Photon statistics without counting photons

Andrea R. Rossi,* Stefano Olivares,† and Matteo G. A. Paris‡

Dipartimento di Fisica and INFM, Università degli Studi di Milano, Milano, Italy

We give a detailed analysis of an indirect method used to obtain the photon distribution of a single-mode field using only on/off avalanche photodetectors. The method is based on measuring the field at different quantum efficiencies and then inferring the photon distribution by maximum-likelihood estimation. We address the case when only a limited range of quantum efficiency is available and when these values are not precisely known. The convergence of the method and its robustness against fluctuations are illustrated by means of numerically simulated experiments.

PACS number(s): 42.50.Ar, 03.65.Wj, 42.50.Dv

I. INTRODUCTION

In this Brief Report, we address a simple method to obtain the photon distribution without directly counting photons. In this scheme, repeated preparations of the signal are revealed through avalanche photodetectors (APD's) at different quantum efficiencies. The resulting on/off statistics is then used to reconstruct the photon distribution through maximum-likelihood estimation. Since the model is linear and the photon distribution is a set of positive numbers, then the maximum of the likelihood functional can be found iteratively by the expectation-maximization (EM) algorithm [1,2]. The method does not require long time stability and involves only simple optical components. The number of experimental runs depends on the signal under investigation, roughly increasing with its nonclassicality.

The idea of inferring photon distribution through on/off detection at different efficiencies has been already analyzed theoretically [3], and implemented to realize a multichannel fiber loop detector [4]. Here we analyze the reconstruction when only a subset of values $0 < \eta_{\min} < \eta < \eta_{\max} < 1$ of the quantum efficiency is available, and discuss in detail the statistical properties of the method: convergence and robustness against fluctuations in the value of the quantum efficiencies.

Given a single-mode state $\varrho = \Sigma_{n,m} \varrho_{nm} |n\rangle\langle m|$ we are interested in the photon distribution, i.e., in the set of positive numbers $\varrho_n \equiv \varrho_{nn} \geq 0$. We assume to have at disposal APD's, which can only discriminate the vacuum from the presence of radiation, with a certain quantum efficiency η. This kind of measurement is described by a two-value probability operator-valued measure (POVM)

$$\Pi_{\mathrm{off}}(\eta) = \sum_{n=0}^{\infty} (1 - \eta)^n |n\rangle\langle n|, \quad \Pi_{\mathrm{on}}(\eta) = \mathbb{I} - \Pi_{\mathrm{off}}(\eta). \quad (1)$$

Therefore the detector does not click with a probability

$$p_{\mathrm{off}}(\eta) = \mathrm{Tr}\{\varrho \Pi_{\mathrm{off}}(\eta)\} = \sum_{n=0}^{\infty} (1 - \eta)^n \varrho_n. \quad (2)$$

From now on we suppress the subscript "off" and always mean p_{off} when we write p. The "off" probabilities for a set of N detectors measuring the same quantum state with different quantum efficiencies are then

$$p_\nu(\eta_\nu) = \sum_n (1 - \eta_\nu)^n \varrho_n \quad (\nu = 0, 1, \ldots, N). \quad (3)$$

If we know all of the η_ν's values, Eq. (3) is a linear system with unknowns $\{\varrho_n\}$. In practice, it is not necessary to have at disposal many detectors with different quantum efficiencies, since a suitable tuning of η can be obtained by optical filters or even through an interferometric setup.

Suppose now that the ϱ_n's are negligible for $n > \bar{n}$ and that we are able to measure the signal with $N = \bar{n}$ different η's. In this case Eq. (3) is a linear system of the form $\mathbf{p} = \mathbb{V} \cdot \varrho$, where $\mathbf{p} = \{p_0, p_1, \ldots, p_{\bar{n}-1}\}^T$ and $\varrho = \{\varrho_0, \varrho_1, \ldots, \varrho_{\bar{n}-1}\}^T$, and the coefficients matrix $\mathbb{V}$ (for $\eta_i \neq \eta_j \ \forall i, j$) is a nonsingular Vandermonde matrix of order $\bar{n}$. Unfortunately, the reconstruction of ϱ_n by matrix inversion cannot be used in practice since it would require an unreasonable number of experimental runs [3]. This problem can be circumvented by considering Eqs. (3) as a statistical model for the parameters ϱ_n to be solved by maximum-likelihood (ML) estimation. We assume $N > \bar{n}$ and define

