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# Distances between Qubits and Sensitivity to Perturbations

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**Abstract**—We address the sensitivity of Bures and Hilbert distances between two qubits in revealing (i) small perturbations occurring to one of the qubits and (ii) small changes of the noise parameter in a noisy channel. A general relation is derived, as well as specific bounds for perturbations induced by relevant noisy quantum channels.

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## 1. INTRODUCTION

The issue of quantifying the distance between two quantum states is a relevant topic in the description of quantum information processing. As a matter of fact, the notion of distance is needed whenever one is faced with the problems of describing the degradation of a signals, the noise of a channel, or the amount of information gained in a measurement. In addition, the complementary notion of similarity between quantum states is essential to assess purification [1] or teleportation [2] protocols, as well as signal cloning [3], remote state preparation [4], and state estimation [5].

If one restricts attention to signals encoded in pure quantum states, then the similarity between two states coincides with the scalar product in the Hilbert space, i.e., with the overlap between the wave vectors describing the two signals. On the other hand, when one deals with mixed quantum states, there is not a unique definition of similarity or distance, and different quantities should be compared in order to find the more convenient one for a given application.

In this paper, we focus our attention on qubit systems and compare a few relevant definitions of distance, that is, Hilbert [6], trace [7], and Bures distance [8–11], in terms of their sensitivity with respect to perturbation that may occur to one of the qubits. As it is well known, Bures and Hilbert distances are monotonic with respect to each other. As we will see, this is no longer true for the sensitivity, i.e., the rate of variation occurring after a perturbation, which depends on the degree of mixing of the involved qubits, as well as on the kind of perturbation. The imbalance between the sensitivity of several different figures of merit has been analyzed [12] for the depolarizing channel. Here we consider a restricted set of figures of merit, corresponding to proper distances, and evaluate sensitivity in revealing general perturbations occurring to one of the qubit and the noise introduced by specific channels.

The paper is structured as follows. In the next section, we establish notation and introduce the different distances in terms of the Bloch vector and of the invariants of the density matrix. In Section 3, we compare the sensitivity of the distances with respect to a general perturbation occurring to one of the qubits and derive inequalities in terms of the mixing of the other one. In Section 4, we consider specific perturbations coming from the evolution in relevant noisy quantum channels and evaluate the sensitivity of the distances in revealing small changes in the noise parameter. In particular, we consider the depolarizing channel and the phase- and amplitude-damping channels. Finally, Section 5 closes the paper by summarizing results.

## 2. DISTANCES BETWEEN QUBITS

The degree of difference between two *pure* states $\varrho = |\varphi\rangle\langle\varphi|$ and $\tau = |\psi\rangle\langle\psi|$ can be quantified by

$$D(\varrho, \tau) = \sqrt{1 - |\langle\varphi|\psi\rangle|^2}, \tag{1}$$

which turns out to be a distance on the set of pure quantum states. When the two states are *not pure*, then there is no unique definition, though the different distances reduce to (1) when applied to pure states. In this paper, we address different definitions for a distance among qubits: the trace distance, the Hilbert–Schmidt distance, and the Bures distance obtained from the fidelity between the two states.

The first distance we consider, the *trace distance* (T-distance), is defined as

$$D_T(\varrho, \tau) \equiv \frac{1}{2}\mathrm{Tr}|\varrho - \tau| = \frac{1}{2}\mathrm{Tr}\sqrt{(\varrho - \tau)^2}. \tag{2}$$

This definition resembles the Kolmogorov distance $D_K(\{p_m\}, \{q_m\}) = \sum_m |p_m - q_m|$ between two probability distributions. Indeed, if we consider $\{p_m\}$ and $\{q_m\}$ as coming from the measurement of a generic positive-operator valued measure (POVM) $\{\Pi_m\}$ on the two states, i.e., $p_m \equiv \mathrm{Tr}[\varrho\Pi_m]$ and $q_m \equiv \mathrm{Tr}[\tau\Pi_m]$, then the trace distance between $\varrho$ and $\tau$ corresponds to the maximum of $D_K$ taken over all the POVMs, namely $D_T(\varrho, \tau) = \max_{\{\Pi_m\}} D_K(\{p_m\}, \{q_m\})$ [7]. The trace distance is invariant under unitary transformations performed on the two states, whereas general quantum operations are contractive, namely, decrease the T-distance.

