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Detecting quantum non-Gaussianity of noisy Schrödinger cat states

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Abstract
Highly quantum nonlinear interactions between different bosonic modes lead to the generation of quantum non-Gaussian states, i.e. states that cannot be written as mixtures of Gaussian states. A paradigmatic example is given by Schrödinger's cat states, that is, coherent superpositions of coherent states with opposite amplitude. We here consider a novel quantum non-Gaussianity criterion recently proposed in the literature and prove its effectiveness on Schrödinger cat states evolving in a lossy bosonic channel. We prove that the quantum non-Gaussianity can be effectively detected for high values of losses and for large coherent amplitudes of the cat states.

Keywords: quantum information, quantum optics, non-classical states, non-Gaussianity

(Some figures may appear in colour only in the online journal)

1. Introduction

The achievement and control of optical nonlinearities at the quantum level (QNL) is arguably one of the central goals of modern quantum optics[3]. Highly Hamiltonian nonlinear processes such as Kerr interactions, or conditional operations such as photon addition/subtraction and Fock state generation [1] have indeed proven to be powerful tools for investigating and exploiting the quantum nature of the electromagnetic field. In light of this, techniques that reliably verify the successful experimental implementation of QNL are highly desirable. Closely linked to the concept of optical nonlinearity is that of the set of *Gaussian states* [2], which can be seen as a collection of states that can be obtained by applying quadratic Hamiltonians to thermal states of radiation. It is easy to realize that, without the use of QNL, one is invariably limited to the preparation of Gaussian states and their convex combinations. Conversely, the successful detection of a state that cannot be written in this form, a *quantum non-Gaussian state*, can only be explained by the presence of a QNL during the preparation stage. The detection and characterization of quantum non-Gaussianity (QNG) thus acquires fundamental importance in the study of continuous-variable quantum states. The literature presents a number of methods for detecting non-classical states [3], defined as states that cannot be written as mixtures of coherent states, or quantifying the deviation of a quantum state from a Gaussian [4–7]. However these methods are, respectively, not able to discriminate between quantum non-Gaussian states and squeezed states, and not suitable for distinguishing between quantum non-Gaussian states and mixtures of Gaussian states. In fact, excluding the case of states with negative Wigner function, which are certainly quantum non-Gaussian, no general method for distinguishing between the two sets is known. This state of knowledge triggered the development of sufficient methods to detect QNG in noisy setups, where no negativity of the Wigner function can be observed [8, 9], allowing one to witness the successful implementation of QNL processes despite the high levels of noise [10, 11]. In this paper we apply the method introduced in [9], to investigate QNG of *Schrödinger cat states* [12–14] undergoing severe optical loss, such that its Wigner function becomes positive everywhere. Focusing on the so-called *odd* and *even* cat states, we find that QNG can be witnessed for any value of the model parameters in the former case, and for a significant but finite range of parameters in the

[3] In this paper, we use the term 'nonlinearity' to indicate any process that cannot be realized with Hamiltonians that are second-degree polynomials in the bosonic operators. More technically, a nonlinear process cannot be obtained as a convex combination of Gaussian operations.

latter. In what follows, we start by briefly summarizing the results of [9] and the basics of the employed physical model, and then use these tools to carry out a detailed analysis of the problem of interest.

2. Quantum non-Gaussianity criteria

We here review QNG criteria based on the Wigner function which have been proposed in [9]. We will restrict ourselves here to single-mode systems, described by bosonic operators satisfying the commutation relation $[a, a^\dagger] = \mathbb{1}$. Any single mode quantum state ϱ can be equivalently described by its characteristic function or its Wigner function, defined respectively as

$$\chi[\varrho](\gamma) = \mathrm{Tr}[\varrho D(\gamma)], \quad W[\varrho](\alpha) = \int \frac{\mathrm{d}^2\gamma}{\pi^2} \mathrm{e}^{\gamma^*\alpha - \gamma\alpha^*} \chi[\varrho](\gamma), \tag{1}$$

where $D(\gamma) = \exp\{\gamma a^\dagger - \gamma^* a\}$ represents the displacement operator. A quantum state is called Gaussian if and only if its Wigner function is a Gaussian function.

