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HOMODYNE CHARACTERIZATION OF CONTINUOUS VARIABLE BIPARTITE STATES

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We suggest a scheme to reconstruct the covariance matrix of continuous variable two-mode states using a single homodyne detector and a few polarization elements. Our method can be used to fully characterize the bipartite Gaussian entangled states obtained at the output of a (nondegenerate) optical parametric amplifier driven below threshold, as well as to extract relevant informations on generic bipartite states made of two frequency degenerate but orthogonally polarized modes.

Keywords: Entanglement; homodyne detection; Gaussian states.

1. Introduction

The class of Gaussian states encompasses most of the states that can be generated in a quantum optical lab. In particular Gaussian entangled states can be obtained, exploiting properties of non-linear optical crystal, at the output of a continuous-wave nondegenerate optical parametric oscillator (OPO) below threshold, injected by seed waves with degenerate frequencies but orthogonal polarization.[1] In continuous variables (CV) quantum information processing, Gaussian states play a crucial role, being the resource at the heart of teleportation, dense coding, cryptography and telecloning protocols.

Besides mean values of the field operators, the most relevant quantity needed to characterize a bipartite state is its covariance matrix σ.[2] Once the covariance matrix is given, the entanglement of the state can be evaluated and, in turn, the performances of the state itself as a support for quantum information protocols are known. Moreover, since entanglement is generally corrupted by the interaction with the environment, it becomes crucial to establish whether or not it has survived the environmental noise. As a consequence, besides being of fundamental interest, a simple characterization technique for bipartite states is important for checking the effective entanglement in a noisy channel[3–8] as well as the corresponding state purity and nonclassicality[9,10] In this paper the covariance matrix measurement based on a single homodyne detector[11] is specialized to the case of the Gaussian state at the output of an OPO, which can be described as a *two-mode squeezed* *thermal state* (TST).[12,13] This is a general expression for a zero-amplitude bipartite Gaussian state, which can also be obtained from an ideal two–mode squeezed state subjected to lossy propagation.[16] The two-mode squeezing (entanglement) and thermal contributions can be evaluated in terms of the parameters characterizing the OPO, say the output squeezing ratio and the distance from threshold, as well as the effect of losses experienced by the state through the transmission channel; this in turns allows estimating the matrix elements of σ for different conditions. In particular, the analysis reported here refers to the experimental parameters for a type II non-monolithic OPO based on a poled KTP crystal (PKTP), which is currently investigated in Napoli.

The paper is structured as follows: at first we review the general form of covariance matrix with focus on the case of two-mode squeezed thermal states and then we illustrate the scheme to reconstruct the covariance matrix using a single homodyne detector. Finally, experimental details concerning the bipartite state generated by a PKTP-OPO are discussed.

2. Covariance Matrix for a Bipartite State

For any bipartite system, made of two different modes of the field, a covariance matrix (CM) can be defined. CM elements are made of combinations of first and second order statistical momenta in the field variables. In the case of Gaussian state, i.e. state whose characteristic functions are Gaussian, CM contains the full information about the state. Given two field modes, a and b, then the CM can be written in terms of the field amplitude and phase quadratures defined for each mode k, namely,

$$X_k = \frac{k + k^\dagger}{\sqrt{2}}, \quad Y_k = \frac{k - k^\dagger}{i\sqrt{2}}, \tag{1}$$

k and $k^\dagger$, $k = a, b$, being the annihilation and creation operators, respectively. In turn, CM σ reads

$$\sigma = \begin{pmatrix} \Delta X_a^2 & \Delta X_a Y_a & \Delta X_a X_b & \Delta X_a Y_b \\ \Delta Y_a X_a & \Delta Y_a^2 & \Delta Y_a X_b & \Delta Y_a Y_b \\ \Delta X_b X_a & \Delta X_b Y_a & \Delta X_b^2 & \Delta X_b Y_b \\ \Delta Y_b X_a & \Delta Y_b Y_a & \Delta Y_b X_b & \Delta Y_b^2 \end{pmatrix}, \tag{2}$$

where $\Delta O_k^2 = \langle O_k^2 \rangle - \langle O_k \rangle^2$ is the variance of the observable O traced over the whole density matrix of the bipartite system ($\langle O_k \rangle = \mathrm{Tr}[\rho\, O_k]$), and $\Delta O_k O'_{k'} = \frac{1}{2}\langle [O_k, O'_{k\prime}]_+ \rangle - \langle O_k \rangle \langle O'_{k'} \rangle$ (k and k' modal index) the mutual correlation between O and O', $[O_k, O'_{k\prime}]_+$ denoting the anti-commutator between the two observables.