Let us now consider the Bloch representation of qubits:

$$\varrho = \frac{1}{2}(\sigma_0 + \underline{r} \cdot \underline{\sigma}), \quad \tau = \frac{1}{2}(\sigma_0 + \underline{t} \cdot \underline{\sigma}), \qquad (3)$$

with $|\underline{r}|, |\underline{t}| \le 1$ and $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\sigma_k$ being the Pauli matrices with $\sigma_0 = \mathbb{1}$. Using Eq. (3), one has $(\varrho - \tau)^2 = \frac{1}{4}|\underline{r} - \underline{t}|^2\sigma_0$, and Eq. (2) becomes

$$D_T(\varrho, \tau) = \frac{1}{2}|\underline{r} - \underline{t}| = \frac{1}{2}\left[\sum_{k=1}^{3}(r_k - t_k)^2\right]^{1/2}, \qquad (4)$$

i.e., half of the *Euclidean* distance in $\mathbb{R}^3$.

The *Hilbert–Schmidt distance* (H-distance) is written as follows:

$$D_H(\varrho, \tau) \equiv \sqrt{\frac{1}{2}\mathrm{Tr}[(\varrho - \tau)^2]} = \sqrt{\frac{1}{2}(\mu_\varrho + \mu_\tau) - \kappa_{\varrho\tau}}, \quad (5)$$

where we introduced the *purity* and the state *overlap*, namely,

$$\mu_\varrho = \mathrm{Tr}[\varrho^2] = \frac{1}{2}(1 + |\underline{r}|^2), \qquad (6)$$

$$\kappa_{\varrho\tau} = \mathrm{Tr}[\varrho\tau] = \frac{1}{2}(1 + \underline{r} \cdot \underline{t}). \qquad (7)$$

Notice that $1/2 \le \mu_\varrho \le 1$ and $0 \le \kappa_{\varrho\tau} \le 1$. In the case of qubits, using Eqs. (4), (6), and (7) one easily obtains that

$$D_H(\varrho, \tau) = D_T(\varrho, \tau). \qquad (8)$$

From now on we only refer to the H-distance. Notice that the Eq. (8) no longer holds if the Hilbert space dimension is larger than 2.

Finally, the *Bures distance* (B-distance), is obtained from the so-called fidelity $F$ between the two states, namely [8, 10, 11],

$$F(\varrho, \tau) \equiv (\mathrm{Tr}\sqrt{\sqrt{\varrho}\tau\sqrt{\varrho}})^2. \qquad (9)$$

The Bures distance is defined as

$$D_B(\varrho, \tau) \equiv \sqrt{1 - F(\varrho, \tau)}. \qquad (10)$$

The properties of Bures distance follow from those of fidelity, which represents the maximum of the overlap $|\langle\langle\varphi|\psi\rangle\rangle|^2$ taken over all the possible purifications $|\varphi\rangle\rangle$ and $|\psi\rangle\rangle$ of $\varrho$ and $\tau$, respectively (Uhlmann's theorem) [11]. By means of this theorem we can deduce the fidelity properties: it is a symmetric, nonnegative, continuous, concave function of the states, and it is equal to unity if and only if the states do coincide.

Focusing our attention on qubits and using the Bloch representation, we have

$$\begin{aligned} D_B(\varrho, \tau) &= \sqrt{\frac{1}{2}[1 - \underline{r} \cdot \underline{t} - \sqrt{(1 - |\underline{r}|^2)(1 - |\underline{t}|^2)}]} \\ &= \sqrt{1 - \kappa_{\varrho\tau} - \sqrt{(1 - \mu_\varrho)(1 - \mu_\tau)}}, \end{aligned} \qquad (11)$$

which can be obtained by explicitly evaluating the fidelity through the diagonalization of the operator $A = \sqrt{\sqrt{\varrho}\tau\sqrt{\varrho}}$, by solving the characteristic equation $A^2 - A\mathrm{Tr}[A] + \mathbb{1}\mathrm{Det}[A] = 0$ using Bloch representation of qubit states.

