

Bounds to precision for quantum interferometry with Gaussian states and operations

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We address high-precision measurements by active and passive interferometric schemes based on Gaussian states and operations. In particular, we look for the best states to be injected into their ports according to the quantum Cramér–Rao bound, i.e., maximizing the quantum Fisher information over all the involved parameters, given a constraint on the overall mean number of photons entering into the interferometer. We found that for passive interferometers involving only beam splitters, the optimal input leading to Heisenberg scaling is a pair of identical squeezed-coherent states with at most one-third of the total energy employed in squeezing. For active interferometers involving optical amplifiers, input coherent signals are enough to achieve Heisenberg scaling, given an optimal value of the amplification gain. For passive schemes our results clarify the role of squeezing in improving both the reference phase and the signal phase of an interferometer. © 2015 Optical Society of America

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1. INTRODUCTION

Optical interferometry is among the most precise measurement techniques available in physics. Therefore, it got involved in the most challenging task of modern cosmology, the direct detection of gravitational waves (GWs) [1]. In an interferometer the space-time ripples induced by a GW change the optical path of the light beams traveling along the setup, thus leading to a measurable fluctuation of the phase shift. The perturbations induced by a GW are extremely small and interferometers involved in GW detection must resolve tiny phase shifts with high precision. In the last decades, the technological developments of laser physics and optical components permitted mechanical and thermal noise, i.e., the noise of classical origin, to be largely suppressed. Yet, interferometric precision is limited by the quantum nature of the radiation field used to probe the presence of GWs.

The general interferometric problem is that of maximizing the precision in the estimation of phase-shift fluctuations at a fixed energy of the involved light beams. Actually, fluctuations associated with the very quantum nature of light pose bounds to the precision. On the other hand, the nonclassical features of the probe beams may also be exploited to improve these bounds compared to the corresponding classical ones, thus leaving room for quantum enhanced interferometry. Much work has been devoted to investigating the ultimate bounds to the

precision in estimating the phase shift in an interferometer and to find the states that achieve these limits [2–18]. Overall, two main lines of research have been pursued: the one devoted to the use of finite superpositions of number states, e.g., the so-called NOON states [19]; the other involving Gaussian states and operations, e.g., squeezing [20]. While NOON states may lead to optimal or near-optimal performances at a fixed number of particles, they are not considered of practical interest because they are very sensitive to detection noise and optical losses and their generation is still experimentally challenging.

For these reasons we focus on interferometric schemes involving Gaussian states and operations, including active and passive elements, and analyze in detail the ultimate quantum bounds to precision. Here the number of particles is not fixed and optimization is performed at a fixed energy of the beams, i.e., fixing the average number of photons. This is relevant, since the energy is directly linked to the radiation pressure, which should be under control in high precision optical interferometers.

An interferometer is a device where the phase shift occurring to one or more modes of the radiation field are mapped onto the variations of some measurable quantity at the output. In other words, interferometers aim at monitoring the stability of a given configuration against perturbations rather than at estimating the value of an unknown phase shift. The general

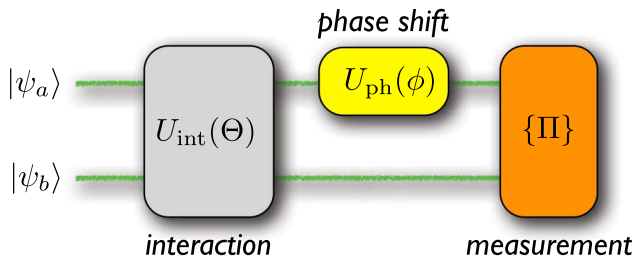


Fig. 1. Schematic diagram of a two-mode interferometric setup. The two modes, prepared in the factorized state $|\psi_a\rangle$ and $|\psi_b\rangle$, are coupled by the unitary operator $U_{\text{int}}(\Theta)$. Then, one of the modes is phase-shifted by an amount ϕ and, finally, the two-mode state undergoes a measurement described by the POVM $\{\Pi\}$.

interferometric scheme we consider is depicted in Fig. 1. It may be described as a unitary operation $U_{\text{int}}(\Theta)$ acting on two input modes $k = a, b$, $[k, k^\dagger] = 1$, excited in the two-mode initial state $|\psi_a\rangle \otimes |\psi_b\rangle$, followed by a phase shift $U_{\text{ph}}(\phi) = U(\phi) \otimes \mathbb{I}$, $U(\phi) = \exp(-i\phi a^\dagger a)$ imposed onto one of them. Finally, a suitable joint measurement, described by a two-mode positive operator-valued measure (POVM), $\{\Pi\}$, is performed on the evolved modes. The vector Θ contains the information about the parameters characterizing the interferometer. Overall, the state before the measurement stage is

$$|\Psi(\Theta, \phi)\rangle = U_{\text{ph}}(\phi)U_{\text{int}}(\Theta)|\psi_a\rangle \otimes |\psi_b\rangle.$$

It depends on the phase shift ϕ and on the specific features of the interferometer through the set of parameters contained in the vector Θ .

