LOCALIZATIONS OF THE CATEGORIES OF A_∞ CATEGORIES AND INTERNAL HOMS OVER A RING

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Abstract. We show that, over an arbitrary commutative ring, the localizations of the categories of dg categories, of cohomologically unital, of unital and of strictly unital A_∞ categories with respect to the corresponding classes of quasi-equivalences are all equivalent. The result is proven at the ∞-categorical level by considering the natural ∞-categorical models of the categories above. As an application of the techniques we develop to compare the localizations mentioned above, we provide a new proof of the existence of internal Homs for the homotopy category of dg categories in terms of the category of (strictly) unital A_∞ functors. This yields a complete proof of a claim by Kontsevich and Keller.

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Introduction

This paper extends and, at the same time, repairs some existing results about the homotopy categories of differential graded (dg from now on) and A_∞ categories. The interest in these homotopy categories has grown during the last two decades and their study has produced several remarkable results. Nonetheless most of them depend on the assumption that such categories are linear over a field. At first sight this might look like a mild assumption but, as soon as we start thinking of applications of dg or A_∞ categories to algebraic or geometric problems such as deformation theory, it becomes a priority to replace the ground field with any commutative ring.

This simple observation was the main incentive to reconsider our previous results in [5] whose proofs deeply used the assumption that the categories are linear over a field. Unfortunately, the

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effort to generalize our previous work drew our attention to the unpleasant presence in the literature of a couple of mistakes with deep repercussions on several papers, including [5]. We will come back to them later in the introduction. For now we want to stress that our effort to find a way out of them was not only successful but provided a wide generalization of all known results along the lines that we would like to outline now.

Let us consider the category $\mathbf{dgCat}$ consisting of (small) dg categories defined over a commutative ring $\Bbbk$. Due to the work of Tabuada [28], $\mathbf{dgCat}$ has a model category structure which allows one to consider its homotopy category $\mathrm{Ho}(\mathbf{dgCat})$, which is nothing but the localization of $\mathbf{dgCat}$ with respect to all quasi-equivalences. The latter being special dg functors which induce an equivalence at the homotopy level. If we replace $\mathbf{dgCat}$ with the corresponding category of A_∞ categories, one can still consider its localization with respect to quasi-equivalences. Note that the category of A_∞ categories famously does not have a model structure with limits and colimits (see, for example, [5, Section 1.5]).

The delicate issue about A_∞ categories is that they can be endowed with various notions of unit and all of them are somehow motivated by the applications. One can indeed take the category $\mathbf{A_\infty Cat}$ of strictly unital A_∞ categories. Or, alternatively, the category $\mathbf{A_\infty Cat^u}$ of unital A_∞ categories. One could go further and consider the category $\mathbf{A_\infty Cat^c}$ of cohomologically unital A_∞ categories. We will discuss these subtleties in detail in Section 1.3. For now it is enough to keep in mind that we have natural faithful (the last one also full) functors

$$\mathbf{dgCat} \hookrightarrow \mathbf{A_\infty Cat} \hookrightarrow \mathbf{A_\infty Cat^u} \hookrightarrow \mathbf{A_\infty Cat^c}. \tag{1}$$

While strictly unital A_∞ categories are natural generalizations of dg categories, unital and cohomologically unital ones are those which appear when dealing with Fukaya categories. When the ground ring is a field the categories $\mathbf{A_\infty Cat^u}$ and $\mathbf{A_\infty Cat^c}$ coincide, but over a commutative ring they are different, in general. More flexible and interesting categorical objects are obtained by considering their localizations

$$\mathrm{Ho}(\mathbf{dgCat}) \longrightarrow \mathrm{Ho}(\mathbf{A_\infty Cat}) \longrightarrow \mathrm{Ho}(\mathbf{A_\infty Cat^u}) \longrightarrow \mathrm{Ho}(\mathbf{A_\infty Cat^c}) \tag{2}$$

with respect to the corresponding classes of quasi-equivalences.

The need for a comparison between the (homotopy) category of dg categories and the one of A_∞ categories is pervasive. We will mention more applications later in the introduction. For now, it is worth recalling that the core of the Homological Mirror Symmetry Conjecture, due to Kontsevich [16], is indeed a comparison between dg enhancements of the bounded derived category of coherent sheaves on a Calabi–Yau threefold and the Fukaya category (hence an A_∞ category) on a mirror Calabi–Yau threefold.

