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SENSITIVITY OF HILBERT AND BURES DISTANCES TO QUBIT PERTURBATIONS

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We compare the sensitivity of Hilbert and Bures distances between two qubits in revealing small perturbations occurring to one of the qubits. We also analyze sensitivity in revealing perturbations to noise parameter of a depolarizing channel.

Keywords: Quantum distance; noisy channels.

The notion of distance between two quantum states is relevant in describing the degradation of a signal, the noise of a channel or the amount of information gained in a measurement. The complementary notion of similarity between quantum states is also important, as, for example, to assess state purification[1,2] or teleportation[3] protocol, as well as signal cloning,[4] remote state preparation[5] or estimation.[6] For pure quantum states the similarity between two states may be quantified by the state overlap. On the other hand, for mixed quantum states, there is not a unique definition of similarity, nor of distance, and the different quantities should be compared, in order to find the more convenient for a given application.

In this paper we focus our attention on qubit systems and compare Hilbert,[7–9] and Bures distances[10–13] in terms of their sensitivity in revealing perturbations that may occur to one of the qubits. As a matter of fact, Bures and Hilbert distances are monotone with respect to each other. However, this is no longer true for the sensitivity i.e. the rate of variation occurring after a perturbation, which depends on the degree of mixing of the qubits, as well as on the kind of perturbation. The imbalance between the sensitivity of several different figures of merit has been analyzed[14] for the depolarizing channel. Here we consider a restricted set of figures of merit, corresponding to proper distances, and evaluate sensitivity in revealing general perturbations occurring to one of the qubits. Application to the detection of perturbations to the parameter of a depolarizing channel will be discussed. Other types of noisy channels have been also analyzed.[15]

The degree of difference between two *pure* states $\varrho = |\varphi\rangle\langle\varphi|$ and $\tau = |\psi\rangle\langle\psi|$ can be quantified by

$$\mathrm{D}\,(\varrho, \tau) = \sqrt{1 - |\langle\varphi|\psi\rangle|^2}\,, \tag{1}$$

which turns out to be a distance on the set of pure quantum states. When the two states are *not* pure, then there is not a unique definition, though the different distances reduce to (1) when applied to pure states. In this paper, we address the Hilbert–Schmidt distance and the Bures distance. Let us consider two qubit states, with Bloch representation given by

$$\varrho = \frac{1}{2}(\sigma_0 + \underline{r}\cdot\underline{\sigma})\,, \quad \tau = \frac{1}{2}(\sigma_0 + \underline{t}\cdot\underline{\sigma})\,, \tag{2}$$

where $|\underline{r}|, |\underline{t}| \le 1$ and $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, σ_k being the Pauli matrices with $\sigma_0 = \mathbb{1}$. The *Hilbert–Schmidt distance* (H-distance) is defined as follows

$$\begin{aligned} \mathrm{D}_H(\varrho,\tau) &\equiv \sqrt{\frac{1}{2}\mathrm{Tr}\,[\,(\varrho-\tau)^2\,]} = \sqrt{\frac{1}{2}(\mu_\varrho + \mu_\tau) - \kappa_{\varrho\tau}}\,, \\ &= \frac{1}{2}\left[\sum_{k=1}^{3}(r_k - t_k)^2\right]^{1/2}\,, \end{aligned} \tag{3}$$

where we introduced the *purity* and the state *overlap*, namely

$$\mu_\varrho = \mathrm{Tr}[\varrho^2] = \frac{1}{2}(1 + |\underline{r}|^2), \tag{4}$$

$$\kappa_{\varrho\tau} = \mathrm{Tr}[\varrho\tau] = \frac{1}{2}(1 + \underline{r}\cdot\underline{t}). \tag{5}$$

Notice that $1/2 \le \mu_\varrho \le 1$ and $0 \le \kappa_{\varrho\tau} \le 1$. For qubits the Hilbert distance is also equal to the so-called trace distance $\mathrm{D}_T\,(\varrho,\tau) \equiv \frac{1}{2}\mathrm{Tr}\,|\,\varrho - \tau\,|$ i.e. half of the *Euclidean* distance in $\mathbb{R}^3$. Notice also that this equality no longer holds if the Hilbert space dimension is larger than 2.