The sensitivity of an interferometer in revealing fluctuations of the phase shift ϕ is governed by the quantum Fisher information (QFI) of the family of states $|\Psi(\Theta, \phi)\rangle$. In recent years many efforts have been made to obtain analytical expressions of the QFI for single- and multimode states in the framework of single- and multi-parameter estimation [21–24] and interferometry [25]. However, given the QFI of a family of states, it is not always straightforward to find the optimal state maximizing it. In our interferometric scenario, upon evaluating analytically the QFI we will be able to optimize the interferometer over all the involved parameters and to find the optimal input states. As we will see, we found that squeezing is indeed a resource and then evaluating exactly how much squeezing is needed to achieve optimal performances in the different configurations. Our analysis extends and generalizes previous results [26], where a squeezed vacuum has been shown to be the best state for the second port of a passive interferometer, when the first port is fed by a laser source, i.e., a coherent state.

In this paper, we investigate the ultimate quantum limit to the sensitivity of interferometers based on Gaussian states and operations. In particular, we focus on two classes of interferometers, sometimes referred to as $SU(2)$ and $SU(1, 1)$ interferometers, respectively [3]. In the first case the interaction operator $U_{\text{int}}(\Theta)$ belongs to the $SU(2)$ group and describes the action of passive devices, e.g., beam splitters. The energy is thus conserved during the evolution. Conversely, $SU(1, 1)$ interferometers also involve active devices such as optical amplifiers, which add some energy to the system besides making the two

modes interfere. As we will see, our analysis indicates that for passive interferometers the optimal input leading to Heisenberg scaling is a pair of identical squeezed-coherent states with at most one-third of the total energy employed in squeezing, whereas for active schemes involving optical amplifiers input coherent signals are enough to achieve Heisenberg scaling, given an optimal value of the amplification gain.

The paper is structured as follows. In the next section we briefly review the quantum Cramér–Rao bound governing the ultimate precision achievable in quantum interferometry. In Section 3 we focus on passive interferometers, whereas in Section 4 we devote attention to active ones. Section 5 closes the paper by discussing results, with some concluding remarks.

2. QUANTUM CRAMÉR–RAO BOUND

The quantum Cramér–Rao bound [27–30]

$$\delta\phi^2 \geq \frac{1}{MH(\phi)}, \quad (1)$$

establishes that apart from the statistical scaling (M is the number of measurements), the variance of any unbiased estimator of the phase shift, i.e., the minimum detectable fluctuation in the estimation of the parameter ϕ , is bound by the inverse of the QFI $H(\phi)$ [31,32]. The quantum Cramér–Rao bound is a convenient tool used to evaluate the performance of an interferometer since it applies to all quantum measurements on the two modes, and to all procedures for estimating the phase shifts from the measurement results. In general, precision depends on the value of the phase shift itself, since the QFI is given by

$$H(\phi) = \text{Tr}[R_\phi L_\phi^2],$$

where R_ϕ is the two-mode state at the output of the interferometer and L_ϕ , the so-called symmetric logarithmic derivative, is the operator satisfying the Lyapunov-like equation,

$$\partial_\phi R_\phi = \frac{1}{2}(L_\phi R_\phi + R_\phi L_\phi).$$

In the present case, where the state of the two modes is pure, i.e., $R_\phi = |\Psi(\Theta, \phi)\rangle\langle\Psi(\Theta, \phi)|$, the solution of the above equation is available explicitly as

$$L_\phi = 2(|\partial_\phi\Psi(\Theta, \phi)\rangle\langle\Psi(\Theta, \phi)| + |\Psi(\Theta, \phi)\rangle\langle\partial_\phi\Psi(\Theta, \phi)|),$$

and the QFI turns out to be independent of the value of the phase shift. In particular, the QFI is proportional to the fluctuations of the phase-shift generator $G = a^\dagger a \otimes \mathbb{I}$, namely, $H = 4(\langle G^2 \rangle - \langle G \rangle^2)$, where the expectation value $\langle \dots \rangle$ is taken over the output state $|\Psi(\Theta, \phi)\rangle$.

