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OPTIMIZED QUBIT PHASE ESTIMATION IN NOISY QUANTUM CHANNELS

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We address the estimation of phase-shifts for qubit systems in the presence of noise. Different sources of noise are considered including bit flip, bit-phase flip and phase flip. We derive the ultimate quantum limits to precision of estimation by evaluating the analytical expressions of the quantum Fisher information and assess performances of feasible measurements by evaluating the Fisher information for realistic spin-like measurements. We also propose an experimental scheme to test our results.

Keywords: Quantum estimation; qubit.

1. Introduction

In this paper, we address the estimation of a phase-shift imposed to a qubit in the presence of noise. The scheme we have in mind is the following: a unitary rotation is applied to a qubit initially prepared in a known state, then, before being detected, the qubit propagates through a noisy channel. We consider a fixed axis for the rotation and three possible kind of noise: bit flip, phase-bit flip and phase flip. Our goal is to find the optimal strategy to estimate the value of the phase-shift, i.e. the measurement achieving the quantum Cramèr-Rao bound to precision, and to compare the performances of feasible qubit measurements with the optimal ones. To this aim, we employ local quantum estimation theory and evaluate the quantum Fisher information for three families of output states, as well as the Fisher information of realistic measurements. As we will see for bit-flip and phase-bit flip noises, upon choosing a suitable initial state, one can always attain the minimum uncertainty in the estimation allowed by quantum mechanics, whereas phase flip noise leads to an unavoidable loss of information.

The paper is structured as follows. In Sec. 2 we review the formalism we use throughout the paper and introduce the description of the different noisy channels. We also introduce the quantum Fisher information and the Fisher information needed to evaluate the ultimate bounds on the precision of the estimation. In Sec. 3 we systematically investigate the phase estimation in the presence of the three possible sources of noise. In particular, we calculate the explicit analytical expressions of the quantum Fisher information and of the Fisher information. Section 4 is devoted to the discussion of our results. In this section we give the "geometrical" interpretation of our results by investigating the dynamics of the system in the Bloch sphere representation. We also propose an experiment to test our predictions in Sec. 5. Section 6 closes the paper with some concluding remarks.

2. The System and the Noisy Evolution

Let us consider a single qubit in the pure state (we use the eigenvectors $|0\rangle$ and $|1\rangle$ of the Pauli matrix σ_3 as basis):

$$\varrho_0 = \begin{pmatrix} \cos^2\vartheta & e^{i\varphi}\cos\vartheta\sin\vartheta \\ e^{-i\varphi}\cos\vartheta\sin\vartheta & \sin^2\vartheta \end{pmatrix}. \tag{1}$$

Upon introducing the Bloch sphere, the state (1) can be also written as: $\varrho_0 = 1/2(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$ where 1 is the 2×2 identity matrix, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli $\mathbf{r} = (\sin 2\vartheta\, \cos\varphi, \cos 2\vartheta\, \sin\varphi, \cos 2\vartheta)$, 2ϑ is the azimuthal angle ($\vartheta = 0$ and $\vartheta = \pi/2$ correspond to the north and south poles of the Bloch sphere, that are the states $|0\rangle$ and $|1\rangle$, respectively).

Now, we apply a phase shift to ϱ_0. The unitary operator associated with the phase shift is $U_3(\phi) = \exp(-i/2\phi\,\sigma_3)$. Since the phase shift imposed by $U_3(\phi)$ is a rotation around the z-axis of the Bloch sphere, we can put $\varphi = 0$ without loss of generality, thus, the shifted qubit state reads:

$$\varrho_\phi = U_3(\phi)\,\varrho_0\,U_3^\dagger(\phi) = \begin{pmatrix} \cos^2\vartheta & e^{i\phi}\cos\vartheta\sin\vartheta \\ e^{-i\phi}\cos\vartheta\sin\vartheta & \sin^2\vartheta \end{pmatrix}. \tag{2}$$

In this paper we address the estimation of ϕ assuming the qubit evolving through a channel affected by one of three possible noises (our analysis can be easily extended to any combination of these): bit flip, phase-bit flip and phase flip noises.[1] These channels are described by the following map acting on ϱ_ϕ:

$$\mathcal{E}_k(\varrho_\phi) = \varepsilon_0\,\varrho_\phi + \varepsilon_1\,\sigma_k\,\varrho_\phi\,\sigma_k, \tag{3}$$

with $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_1 + \varepsilon_2 = 1$. Depending on the value of k, the map (3) describes the bit flip ($k = 1$), the phase-bit flip ($k = 2$) and the phase flip ($k = 3$).[1] By investigating it, we can see that the state is left unchanged with the probability ε_0, while with probability ε_1 it is transformed (undergoes an error) according to the corresponding Pauli matrix.