Taking a higher categorical viewpoint and following [26, 29], we can consider the ∞-categories

$$\mathrm{Ho}_\infty(\mathbf{dgCat}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A_\infty Cat}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A_\infty Cat^u}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A_\infty Cat^c}) \tag{3}$$

which are the natural ∞-categorical enhancements of $\mathrm{Ho}(\mathbf{dgCat})$, $\mathrm{Ho}(\mathbf{A_\infty Cat})$, $\mathrm{Ho}(\mathbf{A_\infty Cat^u})$ and $\mathrm{Ho}(\mathbf{A_\infty Cat^c})$, respectively, with natural ∞-functors induced by (1) (see the beginning of Section 3 for more details). The first main result of this paper is then the following.

Theorem A. *All functors in* (3) *are equivalences of* ∞*-categories.*

We should note that such a result is a generalization of Theorem 1.1 in an earlier version of [22]. Unfortunately, as the authors later realized, the proof in loc. cit. turned out to be wrong. If we stick to dg or strictly unital A_∞ categories which are linear over a field, Theorem A was proved in [26], by using in an essential way the results in [5]. Actually, we will explain in Section 3.1 that the same argument as in [26], together with Proposition 3.5, yields a proof of the part of Theorem A which claims that $\mathrm{Ho}_\infty(\mathbf{dgCat})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$ are equivalent, over an arbitrary commutative ring. The equivalence between the latter category and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$ is established in Section 3.2. This result has already appeared in [29], but here we provide a more direct and simpler proof. The equivalence between $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{c}})$ is rather easily obtained in Section 3.3. It should be noted that, beyond the categories appearing in (1), one can consider the categories $\mathbf{dgCat}^{\mathbf{u}}$ and $\mathbf{dgCat}^{\mathbf{c}}$ of, respectively, unital and cohomologically unital dg categories. As a byproduct of our proofs, it turns out that also the ∞-categories $\mathrm{Ho}_\infty(\mathbf{dgCat}^{\mathbf{u}})$ and $\mathrm{Ho}_\infty(\mathbf{dgCat}^{\mathbf{c}})$ are equivalent to those in (3).

It is worth pointing out that, as observed in [29, Section 1.2], Theorem A combined with the results in [13] shows that the Gepner–Haugseng's model for the collection of all ∞-categories enriched in chain complexes in [12] is equivalent to $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$. And more applications of the above result are discussed in [29, Section 3].

If we pass to the homotopy categories, then Theorem A has a very nice and useful 1-categorical incarnation which we state as follows.

Corollary B. *All functors in* (2) *are equivalences of categories.*

We will discuss later in the introduction other interesting applications of Theorem A. Now we want to present the second important result of the paper: a new proof for the existence of internal Homs in the homotopy category $\mathrm{Ho}(\mathbf{dgCat})$. In order to make this precise, recall that given two dg categories $\mathbf{A}_1$ and $\mathbf{A}_2$, one can form their tensor product $\mathbf{A}_1 \otimes \mathbf{A}_2$ in $\mathbf{dgCat}$. In order to get a well defined tensor product in $\mathrm{Ho}(\mathbf{dgCat})$ we need to *derive* it by setting $\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2 := \mathbf{A}_1 \otimes \mathbf{A}_2^{\mathrm{hp}}$. Here, given a dg category $\mathbf{A}$, we denote by $\mathbf{A}^{\mathrm{hp}}$ (respectively $\mathbf{A}^{\mathrm{hps}}$) a h-projective (respectively h-projective with split units) dg category with the property that $\mathbf{A} \cong \mathbf{A}^{\mathrm{hp}}$ (respectively $\mathbf{A} \cong \mathbf{A}^{\mathrm{hps}}$) in $\mathrm{Ho}(\mathbf{dgCat})$ (see Remark 4.7). Note that, in full generality, an A_∞ category $\mathbf{A}$ is *h-projective* if the complex of morphisms $\mathbf{A}(A, A')$ is such, for all $A, A' \in \mathbf{A}$, and the technical property of having split units is stated in Definition 3.4.