3. PASSIVE INTERFEROMETERS

In a typical passive interferometer, such as the Mach–Zehnder or the Michelson interferometer, the energy of the input $|\psi_a\rangle \otimes |\psi_b\rangle$ is conserved during the evolution and the interaction operator $U_{\text{int}}(\Theta)$ corresponds to the action of a balanced beam splitter (BS), namely, $U_{\text{BS}} \equiv U_{\text{int}}(\Theta) = \exp[\frac{\pi}{4}(a^\dagger b - ab^\dagger)]$. Unbalanced devices may be also considered, which however lead to inferior performances [33,34]. The precision of this kind of interferometer depends on the choice of $|\psi_a\rangle \otimes |\psi_b\rangle$ and on the detection stage. We take the most general pure (and

factorized) Gaussian signals as input $|\psi_a\rangle \otimes |\psi_b\rangle = |\alpha, \xi\rangle \otimes |\gamma, \zeta\rangle$, where $|z, s\rangle = D(z)S(s)|0\rangle$ is a squeezed-coherent state $D(z) = \exp\{z k^\dagger - z^* k\}$ and $S(s) = \exp\{\frac{1}{2}[s(k^\dagger)^2 - s^* k^2]\}$ are the displacement and squeezing operators of modes $k = a, b$, respectively. Without loss of generality, we can assume $\alpha, \xi, \gamma \in \mathbb{R}$ and $\zeta = r e^{-i\theta} \in \mathbb{C}$, and for this choice the QFI $H \equiv H(\alpha, \gamma, \xi, r, \theta)$ is given by

$$H = \frac{1}{4} \{4(\alpha + \gamma)^2 [e^{2\xi} + \cosh 2r] + \cosh 4\xi + \cosh 4r + 2 \cos \theta \sinh 2r [2(\alpha + \gamma)^2 + \sinh 2\xi] + \cosh 2(r - \xi) + \cosh 2(\xi + r) - 4\}. \quad (2)$$

In order to characterize the interferometer and maximize the QFI with respect to the input state it is useful to introduce the following parameterization: the total number of photons in the initial state,

$$N_{\text{tot}} = \alpha^2 + \gamma^2 + \sinh^2 r + \sinh^2 \xi,$$

the signal fraction,

$$\Delta = \alpha^2 / (\alpha^2 + \gamma^2),$$

and the total and partial squeezing fractions,

$$\beta_{\text{tot}} = (\sinh^2 r + \sinh^2 \xi) / N_{\text{tot}} \beta = \sinh^2 \xi / N_{\text{tot}},$$

respectively. Note that $0 \leq \Delta \leq 1$, $0 \leq \beta_{\text{tot}} \leq 1$, and $0 \leq \beta \leq \beta_{\text{tot}}$. Thereafter, given N_{tot} and β_{tot} , the maximum QFI is achieved for $\Delta = 1/2$, $\beta = \beta_{\text{tot}}/2$, and $\theta = 0$, namely, the inputs have the same coherent amplitude $\alpha = \gamma$, and the squeezing parameter $\xi = r$, and explicitly reads

$$H_{\text{max}}(N_{\text{tot}}, \beta_{\text{tot}}) = 2N_{\text{tot}} [2 + N_{\text{tot}} \beta_{\text{tot}} (2 - \beta_{\text{tot}}) + 2(1 - \beta_{\text{tot}}) \sqrt{N_{\text{tot}} \beta_{\text{tot}} (2 + N_{\text{tot}} \beta_{\text{tot}})}], \quad (3)$$

where $N_{\text{tot}} = 2(\alpha^2 + \sinh^2 r)$ and $N_{\text{tot}} \beta_{\text{tot}} = 2 \sinh^2 r$. We may perform a further numerical maximization of Eq. (3) with respect to β_{tot} and the results are reported in the upper panel of Fig. 2, where one sees that the optimal value of β_{tot} increases with N_{tot} . In the large energy regime $N_{\text{tot}} \gg 1$, we have $\beta_{\text{tot}} \simeq 2/3$, corresponding to

$$H_{\text{max}} \simeq \frac{8}{9} N_{\text{tot}}^2 [2 + (1 + 3N_{\text{tot}}^{-1})^{1/2}] + 4N_{\text{tot}} \simeq \frac{8}{3} N_{\text{tot}}^2.$$

Notice that Heisenberg scaling $H \propto N_{\text{tot}}^2$ is achieved with an improved proportionality constant (compared to known results, see, e.g., [2]) due to the fact that we have optimized the input signals over all the state parameters.