$$p_\nu \equiv p_\nu(\eta_\nu), \quad A_{\nu n} \equiv (1 - \eta_\nu)^n, \quad (4)$$

so that Eq. (3) can be rewritten as

$$p_\nu = \sum_n A_{\nu n} \varrho_n. \quad (5)$$

The solution of this linear and positive (LINPOS) model can be obtained using the EM algorithm [1,2]. By imposing the restriction $\Sigma_n \varrho_n = 1$, one obtains the iterative solution

$$\varrho_n^{(i+1)} = \varrho_n^{(i)} \sum_\nu \frac{A_{\nu n}}{\sum_\lambda A_{\lambda n}} \frac{f_\nu}{p_\nu[\{\varrho_n^{(i)}\}]}, \quad (6)$$

where $\varrho_n^{(i)}$ is the value of ϱ_n evaluated at the ith iteration, f_ν are the experimental frequencies of the "off" events in the

*Electronic address: andrea.rossi@mi.infn.it
†Electronic address: stefano.olivares@mi.infn.it
‡Electronic address: matteo.paris@fisica.unimi.it

run with $\eta=\eta_\nu$ and $p_\nu[\{\varrho_n^{(i)}\}]$ are the probabilities p_ν calculated using the reconstructed distribution $\{\varrho_n^{(i)}\}$ at the ith iteration.

The EM algorithm is known to converge unbiasedly to the ML solution. The confidence interval on the determination of the element ϱ_n can be given in terms of the variance

$$\sigma_n = (\mathcal{N}F_n)^{-1/2}, \tag{7}$$

$\mathcal{N}$ being the total number of measurements, F_n the Fisher's information [6],

$$F_n = \sum_\nu \frac{1}{q_\nu}\left(\frac{\partial q_\nu}{\partial \varrho_n}\right)^2, \tag{8}$$

and

$$q_\nu = \frac{p_\nu}{\sum_\nu p_\nu} = \frac{\sum_n A_{\nu n}\varrho_n}{\sum_{\nu n} A_{\nu n}\varrho_n}$$

are the renormalized probabilities of *no-click* with quantum efficiency η_ν. $N_0=\sum_{\nu n}A_{\nu n}\varrho_n$ is the global fraction of no-click events (irrespective of the quantum efficiency).

Notice that Eq. (6) provides a solution once an initial distribution $\{\varrho_n^{(0)}\}$ is chosen. In our simulated experiments we start from the uniform distribution $\varrho_n^{(0)}=(1+\bar{n})^{-1}$ in $[0,\bar{n}]$. Other choices, the only constraint being $\varrho_n^{(0)}\neq 0$, $\forall n$, do not dramatically influence the convergence properties of the algorithm.
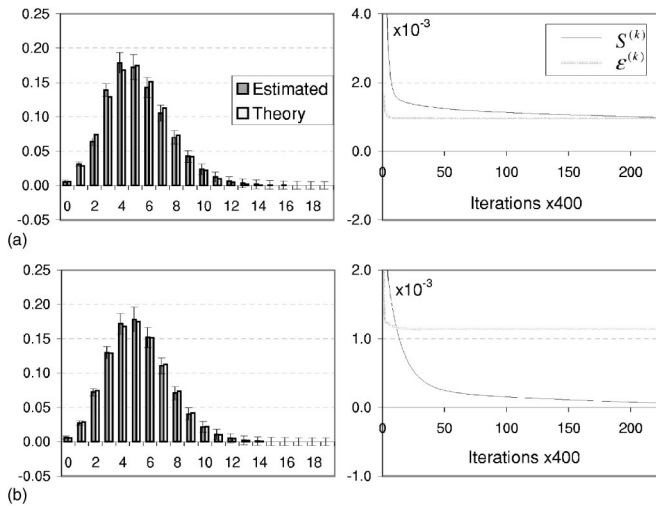

