one finds $\boldsymbol{\sigma}' = \boldsymbol{\sigma}_{\mathrm{out}}$, and, since Gaussian states are completely characterized by their covariance matrix (and first moments), one can conclude that the final states are the same.

APPENDIX B: NO UNAMBIGUOUS DETECTION OF SAMENESS OF TWO STATES

In this Appendix we show that the no-error condition given in Eq. (9b), together with continuity of the involved mappings, implies that we cannot unambiguously detect the sameness of two states. Let us consider a state $|\Phi\rangle = S(re^{i\varphi})|0\rangle \otimes S(se^{i\varphi})|0\rangle \in \mathcal{S}_D^\varphi$ with $r \neq s$. The no-error condition (9b) demands that

$$\mathrm{Tr}(\Pi_S|\Phi\rangle\langle\Phi|) = 0, \quad \forall r \neq s. \tag{B1}$$

Let us now take the limit $s \to r$. Thanks to continuity of the trace and the chosen parametrization of the set of states, we conclude that $\forall r$

$$\langle 0|S^\dagger(re^{i\varphi}) \otimes S^\dagger(re^{i\varphi})\Pi_S S(re^{i\varphi}) \otimes S(re^{i\varphi})|0\rangle = 0. \tag{B2}$$

It follows that Eq. (B1) has to hold for arbitrary r and s. Since Π_S is a positive operator, it should be zero on the relevant part of the Hilbert space spanned by $\mathcal{S}^\varphi \otimes \mathcal{S}^\varphi$, i.e., all the possible pairs of the compared states. Hence, without loss of generality, we can choose $\Pi_S = 0$ on the whole Hilbert space.

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