It is worth noting that, for the optimal input states leading to Eq. (3), we have

$$U_{\text{BS}}|\alpha, r\rangle \otimes |\alpha, r\rangle = |\sqrt{2}\alpha, r\rangle \otimes |0, r\rangle,$$

that is, the state undergoing the phase shift is still factorized and, in particular, the single mode acquiring the phase is the displaced-squeezed state $|\sqrt{2}\alpha, r\rangle$. This result may be better appreciated upon recalling previous findings for the single-mode case [35]. In that case, an optimal estimation of a phase shift imposed onto a Gaussian state is achieved when all the energy is used to squeeze the vacuum, i.e., the displacement is useless. The corresponding QFI reads $H_{\text{sm}} = 8N_{\text{tot}}(1 + N_{\text{tot}})$ [35]. One could have naïvely expected that, for an interferometric setup involving two modes, energy has been used to

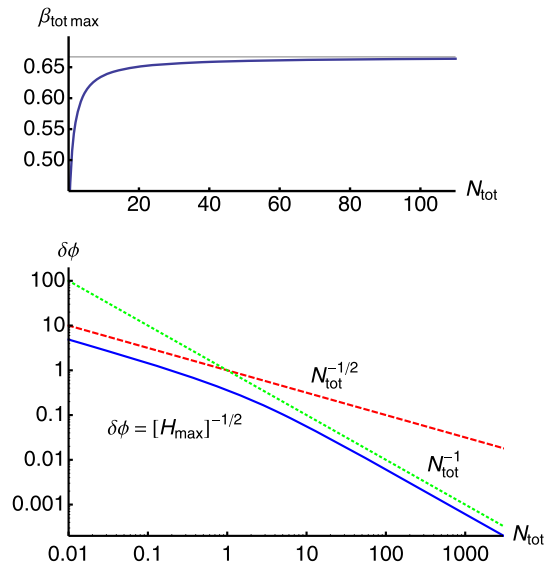


Fig. 2. Top: optimal value of the total squeezing fraction β_{tot} , maximizing the QFI for a passive interferometer, as a function of N_{tot} . Bottom: corresponding phase sensitivity $\delta\phi$ as a function of N_{tot} (blue solid line). The shot-noise limit $N_{\text{tot}}^{-1/2}$ (red dashed line) and the Heisenberg limit N_{tot}^{-1} (green dotted line) are also reported.

generate squeezing as well, leading to $\beta_{\text{tot}} = 1$ and $\beta = 1/2$, where the last condition ensures that $U_{\text{BS}}|0, r\rangle \otimes |0, r\rangle = |0, r\rangle \otimes |0, r\rangle$ [34]. Actually, this is not the case since it would lead to $H'_{\text{max}} = 4N_{\text{tot}}(1 + \frac{1}{2}N_{\text{tot}})$, which follows from H_{sm} halving N_{tot} (just one mode undergoes the phase shift). The need of a displacement is due to the realistic description of the phase reference (the other mode of the interferometer), which is neglected in the single-mode analysis, where the existence of a stable classical reference is somehow assumed. It also shows that the quantum enhancement coming from the use of squeezing results from the improvement of both the reference phase and the signal phase within the interferometric setup.

The lower panel of Fig. 2 shows the behavior of the phase sensitivity $\delta\phi = [H_{\text{max}}]^{-1/2}$ as a function of N_{tot} . We clearly observe two regimes: for $N_{\text{tot}} \lesssim 1$ sensitivity is shot-noise limited, namely, $\delta\phi \propto N_{\text{tot}}^{-1/2}$, while the Heisenberg limit $\delta\phi \propto N_{\text{tot}}^{-1}$ is reached for $N_{\text{tot}} \gg 1$. Overall, our result shows that in order to achieve optimal estimation of a phase shift using a passive interferometer and Gaussian states, we should excite the two input modes with identical signals and distribute the coherent and the squeezing photons according to the value of β_{tot} maximizing Eq. (3). On the one hand, this choice allows one to obtain a precise phase reference [36,37], i.e., the squeezed vacuum after the interaction. On the other hand, it shows that a coherent amplitude is always needed.