FIG. 1. Reconstruction of the photon distribution of a coherent state with $\langle a^\dagger a\rangle=5.20$: reconstructed distribution (left); normalization factor $S^{(k)}=\sum_n\varrho_n^{(k)}-1$ and total error $\varepsilon^{(k)}$ of Eq. (9) as functions of n_{it} (right). The confidence interval has been evaluated using Eq. (7). We set $n_x=n_{it}=10^5$. We use $N=50$ different quantum efficiencies with a minimum efficiency $\eta_{min}=0.02$. The maximum efficiency: (a) $\eta_{max}=0.99$; (b) $\eta_{max}=0.5$. The Hilbert space is truncated at $\bar{n}=20$.

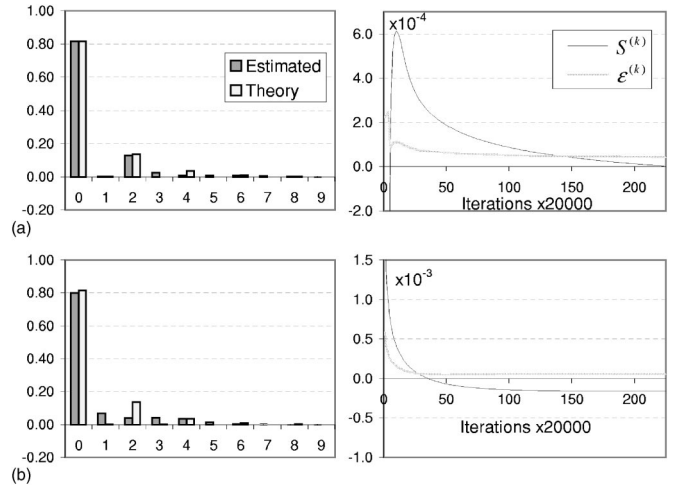FIG. 2. Reconstruction of the photon distribution of a squeezed state with $\zeta=0.99$ and $\langle a^\dagger a\rangle=0.5$. Notice that a larger number of iterations is needed in comparison with the coherent signal's case. We set $n_x=10^6$ and $n_{it}=5\times10^6$. The maximum efficiency is (a) $\eta_{max}=0.99$; (b) $\eta_{max}=0.7$. The other parameters are the same as in Fig. 1.

II. MONTE CARLO SIMULATED EXPERIMENTS AND DISCUSSION

We have performed several numerical simulations in order to check the accuracy and reliability of this method by varying the different parameters. Since the solution of the ML estimation is obtained iteratively, the most important aspect to keep under control is its convergence. As a measure of convergence we use the total absolute error at the kth iteration,

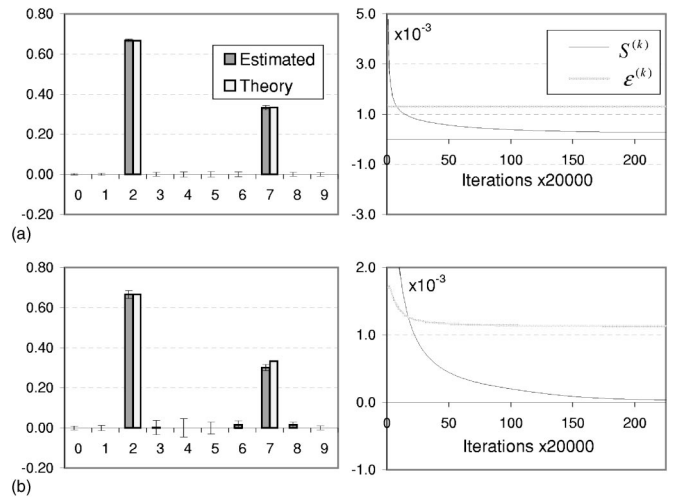$$\varepsilon^{(k)} = \sum_{\nu=0}^{N}|\epsilon_\nu^{(k)}|, \tag{9}$$

where

FIG. 3. Reconstruction of the photon distribution of the superposition of Fock states $|\psi\rangle=(2/3)^{1/2}|2\rangle+(1/3)^{1/2}|7\rangle$. We set $n_x=10^4$ and $n_{it}=5\times10^6$. The maximum efficiency is: (a) $\eta_{max}=0.99$; (b) $\eta_{max}=0.5$. The other parameters are the same as in Fig. 1.