The Gaussian convex hull is the set of states

$$\mathcal{G} = \left\{ \varrho \in \mathcal{H} \,|\, \varrho = \int \mathrm{d}\boldsymbol{\lambda}\, p(\boldsymbol{\lambda})\, |\psi_{\mathsf{G}}(\boldsymbol{\lambda})\rangle\langle\psi_{\mathsf{G}}(\boldsymbol{\lambda})| \right\}, \tag{2}$$

where $\mathcal{H}$ denotes the Hilbert space of continuous-variable quantum states, $p(\boldsymbol{\lambda})$ is a proper probability distribution and $|\psi_{\mathsf{G}}(\boldsymbol{\lambda})\rangle$ are pure Gaussian states. We define a quantum state *quantum non-Gaussian* iff it is not possible to express it as a convex mixture of Gaussian states, that is, iff $\varrho \notin \mathcal{G}$. In [9] it is proved that for any $\varrho \in \mathcal{G}$, the following inequality holds

$$W[\varrho](0) \geqslant \frac{2}{\pi} \mathrm{e}^{-2\bar{n}(\bar{n}+1)}, \tag{3}$$

where $\bar{n} = \mathrm{Tr}\,[\varrho a^\dagger a]$. Together with the observation that the set $\mathcal{G}$ is closed under any Gaussian map $\mathcal{E}_{\mathsf{G}}$, inequality (3) leads to the following generalized QNG criterion.

Criterion 1. *Given a quantum state* ϱ *and a Gaussian map* $\mathcal{E}_{\mathsf{G}}$*, define the QNG witness*

$$\Delta\,[\varrho, \mathcal{E}_{\mathsf{G}}] = W\,[\mathcal{E}_{\mathsf{G}}(\varrho)](0) - \frac{2}{\pi} \exp\{-2\bar{n}_{\mathcal{E}}(\bar{n}_{\mathcal{E}} + 1)\}, \tag{4}$$

where $\bar{n}_{\mathcal{E}} = \mathrm{Tr}[\mathcal{E}_{\mathsf{G}}(\varrho) a^\dagger a]$*. Then,*

$$\exists\, \mathcal{E}_{\mathsf{G}} \text{ s.t. } \Delta[\varrho, \mathcal{E}_{\mathsf{G}}] < 0 \Rightarrow \varrho \notin \mathcal{G}. \tag{5}$$

In the following section, we will apply this criterion to Schrödinger's cat states evolving in a lossy channel. As additional Gaussian maps $\mathcal{E}_{\mathsf{G}}$ we will consider the simplest examples, that is, displacement operations $D(\beta)$, squeezing operations $S(\xi) = \exp\left\{\frac{1}{2}\xi(a^\dagger)^2 - \frac{1}{2}\xi^* a^2\right\}$ and combinations of the two.

3. Detecting quantum non-Gaussianity of Schrödinger's cat states

Schrödinger's cat states are defined as

$$|\psi_{\alpha,\xi}\rangle = \frac{|-\alpha\rangle + \xi|\alpha\rangle}{\mathcal{N}}, \tag{6}$$

where, without losing generality, $\alpha \in \mathbb{R}$ and $\mathcal{N} = \sqrt{1 + \xi^2 + 2\xi \mathrm{e}^{-2\alpha^2}}$ denote the normalization constant. By considering the parameter $\xi = 1$ and -1, one obtains respectively the so-called *even* $|\psi_{\text{even}}\rangle$ and *odd* $|\psi_{\text{odd}}\rangle$ cat states. In the following we will restrict our analysis to these particular classes of states, whose Wigner functions are plotted in figure 1 for $\alpha = 1$. We will consider their evolution in a lossy bosonic channel described by the Markovian master equation

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \gamma a \varrho a^\dagger - \frac{\gamma}{2}(a^\dagger a \varrho + \varrho a^\dagger a). \tag{7}$$

The resulting time evolution is characterized by a single parameter $\epsilon = 1 - \mathrm{e}^{-\gamma t}$ and it models both the incoherent loss of photons in a zero temperature environment, and the performances of detectors having an efficiency parameter $\eta = 1 - \epsilon$. We will denote the evolved state as $\mathcal{E}_\epsilon(\varrho_0)$. In the Wigner function picture, the evolution can be analytically solved by means of the formula

$$W\,[\mathcal{E}_\epsilon(\varrho_0)](\lambda) = \frac{2}{\pi\epsilon} \int \mathrm{d}^2\lambda'\, W[\varrho](\lambda') \times \exp\left\{ -\frac{2\left|\lambda - \lambda'\sqrt{1-\epsilon}\right|^2}{\epsilon} \right\}. \tag{8}$$