In general, H- and B-distances satisfy the relation

$$\begin{aligned} D_B(\varrho, \tau)^2 &= D_H(\varrho, \tau)^2 + 1 - \frac{1}{2}(\mu_\varrho + \mu_\tau) \\ &\quad - \sqrt{(1 - \mu_\varrho)(1 - \mu_\tau)}, \end{aligned} \qquad (12)$$

which implies

$$D_H(\varrho, \tau) \le D_B(\varrho, \tau). \qquad (13)$$

In particular, if $\varrho$ is a pure state we obtain

$$D_H(\varrho, \tau) = \sqrt{\frac{1}{2}(1 + \mu_\tau) - \kappa_{\varrho\tau}}, \qquad (14)$$

$$D_B(\varrho, \tau) = \sqrt{1 - \kappa_{\varrho\tau}}. \qquad (15)$$

## 3. SENSITIVITY TO PERTURBATIONS

In order to compare the effects of a perturbation on the two distances, and in turn to assess their *sensitivity* in revealing the perturbation itself, we now evaluate the distance between a fixed qubit, say $\varrho$, and a slightly perturbed one, $\tau$. Up to first order, one has

$$D(\varrho, \tau + d\tau) = D(\varrho, \tau) + \nabla D(\varrho, \tau) \cdot d\underline{t}, \qquad (16)$$

where

$$\tau + d\tau = \frac{1}{2}[\sigma_0 + (\underline{t} + d\underline{t}) \cdot \underline{\sigma}], \qquad (17)$$

with $|\underline{t} + d\underline{t}| \le 1$.

In the following, we calculate the gradient $\nabla D$ for H- and B-distance and compare the quadratic norm of the two vectors, taken as a measure of the ability in revealing the occurrence of a perturbation, i.e., as a measure of sensitivity. We have

$$\nabla D_H(\varrho, \tau) = \frac{\underline{t} - \underline{r}}{4D_H(\varrho, \tau)} = \frac{1}{2} \qquad (18)$$

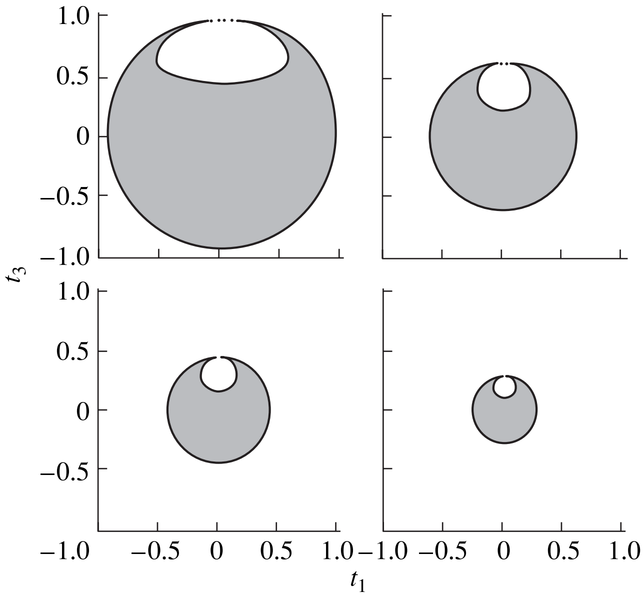


**Fig. 1.** Plot of the region defined by the inequality (24), with $\underline{r} = (0, 0, 2a^2 - 1)$ and $\underline{t} = (t_1, 0, t_3)$, for different values of the parameter $a$: (a) $a = 0.99$, (b) $a = 0.9$, (c) $a = 0.85$, and (d) $a = 0.8$. The gray region corresponds to $|\nabla D_B|^2 \le |D_H|^2$, i.e., to regimes where the H-distance is more sensitive than B-distance to perturbations.

and

$$\nabla D_B(\varrho, \tau) = \frac{\sqrt{\omega_{\varrho\tau}}\underline{t} - \underline{r}}{4D_B(\varrho, \tau)}, \tag{19}$$

with

$$\omega_{\varrho\tau} = \frac{1 - |\underline{r}|^2}{1 - |\underline{t}|^2} = \frac{1 - \mu_\varrho}{1 - \mu_\tau}. \tag{20}$$

In turn, the quadratic norms of (18) and (19) read as follows:

$$|\nabla D_H(\varrho, \tau)|^2 = \frac{1}{4}, \tag{21}$$

$$|\nabla D_B(\varrho, \tau)|^2 = \frac{\omega_{\varrho\tau}(2\mu_\tau - 1) + 2\mu_\varrho + 1 - 4\kappa_{\varrho\tau}\sqrt{\omega_{\varrho\tau}}}{[4D_B(\varrho, \tau)]^2}. \tag{22}$$