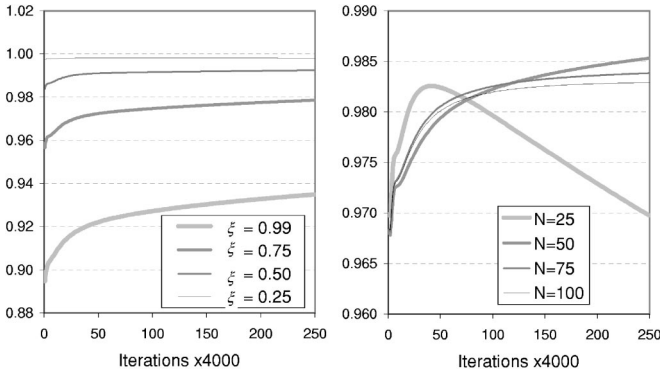FIG. 4. Fidelity $G^{(k)}$ versus the number of iterations: for a squeezed state with $\langle a^\dagger a\rangle=1.0$ and different squeezing fractions ξ (left), and for a squeezed state with $\langle a^\dagger a\rangle=1.0$, $\xi=0.75$ and different numbers N of η's values (right). In both cases $n_x=10^5$.

$$\epsilon_\nu^{(k)} = p_\nu - p_\nu[\{\varrho_n^{(k)}\}] = p_\nu - \sum_{n=0}^{\bar{n}-1} (1-\eta_\nu)^n \varrho_n^{(k)}. \qquad (10)$$

The total error $\varepsilon^{(k)}$ measures the distance of the probabilities $p_\nu[\{\varrho_n^{(k)}\}]$, as calculated at the kth iteration, from the actual experimental probabilities. As a measure of accuracy we adopt the fidelity

$$G^{(k)} = \sum_n \sqrt{\varrho_n \varrho_n^{(k)}} \qquad (11)$$

between the reconstructed distribution and the theoretical one.

The simulated experiments we performed concern coherent states $|\alpha\rangle = D(\alpha)|0\rangle$ (Fig. 1), squeezed states $|\alpha,\zeta\rangle = D(\alpha)S(\zeta)|0\rangle$ (Fig. 2), and superposition of two Fock states (Fig. 3), where $D(\alpha)=\exp\{\alpha a^\dagger - \bar{\alpha}a\}$ is the displacement operator and $S(\zeta)=\exp\{\frac{1}{2}(\zeta a^{\dagger 2} - \bar{\zeta}a^2)\}$ is the squeezing operator. Squeezed states have been parametrized through the total average photon number $\langle a^\dagger a\rangle = |\alpha|^2 + |\zeta|^2/(1-|\zeta|^2)$ and the
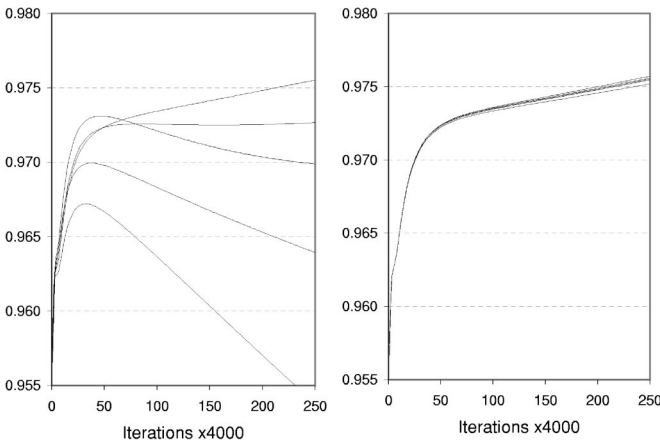

FIG. 5. Fidelity $G^{(k)}$ versus the number of iterations for a squeezed state with $\langle a^\dagger a\rangle=1.5$ and $\zeta=0.75$. Each line represents a different simulated run with $n_x=10^5$ (left) and $n_x=10^6$ (right). The other parameters are the same as in Fig. 1.

squeezing fraction $\xi=1-|\alpha|^2/\langle a^\dagger a\rangle$: $\xi=1$ corresponds to a squeezed vacuum, while $\xi=0$ to a coherent state. The error bars are calculated using Eqs. (7) and (8).

