LOCALIZATIONS OF THE CATEGORY OF A_∞ CATEGORIES AND INTERNAL HOMS OVER A RING

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Abstract. We show that, over an arbitrary commutative ring, the localizations of the categories of dg categories, of unital and of strictly unital A_∞ categories with respect to the corresponding classes of quasi-equivalences are all equivalent. The same result is also proved at the ∞-categorical level in the strictly unital case. As an application, we provide a new proof of the existence of internal Homs for the homotopy category of dg categories in terms of the category of unital A_∞ functors, thus yielding a complete proof of a claim by Kontsevich and Keller.

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Introduction

This paper extends and, at the same time, repairs some existing results about the homotopy categories of differential graded (dg from now on) and A_∞ categories. The study of these homotopy categories has grown during the last two decades and it has produced several remarkable results. Nonetheless most of them depend on the assumption that such categories are linear over a field. At first sight this might look like a mild assumption but, as soon as we start thinking of applications of dg or A_∞ categories to algebraic or geometric problems such as deformation theory, it becomes a priority to replace the ground field with any commutative ring.

This simple observation was the main incentive to reconsider our previous results in [5] whose proofs deeply used the assumption that the categories are linear over a field. Unfortunately, the effort to generalize our previous work drew our attention to the unpleasant presence in the literature

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of a couple of mistakes with deep repercussions on several papers, including [5]. We will come back to them later in the introduction. For now we want to stress that our effort to find a way out of them was not only successful but provided a wide generalization of all known results along the lines that we would like to outline now.

Let us consider the category **dgCat** consisting of (small) dg categories defined over a commutative ring $\Bbbk$. Due to the work of Tabuada [24], **dgCat** has a model category structure which allows one to consider its homotopy category $\mathrm{Ho}(\mathbf{dgCat})$, which is nothing but the localization of **dgCat** with respect to all quasi-equivalences. The latter being special dg functors which induce an equivalence at the homotopy level. If we can replace **dgCat** with the corresponding category of A_∞ categories, one can still consider its localization with respect to quasi-equivalences. Note that the category of A_∞ categories famously does not have a model structure with limits and colimits (see, for example, [5, Section 1.5]).

The delicate issue about A_∞ categories is that various notions of unit are available for them. One can indeed take the category $\mathbf{A_\infty Cat}$ of strictly unital A_∞ categories. Or, alternatively, the category $\mathbf{A_\infty Cat^u}$ of unital A_∞ categories. One could go further and consider cohomologically unital A_∞ categories $\mathbf{A_\infty Cat^c}$. We will discuss these subtleties in detail in Section 1.3. For now it is enough to keep in mind that we have natural faithful functors

$$\mathbf{dgCat} \hookrightarrow \mathbf{A_\infty Cat} \hookrightarrow \mathbf{A_\infty Cat^u} \hookrightarrow \mathbf{A_\infty Cat^c}.$$

While strictly unital A_∞ categories are natural generalizations of dg categories, unital ones are those who appear when dealing with Fukaya categories. If we work with categories linear over a commutative ring then $\mathbf{A_\infty Cat^u}$ and $\mathbf{A_\infty Cat^c}$ are different and the latter is, from many perspectives, hard to deal with and too coarse. But when the ground ring is actually a field, these two categories coincide. Thus, since the aim of this paper is to recover and extend the results in [5] to categories which are linear over a commutative ring, we will stick only to the first three categories in the above sequence of inclusions and to their localizations $\mathrm{Ho}(\mathbf{dgCat})$, $\mathrm{Ho}(\mathbf{A_\infty Cat})$ and $\mathrm{Ho}(\mathbf{A_\infty Cat^u})$ with respect to the corresponding classes of quasi-equivalences.

The need for a comparison between the (homotopy) category of dg categories and the one of A_∞ categories is pervasive. We will mention more applications later in the introduction. For now, it is worth recalling that the core of the Homological Mirror Symmetry Conjecture, due to Kontsevich [15], is indeed a comparison between dg enhancements of the bounded derived category of coherent sheaves on a Calabi–Yau threefold and the Fukaya category (hence an A_∞ category) on a mirror Calabi–Yau threefold.

In order to state our first main result we need to introduce an additional category. If $\mathbf{A}$ and $\mathbf{B}$ are unital A_∞ categories, we can consider the A_∞ category $\mathbf{Fun}_{\mathbf{A_\infty Cat^u}}(\mathbf{A},\mathbf{B})$ which will be carefully defined in Section 1.4. Its objects are the unital A_∞ functors from $\mathbf{A}$ to $\mathbf{B}$ and two unital A_∞ functors $\mathsf{F},\mathsf{G}\colon \mathbf{A}\to\mathbf{B}$ are equivalent $\mathsf{F}\approx\mathsf{G}$ if they are isomorphic in the 0-th cohomology of $\mathbf{Fun}_{\mathbf{A_\infty Cat^u}}(\mathbf{A},\mathbf{A})$. Hence we can take the quotient $\mathbf{A_\infty Cat^u}/\approx$ of $\mathbf{A_\infty Cat^u}$ with respect to this equivalence relation. One can go further and look at all h-projective unital A_∞ categories and form the full subcategory $\mathbf{A_\infty Cat^u_{hp}} \hookrightarrow \mathbf{A_\infty Cat^u}$. Recall that an A_∞ category $\mathbf{A}$ is *h-projective* if the complex of morphism $\mathbf{A}(A,B)$ is such, for all A,B in $\mathbf{A}$. From this it is clear that $\mathbf{A_\infty Cat^u_{hp}}$

and $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ coincide when $\Bbbk$ is a field. Anyway, in complete generality, we can form the quotient $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$.

With this in mind, we can finally state our first main result.

Theorem A. *The faithful functors* $\mathbf{dgCat} \hookrightarrow \mathbf{A}_\infty\mathbf{Cat} \hookrightarrow \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ *induce natural equivalences*

$$\mathrm{Ho}(\mathbf{dgCat}) \cong \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}) \cong \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}).$$

Moreover, these categories are equivalent to $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$.

The existence of the equivalence $\mathrm{Ho}(\mathbf{dgCat}) \cong \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})$ is the content of Theorem 3.6 while the equivalence $\mathrm{Ho}(\mathbf{dgCat}) \cong \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$ and the one between $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx$ are proved in Theorem 4.1. Note that, if $\Bbbk$ is a field, the last part of Theorem A just says that $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx$ are equivalent.

One important application of the result above is about uniqueness of enhancements for algebraic triangulated categories. Roughly, a triangulated category is algebraic if it admits a higher categorical model: either dg or A_∞ or ∞-stable categorical. The quest for the uniqueness of such a model was initiated by a very influential conjecture by Bondal, Larsen and Lunts in [3] for geometric triangulated categories. The conjecture was proved by Lunts and Orlov in the seminal paper [17] and the result was then further extended in [1] and [7] (see also [9]) up to the last and most general result in [4]. All these papers prove uniqueness of enhancements for larger classes of interesting triangulated categories using the language of dg categories. Theorem A immediately implies that such results extend to A_∞ enhancements which are linear over any commutative ring.

Finally, it is important to note that, following [22], Theorem A implies an analogous ∞-categorical version with several remarkable applications. Let us indeed denote by $\mathrm{Ho}(\mathbf{dgCat})_\infty$ and $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})_\infty$ the ∞-categorical enhancements of $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})$, respectively (see Section 3.2 for more details). We then have the following.

Theorem B. *The ∞-categories* $\mathrm{Ho}(\mathbf{dgCat})_\infty$ *and* $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})_\infty$ *are equivalent.*

We should note that such a result is nothing but Theorem 1.1 in [19] for strictly unital A_∞ categories. Unfortunately, as the authors later realized, the proof in loc. cit. turned out to be wrong. If we stick to dg or A_∞ categories which are linear over a field, Theorem B was proved in [22]. Actually, we will explain in Section 3.2 that the same argument as in [22], together with Proposition 3.1, yields a proof of Theorem B over an arbitrary commutative ring. It is worth pointing out that, as observed in [19, Remark 1.2], Theorem B combined with the results in [12] shows that Gepner–Haugseng's model for the collection of all ∞-categories enriched in chain complexes in [11] is equivalent to $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})_\infty$.

We now want to discuss our main and highly nontrivial application of Theorem A: a new proof for the existence of internal Homs in the homotopy category $\mathrm{Ho}(\mathbf{dgCat})$. In order to make this precise, recall that given two dg categories $\mathbf{A}_1$ and $\mathbf{A}_2$, one can form their tensor product $\mathbf{A}_1 \otimes \mathbf{A}_2$ in $\mathbf{dgCat}$. In order to get a well defined tensor product in $\mathrm{Ho}(\mathbf{dgCat})$ we need to *derive* it by setting $\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2 := \mathbf{A}_1 \otimes \mathbf{A}_2^{\mathrm{hp}}$, where $\mathbf{A}_2^{\mathrm{hp}}$ stands for a h-projective resolution of $\mathbf{A}_2$ (see Section 4.1 for more details). The main result in [25], later reproved in [6], shows that the tensor product $\otimes^{\mathbb{L}}$

has a right adjoint $\mathbb{R}\underline{Hom}$ in $\mathrm{Ho}(\mathbf{dgCat})$. Namely, we have a natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbb{R}\underline{Hom}(\mathbf{A}_2, \mathbf{A}_3)),$$

for $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ in $\mathbf{dgCat}$.

The astonishing fact is that, well before the appearance of [25], Kontsevich envisioned that such internal Homs should exist and could be described in terms of A_∞ functors between the corresponding dg categories. Such a claim, originally mentioned in [8], was later recasted by Keller in his ICM talk [14] (see Section 4.3 therein) in the following form.

Claim (Kontsevich, Keller)**.** *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be dg categories such that $\mathbf{A}_1$ is h-projective and the unit map $\Bbbk \to \mathbf{A}_1(A, A)$ admits a retraction as a morphism of complexes, for all $A \in \mathbf{A}_1$. Then the dg category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}_1, \mathbf{A}_2)$, whose objects are strictly unital A_∞ functors, is the category of internal Homs between $\mathbf{A}_1$ and $\mathbf{A}_2$.*

It turns out that, by using Theorem A, we can prove the following result which implies, as a special case, the claim above (see Remark 5.3). At the same time, due to its gorgeous generality, it provides a completely new proof of the result in [25] about the existence of internal Homs.

Theorem C. *Given three dg categories $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$, there is a natural bijection of sets*

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3))$$

proving that the symmetric monoidal category $\mathrm{Ho}(\mathbf{dgCat})$ is closed. In particular, we get a natural bijection of sets

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbf{A}_2) \xleftrightarrow{1:1} \mathrm{Isom}(H^0(\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_1^{\mathrm{hp}}, \mathbf{A}_2)))$$

which is compatible with compositions in the first and second entry.

Some comments are now in order here. First of all, the first part of Theorem C implies that the internal Hom dg category $\mathbb{R}\underline{Hom}(\mathbf{A}_2, \mathbf{A}_3)$ is isomorphic in $\mathrm{Ho}(\mathbf{dgCat})$ to the dg category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)$. The advantage is that, once we know that internal Homs can be described as suitable equivalence classes of A_∞ functors, then the claim about the compatibility with compositions becomes straightforward (see Section 5.2 for more details and a precise description of the composition). This is less easy to achieve if one keeps the description of internal Homs in terms of bimodules as in [25] (and [6]). Such a compatibility is indeed described as an open question in the introduction of [25]. Theorem C provides an easy answer to it.

Related work and further applications. The first comparison has to be made with our previous paper [5]. Besides the obvious observation that our new results imply essentially all the ones in [5], we should go back to our first claim in the introduction: the fact that this paper corrects and overcomes some mistakes in the literature.

It was proved by Lefèvre-Hasegawa [16, Theorem 3.2.1.1] for A_∞ algebras and later by Seidel [23, Lemma 2.1] that any cohomologically unital A_∞ algebra or category can be replaced with a strictly unital one, at least when we work over a field. Similarly, [16, Theorem 3.2.2.1] and [23, Remark 2.2] claim that the same is true for functors: an A_∞ functors between strictly unital A_∞ categories can be replaced by a strictly unital one, up to homotopy. Unfortunately, after carefully thinking about these claims, one realizes that none of them can be true in this generality. And the falsity of the first claim about categories (and algebras) was indeed later observed by Seidel in an erratum to [23].

While many technical parts of [5] remain valid (and will also be used in this paper) others, heavily relying on the two claims above, have to be revisited. For example, [5, Proposition 2.5] is easily seen to be false as soon as we consider A_∞ categories $\mathbf{A}$ such that the complex $\mathbf{A}(A, A)$ has trivial cohomology but it is not trivial, for some some A in $\mathbf{A}$. This produces a cascade of problems in the proof of [5, Theorem A] some of which can be overcome by readjusting the arguments while some of them needs the new (and at the same time more general) approach which we adopt in the present paper. A careful comparison shows that the new Theorem A replaces the old one for most of its parts once we observe that the categories of cohomologically unital and of unital A_∞ categories are the same, over a field. There is only one claim in [5] that is not covered by our new results: the equivalence between $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathbf{A}_\infty\mathbf{Cat}/\approx$. Not only we cannot prove such a claim but we actually expect it to be false (see Remark 4.14).

Moreover, for similar reasons, the description of the internal Homs in terms of strictly unital A_∞ functors has to be replaced by the one which uses unital A_∞ functors. The result is that the new Theorem C replaces and generalizes the old [5, Theorem B].

Finally, as we have already explained before, the techniques we develop to prove Theorem A for categories linear over a commutative ring, allow us to provide a complete proof to Theorem 1.1 for strictly unital A_∞ categories and contained in [19]. Actually this takes the form of Theorem B. The latter result together with Theorem C gives then access to the many very interesting applications discussed in the second part of [19] (see, in particular, Sections 3.3 and 4 therein).

Plan of the paper. In Section 1 we briefly recall the basic definitions and constructions which are used all along the paper. We refer to the existing literature for more details but an issue that we try to analyze carefully is the difference between the various notions of unit for A_∞ categories (see Section 1.3).

In Section 2 we show the existence of a crucial pair of adjoint functors between the category of dg categories and the one of A_∞ categories (for later use in Section 5, the key step of the proof needs to be treated in a more general setting, which makes Section 2 the more technical part of the paper). Here we work with non-unital categories, and the analysis has to be refined in Section 3 and Section 4 in order to deal with strictly unital and unital A_∞ categories, thus proving Theorem A. In Section 3.2 we outline the proof of Theorem B.

As for internal Homs, the proof of Theorem C, is carried out in Section 5 with a preliminary discussion about multifunctors in Section 5.1. One of the aims of Section 5 is to clarify the behaviour of composition for the new description of the dg category of internal Homs, which is part of the statement of Theorem C.

Notation and conventions. We assume that a universe containing an infinite set is fixed. Throughout the paper, we will simply call sets the members of this universe. In general the collection of objects of a category need not be a set: we will always specify if we are requiring this extra condition.

We work over a commutative ring $\Bbbk$. We will always assume that the collection of objects in a $\Bbbk$-linear category is a set.

The shift by an integer n of a graded $\Bbbk$-module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ will be denoted by $M[n] = \bigoplus_{i\in\mathbb{Z}} M^{n+i}$. If $x \in M$, we will often write $x[n]$ to denote the same element in $M[n]$. If x is homogeneous, say $x \in M^i$, then we set $\deg(x) := i$ and $\deg'(x) := i + 1$.

We recall the Koszul sign rule: if $f\colon M \to M'$ and $g\colon N \to N'$ are morphisms of graded $\Bbbk$-modules, with g homogeneous, then $f \otimes g\colon M \otimes N \to M' \otimes N'$ maps $x \otimes y$, with x homogeneous, to $(-1)^{\deg(g)\deg(x)} f(x) \otimes g(y)$.

Complexes (or dg modules) are cohomological (namely, the differential has degree $+1$).

1. Preliminaries on A_∞ categories and functors

In this section we provide a concise introduction to A_∞ categories and functors. In the whole paper the subtle relation between the various notions of unit is crucial. We provide here the basic definitions and properties which will be used later.

As in [5], we will follow the sign conventions in [16], which are different from (but equivalent to) those used in other references, like [2] and [23]. In particular, given graded $\Bbbk$-modules $M_1, \dots, M_i, N$, a $\Bbbk$-linear map $f\colon M_i \otimes \cdots \otimes M_1 \to N$ of degree n is identified with the $\Bbbk$-linear map

$$(1.1)\qquad (-1)^{n+i-1}\iota_N^{-1} \circ f \circ (\iota_{M_i} \otimes \cdots \otimes \iota_{M_1})\colon M_i[1] \otimes \cdots \otimes M_1[1] \to N[1]$$

of degree $n + i - 1$, where $\iota_M\colon M[1] \to M$ is the natural isomorphism of degree 1, for every graded $\Bbbk$-module M.

1.1. Non-unital A_∞ categories, functors and natural transformations.

We start by recalling the explicit definitions of non-unital A_∞ categories, functors and (pre)natural transformations. A conceptual explanation of the otherwise mysterious formulas appearing in this section will be given in Section 1.2.

Definition 1.1. A *non-unital A_∞ category* $\mathbf{A}$ consists of a set of objects $\mathrm{Ob}(\mathbf{A})$, of graded $\Bbbk$-modules $\mathbf{A}(A, A')$ for every $A, A' \in \mathbf{A}$ and of $\Bbbk$-linear maps of degree $2 - i$

$$(1.2)\qquad \mathrm{m}^i = \mathrm{m}^i_{\mathbf{A}}\colon \mathbf{A}(A_{i-1}, A_i) \otimes \cdots \otimes \mathbf{A}(A_0, A_1) \to \mathbf{A}(A_0, A_i),$$

for every $i > 0$ and every $A_0, \dots, A_i \in \mathbf{A}$. The maps must satisfy the A_∞ *associativity relations*

$$(1.3)\qquad \sum_{k=1}^{n}\sum_{i=0}^{n-k} (-1)^{i+k(n-i-k)}\mathrm{m}^{n-k+1} \circ (\mathrm{id}^{\otimes n-i-k} \otimes \mathrm{m}^k \otimes \mathrm{id}^{\otimes i}) = 0$$

for every $n > 0$.