In order to establish which distance is more sensitive, we compare the quadratic norms of distance gradients. The inequality

$$|\nabla D_B|^2 \le |\nabla D_H|^2 \tag{23}$$

corresponds to

$$2(\mu_\varrho - 1) + 2\kappa_{\varrho\tau}(1 - \sqrt{\omega_{\varrho\tau}}) + \sqrt{\omega_{\varrho\tau}} + \frac{1}{2}(\omega_{\varrho\tau} - 3) + 2\sqrt{(1 - \mu_\varrho)(1 - \mu_\tau)} \le 0. \tag{24}$$

If $\varrho$ is a pure state, inequality (24) reduces to

$$\kappa_{\varrho\tau} < \frac{3}{4}, \tag{25}$$

whereas, if $\varrho$ is a completely mixed state, we have

$$\frac{1}{1 - \mu_\tau} + 4\sqrt{2(1 - \mu_\tau)} \le 6. \tag{26}$$

This inequality is saturated for $\mu_\tau = \frac{1}{2}$, whereas it has no solution when $\frac{1}{2} < \mu_\tau \le 1$.

In the general case, using the rotational symmetry of the Bloch sphere and without loss of generality, we may assume the state $\varrho$ having components only along the third axis of the sphere, i.e.,

$$\begin{aligned} \varrho &= a^2|0\rangle\langle 0| + (1 - a^2)|1\rangle\langle 1| \\ &= \frac{1}{2}[\sigma_0 + (2a^2 - 1)\sigma_3], \end{aligned} \tag{27}$$

where $|0\rangle$ and $|1\rangle$ are eigenstates of $\sigma_3$. Using invariance of inequality (24) under rotations we may set $t_2 = 0$ and express its solution in a simple way. In Fig. 1 we plot the region delimited by Eq. (24) with $\underline{r} = (0, 0, 2a^2 - 1)$ and $\underline{t} = (t_1, 0, t_3)$ for different values of the parameter $a$: the gray region corresponds to $|\nabla D_B|^2 \le |\nabla D_H|^2$, i.e., to the regime in which the H-distance is more sensitive than the B-distance.

## 4. PERTURBATIONS IN A NOISY CHANNEL

The evolution of a given signal, as the propagation of a qubit in a real channel, is generally affected by the presence of other systems, globally referred to as the environment. The propagation of the system of interest becomes nonunitary, a sign of the noisy effects of the environment. The most general form of the resulting *quantum operation* involves a set of operators $E_k$ (operation elements) and is given by

$$\mathscr{E}(\varrho) = \sum_k E_k \varrho E_k^\dagger.$$

Trace-preserving operations require $\sum_k E_k^\dagger E_k = \sigma_0$ and turn quantum states $\varrho$ into quantum states $\mathscr{E}(\varrho)$. As a matter of fact, different physical processes can give rise to the same system dynamics, i.e., the same operations. In turn, the set of operations elements is not uniquely determined. Using this degree of freedom, we will describe noisy channels for qubit systems in terms of Pauli operators.

In the following, the perturbations introduced by relevant quantum operations on a qubit, which act by deforming or rotating the Bloch sphere, will be addressed with the aim of assessing the performances of H- and B-distances in revealing small changes in the perturbation parameter. In particular, among the possible noisy channels [13], we consider the depolarizing, the amplitude-damping, and the phase-damping channels.

### *4.1. Depolarizing Channel*

The depolarizing channel reduces a qubit state $\varrho$ to a completely mixed state $\sigma_0/2$ with a certain probability $p$. The operation element for the depolarizing channel are given by $E_0 = \sqrt{1-\frac{3}{4}p}\sigma_0$ and $E_k = \frac{1}{2}\sqrt{p}\sigma_k$, corresponding to the evolution

$$\mathscr{E}_p(\varrho) = p\frac{\sigma_0}{2} + (1-p)\varrho. \quad (28)$$

Under the action of operation (28), the Bloch vector $\underline{r}$ associated with $\varrho$ is contracted by a factor of $1-p$, i.e.,

$$\underline{r} \longrightarrow (1-p)\underline{r}. \quad (29)$$

The H- and B-distance between a given state $\varrho$ and its perturbed version are given by