The action of the map (3) can also be obtained by applying a suitable Gaussian noise to the qubit:

$$\mathcal{E}_k(\varrho_\phi) = \int_R d\beta \frac{e^{-\beta^2/(4\Delta^2)}}{\sqrt{4\pi\Delta^2}} U_k(\beta)\varrho_\phi U_k^\dagger(\beta), \tag{4}$$

where $U_k(\beta) = \exp(-i/2\beta\,\sigma_k)$ and Δ^2 is related to the noise amplitude. The coefficients of the map (3) are related to the Gaussian noise as follows:

$$\varepsilon_0 = g_+(\Delta^2), \quad \varepsilon_1 = g_-(\Delta^2), \quad g_\pm(\zeta) = \frac{1}{2}\left(1 \pm e^{-\zeta}\right). \tag{5}$$

The map in Eq. (4) represents the solution of the noisy dynamics described by the following Master equation[1]:

$$\dot{\varrho_\phi} = \gamma \mathcal{L}[\sigma_k]\,\varrho_\phi, \tag{6}$$

where $\mathcal{L}[\sigma_k]\,\varrho_\phi = 1/2\{[\sigma_k \varrho_\phi, \sigma_k] + [\sigma_k, \varrho_\phi \sigma_k]\}$. Eq. (6) gives the same evolution as above if we put $\Delta^2 = \gamma t/2$.

2.1. *Quantum Fisher information and Fisher information for spin measurements*

The minimum uncertainty $\mathrm{Var}_m[\phi]$ achievable in the phase estimation is given by the quantum Cramér-Rao bound:

$$\mathrm{Var}_m[\phi] = 1/H(\phi), \tag{7}$$

$H(\phi)$ being the quantum Fisher information (QFI),[2–5] which can be calculated starting from the eigenvalues and eigenvectors of the signal carrying the information. In the present case, if we call $\lambda_\pm^{(k)} \equiv \lambda_\pm^{(k)}(\phi)$ the two eigenvalues of the state $\mathcal{E}_k(\varrho_\phi)$ and $|\lambda_\pm^{(k)}\rangle \equiv |\lambda_\pm^{(k)}(\phi)\rangle$ the corresponding eigenvectors, the QFI can be written as[8]:

$$\begin{aligned} H_k = {} & \frac{1}{\lambda_+^{(k)}}[\partial_\phi \lambda_+^{(k)}]^2 + \frac{1}{\lambda_-^{(k)}}[\partial_\phi \lambda_-^{(k)}]^2 \\ & + \frac{[\lambda_+^{(k)} - \lambda_-^{(k)}]^2}{\lambda_+^{(k)} + \lambda_-^{(k)}} ls[|\langle \lambda_+^{(k)}|\partial_\phi \lambda_-^{(k)}\rangle|^2 + |\langle \lambda_-^{(k)}|\partial_\phi \lambda_+^{(k)}\rangle|^2]. \end{aligned} \tag{8}$$

Since we are interested in the estimation of the phase shift around the z-axis, one can show that the optimal states for the estimation are the equatorial states of the Bloch sphere[7]: for this reason we can set $\vartheta = \pi/4$ in Eq. (2).