In particular, (1.3) with $n = 1$ shows that m^1 defines a differential on each $\mathbf{A}(A, A')$, which will always be regarded as a complex in this way. It is also important to observe that m^1 satisfies the graded Leibniz rule with respect to the composition defined by m^2 (by the case $n = 2$) and that m^2 is associative, up to a homotopy defined by m^3 (by the case $n = 3$). This implies that we obtain the non-unital graded *cohomology category* $H(\mathbf{A})$ of $\mathbf{A}$ such that $\mathrm{Ob}(H(\mathbf{A})) = \mathrm{Ob}(\mathbf{A})$,

$$H(\mathbf{A})(A, A') = \bigoplus_i H^i\big(\mathbf{A}(A_1, A_2)\big)$$

for every $A, A' \in \mathbf{A}$ and (associative) composition induced from m^2.

Definition 1.2. A *non-unital A_∞ functor* $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ between two non-unital A_∞ categories $\mathbf{A}$ and $\mathbf{B}$ is a collection $\mathsf{F} = \{\mathsf{F}^i\}_{i\geq 0}$, where $\mathsf{F}^0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ is a map of sets and

$$\mathsf{F}^i\colon \mathbf{A}(A_{i-1}, A_i) \otimes \cdots \otimes \mathbf{A}(A_0, A_1) \to \mathbf{B}(\mathsf{F}^0(A_0), \mathsf{F}^0(A_i)), \tag{1.4}$$

for $i > 0$, are $\Bbbk$-linear maps of degree $1 - i$, for every $A_0, \ldots, A_i \in \mathbf{A}$. The maps must satisfy the following relations

$$\begin{aligned}\sum_{k=1}^{n}\sum_{i=0}^{n-k}(-1)^{i+k(n-i-k)}\mathsf{F}^{n-k+1} \circ (\mathrm{id}^{\otimes n-i-k} \otimes \mathrm{m}^k_{\mathbf{A}} \otimes \mathrm{id}^{\otimes i})& \\ = \sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\ldots,i_r>0}} (-1)^{\sum_{t=1}^{r-1}\sum_{u=t+1}^{r}(1-i_t)i_u}\mathrm{m}^r_{\mathbf{B}} \circ (\mathsf{F}^{i_r} \otimes \cdots \otimes \mathsf{F}^{i_1}),&\end{aligned} \tag{1.5}$$

for every $n > 0$. A non-unital A_∞ functor F is *strict* if $\mathsf{F}^i = 0$ for every $i > 1$.

From (1.5) with $n = 1$ we see that F^1 commutes with the differentials m^1. Moreover, F^1 preserves the compositions m^2, up to a homotopy defined by F^2 (by the case $n = 2$). It follows that F^0 and F^1 induces a non-unital graded functor $H(\mathsf{F})\colon H(\mathbf{A}) \to H(\mathbf{B})$.

Remark 1.3. A non-unital A_∞ category $\mathbf{A}$ such that $\mathrm{m}^i = 0$ for all $i > 2$ is called a *non-unital dg category*; for such categories m^1 and m^2 are usually denoted by d and $\circ$. A strict non-unital A_∞ functor F between two dg categories is called a *non-unital dg functor*; in this case one often writes F instead of F^0 or F^1. There is a category $\mathbf{A_\infty Cat^n}$ (with objects the non-unital A_∞ categories and morphisms the non-unital A_∞ functors) which contains as a subcategory $\mathbf{dgCat^n}$ (with objects the non-unital dg categories and morphisms the non-unital dg functors). While the composition in $\mathbf{dgCat^n}$ is the obvious one, the composition in $\mathbf{A_\infty Cat^n}$ is more subtle (see Section 1.2); however, we will not need its explicit definition.

Definition 1.4. Given $\mathsf{F}, \mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A_\infty Cat^n}$, a *prenatural transformation* $\theta\colon \mathsf{F} \to \mathsf{G}$ of degree p is given by $\Bbbk$-linear maps of degree $p - i$

$$\theta^i\colon \mathbf{A}(A_{i-1}, A_i) \otimes \cdots \otimes \mathbf{A}(A_0, A_1) \to \mathbf{B}\big(\mathsf{F}^0(A_0), \mathsf{G}^0(A_i)\big) \tag{1.6}$$

for every $i \geq 0$ and every $A_0, \ldots, A_i \in \mathbf{A}$. We say that θ is a *natural transformation* if

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=0}^{n-k}(-1)^{i+k(n-i-k)}\theta^{n-k+1} \circ (\mathrm{id}^{\otimes n-i-k} \otimes \mathrm{m}^k_{\mathbf{A}} \otimes \mathrm{id}^{\otimes i}) \\ &+ \sum_{\substack{i_1+\cdots+i_r+k+j_1+\cdots+j_s=n\\ i_1,\ldots,i_r,j_1,\ldots,j_s>0,k\geq 0}} (-1)^{p+r(p-1)+\sum_{t=1}^{r}(1-i_t)(n-\sum_{u=1}^{t-1} i_u)+(p-k)\sum_{t=1}^{s} j_t+\sum_{t=1}^{s-1}\sum_{u=t+1}^{s}(1-j_t)j_u} \\ &\qquad \mathrm{m}^{r+s+1}_{\mathbf{B}} \circ (\mathsf{G}^{j_s} \otimes \cdots \otimes \mathsf{G}^{j_1} \otimes \theta^k \otimes \mathsf{F}^{i_r} \otimes \cdots \otimes \mathsf{F}^{i_1}) = 0\end{aligned} \tag{1.7}$$

for every $n \geq 0$.

Observe that θ^0 can be identified with a collection of elements $\theta^0_A \in \mathbf{B}\big(\mathsf{F}^0(A), \mathsf{G}^0(A)\big)^p$ for every $A \in \mathbf{A}$. Moreover, (1.7) with $n = 0$ shows that these elements are closed, whereas the case $n = 1$

implies that their images in cohomology define a natural transformation $H(\theta)\colon H(\mathsf{F}) \to H(\mathsf{G})$ of degree p.

1.2. **Reminder on the bar and cobar constructions.** This section is a quick reminder about the bar and cobar constructions. In [5, Sections 1.2 and 1.3] the reader can find definitions and properties of some notions which are not recalled here, like those of (graded or dg) quiver, cocategory and cofunctor. For a more detailed presentation see also [2].

We denote by $\mathbf{dgcoCat^n}$ the category whose objects are non-unital cocomplete dg cocategories and whose morphisms are non-unital dg cofunctors.

Given $\mathbf{A} \in \mathbf{A_\infty Cat^n}$, the *bar construction* $\mathrm{B}_\infty(\mathbf{A}) \in \mathbf{dgcoCat^n}$ associated to $\mathbf{A}$ is simply defined to be $\overline{\mathrm{T}}^c(\mathbf{A}[1])$ (where $\mathbf{A}$ is viewed as a graded quiver) as a non-unital graded cocategory. As for the differential, an arbitrary choice of maps $\mathrm{m}^i_{\mathbf{A}}$ as in (1.2) determines a morphism of graded quivers $\overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \mathbf{A}[1]$ of degree 1 (recall (1.1)), which extends uniquely to a $(\mathrm{id}_{\overline{\mathrm{T}}^c(\mathbf{A}[1])}, \mathrm{id}_{\overline{\mathrm{T}}^c(\mathbf{A}[1])})$-coderivation $d_{\mathbf{A}}\colon \overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{A}[1])$ of degree 1. It is easy to see that $d_{\mathbf{A}} \circ d_{\mathbf{A}} = 0$ if and only if (1.3) holds for every $n > 0$, in which case we set $\mathrm{B}_\infty(\mathbf{A}) := (\overline{\mathrm{T}}^c(\mathbf{A}[1]), d_{\mathbf{A}})$.

Similarly, $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A_\infty Cat^n}$ induces $\mathrm{B}_\infty(\mathsf{F})\colon \mathrm{B}_\infty(\mathbf{A}) \to \mathrm{B}_\infty(\mathbf{B})$ in $\mathbf{dgcoCat^n}$. More precisely, an arbitrary choice of maps $\mathsf{F}^0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and F^i for $i > 0$ as in (1.4) determines a morphism of graded quivers $\overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \mathbf{B}[1]$ of degree 0, which extends uniquely to a graded cofunctor $\widehat{\mathsf{F}}\colon \overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{B}[1])$. Then one can check that $d_{\mathbf{B}} \circ \widehat{\mathsf{F}} = \widehat{\mathsf{F}} \circ d_{\mathbf{A}}$ if and only if (1.5) holds for every $n > 0$, in which case we set $\mathrm{B}_\infty(\mathsf{F}) := \widehat{\mathsf{F}}$. Moreover, the composition in $\mathbf{A_\infty Cat^n}$ is defined in such a way that

$$\mathrm{B}_\infty\colon \mathbf{A_\infty Cat^n} \to \mathbf{dgcoCat^n}$$

is a functor, which actually turns out to be fully faithful.

Finally, given $\mathsf{F}, \mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A_\infty Cat^n}$ (more generally, F and G could be given by arbitrary maps $\mathsf{F}^0, \mathsf{G}^0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and $\mathsf{F}^i, \mathsf{G}^i$ for $i > 0$ as in (1.4)), a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{G}$ of degree p determines $\Bbbk$-linear maps $\mathrm{T}^c(\mathbf{A}[1])(A, A') \to \mathbf{B}[1]\big(\mathsf{F}^0(A), \mathsf{G}^0(A')\big)$ of degree $p-1$ for every $A, A' \in \mathbf{A}$, which extend uniquely to a $(\widehat{\mathsf{F}}, \widehat{\mathsf{G}})$-coderivation $\widehat{\theta}\colon \mathrm{T}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{B}[1])$ of degree $p-1$. Here we still denote by $\widehat{\mathsf{F}}, \widehat{\mathsf{G}}\colon \mathrm{T}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{B}[1])$ the extensions by 0 of $\widehat{\mathsf{F}}$ and $\widehat{\mathsf{G}}$. Again, it can be easily proved that $d_{\mathbf{B}} \circ \widehat{\theta} + (-1)^p \widehat{\theta} \circ d_{\mathbf{A}} = 0$ (where we still denote by $d_{\mathbf{A}}\colon \mathrm{T}^c(\mathbf{A}[1]) \to \mathrm{T}^c(\mathbf{A}[1])$ the extensions by 0 of $d_{\mathbf{A}}$) if and only if θ is a natural transformation.

Remark 1.5. Given $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A_\infty Cat^n}$ and a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{F}$ of degree 1 such that $\theta^0 = 0$, we can define $\mathsf{G}^0 := \mathsf{F}^0$ and $\mathsf{G}^i := \mathsf{F}^i + \theta^i$ for $i > 0$. Then we can regard θ as a prenatural transformation $\mathsf{F} \to \mathsf{G}$, and it is not difficult to show that in this way $\widehat{\mathsf{G}} = \widehat{\mathsf{F}} + \widehat{\theta}$. This clearly implies that G is a non-unital A_∞ functor if and only if $\theta\colon \mathsf{F} \to \mathsf{G}$ is a natural transformation.

As a matter of notation, we set

$$\mathrm{B} := \mathrm{B}_\infty|_{\mathbf{dgCat^n}}\colon \mathbf{dgCat^n} \to \mathbf{dgcoCat^n},$$

which is a faithful (but not full) functor. Dually, the *cobar construction* yields a faithful (but not full) functor

$$\Omega\colon \mathbf{dgcoCat^n} \to \mathbf{dgCat^n}.$$

In particular, for $\mathbf{C} \in \mathbf{dgcoCat^n}$, $\Omega(\mathbf{C})$ is simply defined to be $\overline{\mathrm{T}}(\mathbf{C}[-1])$ as a non-unital graded category, with differential induced from the differential and the cocomposition in $\mathbf{C}$.

By [5, Proposition 1.21] there is an adjunction

$$\Omega\colon \mathbf{dgcoCat^n} \rightleftarrows \mathbf{dgCat^n} : \mathrm{B},$$

with counit denoted by $\alpha\colon \Omega\circ \mathrm{B}\to \mathrm{id}_{\mathbf{dgCat^n}}$ and unit denoted by $\beta\colon \mathrm{id}_{\mathbf{dgcoCat^n}}\to \mathrm{B}\circ\Omega$. Since B_∞ is fully faithful, for every $\mathbf{A}\in \mathbf{A_\infty Cat^n}$ there exists unique $\gamma_{\mathbf{A}}\in \mathbf{A_\infty Cat^n}(\mathbf{A},\Omega(\mathrm{B}_\infty(\mathbf{A})))$ such that

$$\beta_{\mathrm{B}_\infty(\mathbf{A})} = \mathrm{B}_\infty(\gamma_{\mathbf{A}})\colon \mathrm{B}_\infty(\mathbf{A})\to \mathrm{B}_\infty(\Omega(\mathrm{B}_\infty(\mathbf{A}))) = \mathrm{B}(\Omega(\mathrm{B}_\infty(\mathbf{A}))).$$

Denoting by $\mathsf{I}^{\mathsf{n}}\colon \mathbf{dgCat^n}\to \mathbf{A_\infty Cat^n}$ the inclusion functor and setting

$$\mathsf{U}^{\mathsf{n}} := \Omega\circ \mathrm{B}_\infty\colon \mathbf{A_\infty Cat^n}\to \mathbf{dgCat^n},$$

it is clear that the A_∞ functors $\gamma_{\mathbf{A}}\colon \mathbf{A}\to \Omega(\mathrm{B}_\infty(\mathbf{A})) = \mathsf{U}^{\mathsf{n}}(\mathbf{A})$ (for $\mathbf{A}\in \mathbf{A_\infty Cat^n}$) define a natural transformation $\gamma\colon \mathrm{id}_{\mathbf{A_\infty Cat^n}}\to \mathsf{I}^{\mathsf{n}}\circ \mathsf{U}^{\mathsf{n}}$.

1.3. **Notions of unity.** Now we need to discuss the various notions of units which will be used in the rest of the paper.

Definition 1.6. A *cohomologically unital A_∞ category* is a non-unital A_∞ category $\mathbf{A}$ such that $H(\mathbf{A})$ is a category (i.e. $H(\mathbf{A})$ is unital).

Definition 1.7 ([18, Definition 7.3 and Lemma 7.4])**.** An A_∞ category $\mathbf{A}$ is *unital* if it is cohomologically unital and, denoting (for every $A\in\mathbf{A}$) by $e_A\in \mathbf{A}(A,A)$ a (closed degree 0) morphism representing $\mathrm{id}_A\in H(\mathbf{A})(A,A)$, the following morphisms of complexes

$$\mathrm{m}^2(-\otimes e_A)\colon \mathbf{A}(A,A')\to \mathbf{A}(A,A') \qquad \mathrm{m}^2(e_A\otimes -)\colon \mathbf{A}(A',A)\to \mathbf{A}(A',A)$$

are homotopic to the identity for every $A,A'\in\mathbf{A}$.

Definition 1.8. A *strictly unital A_∞ category* is a non-unital A_∞ category $\mathbf{A}$ such that for every $A\in\mathbf{A}$ there exists a degree 0 morphisms $\mathrm{id}_A\in\mathbf{A}(A,A)$ satisfying the following properties:

(1) $\mathrm{m}^2(-\otimes \mathrm{id}_A) = \mathrm{id}_{\mathbf{A}(A,A')}$ and $\mathrm{m}^2(\mathrm{id}_A\otimes -) = \mathrm{id}_{\mathbf{A}(A',A)}$ for every $A,A'\in\mathbf{A}$;
(2) $\mathrm{m}^i(f_i\otimes\cdots\otimes f_1) = 0$ if $i\neq 2$ and $f_j = \mathrm{id}_A$ for some $j\in\{1,\dots,i\}$ and some $A\in\mathbf{A}$.

For a non-unital A_∞ category $\mathbf{A}$, its *augmentation* is the strictly unital A_∞ category $\mathbf{A}^+$ such that $\mathrm{Ob}(\mathbf{A}^+) = \mathrm{Ob}(\mathbf{A})$ and

$$\mathbf{A}^+(A,A') = \begin{cases} \mathbf{A}(A,A') & \text{if } A\neq A' \\ \mathbf{A}(A,A')\oplus \Bbbk\, 1_A & \text{if } A = A', \end{cases}$$

with $\mathrm{m}^i_{\mathbf{A}^+}$ the unique extension of $\mathrm{m}^i_{\mathbf{A}}$ such that the additional morphisms 1_A is the unit of A in $\mathbf{A}^+$, for every $A\in\mathbf{A}$ and every $i>0$. To avoid confusion, when $\mathbf{A}$ is strictly unital, the unit in $\mathbf{A}$ is denoted by id_A while the one on $\mathbf{A}^+$ is 1_A, for every $A\in\mathbf{A}$.

Similarly, we get *cohomologically unital dg categories*, *unital dg categories* and *strictly unital dg categories*. In accordance to the existing literature, strictly unital dg categories will be simply referred to as *dg categories*.

Example 1.9. (i) In the special case of (dg or A_∞) categories with only one object, then, for obvious reasons, we will talk about (dg or A_∞) algebras.

(ii) Given two (non-unital, cohomologically unital, unital or strictly unital) dg categories $\mathbf{A}$ and $\mathbf{B}$, we can define a (non-unital, cohomologically unital, unital or strictly unital) dg category $\mathbf{A} \otimes \mathbf{B}$, which is the tensor product of $\mathbf{A}$ and $\mathbf{B}$. Its objects are the pairs (A, B) with $A \in \mathbf{A}$ and $B \in \mathbf{B}$, while $(\mathbf{A} \otimes \mathbf{B})\big((A, B), (A', B')\big) = \mathbf{A}(A, A') \otimes \mathbf{B}(B, B')$. If $\mathbf{A}$ and $\mathbf{B}$ are A_∞ categories, then defining an appropriate tensor product is a more delicate issue which will be discussed in [21].

There is also a notion of homotopy unital A_∞ category (which will not be used in this paper), such that the following implications hold for A_∞ categories (see [18, Section 8.12.]):

$$\text{strictly unital} \implies \text{homotopy unital} \implies \text{unital} \implies \text{cohomologically unital}$$

Remark 1.10. If $\Bbbk$ is a field, then an A_∞ category is unital if and only if it is cohomologically unital. This is simply due to the fact that, over a field, two morphisms of complexes are homotopic if they induce the same map in cohomology.