The main result in [30], later reproved in [6], shows that the tensor product $- \otimes^{\mathbb{L}} \mathbf{A}_2$ has a right adjoint $\mathbb{R}\underline{Hom}(\mathbf{A}_2, -)$ in $\mathrm{Ho}(\mathbf{dgCat})$. Namely, for $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ in $\mathbf{dgCat}$, we have a dg category $\mathbb{R}\underline{Hom}(\mathbf{A}_2, \mathbf{A}_3)$ and a natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbb{R}\underline{Hom}(\mathbf{A}_2, \mathbf{A}_3)). \tag{4}$$

The dg category $\mathbb{R}\underline{Hom}(\mathbf{A}_2, \mathbf{A}_3)$ is the *internal Hom* between $\mathbf{A}_2$ and $\mathbf{A}_3$, and it is uniquely determined (up to isomorphism in $\mathrm{Ho}(\mathbf{dgCat})$) by the bijection in (4).

A direct consequence is that we get a natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Isom}(H^0(\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B}))), \tag{5}$$

for all dg categories $\mathbf{A}, \mathbf{B} \in \mathbf{dgCat}$.

The astonishing fact is that, well before the appearance of [30], Kontsevich envisioned that such internal Homs should exist and could be described in terms of A_∞ functors between the corresponding dg categories. Such a claim, originally mentioned in [8], was later recast by Keller in his ICM talk [15] (see Section 4.3 therein). Its formulation requires some additional notation which we briefly sketch here.

If $\mathbf{A}, \mathbf{B} \in \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}$, we can consider the unital A_∞ category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}, \mathbf{B})$ which will be carefully defined in Section 1.4 and whose objects are the unital A_∞ functors from $\mathbf{A}$ to $\mathbf{B}$. Analogously, if $\mathbf{A}, \mathbf{B} \in \mathbf{A}_\infty\mathbf{Cat}$, we can consider the strictly unital A_∞ category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}, \mathbf{B})$ of stricly unital A_∞ functors from $\mathbf{A}$ to $\mathbf{B}$. It is important to keep in mind that $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}, \mathbf{B})$ and $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}, \mathbf{B})$ are dg categories if $\mathbf{B}$ is such. The precise formulation of the claim mentioned above is then the following.

Claim (Kontsevich, Keller)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be dg categories such that* $\mathbf{A}(A, A')$ *is a cofibrant complex and the unit map* $\Bbbk \to \mathbf{A}(A, A)$ *admits a retraction as a morphism of complexes, for all* $A, A' \in \mathbf{A}$*. Then* $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}, \mathbf{B})$ *is the dg category of internal Homs between* $\mathbf{A}$ *and* $\mathbf{B}$*.*

Actually we can prove the following result which implies, as a special case, the claim above (see Remark 5.4). At the same time, due to its gorgeous generality, it provides a completely new proof of the result in [30] about the existence of internal Homs.

Theorem C. *Given two dg categories* $\mathbf{A}, \mathbf{B}$*, the internal Hom in* $\mathrm{Ho}(\mathbf{dgCat})$ *between* $\mathbf{A}$ *and* $\mathbf{B}$ *is given by the dg category*

$$\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B}) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}^{\mathrm{hps}}, \mathbf{B}),$$

where the isomorphism is in $\mathrm{Ho}(\mathbf{dgCat})$*. Furthermore, we have equivalences of categories*

$$\mathrm{Ho}(\mathbf{dgCat}) \cong \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}/\approx \cong \mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}/\approx .$$

Here we denote by $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}$ the full subcategory of $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}$ whose objects are h-projective. We can then take the quotient $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}/\approx$ of $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}$ with respect to the equivalence relation which states that two unital A_∞ functors are (weakly) equivalent $\mathsf{F} \approx \mathsf{G}$ if they are isomorphic in the 0-th cohomology of $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}}(\mathbf{A}, \mathbf{B})$. Similarly, $\mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}$ denotes the full subcategory of $\mathbf{A}_\infty\mathbf{Cat}$ whose objects are h-projective and have split units and we can take an analogous quotient $\mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}/\approx$ of $\mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}$. Clearly, if $\Bbbk$ is a field, then $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u} = \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}$ (but $\mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}$ is strictly contained in $\mathbf{A}_\infty\mathbf{Cat}$).