On the other hand, if we assume that, after the noisy evolution, we perform the measurement of the two-outcome observable:

$$\Sigma(\alpha) = \sigma_1 \cos\alpha + \sigma_2 \sin\alpha, \tag{9}$$

then the minimum uncertainty on the estimation of ϕ is given by the Fisher information, namely:

$$F_k(\phi,\alpha) = \frac{1}{p_-^{(k)}}[\partial_\phi p_-^{(k)}]^2 + \frac{1}{p_+^{(k)}}[\partial_\phi p_+^{(k)}]^2, \tag{10}$$

with $p_\pm^{(k)} \equiv p_\pm^{(k)}(\phi,\Delta^2,\alpha)$:

$$p_\pm^{(k)} \equiv p_k(\pm 1|\phi) = \mathrm{Tr}[\mathcal{E}_k(\varrho_\phi)\Pi_\pm(\alpha)] \tag{11}$$

where we introduced the projectors:

$$\Pi_\pm(\alpha) = |\Sigma_\pm(\alpha)\rangle\langle\Sigma_\pm(\alpha)| \tag{12}$$

$|\Sigma_\pm(\alpha)\rangle$ being the eigenvectors of $\Sigma(\alpha)$: $\Sigma(\alpha)|\Sigma_\pm(\alpha)\rangle = \pm|\Sigma_\pm(\alpha)\rangle$. Note that $p_\pm^{(k)}$, as defined above, represent the conditional probabilities of the outcomes ± 1 given ϕ.

3. Optimal Phase Estimation

3.1. *Bit flip noise*

For $k = 1$ in Eq. (3) or, equivalently, in Eq. (4), one has the so-called bit-flip noise. The eigenvectors and eigenvalues of $\mathcal{E}_k(\varrho_\phi)$ read (in the eigenbasis of σ_3 and for $\vartheta = \pi/4$):

$$|\lambda_\pm^{(1)}\rangle = \frac{1}{\sqrt{2}}[|1\rangle \pm f^{(1)}(\phi,\Delta^2)|0\rangle], \tag{13}$$

and

$$\lambda_\pm^{(1)} = \frac{1}{2}[1 \pm \sqrt{g_+(2\Delta^2) + g_-(2\Delta^2)\cos 2\phi}], \tag{14}$$

respectively, where the functions $g_\pm(\zeta)$ are given in Eq. (5):

$$f^{(1)}(\phi,\Delta^2) = \frac{\cos\phi - ie^{-\Delta^2}\sin\phi}{\cos^2\phi + e^{-2\Delta^2}\sin^2\phi}[\lambda_+^{(1)} + \lambda_-^{(1)}]. \tag{15}$$

From Eq. (8) one finds:

$$H_1 = 1, \tag{16}$$

that is the QFI is always maximum and, thus, the uncertainty in the estimation is minimum.

The conditional probabilities $p_\pm^{(1)} = \mathrm{Tr}[\mathcal{E}_1(\varrho_\phi)\Pi_\pm(\alpha)]$ in Eq. (11) read:

$$p_\pm^{(1)}(\phi,\Delta^2,\alpha) = \frac{1}{2}[1 \pm (\cos\alpha\,\cos\phi + e^{-\Delta^2}\sin\alpha\,\sin\phi)] \tag{17}$$

in turn, the Fisher information (10) becomes:

$$F_1(\phi,\alpha) = \frac{(\cos\alpha\sin\phi - e^{-\Delta^2}\sin\alpha\cos\phi)^2}{1-(e^{-\Delta^2}\sin\alpha\sin\phi + \cos\alpha\cos\phi)^2}, \tag{18}$$

that leads to $F_1 = 1\ \forall\phi$ if we set $\alpha = 0$, i.e. we measure σ_1: this measurement leads to the optimal estimation of ϕ.

If $\Delta^2 \ll 1$, then the expansion of Eq. (18) reads:

$$F_1(\phi,\alpha) \approx 1 - \frac{2\sin^2(\alpha)}{\sin^2(\alpha-\phi)}\Delta^2. \tag{19}$$

3.2. *Phase-bit flip noise*

The phase-bit flip corresponds to set $k = 2$ in Eq. (3). Now the eigenvectors and eigenvalues are:

$$|\lambda_\pm^{(2)}\rangle = \frac{1}{\sqrt{2}}[|1\rangle \pm f^{(2)}(\phi,\Delta^2)|0\rangle], \tag{20}$$

and

$$\lambda_\pm^{(2)} = \frac{1}{2}[1 \pm \sqrt{g_+(2\Delta^2) - g_-(2\Delta^2)\cos 2\phi}], \tag{21}$$

respectively, where $g_\pm(\zeta)$ are still given in Eq. (5) and:

$$f^{(2)}(\phi,\Delta^2) = \frac{e^{-\Delta^2}\cos\phi - i\sin\phi}{e^{-2\Delta^2}\cos^2\phi + \sin^2\phi}[\lambda_+^{(2)} + \lambda_-^{(2)}]. \tag{22}$$

From Eq. (8) one finds:

$$H_2 = 1, \tag{23}$$

that is the QFI is always maximum as for the bit flip case.