Of course, there are also the corresponding notions of strictly unital, unital and cohomologically unital A_∞ functors (see [23, pp 23] and [18, Definition 8.1. and Proposition 8.2.]).

Definition 1.11. Let $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ be a non-unital A_∞ functor.

F is *cohomologically unital* (respectively *unital*) if $\mathbf{A}$ and $\mathbf{B}$ are cohomologically unital (respectively unital) and $H(\mathsf{F})$ is unital.

F is *strictly unital* if $\mathbf{A}$ and $\mathbf{B}$ are strictly unital and the following properties are satisfied:

(1) $\mathsf{F}^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)}$ for every $A \in \mathbf{A}$;
(2) $\mathsf{F}^i(f_i \otimes \cdots \otimes f_1) = 0$ if $i > 1$ and $f_j = \mathrm{id}_A$ for some $j \in \{1, \dots, i\}$ and some $A \in \mathbf{A}$.

Remark 1.12. Recalling Remark 1.10, we see that over a field there is no distinction between unital and cohomologically unital.

The following definition is only partially standard.

Definition 1.13. A non-unital A_∞ functor $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ is a *quasi-isomorphism* (respectively a *homotopy isomorphism*) if F^0 is bijective and $\mathsf{F}^1\colon \mathbf{A}(A, A') \to \mathbf{B}(\mathsf{F}^0(A), \mathsf{F}^0(A'))$ is a quasi-isomorphism (respectively a homotopy equivalence) of complexes for every $A, A' \in \mathbf{A}$.

When $\mathbf{B}$ is cohomologically unital (respectively unital), $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ is a *quasi-equivalence* (respectively a *homotopy equivalence*) if $H(\mathsf{F})$ is essentially surjective and $\mathsf{F}^1\colon \mathbf{A}(A, A') \to \mathbf{B}(\mathsf{F}^0(A), \mathsf{F}^0(A'))$ is a quasi-isomorphism (respectively a homotopy equivalence) of complexes for every $A, A' \in \mathbf{A}$.

Remark 1.14. Clearly every homotopy isomorphism (respectively homotopy equivalence) is a quasi-isomorphism (respectively quasi-equivalence), and the viceversa holds if $\Bbbk$ is a field.

We will denote by $\mathbf{A_\infty Cat}$ (respectively $\mathbf{A_\infty Cat^u}$) the subcategory of $\mathbf{A_\infty Cat^n}$ whose objects are strictly unital (respectively unital) A_∞ categories and whose morphisms are strictly unital (respectively unital) A_∞ functors. Similarly, $\mathbf{dgCat}$ denotes the subcategory of $\mathbf{dgCat^n}$ whose objects are strictly unital dg categories and whose morphisms are strictly unital dg functors. Moreover, $\mathbf{A_\infty Cat_{dg}}$ will be the full subcategory of $\mathbf{A_\infty Cat}$ whose objects are dg categories.

Remark 1.15. If $\Bbbk$ is a field, $\mathbf{A_\infty Cat^u}$ coincides with what was denoted by $\mathbf{A_\infty Cat^c}$ in [5].

In order to study the relation between $\mathbf{A_\infty Cat^u}$ and $\mathbf{dgCat}$, later we will need the following result (which comes from an A_∞ categorical version of Yoneda's lemma).

Lemma 1.16. *Given* $\mathbf{A} \in \mathbf{A_\infty Cat^u}$*, there exists a homotopy isomorphism* $\mathsf{Y}_\mathbf{A} \colon \mathbf{A} \to \mathbf{R_A}$ *with* $\mathbf{R_A} \in \mathbf{dgCat}$.

Proof. See [2, Corollary 1.4]. □

Coming to prenatural transformations, we will consider the following notion.

Definition 1.17. Given $\mathsf{F}, \mathsf{G} \colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A_\infty Cat^n}$ with $\mathbf{A}$ strictly unital, a prenatural transformation $\theta \colon \mathsf{F} \to \mathsf{G}$ is *strictly unital* if $\theta^i(f_i \otimes \cdots \otimes f_1) = 0$ whenever $i > 0$ and there exists $j \in \{1, \dots, i\}$ such that $f_j = \mathrm{id}_A$ for some $A \in \mathbf{A}$.

1.4. **Category of functors and equivalence relations.** Given $\mathbf{A}, \mathbf{B} \in \mathbf{A_\infty Cat^n}$, there is a natural non-unital A_∞ category $\mathbf{Fun_{A_\infty Cat^n}}(\mathbf{A}, \mathbf{B})$, whose set of objects is $\mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$ and whose morphisms are prenatural transformations (see [5, Section 1.4]). As for the maps $\mathrm{m}^i = \mathrm{m}^i_{\mathbf{Fun_{A_\infty Cat^n}}(\mathbf{A},\mathbf{B})}$, for our aims it is enough to know that $\mathrm{m}^1(\theta)^n$ is given by the left-hand-side of (1.7), for every $\mathsf{F}, \mathsf{G} \in \mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$ and every prenatural transformation $\theta \colon \mathsf{F} \to \mathsf{G}$ of degree p.

Observe that, if $\mathbf{B}$ is (strictly) unital or is a dg category, then $\mathbf{Fun_{A_\infty Cat^n}}(\mathbf{A}, \mathbf{B})$ has the same property. When $\mathbf{A}$ and $\mathbf{B}$ are strictly unital (respectively unital), the full A_∞ subcategory of $\mathbf{Fun_{A_\infty Cat^n}}(\mathbf{A}, \mathbf{B})$ whose set of objects is $\mathbf{A_\infty Cat}(\mathbf{A}, \mathbf{B})$ (respectively $\mathbf{A_\infty Cat^u}(\mathbf{A}, \mathbf{B})$) will be denoted by $\mathbf{Fun_{A_\infty Cat}}(\mathbf{A}, \mathbf{B})$ (respectively $\mathbf{Fun_{A_\infty Cat^u}}(\mathbf{A}, \mathbf{B})$).

Definition 1.18. Let $\mathsf{F}, \mathsf{G} \in \mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$.

(i) F and G are *weakly equivalent* (denoted by $\mathsf{F} \approx \mathsf{G}$) if $\mathbf{B}$ is unital and $\mathsf{F} \cong \mathsf{G}$ in the (unital) category $H^0\big(\mathbf{Fun_{A_\infty Cat^n}}(\mathbf{A}, \mathbf{B})\big)$.

(ii) F and G are *homotopic* (denoted by $\mathsf{F} \sim \mathsf{G}$) if $\mathsf{F}^0 = \mathsf{G}^0$ and there exists a prenatural transformation $\theta \colon \mathsf{F} \to \mathsf{G}$ of degree 0 such that $\theta^0 = 0$ and $\mathsf{G}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for every $i > 0$.

Remark 1.19. As it is proved in [23, Section (1h)], homotopy is an equivalence relation. Since it will be useful later, we also point out the following property, which can be directly deduced from the proof. Let $\mathsf{F}, \mathsf{G}, \mathsf{H} \in \mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$ with $\mathsf{F} \sim \mathsf{G}$ and $\mathsf{G} \sim \mathsf{H}$ through homotopies θ_1 and θ_2, respectively. Assuming that there exists $n > 0$ such that $\theta_2^i = 0$ for $i < n$, then $\mathsf{F} \sim \mathsf{H}$ through a homotopy θ such that $\theta^i = \theta_1^i$ for $i < n$.

Remark 1.20. If $\mathsf{F} \in \mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$ and $\theta \colon \mathsf{F} \to \mathsf{F}$ is a prenatural transformation of degree 0 such that $\theta^0 = 0$, then there exists $\mathsf{G} \in \mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$ such that $\mathsf{F} \sim \mathsf{G}$ with $\mathsf{G}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for every $i > 0$, where θ is regarded as a prenatural transformation $\mathsf{F} \to \mathsf{G}$. Indeed, we set $\mathsf{G}^0 := \mathsf{F}^0$ and, for $n > 0$, we define inductively G^n as the sum of F^n and of the left-hand-side of (1.7) (which involves G^i only with $i < n$, since $\theta^0 = 0$). Then $\mathsf{G}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for $i > 0$ by construction, while the fact that $\mathsf{G} \in \mathbf{A_\infty Cat^n}(\mathbf{A}, \mathbf{B})$ follows from Remark 1.5, taking into account that $\mathrm{m}^1(\theta)$ is a natural transformation (because $\mathrm{m}^1 \circ \mathrm{m}^1 = 0$) of degree 1 and clearly $\mathrm{m}^1(\theta)^0 = 0$.

Since $\approx$ is compatible with compositions, from the category $\mathbf{A}_\infty\mathbf{Cat^u}$ one can obtain a quotient category $\mathbf{A}_\infty\mathbf{Cat^u}/\approx$ with the same objects and whose morphisms are given by

$$\mathbf{A}_\infty\mathbf{Cat^u}/\approx(\mathbf{A},\mathbf{B}) := \mathbf{A}_\infty\mathbf{Cat^u}(\mathbf{A},\mathbf{B})/\approx .$$

Similarly one can construct $\mathbf{A}_\infty\mathbf{Cat}/\approx$ from $\mathbf{A}_\infty\mathbf{Cat}$.

Later we will need the following results.

Lemma 1.21. *Let $\mathsf{F},\mathsf{F}' \in \mathbf{A}_\infty\mathbf{Cat^n}(\mathbf{A},\mathbf{B})$ with $\mathbf{B}$ unital. If there exists a natural transformation $\theta\colon \mathsf{F}\to\mathsf{F}'$ of degree 0 such that $H(\theta)\colon)H(\mathsf{F})\to H(\mathsf{F}')$ is an isomorphism, then $\mathsf{F}\approx\mathsf{F}'$.*

Proof. It follows from [18, Proposition 7.15]. □

Lemma 1.22. *Let $\mathsf{F}\colon \mathbf{A}\to\mathbf{B}$ be a homotopy equivalence (in particular, $\mathbf{B}$ is unital). Then $\mathbf{A}$ and F are also unital and there exists $\mathsf{G}\in\mathbf{A}_\infty\mathbf{Cat^u}(\mathbf{B},\mathbf{A})$ such that $\mathsf{G}\circ\mathsf{F}\approx\mathrm{id}_{\mathbf{A}}$ and $\mathsf{F}\circ\mathsf{G}\approx\mathrm{id}_{\mathbf{B}}$ (hence the image of F is an isomorphism in $\mathbf{A}_\infty\mathbf{Cat^u}/\approx$).*

Proof. It follows from [18, Theorem 8.8]. □

Corollary 1.23. *Let $\mathsf{F},\mathsf{F}' \in \mathbf{A}_\infty\mathbf{Cat^n}(\mathbf{A},\mathbf{B})$ with $\mathbf{B}$ unital. Then $\mathsf{F}\approx\mathsf{F}'$ in each of the following cases.*

1. *$\mathbf{B}$ is strictly unital and $\mathsf{F}\sim\mathsf{F}'$.*
2. *There exists a homotopy equivalence $\mathsf{G}\colon\mathbf{B}\to\mathbf{B}'$ such that $\mathsf{G}\circ\mathsf{F}\approx\mathsf{G}\circ\mathsf{F}'$.*
3. *There exists a homotopy equivalence $\mathsf{H}\colon\mathbf{A}'\to\mathbf{A}$ such that $\mathsf{F}\circ\mathsf{H}\approx\mathsf{F}'\circ\mathsf{H}$.*

Proof. If $\mathbf{B}$ is strictly unital and $\theta\colon\mathsf{F}\to\mathsf{F}'$ is a homotopy, it is straightforward to check that the prenatural transformation $\widetilde{\theta}\colon\mathsf{F}\to\mathsf{F}'$ defined by $\widetilde{\theta}^i := \theta^i$ for $i>0$ and $\widetilde{\theta}^0(A) := \mathrm{id}_{\mathsf{F}^0(A)}$ for every $A\in\mathbf{A}$ is a natural transformation (see also the paragraph before [23, Lemma 2.5]). Thus part (1) follows from Lemma 1.21, whereas parts (2) and (3) are easy consequences of Lemma 1.22. □

2. The non-unital case

Using the notation introduced in Section 1.2, we first state the following result.

Proposition 2.1. *There is an adjunction*

$$\mathsf{U}^{\mathsf{n}}\colon \mathbf{A}_\infty\mathbf{Cat^n} \rightleftarrows \mathbf{dgCat^n} : \mathsf{I}^{\mathsf{n}}, \tag{2.1}$$

whose unit is $\gamma\colon \mathrm{id}_{\mathbf{A}_\infty\mathbf{Cat^n}}\to\mathsf{I}^{\mathsf{n}}\circ\mathsf{U}^{\mathsf{n}}$ and whose counit is $\alpha\colon\mathsf{U}^{\mathsf{n}}\circ\mathsf{I}^{\mathsf{n}} = \Omega\circ\mathrm{B}\to\mathrm{id}_{\mathbf{dgCat^n}}$. Moreover, $\gamma_{\mathbf{A}}$ (for every $\mathbf{A}\in\mathbf{A}_\infty\mathbf{Cat^n}$) and $\alpha_{\mathbf{B}}$ (for every $\mathbf{B}\in\mathbf{dgCat^n}$) are homotopy isomorphisms.

It is easy to see that the same proof of [5, Proposition 1.22] can be adapted to work when $\Bbbk$ is an arbitrary commutative ring and with homotopy isomorphism in place of quasi-isomorphism. However, when dealing with the strictly unital case in Section 3, it will be useful to know the crucial proof of the fact that, for every $\mathbf{A}\in\mathbf{A}_\infty\mathbf{Cat^n}$ and every $A,A'\in\mathbf{A}$,

$$\gamma^1_{\mathbf{A}}\colon \mathbf{A}(A,A')\to\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A,A') = \Omega(\mathrm{B}_\infty(\mathbf{A}))(A,A') \tag{2.2}$$

is a homotopy equivalence of complexes. Actually we will prove a more general statement in Section 2.3, from which we will also deduce a result that will be needed in Section 5. To this aim, we first prove a technical result in Section 2.1 and then introduce the morphism which replaces (2.2) in a more general setting in Section 2.2.

2.1. **A criterion for homotopy equivalence.** We will need the following general and possibly known result about filtered complexes. We include the proof since we could not find a suitable reference.

Lemma 2.2. *Let C be a complex of $\Bbbk$-modules endowed with an ascending and exhaustive filtration $\mathrm{F}^n C$ (with $n \in \mathbb{N}$) such that $\mathrm{F}^0 C = 0$. Assume that for every $n > 1$ the exact sequence of complexes*

$$0 \to \mathrm{F}^{n-1}C \to \mathrm{F}^n C \to \mathrm{gr}^n C := (\mathrm{F}^n C)/(\mathrm{F}^{n-1}C) \to 0$$

splits as a sequence of graded modules and the complex $\mathrm{gr}^n C$ is null-homotopic. Then the inclusion $\mathrm{F}^1 C \hookrightarrow C$ is a homotopy equivalence of complexes.

Proof. We can assume that as a graded module $C = \bigoplus_{n>0} C_n$ with $\mathrm{F}^n C = \bigoplus_{0<m\le n} C_m$ and $\mathrm{gr}^n C = C_n$. We will write $C_{\le n}$ or $C_{<n+1}$ instead of $\mathrm{F}^n C$. For $n > 0$ we will denote by $d_{\le n}$ the differential of $C_{\le n}$ (which is the restriction of the differential d of C) and by d_n the induced differential on C_n. Observe that $d_1 = d_{\le 1}$ and

$$d_{\le n} = \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} : C_{\le n} = C_{<n} \oplus C_n \to C_{\le n} = C_{<n} \oplus C_n$$

for $n > 1$, where $e_n \colon C_n \to C_{<n}$ is a degree 1 map such that

$$d_{<n} \circ e_n = -e_n \circ d_n. \tag{2.3}$$

Denoting by $i_n \colon C_1 \hookrightarrow C_{\le n}$ the inclusion, we claim that there exist morphisms of complexes $p_n \colon C_{\le n} \to C_1$ and degree -1 maps $h_{\le n} \colon C_{\le n} \to C_{\le n}$ such that

$$\mathrm{id}_{C_1} = p_n \circ i_n, \tag{2.4}$$

$$\mathrm{id}_{C_{\le n}} = i_n \circ p_n + d_{\le n} \circ h_{\le n} + h_{\le n} \circ d_{\le n} \tag{2.5}$$

for every $n > 0$, and satisfying the compatibility conditions

$$p_n|_{C_{<n}} = p_{n-1}, \qquad h_{\le n}|_{C_{<n}} = h_{<n} \tag{2.6}$$

for every $n > 1$. Assuming this, one can conclude the proof very easily. Indeed, the maps $p \colon C \to C_1$ and $h \colon C \to C$ such that $p|_{C_{\le n}} = p_n$ and $h|_{C_{\le n}} = h_{\le n}$ for every $n > 0$ are well defined (and unique), thanks to (2.6). Moreover, since each p_n is a morphism of complexes and (2.4) and (2.5) hold, also p is a morphism of complexes and (denoting by $i \colon C_1 \hookrightarrow C$ the inclusion) we obtain

$$\mathrm{id}_{C_1} = p \circ i, \qquad \mathrm{id}_C = i \circ p + d \circ h + h \circ d,$$

thus proving that i is a homotopy equivalence.