$$\mathrm{D}_H(\varrho, \mathscr{E}_p(\varrho)) \equiv \mathrm{D}_H(p, \mu_\varrho) = \frac{p}{2}\sqrt{2\mu_\varrho - 1}, \quad (30)$$

$$\mathrm{D}_B(\varrho, \mathscr{E}_p(\varrho)) \equiv \mathrm{D}_B(p, \mu_\varrho) = \sqrt{\frac{1}{2}[1 + (1-p)(2\mu_\varrho - 1) - 2g(p, \mu_\varrho)]}, \quad (31)$$

where

$$g(p, \mu_\varrho) = \left\{\frac{1}{2}(1-\mu_\varrho)[1-(1-p)^2(2\mu_\varrho - 1)]\right\}^{1/2}. \quad (32)$$

The two distances, in addition to the depolarizing parameter $p$, depend only on the initial purity $\mu_\varrho$. Their derivatives with respect to $p$ quantify the sensitivity in revealing small changes in the strength of the operation $\mathscr{E}_p$. We have

$$\partial_p \mathrm{D}_H(p, \mu_\varrho) = \frac{1}{2}\sqrt{2\mu_\varrho - 1}, \quad (33)$$

$$\partial_p \mathrm{D}_B(p, \mu_\varrho) = \frac{2\mu_\varrho - 1}{4\mathrm{D}_B(p, \mu_\varrho)}\left[1 - \frac{(1-\mu_\varrho)(1-p)}{g(p, \mu_\varrho)}\right]. \quad (34)$$

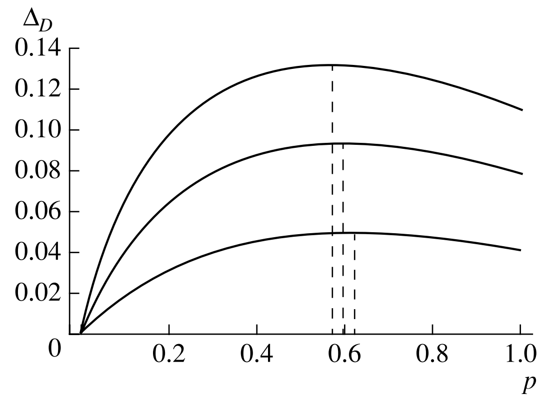


**Fig. 2.** Plot of $\Delta_D(p, \mu_\varrho) = \mathrm{D}_B(p, \mu_\varrho) - \mathrm{D}_H(p, \mu_\varrho)$ as a function of $p$ and different values of the purity: from top to bottom $\mu_\varrho$ = 0.95, 0.90, and 0.80. The dashed vertical lines refer to the values $\bar{p}$ where the maximum of $\Delta_D(p, \mu_\varrho)$ occurs: when $p \geq \bar{p}$ the H-distance is more sensitive than the B-distance.

Let us now introduce the positive quantity (recall that $\mathrm{D}_B(p, \mu_\varrho) \geq \mathrm{D}_H(p, \mu_\varrho)$):

$$\Delta_D(p, \mu_\varrho) = \mathrm{D}_B(p, \mu_\varrho) - \mathrm{D}_H(p, \mu_\varrho). \quad (35)$$

Regions where $\Delta_D(p, \mu_\varrho)$ increases correspond to $\partial_p \mathrm{D}_B(p, \mu_\varrho) \geq \partial_p \mathrm{D}_H(p, \mu_\varrho)$ i.e., to regions where B-distance is more sensitive in revealing small changes in the perturbations parameter $p$. In Fig. 2 we plot $\Delta_D(p, \mu_\varrho)$ as a function of $p$ and for different values of the initial purity $\mu_\varrho$. As is apparent from the plot, there always exists a threshold value $\bar{p}$ above which the H-distance becomes more sensitive. The threshold value depends on the initial purity, slightly increasing for decreasing $\mu_\varrho$.