The conditional probabilities $p_\pm^{(2)} = \mathrm{Tr}[\mathcal{E}_2(\varrho_\phi)\Pi_\pm(\alpha)]$ read:

$$p_\pm^{(2)}(\phi,\Delta^2,\alpha) = \frac{1}{2}[1 \pm (\sin\alpha\,\sin\phi + e^{-\Delta^2}\cos\alpha\,\cos\phi)], \tag{24}$$

and the Fisher information (10) is:

$$F_2(\phi,\alpha) = \frac{(\sin\alpha\cos\phi - e^{-\Delta^2}\cos\alpha\sin\phi)^2}{1-(e^{-\Delta^2}\cos\alpha\cos\phi + \sin\alpha\sin\phi)^2}, \tag{25}$$

that leads to $F_2 = 1\ \forall\phi$ if we set $\alpha = \pi/2$, i.e. we measure σ_2.

If $\Delta^2 \ll 1$, then the expansion of Eq. (25) reads:

$$F_2(\phi,\alpha) \approx 1 - \frac{2\cos^2(\alpha)}{\sin^2(\alpha-\phi)}\Delta^2. \tag{26}$$

3.3. *Phase flip noise*

This case describes a qubit undergoing phase diffusion during its propagation.[7] The eigenvectors and eigenvalues of $\mathcal{E}_k(\varrho_\phi)$ are simply:

$$|\lambda_\pm^{(3)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm e^{-i\phi}|0\rangle) \tag{27}$$

and:

$$\lambda_{\pm}^{(3)} = \frac{1}{2}(1 \pm e^{-\Delta^2}), \tag{28}$$

that lead to the following QFI:

$$H_3 = e^{-2\Delta^2}. \tag{29}$$

Now, since the conditional probabilities $p_{\pm}^{(3)} = \mathrm{Tr}[\mathcal{E}_3(\varrho_\phi)\Pi_{\pm}(\alpha)]$ reads:

$$p_{\pm}^{(3)}(\phi, \Delta^2, \alpha) = \frac{1}{2}[1 \pm e^{-\Delta^2}\cos(\alpha - \phi)], \tag{30}$$

the Fisher information (10) reduces to:

$$F_3(\phi, \alpha) = \frac{e^{-2\Delta^2}\sin^2(\alpha - \phi)}{1 - e^{-2\Delta^2}\cos^2(\alpha - \phi)}, \tag{31}$$

that reaches the maximum $F_3 = e^{-2\Delta^2} = H_3$ for $\alpha = \phi + \pi/2$. Since ϕ is unknown, the achievement of the maximum value requires a two-step adaptive method as described in Ref. 7.

If $\Delta^2 \ll 1$, then the expansion of Eq. (31) reads:

$$F_3(\phi, \alpha) \approx 1 - \frac{2}{\sin^2(\alpha - \phi)}\Delta^2. \tag{32}$$

It is worth to note that now the QFI H_3 is equal to unity only in the absence of noise ($\Delta^2 = 0$): it has been investigated in Ref. 9.

4. Discussion

In Fig. 1 we show the deformation of the Bloch sphere under the action of the three different noises described above. If the initial qubit state is represented by the vector $\mathbf{r} = (r_x, r_y, r_z)$, we have the following evolutions depending on the noise:

$$\mathbf{r} \to (r_x, e^{-\Delta^2} r_y, e^{-\Delta^2} r_z) \quad \text{(bit flip)}; \tag{33}$$

$$\mathbf{r} \to (e^{-\Delta^2} r_x, r_y, e^{-\Delta^2} r_z) \quad \text{(bit-phase flip)}; \tag{34}$$

$$\mathbf{r} \to (e^{-\Delta^2} r_x, e^{-\Delta^2} r_y, r_z) \quad \text{(phase flip)}. \tag{35}$$

If we start from an equatorial state, i.e. $r_z = 0$, then in the presence of bit flip and bit-phase flip the x and y components of $\mathbf{r}$, respectively, are left unchanged, whereas the other is rescaled by the factor $e^{-\Delta^2}$ (left and middle plots in Fig. 1); in the case of the phase flip noise (right plot in Fig. 1) both the x and y components are rescaled. It is now clear why in the case of the first two kinds of noise it is possible to achieve the optimal estimation ($F = H = 1$) by a suitable choice of the measurements: these correspond to measure the component of $\mathbf{r}$, which is not affected by the noise. This is not possible for the phase noise, where both the components x and y are equally degraded.