The *Bures distance* (B-distance), is obtained from the fidelity F between the two states, namely,[10,12,13]

$$F(\varrho,\tau) \equiv \left(\mathrm{Tr}\sqrt{\sqrt{\varrho}\,\tau\sqrt{\varrho}}\right)^2. \tag{6}$$

Bures distance is defined as

$$\mathrm{D}_B(\varrho,\tau) \equiv \sqrt{1 - F(\varrho,\tau)}\,. \tag{7}$$

Properties of Bures distance follow from those of fidelity, which represents the maximum of the overlap $|\langle\langle\varphi|\psi\rangle\rangle|^2$ taken over all the possible purifications $|\varphi\rangle\rangle$ and $|\psi\rangle\rangle$ of ϱ and τ, respectively (Uhlmann's theorem).[13]

Focusing our attention on qubits we have

$$\begin{aligned} \mathrm{D}_B(\varrho,\tau) &= \sqrt{\frac{1}{2}\Big[1 - \underline{r}\cdot\underline{t} - \sqrt{(1-|\underline{r}|^2)(1-|\underline{t}|^2)}\Big]} \\ &= \sqrt{1 - \kappa_{\varrho\tau} - \sqrt{(1-\mu_\varrho)(1-\mu_\tau)}}\,, \end{aligned} \tag{8}$$

which can be obtained by explicitly evaluating the fidelity through the diagonalization of the operator $A = \sqrt{\sqrt{\varrho}\,\tau\sqrt{\varrho}}$, i.e. by solving the characteristic equation $A^2 - A\mathrm{Tr}\,[\,A\,] + \mathbb{1}\mathrm{Det}[A] = 0$ using Bloch representation of qubit states.

In general, H- and B-distances satisfy the relation

$$\mathrm{D}_B\left(\varrho,\tau\right)^2=\mathrm{D}_H\left(\varrho,\tau\right)^2+1-\frac{1}{2}(\mu_\varrho+\mu_\tau)-\sqrt{(1-\mu_\varrho)(1-\mu_\tau)}\,, \tag{9}$$

which implies

$$\mathrm{D}_H\left(\varrho,\tau\right)\leq\mathrm{D}_B\left(\varrho,\tau\right)\,. \tag{10}$$

In particular, if ϱ is a pure state we obtain

$$\mathrm{D}_H(\varrho,\tau)=\sqrt{\frac{1}{2}(1+\mu_\tau)-\kappa_{\varrho\tau}}, \tag{11}$$

$$\mathrm{D}_B(\varrho,\tau)=\sqrt{1-\kappa_{\varrho\tau}}. \tag{12}$$

In order to compare the effects of a perturbation on the two distances, and in turn to assess their *sensitivity* in revealing the perturbation itself, we now evaluate the distance between a fixed qubit, say ϱ, and a slightly perturbed one, τ. Up to first order one has

$$\mathrm{D}\left(\varrho,\tau+d\tau\right)=\mathrm{D}\left(\varrho,\tau\right)+\nabla\mathrm{D}\left(\varrho,\tau\right)\cdot d\underline{t}, \tag{13}$$

where

$$\tau+d\tau=\frac{1}{2}[\sigma_0+(\underline{t}+d\underline{t})\cdot\underline{\sigma}]\,, \tag{14}$$

with $|\underline{t}+d\underline{t}|\leq 1$.

In the following, we calculate the gradient ∇D for H- and B-distance and compare the quadratic norm of the two vectors, taken as a measure of the ability in revealing the occurrence of a perturbation, i.e. as a measure of sensitivity. We have

$$\nabla\mathrm{D}_H(\varrho,\tau)=\frac{\underline{t}-\underline{r}}{4\mathrm{D}_H(\varrho,\tau)}=\frac{1}{2}, \tag{15}$$

$$\nabla\mathrm{D}_B(\varrho,\tau)=\frac{\sqrt{\omega_{\varrho\tau}}\,\underline{t}-\underline{r}}{4\mathrm{D}_B(\varrho,\tau)}, \tag{16}$$

with

$$\omega_{\varrho\tau}=\frac{1-|\underline{r}|^2}{1-|\underline{t}|^2}=\frac{1-\mu_\varrho}{1-\mu_\tau}\,. \tag{17}$$

In turn, the quadratic norms of (15) and (16) read as follows:

$$|\nabla\mathrm{D}_H(\varrho,\tau)|^2=\frac{1}{4}, \tag{18}$$

$$|\nabla\mathrm{D}_B(\varrho,\tau)|^2=\frac{\omega_{\varrho\tau}(2\mu_\tau-1)+2\mu_\varrho+1-4\kappa_{\varrho\tau}\sqrt{\omega_{\varrho\tau}}}{[4\mathrm{D}_B(\varrho,\tau)]^2}. \tag{19}$$