So it remains to prove the claim, and to this purpose we proceed by induction on n. As we can obviously take $p_1 = i_1 = \mathrm{id}_{C_1}$ and $h_{\le 1} = 0$, we assume that $n > 1$ and that the maps p_m and h_m with all the required properties have already been chosen for $0 < m < n$. In particular, p_{n-1} is a morphism of complexes and

$$\mathrm{id}_{C_1} = p_{n-1} \circ i_{n-1}, \tag{2.7}$$

$$\mathrm{id}_{C_{<n}} = i_{n-1} \circ p_{n-1} + d_{<n} \circ h_{<n} + h_{<n} \circ d_{<n}. \tag{2.8}$$

Moreover, since C_n is null-homotopic, there exists a degree -1 map $h_n\colon C_n \to C_n$ such that

$$\mathrm{id}_{C_n} = d_n \circ h_n + h_n \circ d_n. \tag{2.9}$$

Setting

$$p_n := \begin{pmatrix} p_{n-1} & -p_{n-1}\circ e_n \circ h_n \end{pmatrix} : C_{\le n} = C_{<n} \oplus C_n \to C_1,$$
$$h_{\le n} := \begin{pmatrix} h_{<n} & -h_{<n}\circ e_n \circ h_n \\ 0 & h_n \end{pmatrix} : C_{\le n} = C_{<n} \oplus C_n \to C_{\le n} = C_{<n} \oplus C_n,$$

(2.6) is certainly satisfied. Using, beyond the fact that p_{n-1} is a morphism of complexes, (2.3) and (2.9), we obtain also

$$\begin{aligned} d_1 \circ p_n = d_1 \circ \begin{pmatrix} p_{n-1} & -p_{n-1}\circ e_n \circ h_n \end{pmatrix} &= \begin{pmatrix} d_1\circ p_{n-1} & -d_1 \circ p_{n-1}\circ e_n \circ h_n \end{pmatrix} \\ = \begin{pmatrix} p_{n-1}\circ d_{<n} & -p_{n-1}\circ d_{<n}\circ e_n \circ h_n \end{pmatrix} &= \begin{pmatrix} p_{n-1}\circ d_{<n} & p_{n-1}\circ e_n \circ d_n \circ h_n \end{pmatrix} \\ = \begin{pmatrix} p_{n-1}\circ d_{<n} & p_{n-1}\circ e_n \circ (\mathrm{id}_{C_n} - h_n\circ d_n) \end{pmatrix} &= \begin{pmatrix} p_{n-1} & -p_{n-1}\circ e_n \circ h_n \end{pmatrix} \circ \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} = p_n \circ d_{\le n}, \end{aligned}$$

which shows that p_n is a morphism of complexes. Taking into account that $p_n \circ i_n = p_{n-1}\circ i_{n-1}$, (2.4) follows directly from (2.7). Finally, by (2.8), (2.9) and (2.3),

$$\begin{aligned} d_{\le n}\circ h_{\le n} + h_{\le n}\circ d_{\le n} &= \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} \circ \begin{pmatrix} h_{<n} & -h_{<n}\circ e_n\circ h_n \\ 0 & h_n \end{pmatrix} + \begin{pmatrix} h_{<n} & -h_{<n}\circ e_n\circ h_n \\ 0 & h_n \end{pmatrix} \circ \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} \\ &= \begin{pmatrix} d_{<n}\circ h_{<n} + h_{<n}\circ d_{<n} & -d_{<n}\circ h_{<n}\circ e_n\circ h_n + e_n \circ h_n + h_{<n}\circ e_n - h_{<n}\circ e_n \circ h_n \circ d_n \\ 0 & d_n\circ h_n + h_n \circ d_n \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_{C_{<n}} - i_{n-1}\circ p_{n-1} & (h_{<n}\circ d_{<n} + i_{n-1}\circ p_{n-1})\circ e_n \circ h_n + h_{<n}\circ e_n \circ d_n \circ h_n \\ 0 & \mathrm{id}_{C_n} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_{C_{<n}} - i_{n-1}\circ p_{n-1} & i_{n-1}\circ p_{n-1}\circ e_n \circ h_n \\ 0 & \mathrm{id}_{C_n} \end{pmatrix} = \mathrm{id}_{C_{\le n}} - i_n \circ p_n, \end{aligned}$$

which proves (2.5). □

Remark 2.3. The assumption that the sequence splits in Lemma 2.2 is essential. To see this, just consider the case in which C is given by a non-split short exact sequence $0 \to M \xrightarrow{i} N \to P \to 0$ and $\mathrm{F}^1 C$ is the subcomplex $0 \to M \xrightarrow{\sim} i(M) \to 0$, while $\mathrm{F}^n C = C$ for $n > 1$. On the other hand, one can very easily prove that the inclusion $\mathrm{F}^1 C \hookrightarrow C$ is a quasi-isomorphism of complexes, even without that assumption.

2.2. **The relevant morphism.** In this section, we fix two non-unital A_∞ categories $\mathbf{A}$ and $\mathbf{B}$, and we denote by $\mathbf{C}$ the non-unital dg cocategory $\overline{\mathrm{B}_\infty(\mathbf{A})^+ \otimes \mathrm{B}_\infty(\mathbf{B})^+}$. We will also consider the dg quiver $\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}$.

Our aim here is to construct a suitable morphism of complexes

$$\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}\big((A,B),(A',B')\big) \to \Omega(\mathbf{C})\big((A,B),(A',B')\big)$$

where $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$. Its precise definition is in (2.16) below.

Moving in this direction, note that, given $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$, we have

$$\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}\big((A,B),(A',B')\big) = \mathbf{A}(A,A') \otimes \mathbf{B}(B,B') \oplus \mathbf{A}(A,A')^{\delta_{B,B'}} \oplus \mathbf{B}(B,B')^{\delta_{A,A'}} \tag{2.10}$$

as a dg $\Bbbk$-module. In order to explicitly describe also $\Omega(\mathbf{C})\big((A,B),(A',B')\big)$, we first introduce some notation.

For every $i, j \in \mathbb{N}$, we denote by $C_{i,j}\big((A,B),(A',B')\big)$ the graded $\Bbbk$-module

$$\bigoplus_{\substack{A=A_0,A_1,\dots,A_{i-1},A_i=A'\in\mathbf{A}\\ B=B_0,B_1,\dots,B_{j-1},B_j=B'\in\mathbf{B}}} \mathbf{A}(A_{i-1},A_i)[1] \otimes \cdots \otimes \mathbf{A}(A_0,A_1)[1] \otimes \mathbf{B}(B_{j-1},B_j)[1] \otimes \cdots \otimes \mathbf{B}(B_0,B_1)[1],$$

which is meant to be 0 when $i = 0$ and $A \neq A'$ or $j = 0$ and $B \neq B'$ or $i = j = 0$. Given $m_1, n_1, \dots, m_l, n_l \in \mathbb{N}$, we denote by $C_{(m_1,\dots,m_l),(n_1,\dots,n_l)}\big((A,B),(A',B')\big)$ the graded $\Bbbk$-module

$$\bigoplus_{\substack{A=A_0,A_1,\dots,A_{l-1},A_l=A'\in\mathbf{A}\\ B=B_0,B_1,\dots,B_{l-1},B_l=B'\in\mathbf{B}}} C_{m_l,n_l}\big((A_{l-1},B_{l-1}),(A_l,B_l)\big)[-1] \otimes \cdots \otimes C_{m_1,n_1}\big((A_0,B_0),(A_1,B_1)\big)[-1]$$

(in particular, $C_{(i),(j)}\big((A,B),(A',B')\big) = C_{i,j}\big((A,B),(A',B')\big)[-1]$). For every $m, n \geq 0$ we define moreover

$$L_{m,n}\big((A,B),(A',B')\big) := \bigoplus_{\substack{m_1+\cdots+m_l=m\\ n_1+\cdots+n_l=n}} C_{(m_1,\dots,m_l),(n_1,\dots,n_l)}\big((A,B),(A',B')\big).$$

Note that, in particular, $L_{0,0}\big((A,B),(A',B')\big) = 0$ and

$$L_{1,0}\big((A,B),(A',B')\big) = \mathbf{A}(A,A')^{\delta_{B,B'}} \tag{2.11}$$

$$L_{0,1}\big((A,B),(A',B')\big) = \mathbf{B}(B,B')^{\delta_{A,A'}} \tag{2.12}$$

$$\begin{aligned} L_{1,1}\big((A,B),(A',B')\big) = {}&\mathbf{A}(A,A') \otimes \mathbf{B}(B,B') \oplus \mathbf{B}(B,B') \otimes \mathbf{A}(A,A') \\ &\oplus (\mathbf{A}(A,A')[1] \otimes \mathbf{B}(B,B')[1])[-1] \end{aligned} \tag{2.13}$$

Then the non-unital dg category $\Omega(\mathbf{C})$ has the same objects as $\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}$, and

$$\Omega(\mathbf{C})\big((A,B),(A',B')\big) = \bigoplus_{m,n\geq 0} L_{m,n}\big((A,B),(A',B')\big)$$

as a graded $\Bbbk$-module for every $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$. While the composition in $\Omega(\mathbf{C})$ is the natural one given by the tensor product of the cobar construction, the differential on $\Omega(\mathbf{C})\big((A,B),(A',B')\big)$ extends (in such a way that the graded Leibniz rule holds) $\mu + \Delta$, where μ and Δ are determined, respectively, by the differential and the comultiplication on the dg cocategory $\mathbf{C}$. More precisely, given

$$c = (f_m[1] \otimes \cdots \otimes f_1[1] \otimes g_n[1] \otimes \cdots \otimes g_1[1])[-1] \in C_{(m),(n)}\big((A,B),(A',B')\big)$$

with the f_i and the g_j homogeneous, we have

$$\Delta(c) = \sum_{(i,j)\in I_{m,n}} (-1)^{\deg(c_{\leq i,\emptyset})\deg'(c_{\emptyset,>j})+\deg(c_{>i,\emptyset})} c_{>i,>j} \otimes c_{\leq i,\leq j},$$

where $I_{m,n} := \{0,\dots,m\} \times \{0,\dots,n\} \setminus \{(0,0),(m,n)\}$. Here $> i$ and $\leq i$ (respectively $> j$ and $\leq j$) denote the (descending) intervals $[m,i) = [m.i+1]$ and $[i,1]$ (respectively $[n,j)$ and $[j,1]$), and in general

$$c_{[i',i),[j',j)} := (f_{i'}[1] \otimes \cdots \otimes f_{i+1}[1] \otimes g_{j'}[1] \otimes \cdots \otimes g_{j+1}[1])[-1]$$

for $1 \leq i \leq i' \leq m$ and $1 \leq j \leq j' \leq n$. Obviously the empty interval is denoted also by $\emptyset$, while the full interval $[m,1]$ or $[n,1]$ will be denoted by $*$. Clearly

$$\Delta\Big(L_{m,n}\big((A,B),(A',B')\big)\Big) \subseteq L_{m,n}\big((A,B),(A',B')\big)$$

for every $m,n \geq 0$. On the other hand, the component μ^1 of μ induced from $\mathrm{m}^1_{\mathbf{A}}$ and $\mathrm{m}^1_{\mathbf{B}}$ is given on c as above by

$$\mu^1(c) = \sum_{i=1}^{m} (-1)^{\deg'(c_{>i,\emptyset})} \mu^1_{i,0}(c) + \sum_{j=1}^{n} (-1)^{\deg'(c_{*,>j})} \mu^1_{0,j}(c),$$

where

$$\mu^1_{i,0}(c) := (f_m[1] \otimes \cdots \otimes f_{i+1}[1] \otimes \mathrm{m}^1_{\mathbf{A}}(f_i)[1] \otimes f_{i-1}[1] \otimes \cdots \otimes f_1[1] \otimes g_n[1] \otimes \cdots \otimes g_1[1])[-1],$$
$$\mu^1_{0,j}(c) := (f_m[1] \otimes \cdots \otimes f_1[1] \otimes g_n[1] \otimes \cdots \otimes g_{j+1}[1] \otimes \mathrm{m}^1_{\mathbf{B}}(g_j)[1] \otimes g_{j-1}[1] \otimes \cdots \otimes g_1[1])[-1].$$

Hence also in this case

$$\mu^1\Big(L_{m,n}\big((A,B),(A',B')\big)\Big) \subseteq L_{m,n}\big((A,B),(A',B')\big)$$

for every $m,n \geq 0$. As for the other components μ^i of μ induced from $\mathrm{m}^i_{\mathbf{A}}$ and $\mathrm{m}^i_{\mathbf{B}}$ with $i > 1$, for our purposes it is enough to observe that

$$\mu^i\Big(L_{m,n}\big((A,B),(A',B')\big)\Big) \subseteq \bigoplus_{0<m'<m} L_{m',n}\big((A,B),(A',B')\big) \bigoplus_{0<n'<n} L_{m,n'}\big((A,B),(A',B')\big)$$

for every $m,n \geq 0$. This implies that $\mu^1 + \Delta$ is a differential (often denoted simply by d) on each $L_{m,n}\big((A,B),(A',B')\big)$, which will be regarded as a complex in this way. Moreover,

$$L_{*,0}\big((A,B),(A',B')\big) := \bigoplus_{m\geq 0} L_{m,0}\big((A,B),(A',B')\big)$$
$$L_{0,*}\big((A,B),(A',B')\big) := \bigoplus_{n\geq 0} L_{0,n}\big((A,B),(A',B')\big)$$
$$L_{>0}\big((A,B),(A',B')\big) := \bigoplus_{m,n>0} L_{m,n}\big((A,B),(A',B')\big)$$

are subcomplexes of $\Omega(\mathbf{C})\big((A,B),(A',B')\big)$, and obviously there is a decomposition

$$\Omega(\mathbf{C})\big((A,B),(A',B')\big) = L_{*,0}\big((A,B),(A',B')\big) \oplus L_{0,*}\big((A,B),(A',B')\big) \oplus L_{>0}\big((A,B),(A',B')\big).$$

Now, for every $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$, we will consider the maps (see (2.13))

$$\begin{aligned} \mathbf{A}(A,A') \otimes \mathbf{B}(B,B') &\to L_{1,1}\big((A,B),(A',B')\big) \\ f \otimes g &\mapsto (0, (-1)^{\deg(f)\deg(g)} g \otimes f, 0) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} L_{1,1}\big((A,B),(A',B')\big) &\to \mathbf{A}(A,A')\otimes\mathbf{B}(B,B') \\ \big(f\otimes g, g'\otimes f', (f''[1]\otimes g''[1])[-1]\big) &\mapsto f\otimes g+(-1)^{\deg(f')\deg(g')}f'\otimes g' \end{aligned} \tag{2.15}$$

It is easy to check that (2.14) and (2.15) are morphisms of complexes. Remembering (2.10), it is then clear that (2.11), (2.12) and (2.14) define the morphism of complexes

$$\overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big)\to\Omega(\mathbf{C})\big((A,B),(A',B')\big)=\bigoplus_{m,n\geq 0}L_{m,n}\big((A,B),(A',B')\big) \tag{2.16}$$

we are interested in.

2.3. **A general result.** In order to prove the properties of (2.16) we first need the following key result, whose proof is rather technical.

Lemma 2.4. *Let $A,A'\in\mathbf{A}$ and $B,B'\in\mathbf{B}$.*

(1) *The maps* (2.14) *and* (2.15) *are mutually inverse homotopy equivalences of complexes.*
(2) *Given $m,n\in\mathbb{N}$ with $m>1$ or $n>1$, the complex $L_{m,n}\big((A,B),(A',B')\big)$ is null-homotopic.*

Proof. We start by observing that the composition of (2.15) with (2.14) is $\mathrm{id}_{\mathbf{A}(A,A')\otimes\mathbf{B}(B,B')}$, while the composition of (2.14) with (2.15) is the map

$$\begin{aligned} \xi\colon L_{1,1}\big((A,B),(A',B')\big) &\to L_{1,1}\big((A,B),(A',B')\big) \\ \big(f\otimes g, g'\otimes f', (f''[1]\otimes g''[1])[-1]\big) &\mapsto \big(0,(-1)^{\deg(f)\deg(g)}g\otimes f+g'\otimes f',0\big) \end{aligned}$$

We define also $\xi\colon L_{m,n}\big((A,B),(A',B')\big)\to L_{m,n}\big((A,B),(A',B')\big)$ to be 0 for $m>1$, and to be the identity for $m=1$ and $n=0$.

Therefore, in order to prove both (1) and (2) (where, by symmetry, we can assume $m>1$), we just need to find, when $m>1$ or $m=n=1$, a $\Bbbk$-linear map

$$r\colon L_{m,n}\big((A,B),(A',B')\big)\to L_{m,n}\big((A,B),(A',B')\big)$$

of degree -1 such that

$$d\circ r+r\circ d=\mathrm{id}-\xi.$$

More generally, we define r for $m>0$ as follows. By linearity an element of $L_{m,n}\big((A,B),(A',B')\big)$ can be assumed to be of the form

$$c=c^l\otimes\cdots\otimes c^1\in C_{(m_1,\dots,m_l),(n_1,\dots,n_l)}\big((A,B),(A',B')\big),$$

where $m_1+\cdots+m_l=m$, $n_1+\cdots+n_l=n$ and $c^k\in C_{(m_k),(n_k)}\big((A_{k-1},B_{k-1}),(A_k,B_k)\big)$ homogeneous (for $k=1,\dots,l$), with $A_0=A$, $A_l=A'$, $B_0=B$ and $B_l=B'$. Given $1\leq i\leq j\leq l$, we will often use the shorthand $c^{[j,i]}:=c^j\otimes\cdots\otimes c^i$, as well as its variants $c^{[j,i)}$, $c^{(j,i]}$ and $c^{(j,i)}$ (with obvious meanings). Setting

$$\begin{aligned} t &:= \max\big\{k\in\{1,\dots,l\}\mid m_k>0\big\} \\ s' &:= \min\big\{k\in\{1,\dots,t\}\mid m_i=0 \text{ for } k<i<t\big\} \\ s &:= \begin{cases} s' & \text{if } (m_t,n_t)=(1,0) \\ t & \text{otherwise} \end{cases} \end{aligned}$$

(note that they are well defined because $m > 0$), we define recursively

$$r(c) := \sum_{k=s}^{t-1} (-1)^{\deg(c^{[l,t)})+\deg'(c^t)\deg(c^{(t,k)})} c^{[l,t)} \otimes c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]}$$

(so $r(c) = 0$ if $s = t$, in particular if $(m_t, n_t) \neq (1, 0)$). If $(m_t, n_t) = (1, 0)$, $c^t = f \in \mathbf{A}(A_{t-1}, A_t)$ and $c^k = (f_{m_k}[1] \otimes \cdots \otimes f_1[1] \otimes g_{n_k}[1] \otimes \cdots \otimes g_1[1])[-1]$ (with $s \leq k < t$), then

$$r(c^t \otimes c^k) := (-1)^{\deg(f)}(f[1] \otimes f_{m_k}[1] \otimes \cdots \otimes f_1[1] \otimes g_{n_k}[1] \otimes \cdots \otimes g_1[1])[-1].$$