The algorithm converges quite fast for coherent states, as shown in Fig. 1, while for squeezed states (Fig. 2) the number of needed iterations is larger (see the left plot of Fig. 4). In Fig. 3 we show the reconstruction for the unbalanced superpositions of Fock states, namely $|\psi\rangle=(2/3)^{1/2}|2\rangle + (1/3)^{1/2}|7\rangle$.

Moreover, in Figs. 1–3 (right) we report $\varepsilon^{(k)}$ versus the number of iterations for different signals. As it is apparent from the plots, the total error is a good marker for the convergence of the algorithm, while the normalization factor $S^{(k)}=\Sigma_n \varrho_n^{(k)}-1$ (ideally zero at each step) is not. Notice, however, that the minimum of the total error could not always coincide with the maximum fidelity of reconstruction; indeed, fidelity is not the cost function maximized by ML estimation [5]. We have numerically observed that this problem
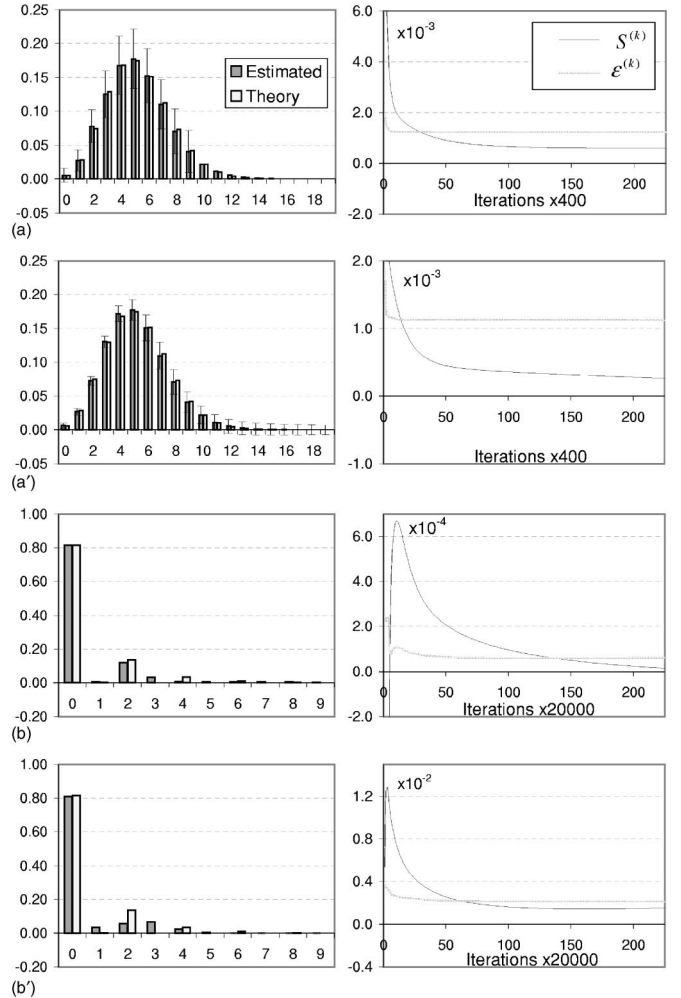

FIG. 6. Reconstruction of the photon distribution for different signals and fluctuating η_ν: (a), (a') coherent state with $\langle a^\dagger a\rangle=5.20$; (b), (b') squeezed state with $\langle a^\dagger a\rangle=0.50$ and $\xi=0.99$. We set $a=2$ in Eq. (12) and: (a), (a') $n_x=n_{it}=10^5$; (b), (b') $n_x=10^6$ and $n_{it}=5\times10^6$. The maximum efficiency is (a), (b) $\eta_{max}=0.99$; (a') $\eta_{max}=0.5$; (b') $\eta_{max}=0.7$. The other parameters are the same as in Fig. 1.

can be circumvented by using a number of iterations, n_{it}, approximately equal to the number of data for each η_ν, n_x. We have no precise explanation for this phenomenon, and provide it as a heuristic prescription leading to best performances for a large class of quantum signals.