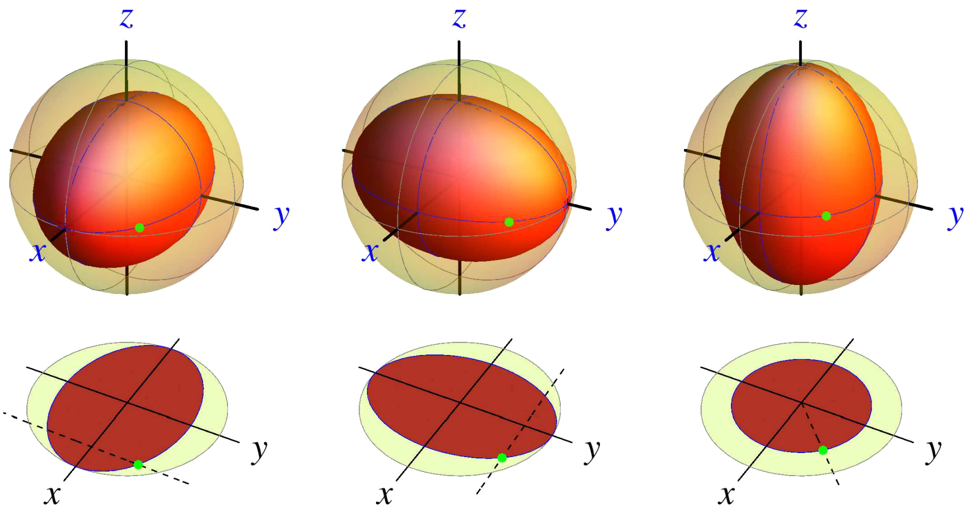

Fig. 1. Deformation of the Bloch sphere and the corresponding equatorial projection under bit flip (left), phase-bit flip (middle) and phase flip (right) noises. In the case of bit flip and phase-bit flip the x and y component, respectively, of the qubit are left unchanged during the evolution (dashed lines). See the text for details. The dot refers to the evolving qubit state.

So far we have addressed the case in which the noise affects the propagation of the qubit *after* the phase-shift. Let us now extend our analysis to the case of noise occurring *before* the phase-shift operation. We can distinguish between two cases: (i) the bit flip and the phase-bit flip noises and (ii) the phase flip noise. In the first two cases (bit and bit-phase flip), we can cancel the effect of noise by choosing suitable input equatorial states (see the left and middle plots of Fig. 1). Since we are interested in the achievement of the optimal estimation, in the case of $\mathcal{E}_1$ we should use $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$ as input signals: they are the eigenstates of σ_1 and, thus, are invariant under the map $\mathcal{E}_1$, i.e.

$$\mathcal{E}_1(|\pm\rangle\langle\pm|) = |\pm\rangle\langle\pm|, \tag{36}$$

Analogously, in the presence of the noise $\mathcal{E}_2$, we should use $|l\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle)$ and $|r\rangle = 1/\sqrt{2}(|0\rangle - i|1\rangle)$, that are the eigenstates of σ_2 and, thus:

$$\mathcal{E}_2(|l\rangle\langle l|) = |l\rangle\langle l|, \quad \mathcal{E}_2(|r\rangle\langle r|) = |r\rangle\langle r|, \tag{37}$$

i.e. they are left unchanged under the action of the map $\mathcal{E}_2$.

For what concerns $\mathcal{E}_3$, the situation is quite different. In this case, as one can see from the right plot in Fig. 1, there is not an equatorial state which is left unchanged [see also Eq. (35)]: a phase noise along this channel leads to a unavoidable loss of information. Nevertheless, one can address this case as if all the noise affects the propagation only after the phase shift, since $\mathcal{E}_3$ and $U_3(\phi)$ commute, being both functions of σ_3, as we described in Refs. 6 and 7.