For the rest of the proof we assume $m > 1$ or $m = n = 1$. First we note that $\xi(c) = c^{[l,t)} \otimes \xi(c^{[t,1]})$. Indeed, we can assume $t < l$, and then $\xi(c) = \xi(c^{[t,1]}) = 0$ if $m > 1$, whereas $\xi(c) = c$ and $\xi(c^{[t,1]}) = c^{[t,1]}$ if $m = n = 1$ (in which case $l = 2$, $(m_2, n_2) = (0, 1)$ and $(m_1, n_1) = (1, 0)$). Since moreover

$$\begin{aligned}
(d \circ r + r \circ d)(c) &= d\big((-1)^{\deg(c^{[l,t)})} c^{[l,t)} \otimes r(c^{[t,1]})\big) + r\big(d(c^{[l,t)}) \otimes c^{[t,1]} + (-1)^{\deg(c^{[l,t)})} c^{[l,t)} \otimes d(c^{[t,1]})\big) \\
&= (-1)^{\deg(c^{[l,t)})} d(c^{[l,t)}) \otimes r(c^{[t,1]}) + c^{[l,t)} \otimes d\big(r(c^{[t,1]})\big) + (-1)^{\deg'(c^{[l,t)})} d(c^{[l,t)}) \otimes r(c^{[t,1]}) + c^{[l,t)} \otimes r\big(d(c^{[t,1]})\big) \\
&= c^{[l,t)} \otimes (d \circ r + r \circ d)(c^{[t,1]}),
\end{aligned}$$

it is enough to prove that

$$(d \circ r + r \circ d)(c^{[t,1]}) = c^{[t,1]} - \xi(c^{[t,1]}). \tag{2.17}$$

We have

$$\begin{aligned}
d\big(r(c^{[t,1]})\big) &= d\Big(\sum_{k=s}^{t-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})} c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]}\Big) \\
&= \sum_{k=s}^{t-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})+\deg(c^{(t,k)})} c^{(t,k)} \otimes d\big(r(c^t \otimes c^k)\big) \otimes c^{(k,1]} \\
&\quad + \sum_{k=s}^{t-1}\sum_{i=1}^{k-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})+\deg'(c^{[t,i)})} c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,i)} \otimes d(c^i) \otimes c^{(i,1]} \\
&\quad + \sum_{k=s}^{t-1}\sum_{i=k+1}^{t-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})+\deg(c^{(t,i)})} c^{(t,i)} \otimes d(c^i) \otimes c^{(i,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]}
\end{aligned}$$

and

$$\begin{aligned}
r\big(d(c^{[t,1]})\big) &= r\Big(\sum_{i=1}^{t} (-1)^{\deg(c^{[t,i)})} c^{[t,i)} \otimes d(c^i) \otimes c^{(i,1]}\Big) \\
&= r\big(d(c^t) \otimes c^{(t,1]}\big) + \sum_{i=s}^{t-1} (-1)^{\deg(c^{[t,i)})+\deg'(c^t)\deg(c^{(t,i)})} c^{(t,i)} \otimes r\big(c^t \otimes d(c^i)\big) \otimes c^{(i,1]} \\
&\quad + \sum_{i=1}^{t-1}\sum_{k=\max\{i+1,s\}}^{t-1} (-1)^{\deg(c^{[t,i)})+\deg'(c^t)\deg(c^{(t,k)})} c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,i)} \otimes d(c^i) \otimes c^{(i,1]} \\
&\quad + \sum_{i=1}^{t-1}\sum_{k=s}^{i-1} (-1)^{\deg(c^{[t,i)})+\deg'(c^t)\deg'(c^{(t,k)})} c^{(t,i)} \otimes d(c^i) \otimes c^{(i,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]},
\end{aligned}$$

whence

$$(2.18)\quad (d\circ r + r\circ d)(c^{[t,1]})$$
$$= r\big(d(c^t)\otimes c^{(t,1]}\big) + \sum_{k=s}^{t-1}(-1)^{\deg(c^t)\deg(c^{(t,k)})}c^{(t,k)}\otimes\Big(d\big(r(c^t\otimes c^k)\big)+(-1)^{\deg(c^t)}r\big(c^t\otimes d(c^k)\big)\Big)\otimes c^{(k,1]}.$$

First we assume $(m_t, n_t) \neq (1,0)$, in which case the right-hand-side of (2.18) is just $r\big(\Delta(c^t)\otimes c^{(t,1]}\big)$. If $m_t > 1$ then

$$(d\circ r + r\circ d)(c^{[t,1]}) = r\big((-1)^{\deg(c^t_{m_t,\emptyset})}c^t_{m_t,\emptyset}\otimes c^t_{<m_t,*}\otimes c^{(t,1]}\big)$$
$$= (-1)^{\deg(c^t_{m_t,\emptyset})}r(c^t_{m_t,\emptyset}\otimes c^t_{<m_t,*})\otimes c^{(t,1]} = c^t\otimes c^{(t,1]} = c^{[t,1]},$$

hence (2.17) holds in this case. If $m_t = 1$ and $n_t > 0$ then

$$(d\circ r + r\circ d)(c^{[t,1]}) = r\Big(\big((-1)^{\deg(c^t_{1,\emptyset})}c^t_{1,\emptyset}\otimes c^t_{\emptyset,*} - (-1)^{\deg(c^t_{1,\emptyset})\deg'(c^t_{\emptyset,*})}c^t_{\emptyset,*}\otimes c^t_{1,\emptyset}\big)\otimes c^{(t,1]}\Big)$$
$$= (-1)^{\deg(c^t_{1,\emptyset})}r(c^t_{1,\emptyset}\otimes c^t_{\emptyset,*})\otimes c^{(t,1]}$$
$$+ (-1)^{\deg(c^t_{1,\emptyset})}\sum_{k=s'}^{t-1}(-1)^{\deg'(c^t_{1,\emptyset})\left(\deg(c^t_{\emptyset,*})+\deg(c^{(t,k)})\right)}c^t_{\emptyset,*}\otimes c^{(t,k)}\otimes r(c^t_{1,\emptyset}\otimes c^k)\otimes c^{(k,1]}$$
$$- (-1)^{\deg(c^t_{1,\emptyset})\deg'(c^t_{\emptyset,*})}\sum_{k=s'}^{t-1}(-1)^{\deg(c^t_{\emptyset,*})+\deg'(c^t_{1,\emptyset})\deg(c^{(t,k)})}c^t_{\emptyset,*}\otimes c^{(t,k)}\otimes r(c^t_{1,\emptyset}\otimes c^k)\otimes c^{(k,1]} = c^{[t,1]},$$

thus proving (2.17) also in this case.

Finally we assume $(m_t, n_t) = (1,0)$. Then we have

$$r\big(d(c^t)\otimes c^{(t,1]}\big) = \sum_{k=s}^{t-1}(-1)^{\deg(c^t)\deg(c^{(t,k)})}c^{(t,k)}\otimes r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)\otimes c^{(k,1]},$$

and so from (2.18) we obtain

$$(2.19)\quad (d\circ r + r\circ d)(c^{[t,1]})$$
$$= \sum_{k=s}^{t-1}(-1)^{\deg(c^t)\deg(c^{(t,k)})}c^{(t,k)}\otimes\Big(r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)+d\big(r(c^t\otimes c^k)\big)+(-1)^{\deg(c^t)}r\big(c^t\otimes d(c^k)\big)\Big)\otimes c^{(k,1]}.$$

Since

$$\mu^1\big(r(c^t\otimes c^k)\big) = \sum_{i=1}^{m_k+1}(-1)^{\deg'\left(r(c^t\otimes c^k)_{>i,\emptyset}\right)}\mu^1_{i,0}\big(r(c^t\otimes c^k)\big) + \sum_{j=1}^{n_k}(-1)^{\deg'\left(r(c^t\otimes c^k)_{*,>j}\right)}\mu^1_{0,j}\big(r(c^t\otimes c^k)\big)$$
$$= \sum_{i=1}^{m_k}(-1)^{\deg(c^t)+\deg(c^k_{>i,\emptyset})}r\big(c^t\otimes\mu^1_{i,0}(c^k)\big) - r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big) + \sum_{j=1}^{n_k}(-1)^{\deg(c^t)+\deg(c^k_{*,>j})}r\big(c^t\otimes\mu^1_{0,j}(c^k)\big)$$
$$= -r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big) - (-1)^{\deg(c^t)}r\Big(c^t\otimes\big(\sum_{i=1}^{m_k}(-1)^{\deg'(c^k_{>i,\emptyset})}\mu^1_{i,0}(c^k) + \sum_{j=1}^{n_k}(-1)^{\deg'(c^k_{*,>j})}\mu^1_{0,j}(c^k)\big)\Big)$$
$$= -r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big) - (-1)^{\deg(c^t)}r\big(c^t\otimes\mu^1(c^k)\big)$$

and

$$\begin{aligned}
&\Delta\big(r(c^t\otimes c^k)\big)\\
&=\sum_{(i,j)\in I_{m_k+1,n_k}}(-1)^{\deg\left(r(c^t\otimes c^k)_{\leq i,\emptyset}\right)\deg'\left(r(c^t\otimes c^k)_{\emptyset,>j}\right)+\deg\left(r(c^t\otimes c^k)_{>i,\emptyset}\right)}r(c^t\otimes c^k)_{>i,>j}\otimes r(c^t\otimes c^k)_{\leq i,\leq j}\\
&\qquad=\sum_{(i,j)\in I_{m_k,n_k}}(-1)^{\deg(c^k_{\leq i,\emptyset})\deg'(c^k_{\emptyset,>j})+\deg(c^k_{>i,\emptyset})+\deg'(c^t)}r(c^t\otimes c^k_{>i,>j})\otimes c^k_{\leq i,\leq j}+c^t\otimes c^k\\
&-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}+\sum_{j=\delta_{m_k,0}}^{n_k-1}(-1)^{\left(\deg(c^k_{*,\emptyset})+\deg'(c^t)\right)\deg'(c^k_{\emptyset,>j})+1}c^k_{\emptyset,>j}\otimes r(c^t\otimes c^k_{*,\leq j})\\
&\qquad=c^t\otimes c^k-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}\\
&\quad-(-1)^{\deg(c^t)}r\Big(c^t\otimes\sum_{(i,j)\in I_{m_k,n_k}}(-1)^{\deg(c^k_{\leq i,\emptyset})\deg'(c^k_{\emptyset,>j})+\deg(c^k_{>i,\emptyset})}c^k_{>i,>j}\otimes c^k_{\leq i,\leq j}\Big)\\
&\qquad=c^t\otimes c^k-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}-(-1)^{\deg(c^t)}r\big(c^t\otimes\Delta(c^k)\big),
\end{aligned}$$

we see that

$$r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)+d\big(r(c^t\otimes c^k)\big)+(-1)^{\deg(c^t)}r\big(c^t\otimes d(c^k)\big)=c^t\otimes c^k-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}.$$

Substituting the last equality in (2.19) and remembering that $m_k=0$ for $s<k<t$, while $m_s=0$ if and only if $m=1$ (in which case $s=1$), we get

$$(d\circ r+r\circ d)(c^{[t,1]})=\begin{cases}c^{[t,1]} & \text{if } m>1\\ c^{[t,1]}-(-1)^{\deg(c^t)\deg(c^{(t,1]})}c^{(t,1]}\otimes c^t & \text{if } m=1,\end{cases}$$

from which we conclude that (2.17) is satisfied also in this case. □

We can finally prove the main result of this section. Note that, when $\mathbf{B}$ is the 0 dg algebra, (2.16) boils down to (2.2). Hence the first part of the following result shows, in particular, that (2.2) is a homotopy equivalence, as wanted.

Proposition 2.5. *For every $A,A'\in\mathbf{A}$ and $B,B'\in\mathbf{B}$ the map* (2.16) *is a homotopy equivalence. Moreover, a morphism of complexes*

$$\Omega(\mathbf{C})\big((A,B),(A',B')\big)\to\overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big)$$

is a homotopy equivalence if and only if its restriction to $\bigoplus_{0\leq m,n\leq 1}L_{m,n}\big((A,B),(A',B')\big)$ is a homotopy equivalence.

Proof. By construction (2.16) is the composition of a map

$$\overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big)\to\bigoplus_{0\leq m,n\leq 1}L_{m,n}\big((A,B),(A',B')\big)\tag{2.20}$$

with the inclusion

$$\bigoplus_{0\leq m,n\leq 1}L_{m,n}\big((A,B),(A',B')\big)\hookrightarrow\Omega(\mathbf{C})\big((A,B),(A',B')\big).\tag{2.21}$$

Now, part (1) of Lemma 2.4 immediately implies that (2.20) is a homotopy equivalence. Therefore it is enough to prove that (2.21) is a homotopy equivalence, as well. Clearly this is true if (and only if) each of the three inclusions

$$\begin{aligned} L_{1,0}\big((A,B),(A',B')\big) &\hookrightarrow L_{*,0}\big((A,B),(A',B')\big)\\ L_{0,1}\big((A,B),(A',B')\big) &\hookrightarrow L_{0,*}\big((A,B),(A',B')\big)\\ L_{1,1}\big((A,B),(A',B')\big) &\hookrightarrow L_{>0}\big((A,B),(A',B')\big)\end{aligned}$$

is a homotopy equivalence. To this aim we apply Lemma 2.2 to the complexes on the right-hand-sides of the above inclusions, endowed with the filtrations

$$\begin{aligned} \mathrm{F}^n L_{*,0}\big((A,B),(A',B')\big) &:= \bigoplus_{0\le m\le n} L_{m,0}\big((A,B),(A',B')\big)\\ \mathrm{F}^n L_{0,*}\big((A,B),(A',B')\big) &:= \bigoplus_{0\le m\le n} L_{0,m}\big((A,B),(A',B')\big)\\ \mathrm{F}^n L_{>0}\big((A,B),(A',B')\big) &:= \bigoplus_{m,m'>0,\, m+m'\le n+1} L_{m,m'}\big((A,B),(A',B')\big)\end{aligned}$$

Note that the assumptions of Lemma 2.2 are satisfied because

$$\begin{aligned} \mathrm{gr}^n L_{*,0}\big((A,B),(A',B')\big) &= L_{n,0}\big((A,B),(A',B')\big)\\ \mathrm{gr}^n L_{0,*}\big((A,B),(A',B')\big) &= L_{0,n}\big((A,B),(A',B')\big)\\ \mathrm{gr}^n L_{>0}\big((A,B),(A',B')\big) &= \bigoplus_{0<m\le n} L_{m,n+1-m}\big((A,B),(A',B')\big)\end{aligned}$$

are null-homotopic for $n>1$ by part (2) of Lemma 2.4. □

Let us now single out the following direct consequence.

Corollary 2.6. *Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are non-unital dg categories. Then there is a natural non-unital dg functor* $\tilde{\mathsf{N}}\colon \Omega(\mathbf{C}) \to \overline{\mathbf{A}^+\otimes\mathbf{B}^+}$*, which is a homotopy isomorphism.*

Proof. The definition of $\tilde{\mathsf{N}}$ can be found in [5, Section 3.1], where it is also proved that it is a quasi-isomorphism (hence a homotopy isomorphism) when $\Bbbk$ is a field. Over an arbitrary commutative ring we can apply the second part of Proposition 2.5. Indeed, it can be readily checked that, for every $A, A'\in\mathbf{A}$ and $B,B'\in\mathbf{B}$, the restriction of

$$\tilde{\mathsf{N}}\colon \Omega(\mathbf{C})\big((A,B),(A',B')\big) \to \overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big)$$

to $\bigoplus_{0\le m,n\le 1} L_{m,n}\big((A,B),(A',B')\big)$ is given by the natural maps (2.11), (2.12) and (2.15). Such restriction is then a homotopy equivalence by part (1) of Lemma 2.4. □

3. The strictly unital case

In this section we prove the equivalence between Ho(**dgCat**) and Ho($\mathbf{A}_\infty$**Cat**). We deal with this in Theorem 3.6, after proving the existence of a natural adjunction in Proposition 3.1. In Section 3.2 we finally prove Theorem B.

3.1. **The adjunction.** In analogy with Proposition 2.1, we have the following result in the strictly unital setting. Its proof is the content of this subsection and at the end of it we will deduce Theorem 3.6.

Proposition 3.1. *There is an adjunction*

$$\mathsf{U}\colon \mathbf{A}_\infty\mathbf{Cat} \rightleftarrows \mathbf{dgCat} : \mathsf{I},$$

where I *is the inclusion functor. Moreover, the unit* $\rho\colon \mathrm{id}_{\mathbf{A}_\infty\mathbf{Cat}} \to \mathsf{I}\circ\mathsf{U}$ *and the counit* $\sigma\colon \mathsf{U}\circ\mathsf{I} \to \mathrm{id}_{\mathbf{dgCat}}$ *are such that* $\rho_{\mathbf{A}}$ *(for every* $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}$*) and* $\sigma_{\mathbf{B}}$ *(for every* $\mathbf{B} \in \mathbf{dgCat}$*) are quasi-isomorphisms. Finally,* $\rho_{\mathbf{A}}$ *is even a homotopy isomorphism if* $\mathbf{A}$ *satisfies the following condition:*

$$\Bbbk \cong \Bbbk\mathrm{id}_A \text{ and the inclusion } \Bbbk\mathrm{id}_A \hookrightarrow \mathbf{A}(A,A)^0 \text{ of } \Bbbk\text{-modules splits, for every } A \in \mathbf{A}. \tag{3.1}$$

This result is proved in [5, Proposition 2.1] assuming that $\Bbbk$ is a field. Actually the first part of the proof works without changes over an arbitrary commutative ring. Only the argument showing that $\rho_{\mathbf{A}}\colon \mathbf{A} \to \mathsf{U}(\mathbf{A})$ is a quasi-isomorphism for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}$ (respectively, a homotopy isomorphism when $\mathbf{A}$ satisfies (3.1)) needs to be modified. We now give a different proof, valid over every commutative ring.