Also, the average values of the operators needed to compute the QNG witnesses $\Delta[\mathcal{E}_\epsilon(\varrho_0), \mathcal{E}_{\mathsf{G}}]$ can be analytically evaluated as

$$\bar{n}_\epsilon = \mathrm{Tr}\,[\mathcal{E}_\epsilon(\varrho_0) a^\dagger a] = (1-\epsilon)\bar{n}_0, \quad \langle a^2\rangle_\epsilon = \mathrm{Tr}\,[\mathcal{E}_\epsilon(\varrho_0) a^2] = (1-\epsilon)\langle a^2\rangle_0, \tag{9}$$

where for an initial Schrödinger's cat state $\varrho_0 = |\psi_{\alpha,\xi}\rangle\langle\psi_{\alpha,\xi}|$ the initial averages read

$$\bar{n}_0 = \frac{\alpha^2(1 + \xi^2 - 2\xi \mathrm{e}^{-2\alpha^2})}{\mathcal{N}^2}, \quad \langle a^2\rangle_0 = \alpha^2. \tag{10}$$

We will focus on large noisy parameters, i.e. $\epsilon > 0.5$ such that no negativity of the Wigner function can be observed [15]. In particular we will determine the maximum values for which we observe a violation

$$\epsilon_{\max}\,[\varrho] = \max\,\{\epsilon\, : \exists \mathcal{E}_{\mathsf{G}} \text{ s.t. } \Delta\,[\mathcal{E}_\epsilon(\varrho), \mathcal{E}_{\mathsf{G}}] \leqslant 0\}. \tag{11}$$

The quantity $\epsilon_{\max}$ is a relevant figure of merit to assess our criterion. In fact, having values of $\epsilon_{\max}$ close to unity corresponds to situations where the criterion is able to detect QNG even in a highly noisy channel or, equivalently, by using highly inefficient detectors.

3.1. Odd cat states

We will start here by considering a odd cat state $|\psi_{\text{odd}}\rangle$. The Wigner function of the initial pure state, plotted in figure 1 (left), is squeezed along the P quadrature and presents a minimum (negative) value at the origin of the phase space. Then we can consider the QNG witness optimized over an additional squeezing operation, i.e.

$$\Delta_{\text{odd}}(s) = \Delta[\mathcal{E}_\epsilon(|\psi_{\text{odd}}\rangle\langle\psi_{\text{odd}}|, S(s)]. \tag{12}$$

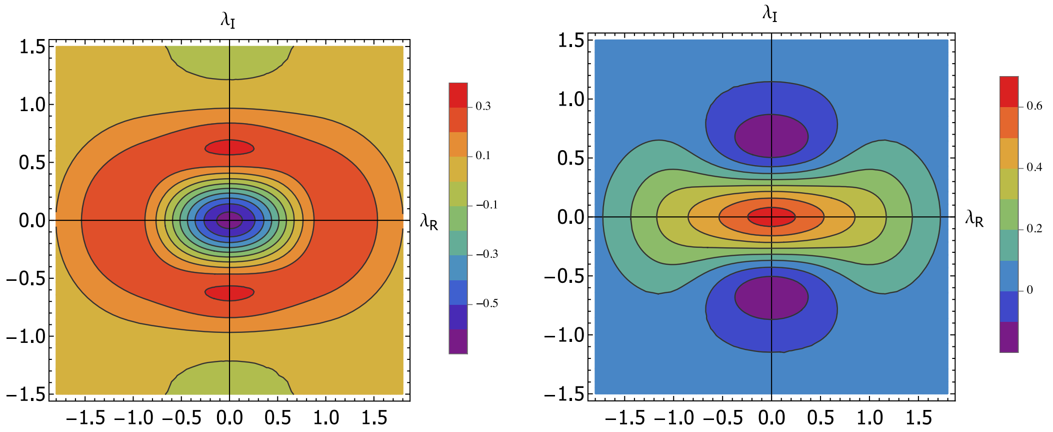

Figure 1. Contour plots of the Wigner functions $W[\varrho](\lambda)$ of the odd cat state (left) and even cat state (right) for $\alpha = 1$.