While the proof of the first part of Theorem C is the content of Section 5, the fact that $\mathrm{Ho}(\mathbf{dgCat})$ is equivalent to $\mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}/\approx$ and $\mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}/\approx$ is established in Proposition 4.12. It is worth pointing out that the equivalence $\mathrm{Ho}(\mathbf{dgCat}) \cong \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}/\approx$ (and similarly the equivalence $\mathrm{Ho}(\mathbf{dgCat}) \cong \mathbf{A}_\infty\mathbf{Cat}_\mathbf{hps}/\approx$) can be seen as a categorification of (5). Indeed, given two dg categories $\mathbf{A}$ and $\mathbf{B}$, we have a natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathbf{A}_\infty\mathbf{Cat}^\mathbf{u}_\mathbf{hp}/\approx (\mathbf{A}^{\mathrm{hp}}, \mathbf{B}^{\mathrm{hp}}) = \mathrm{Isom}(H^0(\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B}^{\mathrm{hp}}))),$$

and the latter is in natural bijection with $\mathrm{Isom}(H^0(\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B})))$ by (4).

Applications and further developments. Our Theorem A should be mainly seen as a foundational result that puts on solid ground the homotopy theory of A_∞ categories. One issue of the category of A_∞ categories which we have mentioned above is the lack of model structure which makes hard to define and study homotopy limits and colimits of A_∞ categories. Our result builds a bridge that allows one to import categorical constructions from the much better behaved world of dg categories where a model structure exists. An interesting discussion on how comparing results similar to Theorem A may be used for the purposes of computing homotopy limits and colimits can be found in [11, Section A.4].

One important incarnation of the ideas mentioned above is the application of Corollary B to the problem of existence and uniqueness of enhancements for triangulated categories. Indeed, from Corollary B we immediately get the following useful result.

Corollary D. *A triangulated category has a (unique) dg enhancement if and only if it has a (unique) A_∞ enhancement.*

Note that for most triangulated categories naturally appearing in algebra or in algebraic geometry the existence of a (dg) enhancement is very easy to prove. On the other hand, the problem of uniqueness of enhancement is usually much harder. The interest for this problem was initiated by an influential conjecture by Bondal, Larsen and Lunts in [3] for geometric triangulated categories. The conjecture was proved by Lunts and Orlov in the seminal paper [19] and the result was then further extended in [7] and [1] (see also [9]), up to the last and most general result in [4], where it was proved that the following triangulated categories have unique dg (hence also A_∞) enhancements:

(a) $\mathrm{D}^?(\mathbf{A})$, where $\mathbf{A}$ is an abelian category and $? = b, +, -, \emptyset$;
(b) The category $\mathrm{D}^?_{\mathrm{qc}}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology and the category $\mathrm{Perf}(X)$ of perfect complexes on a quasi-compact and quasi-separated scheme, for $? = b, +, -, \emptyset$.

Another interesting feature of Theorem A is that its proof can be very easily adapted to prove a similar statement for algebras. More precisely, we can consider the categories $\mathbf{A_\infty Alg}$, $\mathbf{A_\infty Alg^u}$, $\mathbf{A_\infty Alg^c}$ and $\mathbf{A_\infty Alg^n}$ of strictly unital, unital, cohomologically unital and non-unital A_∞ algebras, respectively. Similarly, $\mathbf{dgAlg}$ and $\mathbf{dgAlg^n}$ will be the categories of strictly unital and non-unital dg algebras, respectively. If $\mathbf{C}$ is any of these categories, we denote by $\mathrm{Ho}(\mathbf{C})$ its localization with respect to quasi-isomorphisms, and by $\mathrm{Ho}_\infty(\mathbf{C})$ the ∞-categorical enhancement of $\mathrm{Ho}(\mathbf{C})$. Here it is important to observe that, unlike quasi-equivalences for (dg or A_∞) categories, quasi-isomorphisms for (dg or A_∞) algebras make perfect sense also in the non-unital case. We then immediately get the following result (see Remark 3.3 for the corresponding discussion).

Theorem E. *The natural functors*

$$\mathbf{dgAlg} \hookrightarrow \mathbf{A_\infty Alg} \hookrightarrow \mathbf{A_\infty Alg^u} \hookrightarrow \mathbf{A_\infty Alg^c},$$
$$\mathbf{dgAlg^n} \hookrightarrow \mathbf{A_\infty Alg^n}$$

induce equivalences of ∞-categories

$$\mathrm{Ho}_\infty(\mathbf{dgAlg}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Alg}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Alg^u}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Alg^c}),$$
$$\mathrm{Ho}_\infty(\mathbf{dgAlg^n}) \longrightarrow \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Alg^n}).$$

We avoid to state the analogous of Corollary B for algebras, which of course can be directly deduced from Theorem E. We point out that, in the non-unital case and when $\Bbbk$ is a field of characteristic 0, such a result also follows from [18, Theorem 11.4.8], which is formulated in the more general language of operads.