To this aim, we fix a strictly unital A_∞ category $\mathbf{A}$. As it is explained at the beginning of the proof of [5, Proposition 2.1], $\rho_{\mathbf{A}} = \pi_{\mathbf{A}} \circ \gamma_{\mathbf{A}}$, where the non-unital A_∞ functor $\gamma_{\mathbf{A}}\colon \mathbf{A} \to \mathsf{U}^{\mathsf{n}}(\mathbf{A})$ is a homotopy isomorphism by Proposition 2.1, and the non-unital dg functor $\pi_{\mathbf{A}}$ is defined to be the composition

$$\pi_{\mathbf{A}}\colon \mathsf{U}^{\mathsf{n}}(\mathbf{A}) \hookrightarrow \mathsf{U}^{\mathsf{n}}(\mathbf{A})^+ \twoheadrightarrow \mathsf{U}^{\mathsf{n}}(\mathbf{A})^+/J = \mathsf{U}(\mathbf{A}),$$

with $J = J_{\mathbf{A}}$ the smallest dg ideal of $\mathsf{U}^{\mathsf{n}}(\mathbf{A})^+$ such that $\rho_{\mathbf{A}}$ is a strictly unital A_∞ functor.

First we recall from Section 2.2 (whose notation we adapt and simplify in an obvious way to the setting where $\mathbf{B}$ is the 0 dg algebra) that, for every $A, A' \in \mathbf{A}$,

$$\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A,A') = \bigoplus_{n\geq 0} L_n(A,A')$$

as a graded $\Bbbk$-module, where

$$L_n(A,A') := \bigoplus_{n_1+\cdots+n_l=n} C_{(n_1,\dots,n_l)}(A,A'),$$

with

$$C_{(n_1,\dots,n_l)}(A,A') := \bigoplus_{A=A_0,A_1,\dots,A_{l-1},A_l=A'\in\mathbf{A}} C_{n_l}(A_{l-1}A_l)[-1] \otimes \cdots \otimes C_{n_1}(A_0,A_1)[-1]$$

and, for every $i \geq 0$,

$$C_i(A,A') := \bigoplus_{A=A_0,A_1,\dots,A_{i-1},A_i=A'\in\mathbf{A}} \mathbf{A}(A_{i-1},A_i)[1] \otimes \cdots \otimes \mathbf{A}(A_0,A_1)[1].$$

The differential d on $\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A,A')$ extends $\mu + \Delta$, where μ and Δ are determined, respectively, by the differential and the comultiplication on the dg cocategory $\mathrm{B}_\infty(\mathbf{A})$. Explicitly, given

$$c = (f_n[1] \otimes \cdots \otimes f_1[1])[-1] \in C_{(n)}(A,A') \tag{3.2}$$

with the f_i homogeneous, we have

$$\Delta(c) = \sum_{i=1}^{n-1} (-1)^{\deg(c_{>i})} c_{>i} \otimes c_{\leq i}.$$

The components μ^k of μ induced from $\mathrm{m}^k = \mathrm{m}^k_{\mathbf{A}}$ are given by

$$\mu^k(c) = \sum_{i=1}^{n+1-k} \pm \mu^k_i(c), \tag{3.3}$$

(with $1 \leq k \leq n$), where

$$\mu^k_i(c) := \big(f_n[1] \otimes \cdots \otimes f_{i+k}[1] \otimes \mathrm{m}^k_{\mathbf{A}}(f_{i+k-1} \otimes \cdots \otimes f_i)[1] \otimes f_{i-1}[1] \otimes \cdots \otimes f_1[1]\big)[-1].$$

As for the signs in (3.3), we just need to know that they are $(-1)^{\deg'(c_{>i})}$ for $k = 2$.

Now we can give the following more explicit description of J.

Lemma 3.2. *The dg ideal J coincides with the (a priori not necessarily dg) ideal J' of $\mathsf{U}^{\mathsf{n}}(\mathbf{A})^+$ generated by all the elements of one of the following two forms:*

1. $1_A - \mathrm{id}_A$*, where* $A \in \mathbf{A}$*;*
2. c *as in* (3.2) *with* $n > 1$ *and such that* $f_j = \mathrm{id}_{\tilde{A}}$ *for some* $j \in \{1, \ldots, n\}$ *and some* $\tilde{A} \in \mathbf{A}$.

Proof. If c is as in (3.2), we have

$$\rho^n_{\mathbf{A}}(f_n \otimes \cdots \otimes f_1) = \pi_{\mathbf{A}}\big(\gamma^n_{\mathbf{A}}(f_n \otimes \cdots \otimes f_1)\big) = \pm\pi_{\mathbf{A}}(c).$$

Since $\rho_{\mathbf{A}}$ is strictly unital, it follows that $\pi_{\mathbf{A}}(c) = 0$ if c is a generator of J' of the form (2). On the other hand, $\pi_{\mathbf{A}}(\mathrm{id}_A) = \mathrm{id}_A$ coincides with the image of 1_A through the projection $\mathsf{U}^{\mathsf{n}}(\mathbf{A})^+ \twoheadrightarrow \mathsf{U}(\mathbf{A})$, for every $A \in \mathbf{A}$. Therefore J contains also the generators of J' of the form (1), hence $J' \subseteq J$. To prove the other inclusion it is clearly enough to show that $d(c) \in J'$ for every generator c of J'. As this is obviously true when c is of the form (1), we can assume that c is of the form (2). Then it is clear from the definition that $\mu^k(c) \in J'$ if $k \neq 2$, and so it remains to prove that J' contains

$$\mu^2(c)+\Delta(c) = \sum_{i=1}^{n-1}(-1)^{\deg'(c_{>i})}\mu^2_i(c)+\sum_{i=1}^{n-1}(-1)^{\deg(c_{>i})}c_{>i}\otimes c_{\leq i} = \sum_{i=1}^{n-1}(-1)^{\deg'(c_{>i})}\big(\mu^2_i(c)-c_{>i}\otimes c_{\leq i}\big).$$

Now, it is immediate to see that (for $0 < i < n$) $\mu^2_i(c) \in J'$ if $i \neq j, j-1$ and $c_{>i} \otimes c_{\leq i} \in J'$ if $1 < i < n-1$ or $i = 1 \neq j$ or $i = n-1 \neq j-1$. Moreover, if $j = 1$ then

$$\mu^2_1(c) - c_{>1} \otimes c_{\leq 1} = c_{>1} \otimes (1_{\tilde{A}} - \mathrm{id}_{\tilde{A}}) \in J'.$$

Similarly, if $j = n$ then

$$\mu^2_{n-1}(c) - c_{\geq n} \otimes c_{<n} = (1_{\tilde{A}} - \mathrm{id}_{\tilde{A}}) \otimes c_{<n} \in J'.$$

Finally, if $1 < j < n$ then

$$\mu^2_{j-1}(c) = \mu^2_j(c) = (f_n[1] \otimes \cdots f_{j+1}[1] \otimes f_{j-1}[1] \otimes \cdots \otimes f_1[1])[-1],$$

whence $(-1)^{\deg'(c_{\geq j})}\mu^2_{j-1}(c) + (-1)^{\deg'(c_{>j})}\mu^2_j(c) = 0$. □

From Lemma 3.2 we immediately deduce the following result.

Corollary 3.3. *The non-unital dg functor* $\pi_{\mathbf{A}}\colon \mathsf{U}^{\mathsf{n}}(\mathbf{A}) \to \mathsf{U}(\mathbf{A})$ *is full and its kernel* $I = I_{\mathbf{A}}$ *is a dg ideal of* $\mathsf{U}^{\mathsf{n}}(\mathbf{A})$ *such that* $I(A, A')$ *(for every* $A, A' \in \mathbf{A}$*) is the* $\Bbbk$*-subspace of* $\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')$ *generated by all the elements of one of the following two forms, where* $c^l_{n_l} \otimes \cdots \otimes c^1_{n_1} \in C_{(n_1,\dots,n_l)}(A, A')$*:*

(1) $c^l_{n_l}\otimes\cdots\otimes c^1_{n_1} - c^l_{n_l}\otimes\cdots\otimes c^{i+1}_{n_{i+1}}\otimes \mathrm{id}_{\tilde{A}}\otimes c^i_{n_i}\otimes\cdots\otimes c^1_{n_1}$ *(for suitable* $\tilde{A} \in \mathbf{A}$*), with* $n_1+\cdots+n_l > 0$ *and* $i \in \{0, \dots, l\}$*;*

(2) $c^l_{n_l} \otimes \cdots \otimes c^1_{n_1}$*, with* $c^i_{n_i}$ *of the form (2) in Lemma 3.2 for some* $i \in \{1, \dots, l\}$*.*

For every $A, A' \in \mathbf{A}$ the filtration $L_{\le n}(A, A') := \bigoplus_{m\le n} L_m(A, A')$ on $\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')$ (where $n \ge 0$) induces a filtration $I_{\le n}(A, A') := L_{\le n}(A, A') \cap I(A, A')$ on $I(A, A')$ and a filtration

$$\mathrm{F}^n\mathsf{U}(\mathbf{A})(A, A') := \big(L_{\le n}(A, A') + I(A, A')\big)/I(A, A') \cong L_{\le n}(A, A')/I_{\le n}(A, A')$$

on $\mathsf{U}(\mathbf{A})(A, A') \cong \mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')/I(A, A')$.

Since $\gamma^1_{\mathbf{A}}\colon \mathbf{A}(A, A') \to L_{\le 1}(A, A')$ is an isomorphism of complexes and $I_{\le 1}(A, A') = 0$, we see that $\rho^1_{\mathbf{A}}\colon \mathbf{A}(A, A') \to \mathrm{F}^1\mathsf{U}(\mathbf{A})(A, A')$ is an isomorphism, as well. Therefore we just need to show that the inclusion $\mathrm{F}^1\mathsf{U}(\mathbf{A})(A, A') \hookrightarrow \mathsf{U}(\mathbf{A})(A, A')$ is a quasi-isomorphism, and even a homotopy equivalence if $\mathbf{A}$ satisfies (3.1). By Lemma 2.2 and Remark 2.3 it is enough to prove that for every $n > 1$ the complex $\mathrm{gr}^n\mathsf{U}(\mathbf{A})(A, A')$ is null-homotopic, and also that the inclusion $\mathrm{F}^{n-1}\mathsf{U}(\mathbf{A})(A, A') \hookrightarrow \mathrm{F}^n\mathsf{U}(\mathbf{A})(A, A')$ splits as a morphism of graded $\Bbbk$-modules if $\mathbf{A}$ satisfies (3.1).

Now, recall from Lemma 2.4 and its proof that, for $n > 1$, the complex $L_n(A, A')$ (endowed with the differential d extending $\mu^1 + \Delta$) is null-homotopic, and a map $r\colon L_n(A, A') \to L_n(A, A')$ of degree -1 satisfying $d \circ r + r \circ d = \mathrm{id}$ can be defined (also for $n = 1$) as follows. By linearity an element of $L_n(A, A')$ can be assumed to be of the form

$$c = c^l \otimes \cdots \otimes c^1 \in C_{(n_1,\dots,n_l)}(A, A'), \tag{3.4}$$

where $n_1 + \cdots + n_l = n$ and $c^k \in C_{(n_k)}(A_{k-1}, A_k)$ homogeneous (for $k = 1, \dots, l$), with $A_0 = A$ and $A_l = A'$. Then

$$r(c) := \begin{cases} 0 & \text{if } n_l > 1 \text{ or } n = 1 \\ r(c^l \otimes c^{l-1}) \otimes c^{l-2} \otimes \cdots \otimes c^1 & \text{if } n_l = 1 < n, \end{cases}$$

where, if $n_l = 1 < n$, $c^t = f \in \mathbf{A}(A_{l-1}, A_l)$ and $c^{l-1} = (f_{n_{l-1}}[1] \otimes \cdots \otimes f_1[1])[-1]$, then

$$r(c^l \otimes c^{l-1}) := (-1)^{\deg(f)}(f[1] \otimes f_{n_{l-1}}[1] \otimes \cdots \otimes f_1[1])[-1].$$

Since

$$\mathrm{gr}^n\mathsf{U}(\mathbf{A})(A, A') \cong \big(L_{\le n}(A, A') + I(A, A')\big)/\big(L_{<n}(A, A') + I(A, A')\big) \cong L_n(A, A')/I_n(A, A'),$$

where

$$I_n(A, A') := L_n(A, A') \cap \big(L_{<n}(A, A') + I_{\le n}(A, A')\big),$$

from Lemma 3.4 we deduce that $\mathrm{gr}^n\mathsf{U}(\mathbf{A})(A, A')$ is null-homotopic for $n > 1$.

Lemma 3.4. *The map* $r\colon L_n(A, A') \to L_n(A, A')$ *preserves the subcomplex* $I_n(A, A')$ *for every* $n > 1$ *and every* $A, A' \in \mathbf{A}$*.*

Proof. As r preserves both $L_n(A,A')$ and $L_{<n}(A,A')$, it is enough to prove that, if $c \in I_{\leq n}(A,A')$, then $r(c) \in L_{<n}(A,A') + I_{\leq n}(A,A')$. We can clearly assume that c is as in part (1) or (2) of Corollary 3.3. In the latter case it is obvious from the definition that $r(c)$ is either 0 or a generator of the same form in $I_{\leq n}(A,A')$. So we can assume c to be of the form (1) with $n_1+\cdots+n_l = n-1$, and it is enough to show that $r(c') \in L_{<n}(A,A') + I_{\leq n}(A,A')$, where

$$c' := c^l_{n_l} \otimes \cdots \otimes c^{i+1}_{n_{i+1}} \otimes \mathrm{id}_{\tilde{A}} \otimes c^i_{n_i} \otimes \cdots \otimes c^1_{n_1}.$$

Now, if $i \geq l-1$, then $r(c')$ is either 0 or a generator of the form (2) in $I_{\leq n}(A,A')$. On the other hand, if $i < l-1$, then $r(c') \in L_{<n}(A,A') + I_{\leq n}(A,A')$ because $r(c^l_{n_l} \otimes \cdots \otimes c^1_{n_1}) \in L_{<n}(A,A')$ and $r(c^l_{n_l} \otimes \cdots \otimes c^1_{n_1}) - r(c')$ is either 0 or a generator of the form (1) in $I_{\leq n}(A,A')$. □

Finally, Lemma 3.5 easily implies that the inclusion $\mathrm{F}^{n-1}\mathsf{U}(\mathbf{A})(A,A') \hookrightarrow \mathrm{F}^n\mathsf{U}(\mathbf{A})(A,A')$ splits as a morphism of graded $\Bbbk$-modules if $\mathbf{A}$ satisfies (3.1) and $n > 1$.

Lemma 3.5. *If* $\mathbf{A}$ *satisfies* (3.1)*, then for every* $n > 1$ *and every* $A, A' \in \mathbf{A}$ *there exists a morphism of graded* $\Bbbk$*-modules* $u\colon L_n(A,A') \to L_{<n}(A,A')$ *such that the map*

$$\tilde{u} := \begin{pmatrix} \mathrm{id} & u \end{pmatrix} : L_{<n}(A,A') \oplus L_n(A,A') = L_{\leq n}(A,A') \to L_{<n}(A,A')$$

sends $I_{\leq n}(A,A')$ *to* $I_{<n}(A,A')$.

Proof. By hypothesis for every $\tilde{A} \in \mathbf{A}$ there exists a morphism of graded $\Bbbk$-modules $p\colon \mathbf{A}(\tilde{A},\tilde{A}) \to \Bbbk$ such that $p(\mathrm{id}_{\tilde{A}}) = 1$. First, by linearity, every $c \in L_n(A,A')$ can be assumed to be as in (3.4). Setting

$$S(c) := \{i = 1,\ldots,l \mid n_i = 1 \text{ and } A_{i-1} = A_i\},$$

we denote, for every subset S of $S(c)$, by $u_S(c)$ the expression obtained from c by deleting the terms c^i with $i \in S$. In case $S = S(c) = \{1,\ldots,l\}$ (which implies $A = A'$), we mean $u_S(c) = \mathrm{id}_A$. Now we can define

$$u(c) := \sum_{\emptyset \neq S \subseteq S(c)} (-1)^{|S|-1} \prod_{i\in S} p(c^i) u_S(c).$$

It is immediate from the definition that $\tilde{u}$ sends a generator of the form (2) in Corollary 3.3 to a linear combination of generators of the same form. Hence, given c as above with the additional assumption that there exists $j \in \{1,\ldots,l\}$ such that $c^j = \mathrm{id}_{A_j}$ (in particular, $j \in S(c)$), we just need to show that $\tilde{u}(\tilde{c}) \in I(A,A')$, where $\tilde{c} := u_{\{j\}}(c) - c \in I_{\leq n}(A,A')$ is a generator of the form (1). Equivalently, we must prove that $-\tilde{c} + \tilde{u}(\tilde{c}) \in I(A,A')$. In fact we have

$$\begin{aligned} -\tilde{c} + \tilde{u}(\tilde{c}) &= -u_{\{j\}}(c) + c + u_{\{j\}}(c) - u(c) = c - \sum_{\emptyset \neq S \subseteq S(c)} (-1)^{|S|-1} \prod_{i\in S} p(c^i) u_S(c) \\ &= \sum_{S \subseteq S(c)} (-1)^{|S|} \prod_{i\in S} p(c^i) u_S(c) = \sum_{S \subseteq S(c)\setminus\{j\}} (-1)^{|S|} \prod_{i\in S} p(c^i)(u_{S\cup\{j\}}(c) - u_S(c)), \end{aligned}$$

and each $u_{S\cup\{j\}}(c) - u_S(c)$ is a generator of the form (1) (or 0 if $S \cup \{j\} = S(c) = \{1,\ldots,l\}$). □

This concludes the proof of Proposition 3.1. Now, from this result, with the same proof of [5, Theorem 2.2], we get the following.

Theorem 3.6. *The functors* I *and* U *induce the functors*

$$\mathrm{Ho}(\mathsf{I})\colon \mathrm{Ho}(\mathbf{dgCat}) \to \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}) \qquad \textit{and} \qquad \mathrm{Ho}(\mathsf{U})\colon \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}) \to \mathrm{Ho}(\mathbf{dgCat})$$

which are quasi-inverse equivalences of categories.

Remark 3.7. As in [5, Remark 2.3], it can also be proved that there is an equivalence of categories between $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}})$. Hence $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})$ and $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}})$ are equivalent, as well.