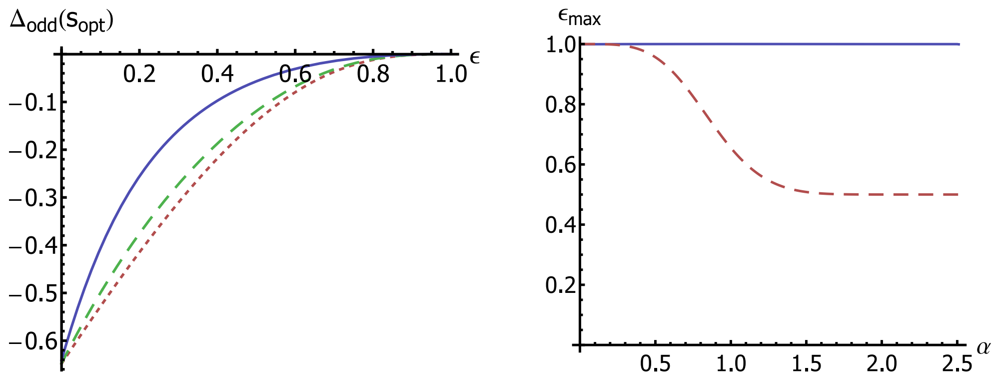

Figure 2. (Left) Optimized QNG witness $\Delta_{odd}(s_{opt})$ for odd cat states evolving in a lossy channel, as a function of ϵ and for different values of α: red dotted line: $\alpha = 0.5$; green dashed line: $\alpha = 1.0$; blue solid line: $\alpha = 1.5$. (Right) maximum value of the noise parameter ϵ_{max} such that the optimized QNG witness $\Delta_{odd}(s_{opt})$ takes negative values, as a function of the coherent states amplitude α (blue solid line). The dashed red line corresponds to the maximum value of the noise parameter ϵ_{max} obtained without considering an additional squeezing (i.e. for $s = 0$).

The average photon number of the squeezed evolved odd cat state, needed to determine $\Delta_{odd}(s)$, can be evaluated as

$$\bar{n}^{(odd)}(s) = (\mu_s^2 + \nu_s^2)\bar{n}_\epsilon + 2\mu_s\nu_s\langle a^2\rangle_\epsilon + \nu_s^2 \,, \tag{13}$$

where $\mu_s = \cosh s$, $\nu_s = \sinh s$, and the values of $\bar{n}_\epsilon$ and $\langle a^2\rangle_\epsilon$ can be obtained from equations (9) and (10) by setting $\xi = -1$. We will then look for values of the additional squeezing parameter s, such that the criterion 1 is fulfilled. In particular one can then try to optimize over the additional squeezing s, for each value of α and ϵ. By exploiting the invariance of the Wigner function in the origin under squeezing operation, this optimization corresponds to the minimization of the average photon number in equation (13). An analytic solution can be obtained, yielding

$$s_{opt} = -\frac{1}{4}\log\frac{1 - e^{2\alpha^2} - 4\alpha^2 e^{2\alpha^2}(1-\epsilon)}{1 - e^{2\alpha^2} - 4\alpha^2(1-\epsilon)}. \tag{14}$$

The behaviour of the resulting optimized witness $\Delta_{odd}(s_{opt})$ is plotted as a function of ϵ for different values of α in figure 2 (left): for the values of α considered, QNG of odd cat states can be detected by the criterion for all the values of ϵ. As pictured in figure 2 (right), it is possible to prove numerically that ϵ_{max} is equal to one, that is, QNG can also be detected for all values of noise, for larger values of α, in cases without an additional squeezing, by considering $\Delta_{odd}(0)$, $\epsilon_{max} \approx 0.5$.

3.2. Even cat states

We now consider the problem of detecting QNG for even cat states $|\psi_{even}\rangle$ evolving in the lossy channel $\mathcal{E}_\epsilon$. By inspecting the plot in figure 1 (right), we notice that the Wigner function of the initial pure state is squeezed and that its minimum is along the P quadrature axis. As a consequence we will consider a combination of displacement and squeezing operations, in order to construct the following QNG witness:

$$\Delta_{even}(\beta, s) = \Delta[\mathcal{E}_\epsilon(|\psi_{even}\rangle\langle\psi_{even}|, D(i\beta)S(s)]. \tag{15}$$

The average photon number $\bar{n}^{(even)}_{\beta,s}$ to be used in the calculation of $\Delta_{even}(\beta, s)$ reads

$$\bar{n}^{(even)}_{\beta,s} = (\mu_,^2 + \nu_s^2)\bar{n}_\epsilon + 2\mu_s\nu_s\langle a^2\rangle_\epsilon + \nu_s^2 + \beta^2, \tag{16}$$

where in this case the values of $\bar{n}_\epsilon$ and $\langle a^2\rangle_\epsilon$ can be obtained from equations (9) and (10) by setting $\xi = 1$.