Let us now discuss the applications of Theorem C and let us go back to the original description in [30] of the internal Homs in terms of special dg bimodules. The fact that dg bimodules can be easily composed via convolutions, led Toën to state the following as an open question in the introduction to [30].

Question (Toën)**.** *Is the bijection* (5) *compatible with compositions?*

Our new description of the internal Homs and the full power of Theorem C immediately yields a positive answer to this question (see Remark 5.3 for a detailed discussion). Finally, we hope that our new description might clarify the still partially mysterious relation between objects in $\mathrm{Isom}(H^0(\mathbb{R}\underline{Hom}(\mathbf{A},\mathbf{B})))$ and Fourier–Mukai kernels of exact functors when $\mathbf{A}$ and $\mathbf{B}$ are dg enhancements of the derived categories of quasi-coherent sheaves on two quasi-compact and separated schemes. In [30, Section 8.3] it was claimed that such a relation should be encoded in a natural way by the bijection (5). However, a (far from trivial) proof of this claim is available only under restrictive assumptions on the schemes (see [21]).

Related work. The first comparison has to be made with our previous paper [5]. Besides the obvious observation that our new results imply essentially all the ones in [5], we should go back to our first claim in the introduction: the fact that this paper corrects and overcomes some mistakes in the literature.

It was proved by Lefèvre-Hasegawa [17, Proposition 3.2.4.1] for A_∞ algebras and later by Seidel [27, Lemma 2.1] for A_∞ categories that any cohomologically unital A_∞ algebra or category can be replaced with a strictly unital one, at least when we work over a field. Similarly, [17, Theorem 3.2.2.1] and [27, Remark 2.2] claim that the same is true for functors: an A_∞ functor between strictly unital A_∞ categories can be replaced by a strictly unital one, up to homotopy. Unfortunately, after carefully thinking about these claims, one realizes that none of them can be true in this generality for trivial reasons. And the falsity of the first claim about categories (and algebras) was indeed later observed by Seidel in an erratum to [27].

While many technical parts of [5] remain valid (and will also be used in this paper) others, heavily relying on the two claims above, have to be revisited. For example, [5, Proposition 2.5] is easily seen to be false as soon as we consider A_∞ categories $\mathbf{A}$ such that the complex $\mathbf{A}(A,A)$ has trivial cohomology but it is not trivial, for some some A in $\mathbf{A}$. This produces a cascade of problems in the proof of [5, Theorem A], some of which can be overcome by readjusting the arguments, while some of them needs the new (and at the same time more general) approach which we adopt in the present paper. A careful comparison shows that the new Corollary B replaces the old one for most

of its parts once we recall that the categories of cohomologically unital and of unital A_∞ categories are the same, over a field. There is only one claim in [5, Theorem A] that is not covered by our new results: the equivalence between $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathbf{A}_\infty\mathbf{Cat}/\approx$. In fact we expect that such an equivalence does not hold, as Corollary B shows that $\mathrm{Ho}(\mathbf{dgCat})$ is equivalent to $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}/\approx$ and it is easy to see that the inclusion of $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}/\approx$ in $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}/\approx = \mathbf{A}_\infty\mathbf{Cat}/\approx$ is not an equivalence (see Remark 4.13).

Moreover, for similar reasons, the description of the internal Homs in terms of strictly unital A_∞ functors has to be corrected or replaced by the one which uses unital A_∞ functors. The result is that the new Theorem C replaces and generalizes the old [5, Theorem B].

Finally, as we have already explained before, Theorem A for categories linear over a commutative ring provides a complete proof of Theorem 1.1 in an earlier version of [22]. The latter result together with Theorem C gives then access to the many very interesting applications discussed in the second part of [22] (see, in particular, Sections 4 and 5 therein) and in [29, Section 1.3].

Plan of the paper. In Section 1 we briefly recall the basic definitions and constructions which are used all along the paper. We refer to the existing literature for more details but an issue that we try to analyze carefully is the difference between the various notions of unit for A_∞ categories (see Section 1.3).