3.2. **Proof of Theorem B.** We refer to [22] for the (few) basic notions about ∞-categories which are needed in this section. We denote by $\mathrm{Ho}(\mathbf{dgCat})_\infty$ (resp. $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat})_\infty$) the ∞-category obtained by localizing the nerve of the category $\mathbf{dgCat}$ (resp. $\mathbf{A}_\infty\mathbf{Cat}$) by the image under the nerve functor of the class $\mathcal{W}^{\mathrm{dg}}$ of quasi-equivalences in $\mathbf{dgCat}$ (resp. the class $\mathcal{W}^{\mathrm{A}_\infty}$ of quasi-equivalences in $\mathbf{A}_\infty\mathbf{Cat}$).

Now, as it was pointed out in [22], from Proposition 3.1 one can also formally deduce Theorem B which is a stronger ∞-categorical version of Theorem 3.6 in the form of [22, Corollary 5.2]. This is due to the fact that the adjunction of Proposition 3.1 is a Dwyer-Kan adjunction, meaning that the following five conditions hold (see [22, Definition 2.1, Theorem 2.2]):

(1) U is left adjoint to I;
(2) $\mathsf{I}(\mathcal{W}^{\mathrm{dg}}) \subseteq \mathcal{W}^{\mathrm{A}_\infty}$;
(3) $\mathsf{U}(\mathcal{W}^{\mathrm{A}_\infty}) \subseteq \mathcal{W}^{\mathrm{dg}}$;
(4) the component of the unit $\rho_{\mathbf{A}} \in \mathcal{W}^{\mathrm{A}_\infty}$ for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}$;
(5) the component of the counit $\sigma_{\mathbf{B}} \in \mathcal{W}^{\mathrm{dg}}$ for every $\mathbf{B} \in \mathbf{dgCat}$.

Observe that, replacing [5, Proposition 2.1] with Proposition 3.1 everything works when $\Bbbk$ is an arbitrary commutative ring (and not just a field as in [22]).

Remark 3.8. As it is pointed out in [22], one can consider different models for ∞-categories and for all of them there is an analogue of [22, Corollary 5.2]. Namely, one gets [22, Corollaries 2.5, 3.2, 4.3, 4.5, 5.1]. All their proofs rely on Proposition 3.1 and thus remain valid over an arbitrary commutative ring. As a consequence, Theorem B could be restated and proved by indifferently using each of these models.

Actually we can say that

$$\mathsf{U}\colon (\mathbf{A}_\infty\mathbf{Cat}, \mathcal{W}^{\mathrm{A}_\infty}) \rightleftarrows (\mathbf{dgCat}, \mathcal{W}^{\mathrm{dg}}) :\mathsf{I}$$

is a Dwyer-Kan adjunction even if $\mathcal{W}^{\mathrm{A}_\infty}$ and $\mathcal{W}^{\mathrm{dg}}$ are the classes of pretriangulated (or Morita) equivalences in $\mathbf{A}_\infty\mathbf{Cat}$ and in $\mathbf{dgCat}$, respectively (see [24, §1.4 and Definition 1.36] or [20, Definition 1.4.7.]). Indeed, every quasi-equivalence is a pretriangulated (and Morita) equivalence so (4) and (5) are satisfied. To prove (2) it suffices to notice that $\mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{A}) = \mathrm{pretr}_{\mathrm{dg}}(\mathbf{A})$ if $\mathbf{A} \in \mathbf{dgCat}$ (see [20, Remark 1.7]). As for (3), suppose that $\mathsf{F} : \mathbf{A} \to \mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}$ induces a quasi-equivalence $\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{F})\colon \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{A}) \to \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{B})$. Then

$$\begin{array}{ccc} \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{A}) & \xrightarrow{\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{F})} & \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{B}) \\ {\scriptstyle \mathrm{pretr}_{\mathrm{A}_\infty}(\rho_{\mathbf{A}})}\downarrow & & \downarrow{\scriptstyle \mathrm{pretr}_{\mathrm{A}_\infty}(\rho_{\mathbf{B}})} \\ \mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathbf{A})) & \xrightarrow[\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathsf{F}))]{} & \mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathbf{B})) \end{array}$$

is a commutative diagram in $\mathbf{A}_\infty\mathbf{Cat}$ in which the upper and the vertical arrows are quasi-equivalences. Hence $\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathsf{F})) = \mathrm{pretr}_{\mathrm{dg}}(\mathsf{U}(\mathsf{F}))$ is a quasi-equivalence, as well.

4. The unital case

In this section we prove the following result, where $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}$ denotes the full subcategory of $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ with objects the h-projective unital A_∞ categories (as in the case of dg categories, we say that $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ is *h-projective* if $\mathbf{A}(A,B)$ is a h-projective complex of $\Bbbk$-modules for every $A, B \in \mathbf{A}$).

Theorem 4.1. *The inclusion functor* $\mathsf{J}\colon \mathbf{dgCat} \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ *induces an equivalence of categories* $\mathrm{Ho}(\mathsf{J})\colon \mathrm{Ho}(\mathbf{dgCat}) \to \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$*. Furthermore, the categories* $\mathrm{Ho}(\mathbf{dgCat})$ *and* $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ *are equivalent (hence* $\mathrm{Ho}(\mathbf{dgCat})$ *and* $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx$ *are equivalent if* $\Bbbk$ *is a field).*

The proof is contained in Section 4.2 and it is based on some preliminary results which are discussed in Section 4.1.

4.1. **Preliminary results.** We begin with the following result about A_∞ functors which corrects a similar statement in [16].

Lemma 4.2. *Let* $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ *be a unital A_∞ functor between two strictly unital A_∞ categories. If* $\mathbf{A}$ *satisfies* (3.1)*, then* F *is homotopic to a strictly unital A_∞ functor.*

Proof. Since F is unital, for every $A \in \mathbf{A}$ there exists $h_A \in \mathbf{B}(\mathsf{F}^0(A), \mathsf{F}^0(A))^{-1}$ such that $\mathsf{F}^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)} - \mathrm{m}^1_{\mathbf{B}}(h_A)$. Then we define a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{F}$ of degree 0 as follows: for every $f \in \mathbf{A}(A, A')$ we set

$$\theta^1(f) := \begin{cases} p(f)h_A & \text{if } A = A' \\ 0 & \text{if } A \neq A' \end{cases}$$

(where $p\colon \mathbf{A}(A,A) \to \Bbbk$ is as in the proof of Lemma 3.5) and $\theta^i := 0$ for $i \neq 1$. By Remark 1.20 we can find $\widetilde{\mathsf{F}} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A},\mathbf{B})$ such that $\mathsf{F} \sim \widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{F}}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for $i > 0$. By definition for every $A \in \mathbf{A}$ we have

$$\widetilde{\mathsf{F}}^1(\mathrm{id}_A) = \mathsf{F}^1(\mathrm{id}_A) + \mathrm{m}^1(\theta)^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)} - \mathrm{m}^1_{\mathbf{B}}(h_A) + \theta^1\big(\mathrm{m}^1_{\mathbf{A}}(\mathrm{id}_A)\big) + \mathrm{m}^1_{\mathbf{B}}\big(\theta^1(\mathrm{id}_A)\big) = \mathrm{id}_{\mathsf{F}^0(A)}.$$

To conclude, using Remark 1.19 and an easy recursive argument, it should be clear that it is enough to prove the following statement. Assume that $\mathsf{F}^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)}$ for every $A \in \mathbf{A}$ and that there exist $n > 1$ and $1 \leq m \leq n$ such that $\mathsf{F}^i(f_i \otimes \cdots \otimes f_1) = 0$ if there exists $j \in \{1, \dots, i\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and either $1 < i < n$ or $i = n$ and $j < m$. Then we can find $\mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A},\mathbf{B})$ such that $\mathsf{F} \sim \mathsf{G}$ through a homotopy θ with $\theta^i = 0$ for $i < n-1$, $\mathsf{G}^i = \mathsf{F}^i$ for

$i < n$ and $\mathsf{G}^n(f_n \otimes \cdots \otimes f_1) = 0$ if there exists $j \in \{1, \ldots, n\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and $j \leq m$.

To this aim, a direct but tedious check shows that we can define θ by

$$\theta^{n-1}(f_{n-1} \otimes \cdots \otimes f_1) := (-1)^m \mathsf{F}^n(f_{n-1} \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$
$$\theta^n(f_n \otimes \cdots \otimes f_1) := (-1)^m \mathsf{F}^{n+1}(f_n \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$

and $\theta^i := 0$ for $i \neq n-1, n$. See also [20, Lemma 3.7] (where the assumption (3.1) is erroneously missing) for more details of the computation. □

In a similar fashion we have the following result about natural transformations.

Lemma 4.3. *Let $\mathsf{F}, \mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}(\mathbf{A}, \mathbf{B})$ and let $\theta\colon \mathsf{F} \to \mathsf{G}$ be a natural transformation of degree p. Then there exists a prenatural transformation $\widetilde{\theta}\colon \mathsf{F} \to \mathsf{G}$ of degree $p-1$ such that $\theta - \mathrm{m}^1(\widetilde{\theta})\colon \mathsf{F} \to \mathsf{G}$ is a strictly unital natural transformation.*

Proof. The argument is similar (and a bit simpler) to the one of Lemma 4.2. In this case the only key step consists in the proof of the following statement. Assume that there exist $n > 0$ and $1 \leq m \leq n$ such that $\theta^i(f_i \otimes \cdots \otimes f_1) = 0$ if there exists $j \in \{1, \ldots, i\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and either $0 < i < n$ or $i = n$ and $j < m$. Then we can find a prenatural transformation $\overline{\theta}\colon \mathsf{F} \to \mathsf{G}$ of degree $p-1$ such that $\overline{\theta}^i = 0$ for $i < n-1$, $\mathrm{m}^1(\overline{\theta})^i = 0$ for $i < n$ and $\mathrm{m}^1(\overline{\theta})^n(f_n \otimes \cdots \otimes f_1) = \theta^n(f_n \otimes \cdots \otimes f_1)$ if there exists $j \in \{1, \ldots, n\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and $j \leq m$. Here we can define $\overline{\theta}$ by

$$\overline{\theta}^{n-1}(f_{n-1} \otimes \cdots \otimes f_1) := (-1)^m \theta^n(f_{n-1} \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$
$$\overline{\theta}^n(f_n \otimes \cdots \otimes f_1) := (-1)^m \theta^{n+1}(f_n \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$

and $\overline{\theta}^i := 0$ for $i \neq n-1, n$. See also [20, Lemma 3.8] for more details. □

We can then prove the following.

Lemma 4.4. *If $\mathsf{F}, \mathsf{F}' \in \mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}}(\mathbf{A}, \mathbf{B})$ are such that $\mathsf{F} \approx \mathsf{F}'$, then F and F' have the same image in $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}})$.*

Proof. This is [5, Lemma 2.10]. The only point of the proof that must be modified is the existence of a suitable strictly unital natural tranformation $\mathsf{F} \to \mathsf{F}'$, for which we invoke Lemma 4.3. □

In the following we will need to use the fact that **dgCat** admits a model structure, where the weak equivalences are the quasi-equivalences and the fibrations are the full dg functors whose H^0 is an isofibration (see [24]). Recall that, in general, if $\mathbf{C}$ is a model category, $X \in \mathbf{C}$ is cofibrant and $Y \in \mathbf{C}$ is fibrant, then the natural map $\mathbf{C}(X, Y) \to \mathrm{Ho}(\mathbf{C})(X, Y)$ induces a bijection

$$\mathbf{C}(X, Y)/\asymp \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{C})(X, Y) \tag{4.1}$$

(see [13, Theorem 1.2.10]), where the equivalence relation $\asymp$ on $\mathbf{C}(X, Y)$ can be defined as follows.[1] First a *cylinder object* for X is given by morphisms $i_0, i_1\colon X \to X'$ and a weak equivalence $s\colon X' \to X$ such that $s \circ i_0 = s \circ i_1 = \mathrm{id}_X$ and $(i_0, i_1)\colon X \coprod X \to X'$ is a cofibration. Then, given

[1] Usually this equivalence relation is called homotopy and is denoted by $\sim$, but we will not do that, in order to avoid confusion with the already defined notion of homotopy for A_∞ functors.

$f_0, f_1 \in \mathbf{C}(X,Y)$, we have $f_0 \asymp f_1$ if and only if there exist a cylinder object (X', i_0, i_1, s) for X and $h \in \mathbf{C}(X',Y)$ such that $f_k = h \circ i_k$, for $k = 0,1$ (see [13, Definition 1.2.4 and Corollary 1.2.6]). Moreover, if $f \in \mathbf{C}(X,Y)$ is a weak equivalence between two fibrant and cofibrant objects, then (always by [13, Theorem 1.2.10]) there exists $g \in \mathbf{C}(Y,X)$ such that $g \circ f \asymp \mathrm{id}_X$ and $f \circ g \asymp \mathrm{id}_Y$.

Remark 4.5. One can easily see that, by construction, the bijection in (4.1) is indeed natural with respect to pre and post composition, if one restricts to fibrant and cofibrant objects of $\mathbf{C}$.

Remark 4.6. If (X', i_0, i_1, s) is a cylinder object for a cofibrant object X, then X' is cofibrant, as well: this follows immediately from the fact that cofibrations are stable under composition and pushouts (see [13, Corollary 1.1.11]).

Remark 4.7. It is clear from the definition that every dg category is fibrant. On the other hand, if $\mathbf{A} \in \mathbf{dgCat}$ is cofibrant, then $\mathbf{A}$ is also h-projective. Indeed, for every $A, B \in \mathbf{A}$ the complex of $\Bbbk$-modules $\mathbf{A}(A,B)$ is cofibrant by [25, Proposition 2.3], hence h-projective by [13, Lemma 2.3.8]. It follows that every dg category admits a h-projective resolution, namely a quasi-equivalence from a h-projective dg category.

Remark 4.8. If $\mathbf{A}$ and $\mathbf{B}$ are h-projective dg-categories, then so is $\mathbf{A} \otimes \mathbf{B}$.

Lemma 4.9. *If* $\mathsf{F}_0, \mathsf{F}_1 \in \mathbf{dgCat}(\mathbf{A},\mathbf{B})$ *are such that* $\mathbf{A}$ *is cofibrant and* $\mathsf{F}_0 \asymp \mathsf{F}_1$*, then* $\mathsf{F}_0 \approx \mathsf{F}_1$.

Proof. By definition there exist a cylinder object $(\mathbf{A}', \mathsf{I}_0, \mathsf{I}_1, \mathsf{S})$ for $\mathbf{A}$ and $\mathsf{H} \in \mathbf{dgCat}(\mathbf{A}',\mathbf{B})$ such that $\mathsf{F}_k = \mathsf{H} \circ \mathsf{I}_k$, for $k = 0,1$. Note that both $\mathbf{A}'$ (by Remark 4.6) and $\mathbf{A}$ are cofibrant, hence h-projective by Remark 4.7. So the quasi-equivalence S is actually a homotopy equivalence. Since $\mathsf{S} \circ \mathsf{I}_0 = \mathsf{S} \circ \mathsf{I}_1$, part (2) of Corollary 1.23 implies $\mathsf{I}_0 \approx \mathsf{I}_1$. It follows that $\mathsf{F}_0 = \mathsf{H} \circ \mathsf{I}_0 \approx \mathsf{H} \circ \mathsf{I}_1 = \mathsf{F}_1$. □

Lemma 4.10. *Given* $\mathsf{F} \in \mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}}(\mathbf{A},\mathbf{B})$ *with* $\mathbf{A}$ *cofibrant and satisfying* (3.1)*, there exists* $\mathsf{F}' \in \mathbf{dgCat}(\mathbf{A},\mathbf{B})$ *such that* $\mathsf{F} \approx \mathsf{F}'$.

Proof. By Proposition 3.1 the diagram in $\mathbf{A}_\infty\mathbf{Cat}$

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{\rho_{\mathbf{A}}} & \mathsf{U}(\mathbf{A}) & \xrightarrow{\sigma_{\mathbf{A}}} & \mathbf{A} \\ {\scriptstyle \mathsf{F}}\downarrow & & \downarrow{\scriptstyle \mathsf{U}(\mathsf{F})} & & \downarrow{\scriptstyle \mathsf{F}} \\ \mathbf{B} & \xrightarrow[\rho_{\mathbf{B}}]{} & \mathsf{U}(\mathbf{B}) & \xrightarrow[\sigma_{\mathbf{B}}]{} & \mathbf{B} \end{array}$$

is such that the square on the left commutes. Instead the square on the right commutes when F is a dg functor, but not in general. Taking into account that $\sigma_{\mathbf{A}} \circ \rho_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}}$ and $\sigma_{\mathbf{B}} \circ \rho_{\mathbf{B}} = \mathrm{id}_{\mathbf{B}}$, in any case we have

$$\mathsf{F} \circ \sigma_{\mathbf{A}} \circ \rho_{\mathbf{A}} = \mathsf{F} = \sigma_{\mathbf{B}} \circ \rho_{\mathbf{B}} \circ \mathsf{F} = \sigma_{\mathbf{B}} \circ \mathsf{U}(\mathsf{F}) \circ \rho_{\mathbf{A}}.$$

Since $\rho_{\mathbf{A}}$ is a homotopy equivalence, from part (3) of Corollary 1.23 we obtain

$$\mathsf{F} \circ \sigma_{\mathbf{A}} \approx \sigma_{\mathbf{B}} \circ \mathsf{U}(\mathsf{F}). \tag{4.2}$$

Now, let $\mathsf{S} \colon \mathbf{C} \to \mathsf{U}(\mathbf{A})$ be a quasi-equivalence in $\mathbf{dgCat}$ with $\mathbf{C}$ cofibrant (given, for instance, by a cofibrant replacement of $\mathsf{U}(\mathbf{A})$). Then $\sigma_{\mathbf{A}} \circ \mathsf{S} \colon \mathbf{C} \to \mathbf{A}$ is a quasi-equivalence in $\mathbf{dgCat}$ between two fibrant and cofibrant objects. Therefore there exists $\mathsf{G} \in \mathbf{dgCat}(\mathbf{A}, \mathbf{C})$ such that $\sigma_{\mathbf{A}} \circ \mathsf{S} \circ \mathsf{G} \asymp \mathrm{id}_{\mathbf{A}}$. By Lemma 4.9 this implies

$$\sigma_{\mathbf{A}} \circ \mathsf{S} \circ \mathsf{G} \approx \mathrm{id}_{\mathbf{A}}. \tag{4.3}$$

Using (4.3) and (4.2) we obtain

$$\mathsf{F} = \mathsf{F} \circ \mathrm{id}_{\mathbf{A}} \approx \mathsf{F} \circ \sigma_{\mathbf{A}} \circ \mathsf{S} \circ \mathsf{G} \approx \sigma_{\mathbf{B}} \circ \mathsf{U}(\mathsf{F}) \circ \mathsf{S} \circ \mathsf{G}.$$

To conclude, just observe that $\sigma_{\mathbf{B}}$, $\mathsf{U}(\mathsf{F})$, S and G are all dg functors, hence the same is true for their composition. □

The following is the crucial technical result of this section. It will also play an important role in the description of the internal Homs which is the content of the next section.