In order to establish which distance is more sensitive, we compare the quadratic norms of distance gradients. The inequality $|\nabla \mathrm{D}_B|^2 \leq |\nabla \mathrm{D}_H|^2$ corresponds to

$$2(\mu_\varrho - 1) + 2\kappa_{\varrho\tau}(1 - \sqrt{\omega_{\varrho\tau}}) + \sqrt{\omega_{\varrho\tau}} + \frac{1}{2}(\omega_{\varrho\tau} - 3) + 2\sqrt{(1-\mu_\varrho)(1-\mu_\tau)} \leq 0\,. \tag{20}$$

If ϱ is a pure state the inequality reduces to $\kappa_{\varrho\tau} \leq \frac{3}{4}$, whereas, if ϱ is a completely mixed state, we have

$$\frac{1}{1-\mu_\tau} + 4\sqrt{2(1-\mu_\tau)} \leq 6\,. \tag{21}$$

This inequality is saturated for $\mu_\tau = \frac{1}{2}$ whereas it has no solution when $\frac{1}{2} < \mu_\tau \leq 1$. For the general case, we illustrate the regions where inequality (20) holds in Fig. 1 as a function of the initial purities and for different values of the overlap: the gray regions correspond to $|\nabla \mathrm{D}_B|^2 \leq |\nabla \mathrm{D}_H|^2$, i.e. where the H-distance is more sensitive than the B-distance.

The evolution of a given signal, as the propagation of a qubit in a real channel, is usually affected by the interaction with the environment, which makes the evolution non-unitary. A relevant example is given by the depolarizing channel, which reduces a qubit state ϱ to the completely mixed state $\sigma_0/2$ with a certain probability p. The depolarizing map reads as follows

$$\mathcal{E}_p(\varrho) = p\,\frac{\sigma_0}{2} + (1-p)\,\varrho\,. \tag{22}$$

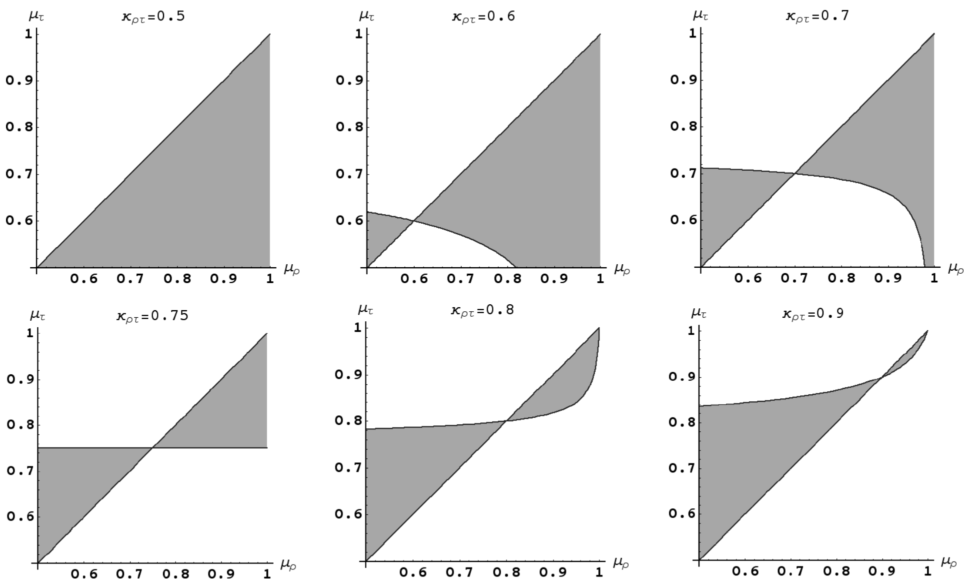

Fig. 1. Comparison of sensitivities as a function of the initial purities and different values of the overlap. The gray regions correspond to $|\nabla \mathrm{D}_B|^2 \leq |\nabla \mathrm{D}_H|^2$, i.e. where the H-distance is more sensitive than the B-distance.