In Section 2 we show the existence of a crucial pair of adjoint functors between the category of non-unital dg categories $\mathbf{dgCat}^{\mathbf{n}}$ and $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ (for later use in Section 5, the key step of the proof needs to be treated in a more general setting, which makes Section 2 the more technical part of the paper). The analysis has to be refined in Section 3 in order to deal with strictly unital, unital and cohomologically unital A_∞ categories, thus proving Theorem A.

As for Theorem C, the proof of the last part is carried out in Section 4. The first part of Theorem C is proved in Section 5, after a preliminary discussion about multifunctors in Section 5.1.

Notation and conventions. We assume that a universe containing an infinite set is fixed, and we will simply call sets the members of this universe. In general the collection of objects of a category need not be a set: we will always specify if we are requiring this extra condition.

We work over a commutative ring $\Bbbk$. We will always assume that the collection of objects in a $\Bbbk$-linear category is a set.

The shift by an integer n of a graded $\Bbbk$-module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ will be denoted by $M[n] = \bigoplus_{i\in\mathbb{Z}} M^{n+i}$. If $x \in M$, we will often write $x[n]$ to denote the same element in $M[n]$. If x is homogeneous, say $x \in M^i$, then we set $\deg(x) := i$ and $\deg'(x) := i + 1$.

We recall the Koszul sign rule: if $f\colon M \to M'$ and $g\colon N \to N'$ are morphisms of graded $\Bbbk$-modules, with g homogeneous, then $f \otimes g\colon M \otimes N \to M' \otimes N'$ maps $x \otimes y$, with x homogeneous, to $(-1)^{\deg(g)\deg(x)} f(x) \otimes g(y)$.

Complexes (or dg modules) are cohomological (namely, the differential has degree $+1$).

1. Preliminaries on A_∞ categories and functors

In this section we provide a concise introduction to A_∞ categories and functors. In the whole paper the subtle relation between the various notions of unit is crucial. We provide here the basic definitions and properties which will be used later.

As in [5], we will follow the sign conventions in [17], which are different from (but equivalent to) those used in other references, like [2] and [27]. In particular, given graded $\Bbbk$-modules $M_1, \dots, M_i, N$, a $\Bbbk$-linear map $f\colon M_i \otimes \cdots \otimes M_1 \to N$ of degree n is identified with the $\Bbbk$-linear map

$$(-1)^{n+i-1}\iota_N^{-1} \circ f \circ (\iota_{M_i} \otimes \cdots \otimes \iota_{M_1})\colon M_i[1] \otimes \cdots \otimes M_1[1] \to N[1] \tag{1.1}$$

of degree $n+i-1$, where $\iota_M\colon M[1] \to M$ is the natural isomorphism of degree 1, for every graded $\Bbbk$-module M.

1.1. **Non-unital A_∞ categories, functors and natural transformations.** We start by recalling the explicit definitions of non-unital A_∞ categories, functors and (pre)natural transformations. A conceptual explanation of the otherwise mysterious formulas appearing in this section will be given in Section 1.2.

Definition 1.1. A *non-unital A_∞ category* $\mathbf{A}$ consists of a set of objects $\mathrm{Ob}(\mathbf{A})$, of graded $\Bbbk$-modules $\mathbf{A}(A, A')$ for every $A, A' \in \mathbf{A}$ and of $\Bbbk$-linear maps of degree $2-i$

$$\mathrm{m}^i = \mathrm{m}^i_{\mathbf{A}}\colon \mathbf{A}(A_{i-1}, A_i) \otimes \cdots \otimes \mathbf{A}(A_0, A_1) \to \mathbf{A}(A_0, A_i), \tag{1.2}$$

for every $i > 0$ and every $A_0, \dots, A_i \in \mathbf{A}$. The maps must satisfy the A_∞ *associativity relations*

$$\sum_{k=1}^{n}\sum_{i=0}^{n-k}(-1)^{i+k(n-i-k)}\mathrm{m}^{n-k+1} \circ (\mathrm{id}^{\otimes n-i-k} \otimes \mathrm{m}^k \otimes \mathrm{id}^{\otimes i}) = 0 \tag{1.3}$$

for every $n > 0$.