A two-mode squeezed state, as the state generated by a lossless (ideal) sub-threshold frequency degenerate but polarization non-degenerate OPO[17] can be formally obtained by the action of the two-mode squeezing operator $S_2(r_0) = \exp\{r_0(a^\dagger b^\dagger - a\, b)\}$ on the vacuum state of the two modes a and b, i.e. $\rho_0 = |0\rangle_{aa}\langle 0| \otimes |0\rangle_{bb}\langle 0|$. The density matrix is thus given by $\rho = S_2(r_0)\rho_0\, S_2^\dagger(r_0)$.

In this case CM takes the form

$$\sigma = \frac{1}{2}\begin{pmatrix} A\,\mathbb{I} & C\,\sigma_3 \\ C\,\sigma_3 & B\,\mathbb{I} \end{pmatrix}, \tag{3}$$

with $\mathbb{I}$ the 2×2 identity matrix and $\sigma_3 = \mathrm{Diag}(1, -1)$, $A = B = \cosh 2r_0$ and $C = \sinh 2r_0$.

As matter of fact, the actual state generated by a realistic sub-threshold frequency degenerate but polarization non-degenerate OPO is given by a two-mode squeezed state subjected to a lossy propagation,[16] i.e. a TST that can be represented as the action of a two-mode squeezing operator $S_2\,(r_\eta)$ on the tensor product $\rho_\nu = \nu_a \otimes \nu_b$ where $\nu_k = (1+N_k)^{-1}[N_k/(1+N_k)]^{k^\dagger k}$, $k = a, b$, are thermal states with N_k average photons, respectively. The N_k's depend on the ideal squeezing ratio r_0 and on the collection (η_c) and detection (η_d) efficiencies where η_c accounts for the coupling of the state out of the cavity and η_d for the homodyne and the detectors quantum efficiencies ($\eta_{\mathrm{tot}} = \eta_c\eta_d$). In general η_{tot} is defined for each mode, but in the present work the modes are supposed to be perfectly symmetric from the point of view of collection and detection efficiencies, thus giving a *symmetric* bipartite state. The average thermal photon number N_η and the squeezing ratio r_η are given by[16]

$$(2N_\eta + 1)\cosh 2r_\eta = 1 + \eta_{\mathrm{tot}}(\cosh 2r_0 - 1)\,, \tag{4}$$

$$(2N_\eta + 1)\sinh 2r_\eta = \eta_{\mathrm{tot}}\sinh 2r_0\,. \tag{5}$$

In turn, the covariance matrix σ of the actual state is of the form (3) with

$$A = B = (2N_\eta + 1)\cosh 2r_\eta, \tag{6}$$

$$C = (2N_\eta + 1)\sinh 2r_\eta. \tag{7}$$

It can be proved[15] that the state is entangled if and only if

$$A^2 + B^2 + 2|C|^2 - 1 > (AB - C^2)^2. \tag{8}$$

3. Measurement Scheme and Experimental Expectations

CM can be recovered by using a single homodyne detector.[11] The measurement method relies on the possibility of decomposing σ into the linear combination of two independent matrices $\mathbf{V}$ and $\mathbf{M}$. On one hand, the matrix $\mathbf{M}$ contains only quadrature first moments and can recovered by measuring the average value for the four quadrature X_k and Y_k ($k = a, b$). On the other hand, $\mathbf{V}$ contains second order moments and is given by

$$\mathbf{V} = \begin{pmatrix} \langle X_a^2\rangle & \frac{1}{2}\langle[Y_a, X_a]_+\rangle & \langle X_aX_b\rangle & \langle X_aY_b\rangle \\ \frac{1}{2}\langle[Y_a, X_a]_+\rangle & \langle Y_a^2\rangle & \langle Y_aX_b\rangle & \langle Y_aY_b\rangle \\ \langle X_bX_a\rangle & \langle X_bX_a\rangle & \langle X_b^2\rangle & \frac{1}{2}\langle[X_b, Y_b]_+\rangle \\ \langle Y_bX_a\rangle & \langle Y_bY_a\rangle & \frac{1}{2}\langle[Y_b, X_b]_+\rangle & \langle Y_b^2\rangle \end{pmatrix}, \tag{9}$$

It can be recovered by measuring at least fourteen quadratures, e.g. X_k, Y_k ($k = a$, b, c, d, e) together with Z_a, T_a and Z_b, T_b. The quadratures Z_k, T_k reflect the following conventions

$$Z_k \equiv x_{k,\pi/4}\,, \quad T_k \equiv x_{k,-\pi/4}, \tag{10}$$

where $x_{k,\phi}$ is the generic quadrature operator for the mode k

$$x_{k,\phi} = \frac{k\,e^{-i\phi} + k^\dagger\,e^{i\phi}}{\sqrt{2}}. \tag{11}$$

The mode label k refers to five field modes obtained as linear combinations of the initial pair, namely,

$$a, \quad b, \quad c = \frac{a+b}{\sqrt{2}}, \quad d = \frac{a-b}{\sqrt{2}}, \quad e = \frac{ia+b}{\sqrt{2}}. \tag{12}$$

From the polarization point of view, a and b correspond to vertical and horizontal polarizations, respectively, c and d are rotated polarization modes at $\pm\pi/4$, whereas e corresponds to left-handed circular polarization. Notice that the number of parameters needed to characterize a bipartite Gaussian state is also equal to fourteen.