Concerning the values of the quantum efficiency, we used N values of η uniformly distributed in $[\eta_{min}, \eta_{max}]$ with $\eta_{min} \simeq 0$ and $\eta_{max} < 1$. In principle, a different distribution (not uniform) may influence the performances of the algorithm. We found, however, that both convergence and accuracy are not much affected by a different choice, which may become relevant only if the spacing between the efficiency values becomes smaller. It should be noticed that the algorithm works well also when η_{max} is considerably smaller than unit (see Figs. 1–3): this is a relevant feature of the method in view of its experimental implementations in different working regimes.

In experiments where we have no *a priori* information on the state under investigation it could happen that part, or even most, of the photon distribution ϱ_n lies outside the reconstruction region (from 0 to $\bar{n}$). In this case we have checked that the algorithm is able to reconstruct accurately the norm of the included part, such that a simple check of the distribution norm allows us to optimize $\bar{n}$ (and in turn N) in few steps. This is a remarkable feature of the algorithm, since in general a large N improves convergence but does not guarantee better accuracy (right plot of Fig. 4), which is achievable by increasing the number of experimental data for each η, n_x, as shown in Fig. 5.

A question may arise about the robustness of the method against fluctuations in the value of the η_ν, i.e., whether or not their precise knowledge is needed. In order to check robustness we have performed simulated experiments where, during the run, the quantum efficiency may fluctuate. In particular, we assumed each η_ν uniformly distributed in the range $(\bar{\eta}_\nu - \sigma, \bar{\eta}_\nu + \sigma)$, where

$$\sigma = \frac{\bar{\eta}_{max} - \bar{\eta}_{min}}{aN} \quad (a > 0), \tag{12}$$

$\bar{\eta}_\nu$ being the expected value. The value $a = 2$ corresponds to each η_ν fluctuating in an interval as large as the spacing $(\bar{\eta}_{\nu+1} - \bar{\eta}_\nu)$ around its expected value. The values of p_ν change accordingly during the run. Our results are summarized in Fig. 6. The reconstruction is not dramatically affected by fluctuations, though errors bars are slightly larger. We conclude that the method is robust against fluctuations.

III. CONCLUSIONS

We analyzed in details an iterative algorithm to infer the photon distribution of a single-mode radiation field using only avalanche photodetectors. The method is accurate and statistically reliable for a large class of Gaussian (coherent and squeezed) and non-Gaussian states (superpositions and mixtures of $\langle n \rangle$ states), provided that on/off photodetection may be performed at different quantum efficiencies. The scheme involves only simple optical components, and allows reconstruction with APD quantum efficiency considerably smaller than unit. The convergence of the method, and its robustness against fluctuations of quantum efficiency have been demonstrated numerically, by means of Monte Carlo simulated experiments.

[1] A. P. Dempster, N. M. Laird, and D. B. Rubin, J. R. Stat. Soc. Ser. B. Methodol. **39**, 1 (1977); Y. Vardi and D. Lee, *ibid.* **55**, 569 (1993).

[2] R. A. Boyles, J. R. Stat. Soc. Ser. B. Methodol. **45**, 47 (1983); C. F. J. Wu, Ann. Stat. **11**, 95 (1983).

[3] D. Mogilevtsev, Opt. Commun. **156**, 307 (1998); Acta Phys. Slov. **49**, 743 (1999).

[4] J. Řeháček, Z. Hradil, O. Haderka, J. Peřina, Jr., and M. Hamar, Phys. Rev. A **67**, 061801(R) (2003); O. Haderka, M. Hamar, and J. Peřina, Eur. Phys. J. D **28**, 149 (2004).

[5] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[6] H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).