### *4.2. Amplitude Damping*

The process of energy dissipation is described by the following quantum operation:

$$\mathscr{E}_\gamma(\varrho) = E_0 \varrho E_0 + E_1 \varrho E_1, \quad (36)$$

which is usually referred to as the amplitude-damping channel. The operation elements are given by $E_0 = \frac{1}{2}(1 + \sqrt{1-\gamma})\sigma_0 + \frac{1}{2}(1 - \sqrt{1-\gamma})\sigma_3$, and $E_1 = \frac{1}{2}\sqrt{\gamma}(\sigma_1 - i\sigma_2)$, $\gamma$ being probability of losing an energy quantum. Notice that only the state $|0\rangle$ is left unchanged by the evolution (36). The effect of $\mathscr{E}_\gamma$ corresponds to the following Bloch vector transformation:

$$\underline{r} \longrightarrow (r_1\sqrt{1-\gamma}, r_2\sqrt{1-\gamma}, \gamma + (1-\gamma)r_3). \quad (37)$$

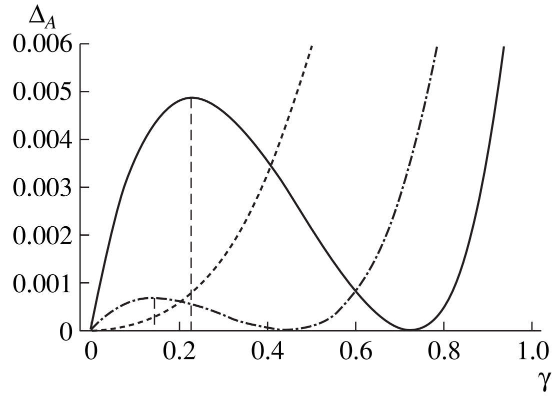


**Fig. 3.** Plot of $\Delta_A(\gamma, \mu_\varrho, r_3) = D_B(\gamma, \mu_\varrho, r_3) - D_H(\gamma, \mu_\varrho, r_3)$ as a function of $\gamma$ with $r_3 = 0.4$ and different values of the purity: $\mu_\varrho = 0.95$ (solid line), 0.9 (dot-dashed line), and 0.8 (dashed line). The dashed vertical lines refer to the values $\bar{\gamma}$ for which the maximum of $\Delta_A(\gamma, \mu_\varrho, r_3)$ occurs. See the text for details.

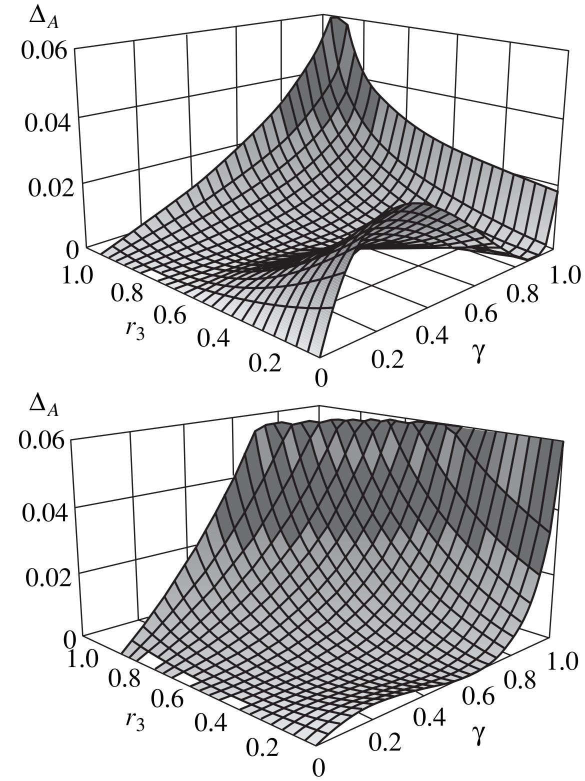


**Fig. 4.** Plot of $\Delta_A(\gamma, \mu_\varrho, r_3) = D_B(\gamma, \mu_\varrho, r_3) - D_H(\gamma, \mu_\varrho, r_3)$ as a function of $\gamma$ and $r_3$ with $\mu_\varrho = 0.95$ (left) and $\mu_\varrho = 0.95$ (right). Notice that as $r_3$ increases the minimum disappears.