In particular, (1.3) with $n = 1$ shows that m^1 defines a differential on each $\mathbf{A}(A, A')$, which will always be regarded as a complex in this way. It is also important to observe that m^1 satisfies the graded Leibniz rule with respect to the composition defined by m^2 (by the case $n = 2$) and that m^2 is associative, up to a homotopy defined by m^3 (by the case $n = 3$). This implies that we obtain the non-unital graded *cohomology category* $H(\mathbf{A})$ of $\mathbf{A}$ such that $\mathrm{Ob}(H(\mathbf{A})) = \mathrm{Ob}(\mathbf{A})$,

$$H(\mathbf{A})(A, A') = \bigoplus_i H^i\big(\mathbf{A}(A, A')\big)$$

for every $A, A' \in \mathbf{A}$ and (associative) composition induced from m^2.

Definition 1.2. A *non-unital A_∞ functor* $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ between two non-unital A_∞ categories $\mathbf{A}$ and $\mathbf{B}$ is a collection $\mathsf{F} = \{\mathsf{F}^i\}_{i\geq 0}$, where $\mathsf{F}^0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ is a map of sets and

$$\mathsf{F}^i\colon \mathbf{A}(A_{i-1}, A_i) \otimes \cdots \otimes \mathbf{A}(A_0, A_1) \to \mathbf{B}(\mathsf{F}^0(A_0), \mathsf{F}^0(A_i)), \tag{1.4}$$

for $i > 0$, are $\Bbbk$-linear maps of degree $1-i$, for every $A_0, \dots, A_i \in \mathbf{A}$. The maps must satisfy the following relations

$$\begin{aligned}\sum_{k=1}^{n}\sum_{i=0}^{n-k}(-1)^{i+k(n-i-k)}\mathsf{F}^{n-k+1} \circ (\mathrm{id}^{\otimes n-i-k} \otimes \mathrm{m}^k_{\mathbf{A}} \otimes \mathrm{id}^{\otimes i}) \\ = \sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r>0}} (-1)^{\sum_{t=1}^{r-1}\sum_{u=t+1}^{r}(1-i_t)i_u}\mathrm{m}^r_{\mathbf{B}} \circ (\mathsf{F}^{i_r} \otimes \cdots \otimes \mathsf{F}^{i_1}),\end{aligned} \tag{1.5}$$

for every $n > 0$. A non-unital A_∞ functor F is *strict* if $\mathsf{F}^i = 0$ for every $i > 1$.

From (1.5) with $n = 1$ we see that F^1 commutes with the differentials m^1. Moreover, F^1 preserves the compositions m^2, up to a homotopy defined by F^2 (by the case $n = 2$). It follows that F^0 and F^1 induce a non-unital graded functor $H(\mathsf{F})\colon H(\mathbf{A}) \to H(\mathbf{B})$.

Remark 1.3. A non-unital A_∞ category $\mathbf{A}$ such that $\mathrm{m}^i = 0$ for all $i > 2$ is called a *non-unital dg category*; for such categories m^1 and m^2 are usually denoted by d and $\circ$. A strict non-unital A_∞ functor F between two dg categories is called a *non-unital dg functor*; in this case one often writes F instead of F^0 or F^1. There is a category $\mathbf{A_\infty Cat^n}$ (with objects the non-unital A_∞ categories and morphisms the non-unital A_∞ functors) which contains as a subcategory $\mathbf{dgCat^n}$ (with objects the non-unital dg categories and morphisms the non-unital dg functors). While the composition in $\mathbf{dgCat^n}$ is the obvious one, the composition in $\mathbf{A_\infty Cat^n}$ is more subtle (see Section 1.2); however, we will not need its explicit definition.

Definition 1.4. Given $\mathsf{F}, \mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A_\infty Cat^n}$, a *prenatural transformation* $\theta\colon \mathsf{F} \to \mathsf{G}$ of degree p is given by $\Bbbk$-linear maps of degree $p - i$

$$\theta^i\colon \mathbf{A}(A_{i-1}, A_i) \otimes \cdots \otimes \mathbf{A}(A_0, A_1) \to \mathbf{B}\big(\mathsf{F}^0(A_0), \mathsf{G}^0(A_i)\big) \tag{1.6}$$

for every $i \geq 0$ and every $A_0, \dots, A_i \in \mathbf{A}$. We say that θ is a *natural transformation* if