We proved above that it is possible to fully reconstruct the covariance matrix σ by measuring fourteen different quadratures of five field modes obtained as linear combination of the initial pair. Now, we consider an implementation on the bright continuous-wave beams generated by a seeded degenerate OPO below threshold based on a type-II nonlinear crystal.[17] The two collinear beams (a and b) exiting the OPO are orthogonally polarized and excited in a CV bipartite entangled state. In the following we assume a as vertically polarized and b as horizontally polarized.

The mode under scrutiny is selected by inserting suitable components on the optical path of fields a and b, before the homodyne detector. Modes a–d are obtained by means of a rotator of polarization R_ϑ (namely a $\lambda/2$ wave-plate) and a polarizing beam splitter (PBS), which reflects toward the detector the vertically polarized component of the impinging beam. The action of the rotator R_ϑ on the basis $\{|V\rangle, |H\rangle\}$ is given by

$$R_\vartheta|V\rangle = \cos\vartheta\,|V\rangle - \sin\vartheta\,|H\rangle, \tag{13}$$

$$R_\vartheta|H\rangle = \sin\vartheta\,|V\rangle + \cos\vartheta\,|H\rangle. \tag{14}$$

In order to select mode e a $\lambda/4$ wave-plate should be inserted just before the rotator R_ϑ (see Fig. 1). The $\lambda/4$ wave-plate produces a $\pi/2$ shift between horizontal and vertical polarization components, thus turning the polarization from linear into circular. Overall, the vertically polarized mode k arriving at the detector can be expressed in terms of the initial modes as follows $k = \exp\{i\varphi\}\cos\vartheta\,a + \sin\vartheta\,b$, where $\varphi = \pi/2$ when the $\lambda/4$ wave-plate is inserted, $\varphi = 0$ otherwise.

Once the mode k has been selected, a homodyne detector is used to measure the generic quadrature $x_{k,\phi}$. Homodyne relies on the controlled interference between the quantum beam (signal) to be analyzed and a strong "classical" local oscillator

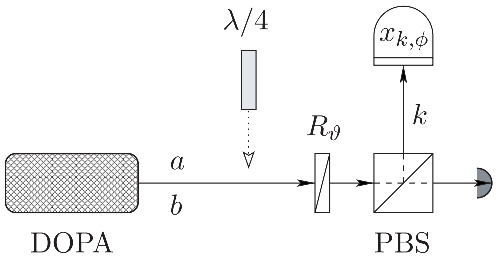

Fig. 1. Scheme to measure the covariance matrix of the bipartite (entangled) state generated by an OPO. The two modes, a (vertical polarization) and b (horizontal polarization), pass through a (removable) $\lambda/4$ wave-plate and a rotator of polarization R_ϑ; finally, a PBS reflects the vertically polarized component of its input toward a homodyne detector, which measures the $x_{k,\phi}$ quadrature. See text for details.

(LO) beam of phase ϕ. Indeed, to access $x_{k,\phi}$ one have to suitably tune the phase ϕ. The optimization of the efficiency is provided by matching the LO mode to the mode k. The mode matching requires precise control of the LO frequency, spatial and polarization properties. Remarkably, the detected mode is always vertically polarized, thus avoiding any need of tuning the LO polarization.

The system actually under study at the University of Napoli is made of a OPO based on a periodically poled KTP crystal (PPKTP). The system is designed to have a collection efficiency $\eta_c = 0.75$, while the homodyne and detection efficiency is $\eta_d = 0.89$ thus giving a total efficiency $\eta_{\text{tot}} = 0.67$. The ideal squeezing ratio r_0, for frequencies well inside the cavity bandwidth, depends on the distance from the threshold $E = P_p/P_{th}$, i.e. the ratio between the actual pump power driving the OPO and the threshold,[18] namely

$$r_0 \simeq -\frac{1}{2}\ln\left(1 - \frac{4E}{(1+E)^2}\right). \tag{15}$$

Working at $E = 0.5$, the system is expected to deliver a Gaussian bipartite state with the following CV parameters

$$A = B = 2.851, \quad C = 2.430. \tag{16}$$

This state would be entangled being satisfied the condition of Eq. (8). Higher pump powers would increase the squeezing though, on the other hand, would make the state slightly non-Gaussian.[19]

4. Conclusions

In conclusion, the covariance matrix of the bipartite two-modes squeezed thermal state generated by a sub-threshold OPO can be characterized by a single homodyne detector by measuring field quadratures relative to different combinations of the two orthogonally polarized modes. The measurement scheme will be implemented at the output of a PPKTP type-II OPO designed to deliver an entangled state.

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