Under the action of the operation (22), the Bloch vector $\underline{r}$ associate with ϱ is contracted by a factor $1-p$, i.e. $\underline{r} \to (1-p)\,\underline{r}$. The H- and B-distance between a given state ϱ and its perturbed version are given by

$$\mathrm{D}_H(\varrho, \mathcal{E}_p(\varrho)) \equiv \mathrm{D}_H(p, \mu_\varrho) = \frac{p}{2}\sqrt{2\mu_\varrho - 1}\,, \tag{23}$$

$$\mathrm{D}_B(\varrho, \mathcal{E}_p(\varrho)) \equiv \mathrm{D}_B(p, \mu_\varrho) = \sqrt{\frac{1}{2}[1-(1-p)(2\mu_\varrho - 1) - 2g(p, \mu_\varrho)]}\,, \tag{24}$$

where

$$g(p, \mu_\varrho) = \left\{\frac{1}{2}(1-\mu_\varrho)\left[1-(1-p)^2\,(2\mu_\varrho - 1)\right]\right\}^{1/2}\,. \tag{25}$$

The two distances, besides the depolarizing parameter p, depend only on the initial purity μ_ϱ. Their derivatives with respect to p quantify the sensitivity in revealing small changes in the strength of the operation $\mathcal{E}_p$. We have

$$\partial_p \mathrm{D}_H(p, \mu_\varrho) = \frac{1}{2}\sqrt{2\mu_\varrho - 1}, \tag{26}$$

$$\partial_p \mathrm{D}_B(p, \mu_\varrho) = \frac{2\mu_\varrho - 1}{4\,\mathrm{D}_B(p, \mu_\varrho)}\left[1 - \frac{(1-\mu_\varrho)(1-p)}{g(p, \mu_\varrho)}\right]\,. \tag{27}$$

The region for which $\partial_p \mathrm{D}_B \leq \partial_p \mathrm{D}_H$, namely where H-distance is more sensitive than the B-distance, corresponds to the values of p and μ satisfying the inequality

$$\frac{1}{\mathrm{D}_B(p, \mu_\varrho)}\sqrt{2\mu_\varrho - 1}\left[1 - \frac{(1-\mu_\varrho)(1-p)}{g(p, \mu_\varrho)}\right] \leq 1\,. \tag{28}$$

Inequality (28) holds for $p > p^*$ where $1/2 < p^* < 2/3$ is a monotonically decreasing function of μ, shown in Fig. 2. We conclude that H-distance is more sensitive in the high noise regime, whereas the influence of the initial purity is less pronounced.

In conclusion, we addressed the detection of small perturbations occurring to a qubit through the change of the distance from a fixed qubit. In addition, we analyzed the detection of small changes in the noise parameter of a depolarizing channel by evaluating the distance of the perturbed state to the initial one. In both cases

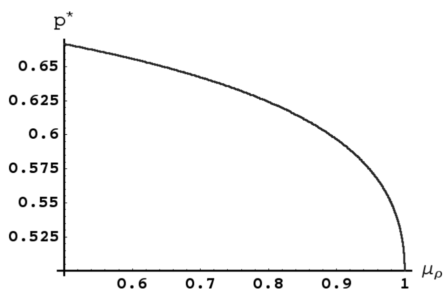

Fig. 2. The function $p^*(\mu_\varrho)$. For $p > p^*$ the H-distance is more sensitive than B-distance in revealing changes to the noise parameter of a depolarizing channel.

we have compared Bures and Hilbert distances in terms of their sensitivity to perturbation. A general relation is derived, as well as a specific bound for depolarizing channel, where the H-distance becomes more sensitive for large values of the noise parameter.

Acknowledgments

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References

1. C. H. Bennett *et al.*, *Phys. Rev. Lett.* **76** (1996) 722.
2. A. Kent, *Phys. Rev. Lett.* **81** (1998) 2839.
3. F. Verstraete and H. Verschelde, *Phys. Rev. Lett.* **90** (2003) 097901.
4. N. Gisin and S. Massar, *Phys. Rev. Lett.* **79** (1997) 2153.
5. A. X. Chen, J. H. Li and J. X. Jin, *Commun. Theor. Phys.* **43** (2005) 631.
6. M. G. A. Paris and J. Rehacek (eds.) *Quantum State Estimation*, Lecture Notes in Physics, Vol. 649 (Springer, Berlin, 2004).
7. V. Buzek and M. Hillery, *Phys. Rev. A* **54** (1996) 1844.
8. V. Buzek, G. Adam and G. Drobny, *Phys. Rev A* **54** (1996) 804.
9. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Press, Cambridge, 2000).
10. M. Hubner, *Phys. Lett. A* **163** (1992) 239.
11. R. Josza, *J. Mod. Opt.* **41** (1994) 2315.
12. V. Belavkin, *Rep. Math. Phys.* **55** (2005) 61.
13. A. Uhlmann, *Rep. Math. Phys.* **9** (1976) 273.
14. N. A. Peters, T.-C. Wei and P. G. Kwiat, *Phys. Rev. A* **70** (2004) 052309.
15. S. Salvini, S. Olivares and M. G. A. Paris, *Laser Phys.* **17** (2007) 552.