The H-distance between a given qubit $\varrho$ and its perturbed version is now given by $D_H(\varrho, \mathscr{E}_\gamma(\varrho)) \equiv D_H(\gamma, \mu_\varrho, r_3)$ where

$$D_H(\gamma, \mu_\varrho, r_3) = \frac{1}{2}\sqrt{(1-\sqrt{1-\gamma})^2(2\mu_\varrho - 1 - r_3^2) + \gamma^2(r_3-1)^2}, \quad (38)$$

whereas the B-distance reads as follows:

$$D_B(\varrho, \mathscr{E}_\gamma(\varrho)) \equiv D_B(\gamma, \mu_\varrho, r_3) = \left\{\frac{1}{2}[1 - \sqrt{1-\gamma}(2\mu_\varrho - 1 - r_3^2), + (1-\gamma)r_3^2 - \gamma r_3] - f(\gamma, \mu_\varrho, r_3)\right\}^{1/2}, \quad (39)$$

with

$$f(\gamma, \mu_\varrho, r_3) = \sqrt{(1-\mu_\varrho)} \times \sqrt{1 - (1-\gamma)(2\mu_\varrho - 1 - r_3^2) - [\gamma + (1-\gamma)r_3]^2}. \quad (40)$$

Notice that the two distances depend explicitly on a third component $r_3$ of the initial Bloch vector. The corresponding derivatives are given by

$$\partial_\gamma D_H(\gamma, \mu_\varrho, r_3) = \frac{1}{8D_H(\gamma, \mu_\varrho, r_3)}\left\{2\gamma(1-r_3)^2 + \frac{(2\mu_\varrho - 1 - r_3^2)(1-\sqrt{1-\gamma})}{\sqrt{1-\gamma}}\right\} \quad (41)$$

and

$$\partial_\gamma D_B(\gamma, \mu_\varrho, r_3) = \frac{1}{4D_B(\gamma, \mu_\varrho, r_3)}\left\{\frac{2\mu_\varrho - 1 - r_3^2}{2\sqrt{1-\gamma}} - (1-r_3)r_3 - \frac{(1-\mu_\varrho)[2\mu_\varrho - 1 - 2r_3 - r_3^2 - 2\gamma(1-r_3)^2]}{\sqrt{2}f(\gamma, \mu_\varrho, r_3)}\right\}. \quad (42)$$

By introducing the positive quantity

$$\Delta_A(\gamma, \mu_\varrho, r_3) = D_B(\gamma, \mu_\varrho, r_3) - D_H(\gamma, \mu_\varrho, r_3), \quad (43)$$

we may compare the sensitivity of the two distances as a function of the involved parameters. For a given value of $r_3$ (see Fig. 3), one finds an interval of $\gamma$ values in which the H-distance is more sensitive than the B one if the initial purity $\mu_\varrho$ is above a threshold value, which itself depends on $r_3$. As the initial purity falls, below the threshold the B-distance is always more sensitive, independent of the value of the noise parameter $\gamma$. The behavior of $\Delta_A(\gamma, \mu_\varrho, r_3)$ as a function of both $r_3$ and $\gamma$ is shown in Fig. 4, where the quantity is plotted for two different values of the purity: we can see that, as $r_3$ increases, the region of $\gamma$ values in which the H-distance is more sensitive than the B one is reduced and, finally, disappears.

### 4.3. Phase Damping

This channel describes the loss of the relative phase between the energy eigenstates without amplitude damping. The quantum operation is given by

$$\mathscr{E}_\zeta(\varrho) = E_0\varrho E_0 + E_1\varrho E_1, \tag{44}$$

with the operation elements given by $E_0 = \frac{1}{2}(1 + \sqrt{1-\zeta})\sigma_0 + \frac{1}{2}(1 - \sqrt{1-\zeta})\sigma_3$ and $E_1 = \frac{1}{2}\sqrt{\zeta}(\sigma_0 - \sigma_3)$. $\zeta$ denotes the probability of the transition. As in the previous case, only the eigenstate $|0\rangle$ is left unchanged. Because of operation (44), the Bloch vector transforms as

$$\underline{r} \longrightarrow (r_1\sqrt{1-\zeta}, r_2\sqrt{1-\zeta}, r_3), \tag{45}$$

and, therefore, the H-distance is given by

$$\begin{aligned} D_H(\varrho, \mathscr{E}_\zeta(\varrho)) &\equiv D_H(\zeta, \mu_\varrho, r_3) \\ &= \frac{1-\sqrt{1-\zeta}}{2}\sqrt{2\mu_\varrho - 1 - r_3^2}, \end{aligned} \tag{46}$$