The optimization over all the state parameters allows one to improve the proportionality constant in the Heisenberg scaling. In order to see this explicitly, let us consider the case where the input signals are given by $|\alpha, 0\rangle \otimes |0, \zeta\rangle$, with $\zeta = r e^{-i\theta}$, which is a coherent state and a squeezed vacuum state (we still assume $\alpha \in \mathbb{R}$). This is a configuration often considered for GW detectors [38] and has been recently addressed in [26] (see also [2,39]). In Ref. [26] the coherent amplitude is fixed and the

optimality of a squeezed vacuum for the second port has been shown. Here, we optimize the QFI:

$$H^{(\text{gw})}(N_{\text{tot}}, \beta) = 2N_{\text{tot}} \left[1 + \left(\frac{1}{2} + N_{\text{tot}} \right) \beta + (1 - \beta) \sqrt{N_{\text{tot}} \beta (1 + N_{\text{tot}} \beta)} \right], \quad (4)$$

also over the squeezing fraction $\beta = \sinh^2 r / N_{\text{tot}}$ at fixed $N_{\text{tot}} = \alpha^2 + \sinh^2 r$. We found the maximum for $\beta = 1$, corresponding to

$$H_{\text{max}}^{(\text{gw})} = 2N_{\text{tot}} \left(\frac{3}{2} + N_{\text{tot}} \right) \stackrel{N_{\text{tot}} \gg 1}{\simeq} 2N_{\text{tot}}^2.$$

4. ACTIVE INTERFEROMETERS

As a paradigmatic example of interferometers involving amplification we consider the so-called coherent-light-boosted interferometer (CLBI) [40], in which $U_{\text{int}}(\Theta) = \exp(\xi a^\dagger b^\dagger - \xi^* ab)$, i.e., $\Theta = \xi = r e^{-i\theta}$ (see Fig. 1). The interaction imposes phase-sensitive amplification (i.e. two-mode squeezing) and introduces quantum correlations between the two modes, which suggests that quantum enhancement may already be achieved starting from classical input signals. In order to check this conjecture we consider as input just a couple of coherent states $|\psi_a\rangle \otimes |\psi_b\rangle = |\alpha\rangle \otimes |\gamma\rangle$, with $\alpha, \gamma \in \mathbb{R}$. This also makes the interferometer feasible with current technology, since it minimizes the impact of phase-matching imperfections at the input of the amplifier [41–43]. The QFI $H^{(\text{CLBI})} \equiv H^{(\text{CLBI})}(\alpha, \gamma, r, \theta)$ for the CLBI is given by

$$H^{(\text{CLBI})} = \left(\alpha^2 + \gamma^2 + \frac{1}{2} \right) (1 + \cosh 4r) - 1 + 2\alpha\gamma \cos \theta \sinh 4r + 2(\alpha^2 - \gamma^2) \cosh 2r. \quad (5)$$

In order to simplify the notation, we observe that the maximum over θ is obtained for $\theta = \pi$, and introduce the following parameters characterizing the interferometer: the signal fraction $\Delta = \alpha^2 / (\alpha^2 + \gamma^2)$, the total number of photons after the interaction, namely,

$$N_{\text{tot}} = (\alpha^2 + \gamma^2 + 1) \cosh(2r) + 2\alpha\gamma \cos(\theta) \sinh(2r) - 1,$$

and the squeezing fraction $\beta = 2 \sinh^2 r / N_{\text{tot}}$. The analytical expression of the corresponding $H^{(\text{CLBI})}(N_{\text{tot}}, \Delta, \beta)$ is clumsy and is not explicitly reported here. If we fix the value of N_{tot} and focus on $\Delta > 1/2$, then it is possible to find numerically the values Δ_{max} and β_{max} maximizing $H^{(\text{CLBI})}$. These are reported in the upper panel of Fig. 3, where we can see that they are both an increasing function of N_{tot} , with $\Delta_{\text{max}} \rightarrow 1/2$ and $\beta_{\text{max}} \rightarrow 2/3$ for $N_{\text{tot}} \gg 1$.