We will look for the optimal values $\{\beta_{opt}, s_{opt}\}$ which minimize $\Delta_{even}(\beta, s)$ for given values of the noise parameter

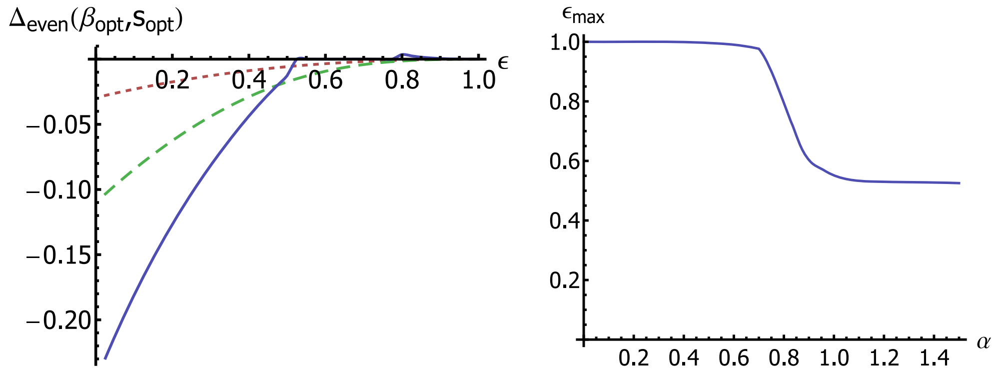

Figure 3. (Left) Optimized QNG witness $\Delta_{\text{even}}(\beta_{\text{opt}}, s_{\text{opt}})$ for even cat states evolving in a lossy channel, as a function of ϵ and for different values of α: red dotted line: $\alpha = 0.4$; green dashed line: $\alpha = 0.6$; blue solid line: $\alpha = 1.0$. (Right) maximum value of the noise parameter ϵ_{max} such that the optimized QNG witness $\Delta_{\text{even}}(\beta_{\text{opt}}, s_{\text{opt}})$ takes negative values, as a function of the coherent states amplitude α. Notice that for $\beta = 0$ and $s = 0$, one would obtain $\epsilon_{\text{max}} = 0$ for all values of α.

ϵ and the coherent states amplitude α. Unfortunately, in this case an analytical approach cannot be pursued, since the displacement operation changes both the value of the Wigner function in the origin of the evolved state and the average photon number in equation (16). As a consequence the optimal values will be obtained numerically for each couple of values of ϵ and α. The corresponding optimized QNG witness $\Delta_{\text{even}}(\beta_{\text{opt}}, s_{\text{opt}})$ has been evaluated and plotted in figure 3 (left). We can clearly observe that, thanks to the additional Gaussian operations, we are able to detect QNG for non-trivial values of the noise parameter, that is for $\epsilon > 0.5$. The maximum value of the noise parameter ϵ_{max} for which we observe negative values of Δ_{even} has been obtained numerically and it is plotted in figure 3 (right). For small values of the coherent states amplitude α, one can detect QNG for practically all the possible values of noise: in particular for $\alpha \leqslant 0.1$ we have $\epsilon_{\text{max}} \approx 1$ (up to numerical precision), and for $\alpha < 0.6$ we still have $\epsilon_{\text{max}} > 0.99$. Unfortunately, by further increasing the amplitude up to $\alpha = 1$, the witness performances are drastically reduced and ϵ_{max} approaches its limiting value $\epsilon_{\text{max}} \approx 0.5$. Notice that if we do not consider additional operations, that is by setting $\beta = 0$ and $s = 0$, one obtains $\epsilon_{\text{max}} = 0$ for all values of α, that is QNG cannot be detected.

4. Conclusions

We have applied a recently proposed QNG criterion to Schrödinger's cat states evolving in a lossy bosonic channel. We observe that by optimizing the witness by additional Gaussian operations, one can detect QNG, and thus a QNL process, for non-trivial values of the noise parameter, that is for severe optical loss yielding a positive Wigner function. In particular, the criterion works really well for *odd* cat states, while it is effective only for small amplitude *even* cat states.

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