whereas the B-distance reads $D_B(\varrho, \mathscr{E}_\zeta(\varrho)) \equiv D_B(\zeta, \mu_\varrho, r_3)$, where

$$\begin{aligned} &D_B(\zeta, \mu_\varrho, r_3) \\ &= \sqrt{\frac{1}{2}[1 - r_3^2 - \sqrt{1-\zeta}(2\mu_\varrho - 1 - r_3^2)] - h(\zeta, \mu_\varrho, r_3)}, \end{aligned} \tag{47}$$

with

$$\begin{aligned} &h(\zeta, \mu_\varrho, r_3) \\ &= \sqrt{(1-\mu_\varrho)}\sqrt{(1-\mu_\varrho) + \frac{\zeta}{2}(2\mu_\varrho - 1 - r_3^2)}. \end{aligned} \tag{48}$$

In turn, derivatives with respect to $\zeta$ are

$$\partial_\zeta D_H(\zeta, \mu_\varrho, r_3) = \frac{1}{4}\sqrt{\frac{2\mu_\varrho - 1 - r_3^2}{1-\zeta}}, \tag{49}$$

and

$$\begin{aligned} \partial_\zeta D_B(\zeta, \mu_\varrho, r_3) = \frac{1}{8D_B(\zeta, \mu_\varrho, r_3)}\Bigg[&\frac{2\mu_\varrho - 1 - r_3^2}{\sqrt{1-\zeta}} \\ &- \frac{(1-\mu_\varrho)(2\mu_\varrho - 1 - r_3^2)}{2h(\zeta, \mu_\varrho, r_3)}\Bigg]. \end{aligned} \tag{50}$$

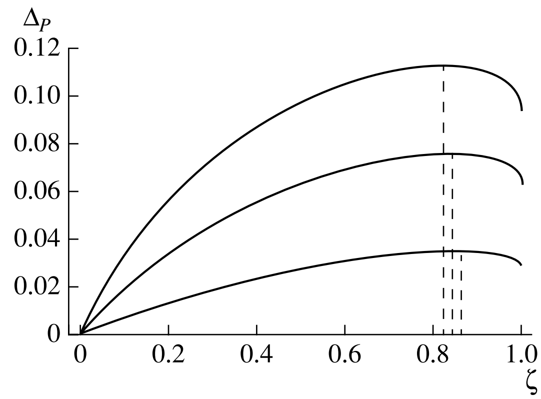


**Fig. 5.** Plot of $\Delta_P(\zeta, \mu_\varrho, r_3) = D_B(\zeta, \mu_\varrho, r_3) - D_H(\zeta, \mu_\varrho, r_3)$ as a function of $\zeta$ with $r_3 = 0.4$ and different values of the purity: from top to bottom $\mu_\varrho = 0.95$, 0.90, and 0.80. The dashed vertical lines refer to the values $\bar{\zeta}$ for which the maximum of $\Delta_P(\zeta, \mu_\varrho, r_3)$ occurs. See the text for details.

In Fig. 5, we plot the difference

$$\Delta_P(\zeta, \mu_\varrho, r_3) = D_B(\zeta, \mu_\varrho, r_3) - D_H(\zeta, \mu_\varrho, r_3) \tag{51}$$

as a function of the parameter $\zeta$ with $r_3 = 0.4$ and different values of the purity. As in the case of the depolarizing channel (see Fig. 2), for each couple of $r_3$ and $\mu_\varrho$ we can identify two different regions divided by a threshold value $\bar{\zeta}$: if $\zeta \geq \bar{\zeta}$, then the H-distance is more sensitive than the B-distance.

## 5. CONCLUSIONS

In this paper, we have addressed two issues: (i) the detection of generic (small) perturbation of the Bloch vector occurring to a qubit through the change of the distance to a fixed qubit and (ii) the detection of small changes in the noise parameter of a channel by evaluating the distance of the perturbed state to the initial one. In both cases, we compared the Bures and Hilbert distance in terms of their sensitivity to perturbations. A general relation is derived, as well as specific bounds for perturbations induced by relevant noisy quantum channels. We found that, for the depolarizing channel and the phase-damping channel, H-distance becomes more sensitive for high noise and high degree of mixing, whereas for the amplitude-damping channel the B-distance is always more sensitive, as far as the mixing exceeds a threshold depending on the third component of the initial Bloch vector.


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