In Fig. 3 we show the behavior of the phase sensitivity $\delta\phi = [H_{\text{max}}^{(\text{CLBI})}]^{-1/2}$ as a function of N_{tot} . Two regimes may be easily identified: for $N_{\text{tot}} \lesssim 1$ we find shot-noise-limited precision $\delta\phi \propto N_{\text{tot}}^{-1/2}$, while for $N_{\text{tot}} \gg 1$ we obtain $H_{\text{max}}^{(\text{CLBI})} \simeq \frac{4}{3} N_{\text{tot}} (N_{\text{tot}} + 2)$ and thus the Heisenberg scaling $\delta\phi \propto N_{\text{tot}}^{-1}$. The comparison between the lower panels of Figs. 2 and 3 reveals that for $N_{\text{tot}} \gg 1$ passive interferometers offer better performance compared to the active ones, at least for the class

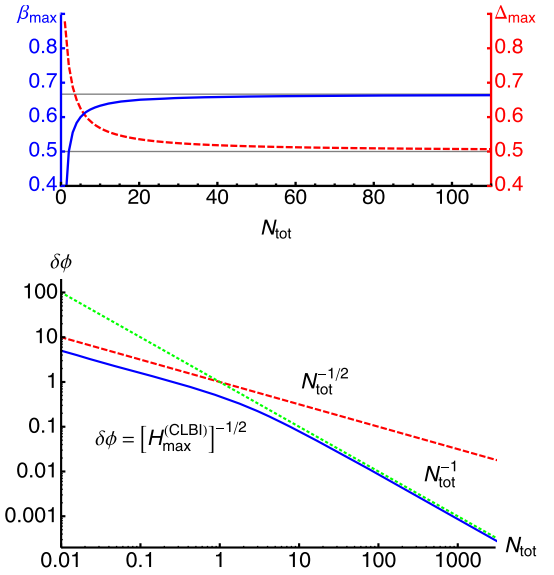


Fig. 3. Top: optimal value of the signal fraction Δ_{max} (red dashed line) and of the squeezing fraction β_{max} (blue solid line), maximizing the QFI for the CLBI, as functions of N_{tot} . Bottom: corresponding sensitivity $\delta\phi$ as a function of N_{tot} . The shot-noise limit $N_{\text{tot}}^{-1/2}$ (red dashed line) and the Heisenberg limit N_{tot}^{-1} (green dotted line) are also reported.

of setup considered here. On the other hand, active interferometers are worth being investigated because there is evidence that the amplification mechanism is suitable to fight the possible photon loss occurring in the interferometer [44].

5. DISCUSSION AND CONCLUSIONS

One may wonder whether and how the ultimate bound to precision may be achieved in practice, i.e., whether there exists a feasible measurement in the final stage of the interferometer whose Fisher information is equal to the QFI or approaches it in some limit. In principle, an observation of this kind is provided by the spectral measure of the symmetric logarithmic derivative L_ϕ . However, even for pure states where the solution is available explicitly it is usually very challenging to implement this kind of measurement. As a consequence, the performance of feasible measurements, e.g., photodetection after a further mixing or amplification stage, have been explored. Also, in these realistic scenarios, squeezing has been shown to represent a resource in achieving the Heisenberg limit [45], or at least to beat the shot-noise limit in some regimes [46]. It should also be mentioned that the use of Bayesian analysis [47,48] or maximum-likelihood estimators [49,50] allows one to achieve the Cramér–Rao bound (either classical or quantum) for any value of the phase shift itself, not only for a specific setting.

Our results show that precision may be indeed improved by optimizing over all the degrees of freedom and indicates which parameters are relevant in the different scenarios. Our results also provide a set of general benchmarks for Gaussian interferometry, which are independent of the measurement scheme used at the output and may be compared to performances of realistic interferometric schemes.

In conclusion, we have analyzed two-mode interferometers based on Gaussian states and operations and found the best input states, optimizing the precision given a constraint on the total mean number of photons entering the setup. For passive interferometers, e.g., Mach–Zehnder-like devices, the optimal input is made of a pair of identical squeezed-coherent states with at most one-third of the energy used in squeezing. Our results clarify the role of squeezing, which improves both the reference phase and the signal phase in the interferometer. For active interferometers, i.e., setups involving optical amplifiers, we focused on the CLBI. In this case coherent signals at the input are enough to achieve Heisenberg scaling, provided that the amplification gain is tuned to an optimal value. Our results go beyond the traditional analysis of laser interferometry and may be relevant for the development of novel quantum-enhanced interferometric schemes.

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