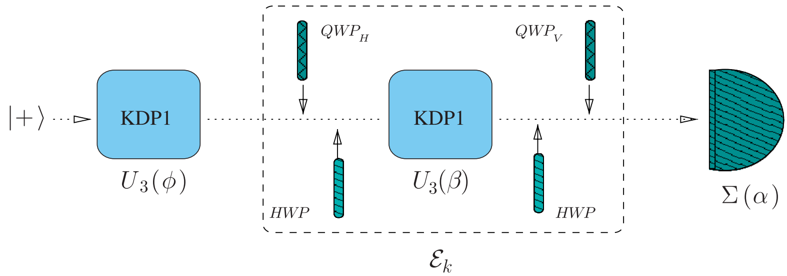

Fig. 2. A polarization qubit state, initially in the state $|+\rangle$, undergoes a phase shift of an amount ϕ by a first KDP crystal (KDP1). In order to simulate the difference sources of noise, a second KDP crystal (KDP2) imposes the phase shift β, with β distributed according to a normal distribution with zero mean and $2\Delta^2$ variance [see Eq. (4)]. The noisy channels $\mathcal{E}_k$ are simulated as follows. To simulate the bit flip noise ($k = 1$) we need to insert a half wave plate @ 22.5° (HWP) before and after the KDP2, while to simulate the bit-phase flip noise ($k = 2$) we insert a quarter wave plate with fast axis horizontal (QWP_H) before the KDP2 and a quarter wave plate with fast axis vertical (QWP_V) after the KDP2, together with the two HWPs. The phase flip noise ($k = 3$) is implemented by means of the KDP2, i.e. removing the HWPs and the QWPs. Depending on the noisy channel, the measurement $\Sigma(\alpha)$ is implemented by a polarizing beam splitter (PBS), together with a HWP @ 22.5°, in the case of $\mathcal{E}_1$, or by a QWP with fast axis horizontal and a HWP @ 22.5°, for $\mathcal{E}_1$ or $\mathcal{E}_2$, respectively.

5. Experimental Proposal

In this section we discuss the experimental implementation of the phase estimation in the presence of bit and phase-bit flip noises (we thoroughly investigated the last case, the phase flip noise, in Ref. 7). As qubit we take the polarization degree of freedom of a single photon state: in this case $|0\rangle \to |H\rangle$ and $|1\rangle \to |V\rangle$, $|H\rangle$ and $|V\rangle$ denoting the horizontal and vertical polarization states, respectively), thus, $|\pm\rangle$ correspond to the linear-polarized states at $\pm 45°$, while $|l\rangle$ and $|r\rangle$ to left- and right-handed circular polarized states, respectively.

As discussed above, without loss of generality we can focus on the presence of the noisy channel $\mathcal{E}_k, k = 1, 2$, *after* the phase shift operation. In Fig. 2 we sketched the schemes referring to the two different noisy channels. The phase shift $U_3(\phi)$ is imposed to the initial qubit in the (equatorial) state $|+\rangle$ by using a KDP crystal, as described in Ref. 7, while the noise is simulated following Eq. (4), that is by applying to the phase-shifted qubit the operation $U_k(\beta), k = 1, 2$, with the value of the real parameter β distributed according to a normal distribution with zero mean and $2\Delta^2$ variance [see Eq. (4)]. Operations $U_1(\beta)$ and $U_2(\beta)$ may be experimentally implemented by means of a KDP crystal preceded and followed by a half-wave plate (HWP), in the case of $U_1(\beta)$, or by a suitable quarter-wave plate (QWP) and a HWP, in the case of $U_2(\beta)$, as depicted in Fig. 2.

6. Conclusion

In this paper we discussed the estimation of the phase shift imposed to a qubit in the presence of different sources of noises. We showed that in the case of bit and phase-bit flip noises, upon choosing a suitable initial state, one can always attain the minimum uncertainty in the estimation allowed by quantum mechanics (the quantum Cramér-Rao bound). On the other hand, the presence of the phase flip noise leads to an unavoidable loss of information and, thus, to a decrease of the QFI, that has been experimentally verified in Ref. 7. Experimental schemes to test our predictions have been also proposed.

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