Proposition 4.11. *For every $\mathbf{A}, \mathbf{B} \in \mathbf{dgCat}$ with $\mathbf{A}$ h-projective there is a natural bijection*

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}).$$

Proof. First we claim that we can assume that $\mathbf{A}$ is semi-free (hence cofibrant by [8, Lemma B.6]). Indeed, there exists a quasi-equivalence $\tilde{\mathbf{A}} \to \mathbf{A}$ in $\mathbf{dgCat}$ with $\tilde{\mathbf{A}}$ semi-free (see [8, Lemma B.5]). Then $\mathbf{A} \cong \tilde{\mathbf{A}}$ in $\mathrm{Ho}(\mathbf{dgCat})$, whence there is a natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\tilde{\mathbf{A}}, \mathbf{B}).$$

Taking into account that $\tilde{\mathbf{A}}$ (by Remark 4.7) and $\mathbf{A}$ are h-projective, Lemma 1.22 implies $\mathbf{A} \cong \tilde{\mathbf{A}}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx$. Thus there is a natural bijection

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\tilde{\mathbf{A}}, \mathbf{B}).$$

It follows that the existence of the required bijection is equivalent to the existence of a natural bijection

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\tilde{\mathbf{A}}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\tilde{\mathbf{A}}, \mathbf{B}).$$

This proves the claim, and so for the rest of the proof we assume that $\mathbf{A}$ is semi-free. Note that, by construction, under this assumption $\mathbf{A}$ also satisfies (3.1).

By Lemma 4.9 the inclusion $\mathbf{dgCat}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}(\mathbf{A}, \mathbf{B})$ induces a map

$$\varphi\colon \mathbf{dgCat}(\mathbf{A}, \mathbf{B})/\asymp\; \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}, \mathbf{B}). \tag{4.4}$$

By (4.1) it is enough to prove that φ is bijective. Indeed, given $\mathsf{F}, \mathsf{F}' \in \mathbf{dgCat}(\mathbf{A}, \mathbf{B})$ such that $\mathsf{F} \approx \mathsf{F}'$, by Lemma 4.4 F and F' have the same image in $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}})$, hence also in $\mathrm{Ho}(\mathbf{dgCat})$ (see Remark 3.7). Therefore $\mathsf{F} \asymp \mathsf{F}'$, again by (4.1), and this proves that φ is injective. Finally, since $\mathbf{A}$ is semi-free, Lemma 4.2 implies that the natural injective map

$$\mathbf{A}_\infty\mathbf{Cat}/\approx(\mathbf{A}, \mathbf{B}) \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}, \mathbf{B})$$

is also surjective. We conclude that φ is surjective by Lemma 4.10. □

4.2. **The equivalences.** We are now ready to prove Theorem 4.1. The argument is split in a couple of steps.

Proposition 4.12. *The inclusion functor* $\mathsf{J}\colon \mathbf{dgCat} \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ *induces an equivalence of categories* $\mathrm{Ho}(\mathsf{J})\colon \mathrm{Ho}(\mathbf{dgCat}) \to \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$.

Proof. Obviously J preserves quasi-equivalences, hence it induces the functor $\mathrm{Ho}(\mathsf{J})$. Now we want to define a functor $\mathsf{K}\colon \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}} \to \mathrm{Ho}(\mathbf{dgCat})$. To this aim, recalling Lemma 1.16 and Lemma 1.22, first we choose, for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$, homotopy equivalences $\mathsf{Y}_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{R}_{\mathbf{A}}$ and $\mathsf{Z}_{\mathbf{A}}\colon \mathbf{R}_{\mathbf{A}} \to \mathbf{A}$, with $\mathbf{R}_{\mathbf{A}} \in \mathbf{dgCat}$, such that $\mathsf{Z}_{\mathbf{A}} \circ \mathsf{Y}_{\mathbf{A}} \approx \mathrm{id}_{\mathbf{A}}$ and $\mathsf{Y}_{\mathbf{A}} \circ \mathsf{Z}_{\mathbf{A}} \approx \mathrm{id}_{\mathbf{R}_{\mathbf{A}}}$. We also choose, for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$, a quasi-equivalence $\mathsf{S}_{\mathbf{A}}\colon \mathsf{K}(\mathbf{A}) \to \mathbf{R}_{\mathbf{A}}$ in $\mathbf{dgCat}$ with $\mathsf{K}(\mathbf{A})$ h-projective. This defines K on objects. As for morphisms, given $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$, we define $\mathsf{K}(\mathsf{F}) \in \mathrm{Ho}(\mathbf{dgCat})(\mathsf{K}(\mathbf{A}), \mathsf{K}(\mathbf{B}))$ as follows. The image in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathsf{K}(\mathbf{A}), \mathbf{R}_{\mathbf{B}})$ of the composition in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$

$$\mathsf{K}(\mathbf{A}) \xrightarrow{\mathsf{S}_{\mathbf{A}}} \mathbf{R}_{\mathbf{A}} \xrightarrow{\mathsf{Z}_{\mathbf{A}}} \mathbf{A} \xrightarrow{\mathsf{F}} \mathbf{B} \xrightarrow{\mathsf{Y}_{\mathbf{B}}} \mathbf{R}_{\mathbf{B}}$$

corresponds (by Proposition 4.11) to a unique $f \in \mathrm{Ho}(\mathbf{dgCat})(\mathsf{K}(\mathbf{A}), \mathbf{R}_{\mathbf{B}})$. Then we can define $\mathsf{K}(\mathsf{F}) := [\mathsf{S}_{\mathbf{B}}]^{-1} \circ f$, where $[\mathsf{S}_{\mathbf{B}}] \in \mathrm{Ho}(\mathbf{dgCat})(\mathsf{K}(\mathbf{B}), \mathbf{R}_{\mathbf{B}})$ denotes the image of $\mathsf{S}_{\mathbf{B}}$. It is immediate to see that K is really a functor and that it takes quasi-equivalences in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ to isomorphisms in $\mathrm{Ho}(\mathbf{dgCat})$. Thus K induces a functor $\mathsf{K}'\colon \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}) \to \mathrm{Ho}(\mathbf{dgCat})$. In order to conclude that $\mathrm{Ho}(\mathsf{J})$ is an equivalence with quasi-inverse K', it remains to show that there exist natural isomorphisms

$$\phi\colon \mathsf{K}' \circ \mathrm{Ho}(\mathsf{J}) \to \mathrm{id}_{\mathrm{Ho}(\mathbf{dgCat})}, \qquad \psi\colon \mathrm{Ho}(\mathsf{J}) \circ \mathsf{K}' \to \mathrm{id}_{\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})}.$$

It is easy to check that, for every $\mathbf{A} \in \mathrm{Ho}(\mathbf{dgCat})$, we can define $\phi_{\mathbf{A}} \in \mathrm{Ho}(\mathbf{dgCat})(\mathsf{K}(\mathbf{A}), \mathbf{A})$ as the unique morphism corresponding (again by Proposition 4.11) to the image of $\mathsf{Z}_{\mathbf{A}} \circ \mathsf{S}_{\mathbf{A}}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathsf{K}(\mathbf{A}), \mathbf{A})$. On the other hand, for every $\mathbf{A} \in \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$, we can directly define $\psi_{\mathbf{A}} \in \mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})(\mathsf{K}(\mathbf{A}), \mathbf{A})$ as the image of $\mathsf{Z}_{\mathbf{A}} \circ \mathsf{S}_{\mathbf{A}}$. □

Proposition 4.13. *The categories* $\mathrm{Ho}(\mathbf{dgCat})$ *and* $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ *are equivalent.*

Proof. Let $\mathbf{C}$ be the full subcategory of $\mathrm{Ho}(\mathbf{dgCat})$ whose objects are h-projective, and let $\mathbf{C}'$ be the full subcategory of $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ whose objects are (strictly unital) dg categories. The inclusion $\mathbf{C} \hookrightarrow \mathrm{Ho}(\mathbf{dgCat})$ is clearly an equivalence, and we claim that the same is true for the inclusion $\mathbf{C}' \hookrightarrow \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$. Indeed, using the notation of the proof of Proposition 4.12, for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}$ there exists a quasi-equivalence $\mathsf{Z}_{\mathbf{A}} \circ \mathsf{S}_{\mathbf{A}}\colon \mathsf{K}(\mathbf{A}) \to \mathbf{A}$ with $\mathsf{K}(\mathbf{A}) \in \mathbf{C}'$. Since both $\mathbf{A}$ and $\mathsf{K}(\mathbf{A})$ are h-projective, $\mathsf{Z}_{\mathbf{A}} \circ \mathsf{S}_{\mathbf{A}}$ is actually a homotopy equivalence, hence its image in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ is an isomorphism by Lemma 1.22. The conclusion follows from the fact that, as an easy consequence of Proposition 4.11, $\mathbf{C}$ and $\mathbf{C}'$ are isomorphic categories. □

Remark 4.14. One could hope to prove in a similar way that the categories $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}/\approx$ are equivalent (where $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}$ denotes the full subcategory of $\mathbf{A}_\infty\mathbf{Cat}$ with objects the h-projective strictly unital A_∞ categories). Unfortunately the above proof cannot be adapted to work in this setting (even when $\Bbbk$ is a field, in which case $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}/\approx$ coincides with $\mathbf{A}_\infty\mathbf{Cat}/\approx$). Indeed, one would need a variant of Proposition 4.11, with $\mathbf{A}_\infty\mathbf{Cat}$ in place of $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}$. But such a statement is clearly false (just take $\mathbf{A}$ and $\mathbf{B}$ two dg algebras with $\mathbf{A} = H(\mathbf{B}) = 0$ and $\mathbf{B} \neq 0$).

5. Internal Homs via A_∞ functors

In this section we prove Kontsevich–Keller's claim about internal Homs with no restrictions on the base ring. Namely, we provide a completely new proof of Theorem C in Section 5.2 which is preceded by some preliminary results about multifunctors in Section 5.1.

5.1. A_∞ **multifunctors.** Let us briefly recall some constructions which are carefully described in Sections 1.2 and 1.4 in [5] and which were originally introduced in [2]. For this reason we will be concise in the presentation and we will refer to these original sources for more details.

More specifically, given $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}$, an A_∞ *multifunctor* from $\mathbf{A}_1, \dots, \mathbf{A}_n$ to $\mathbf{A}$ is a morphism of graded quivers

$$\mathsf{F} \colon \overline{\mathrm{B}_\infty(\mathbf{A}_1)^+ \otimes \cdots \otimes \mathrm{B}_\infty(\mathbf{A}_n)^+} \to \mathbf{A}[1]$$

such that the natural extension

$$\overline{\mathrm{B}_\infty(\mathbf{A}_1)^+ \otimes \cdots \otimes \mathrm{B}_\infty(\mathbf{A}_n)^+} \to \mathrm{B}_\infty(\mathbf{A})$$

of F as graded cofunctor commutes with the differentials. Furthermore, an A_∞ multifunctor is *unital* if all its restrictions are unital. The set of all unital A_∞ multifunctors from $\mathbf{A}_1, \dots, \mathbf{A}_n$ to $\mathbf{A}$ will be denoted by $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$. It is also important to know that, by[2, Proposition 8.15], there is a unital A_∞ category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$ whose set of objects is $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$ (morphisms are suitably defined prenatural trasformations). Note that, if $\mathbf{A}$ is a dg category, then $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$ is a dg category as well.

Proposition 5.1. *For every* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}$ *there is an isomorphism in* $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}$

$$\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_2, \mathbf{A}_3)).$$

Proof. It follows from [2, Proposition 9.18] together with [2, Proposition 4.12]. □

In complete analogy with the case of A_∞ functors (see Section 1.4), if F_1 and F_2 are in $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$, we say that F_1 and F_2 are *weakly equivalent* (denoted by $\mathsf{F}_1 \approx \mathsf{F}_2$) if they are isomorphic in the category $H^0(\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A}))$. The relation $\approx$ is clearly compatible with compositions and then we can define a quotient (multi)category $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}/\approx$ with the same objects and whose morphisms are given by

$$\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}/\approx(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A}) := \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})/\approx.$$

Proposition 5.2. *For every* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbf{dgCat}$ *there is a natural bijection*

$$\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}/\approx(\mathbf{A}_1 \otimes \mathbf{A}_2, \mathbf{A}_3) \xleftrightarrow{1:1} \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}/\approx(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3).$$

Proof. We just sketch the proof, which is essentially the same as that of [5, Proposition 3.8], with a few adjustments at some points. Keeping the same notation of [5, Section 3], $\tilde{\mathsf{N}} \in \mathbf{dgCat}^\mathbf{n}(\Omega(\mathbf{C}), \overline{\mathbf{A}_1^+ \otimes \mathbf{A}_2^+})$ (where $\mathbf{C} := \overline{\mathrm{B}(\mathbf{A}_1)^+ \otimes \mathrm{B}(\mathbf{A}_2)^+}$) is actually a homotopy isomorphism by Corollary 2.6. Then, as $\overline{\mathbf{A}_1^+ \otimes \mathbf{A}_2^+} \in \mathbf{dgCat}$ when $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{dgCat}$ (see the proof of [5, Lemma 3.7]), by Lemma 1.22 $\tilde{\mathsf{N}}$ is unital and there exists $\mathsf{H} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}(\overline{\mathbf{A}_1^+ \otimes \mathbf{A}_2^+}, \Omega(\mathbf{C}))$ such that $\mathsf{H} \circ \tilde{\mathsf{N}} \approx \mathrm{id}_{\Omega(\mathbf{C})}$. No further change is needed in the rest of the proof, except that in the end we use Lemma 1.21 instead of [23, Lemma 1.6]. □

5.2. **Proof of Theorem C.** Given $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbf{dgCat}$, it is enough to prove that there are the following natural bijections (where $\mathbf{A}_i^{\mathrm{hp}}$ denotes a h-projective resolution of $\mathbf{A}_i$, for $i = 1, 2$)

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) & & \\
(A)\ \updownarrow\ 1{:}1 & & \\
\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3) & & \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)) \\
(B)\ \updownarrow\ 1{:}1 & & 1{:}1\ \updownarrow\ (F) \\
\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3) & & \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1^{\mathrm{hp}}, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)) \\
(C)\ \updownarrow\ 1{:}1 & & 1{:}1\ \updownarrow\ (E) \\
\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}_1^{\mathrm{hp}}, \mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3) & \xleftrightarrow[(D)]{1:1} & \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A}_1^{\mathrm{hp}}, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)),
\end{array}
\tag{5.1}
$$

since this would imply the wanted natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)).$$

Now, the existence of (A), and (F) follows from the isomorphisms $\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2 \cong \mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}$ and $\mathbf{A}_1^{\mathrm{hp}} \cong \mathbf{A}_1$ in $\mathrm{Ho}(\mathbf{dgCat})$. Taking into account that $\mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}$ is h-projective by Remark 4.8, (B) and (E) are due to Proposition 4.11. Finally, Proposition 5.2 implies (C), whereas (D) is a direct consequence of Proposition 5.1.

This clearly implies that $\mathrm{Ho}(\mathbf{dgCat})$ is symmetric monoidal category whose internal Hom $\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B})$, for two dg categories $\mathbf{A}$ and $\mathbf{B}$ is, up to isomorphism in $\mathrm{Ho}(\mathbf{dgCat})$, the dg category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B})$.

Remark 5.3. It is worth pointing out that the above conclusion is enough to prove Kontsevich–Keller's Claim in the introduction. Indeed, if $\mathbf{A}$ is a h-projective dg category with the additional property that the unit map $\mathbb{k} \to \mathbf{A}(A, A)$ admits a retraction as a morphism of complexes, for all $A \in \mathbf{A}$, then the fully faithful embedding $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}, \mathbf{B})$ is indeed a quasi-equivalence. The argument is the same as in [5, Corollary 2.6], where we replace (the erroneous) [5, Proposition 2.5] with Lemma 4.2 (note that (3.1) is clearly satisfied in our assumptions).

On the other hand, as $\mathbf{B}$ and its cofibrant replacements $\tilde{\mathbf{B}}$ are isomorphic in $\mathrm{Ho}(\mathbf{dgCat})$, the universal property of the internal Hom yields the isomorphism

$$\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B}) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \tilde{\mathbf{B}})$$

in $\mathrm{Ho}(\mathbf{dgCat})$. Finally, as we explained in the proof of Proposition 4.11, $\mathbf{A}^{\mathrm{hp}}$ and the cofibrant replacement $\tilde{\mathbf{A}}$ are isomorphic in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx$, thus we can simply set

$$\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B}) := \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

yielding a bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A},\mathbf{B}) \xleftrightarrow{1:1} \mathrm{Isom}(H^0(\mathbb{R}\underline{Hom}(\mathbf{A},\mathbf{B}))) = \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}/\approx(\tilde{\mathbf{A}},\tilde{\mathbf{B}}), \tag{5.2}$$

where the latter equality is by definition.

As in the proof Proposition 4.11, the bijection (5.2) boils down to the composition of bijections

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A},\mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\tilde{\mathbf{A}},\tilde{\mathbf{B}}) \xleftrightarrow{1:1} \mathbf{dgCat}(\tilde{\mathbf{A}},\tilde{\mathbf{B}})/\asymp \xleftrightarrow{1:1} \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}/\approx(\tilde{\mathbf{A}},\tilde{\mathbf{B}}).$$

Now, the first and the last bijections are compatible with pre and post compositions by definition, while the second one is such in view of Remark 4.5. This concludes the proof.

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