$$\begin{aligned}\sum_{k=1}^{n}\sum_{i=0}^{n-k}(-1)^{i+k(n-i-k)}\theta^{n-k+1} \circ (\mathrm{id}^{\otimes n-i-k} \otimes \mathrm{m}^k_{\mathbf{A}} \otimes \mathrm{id}^{\otimes i}) \\ + \sum_{\substack{i_1+\cdots+i_r+k+j_1+\cdots+j_s=n \\ i_1,\dots,i_r,j_1,\dots,j_s>0, k\geq 0}} (-1)^{p+r(p-1)+\sum_{t=1}^{r}(1-i_t)(n-\sum_{u=1}^{t-1} i_u)+(p-k)\sum_{t=1}^{s} j_t+\sum_{t=1}^{s-1}\sum_{u=t+1}^{s}(1-j_t)j_u} \\ \mathrm{m}^{r+s+1}_{\mathbf{B}} \circ (\mathsf{G}^{j_s} \otimes \cdots \otimes \mathsf{G}^{j_1} \otimes \theta^k \otimes \mathsf{F}^{i_r} \otimes \cdots \otimes \mathsf{F}^{i_1}) = 0\end{aligned} \tag{1.7}$$

for every $n \geq 0$.

Observe that θ^0 can be identified with a collection of elements $\theta^0_A \in \mathbf{B}\big(\mathsf{F}^0(A), \mathsf{G}^0(A)\big)^p$ for every $A \in \mathbf{A}$. Moreover, (1.7) with $n = 0$ shows that these elements are closed, whereas the case $n = 1$ implies that their images in cohomology define a natural transformation $H(\theta)\colon H(\mathsf{F}) \to H(\mathsf{G})$ of degree p.

1.2. **Reminder on the bar and cobar constructions.** This section is a quick reminder about the bar and cobar constructions. In [5, Sections 1.2 and 1.3] the reader can find definitions and properties of some notions which are not recalled here, like those of (graded or dg) quiver, cocategory and cofunctor. For a more detailed presentation see also [2].

We denote by $\mathbf{dgcoCat^n}$ the category whose objects are non-unital cocomplete dg cocategories and whose morphisms are non-unital dg cofunctors.

Given $\mathbf{A} \in \mathbf{A_\infty Cat^n}$, the *bar construction* $\mathrm{B}_\infty(\mathbf{A}) \in \mathbf{dgcoCat^n}$ associated to $\mathbf{A}$ is simply defined to be $\overline{\mathrm{T}}^c(\mathbf{A}[1])$ (where $\mathbf{A}$ is viewed as a graded quiver) as a non-unital graded cocategory. As for the differential, an arbitrary choice of maps $\mathrm{m}^i_{\mathbf{A}}$ as in (1.2) determines a morphism of graded quivers $\overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \mathbf{A}[1]$ of degree 1 (recall (1.1)), which extends uniquely to a $(\mathrm{id}_{\overline{\mathrm{T}}^c(\mathbf{A}[1])}, \mathrm{id}_{\overline{\mathrm{T}}^c(\mathbf{A}[1])})$-coderivation $d_{\mathbf{A}}\colon \overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{A}[1])$ of degree 1. It is easy to see that $d_{\mathbf{A}} \circ d_{\mathbf{A}} = 0$ if and only if (1.3) holds for every $n > 0$, in which case we set $\mathrm{B}_\infty(\mathbf{A}) := (\overline{\mathrm{T}}^c(\mathbf{A}[1]), d_{\mathbf{A}})$.

Similarly, $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ induces $\mathrm{B}_\infty(\mathsf{F})\colon \mathrm{B}_\infty(\mathbf{A}) \to \mathrm{B}_\infty(\mathbf{B})$ in $\mathbf{dgcoCat}^{\mathbf{n}}$. More precisely, an arbitrary choice of maps $\mathsf{F}^0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and F^i for $i > 0$ as in (1.4) determines a morphism of graded quivers $\overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \mathbf{B}[1]$ of degree 0, which extends uniquely to a graded cofunctor $\widehat{\mathsf{F}}\colon \overline{\mathrm{T}}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{B}[1])$. Then one can check that $d_{\mathbf{B}} \circ \widehat{\mathsf{F}} = \widehat{\mathsf{F}} \circ d_{\mathbf{A}}$ if and only if (1.5) holds for every $n > 0$, in which case we set $\mathrm{B}_\infty(\mathsf{F}) := \widehat{\mathsf{F}}$. Moreover, the composition in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ is defined in such a way that

$$\mathrm{B}_\infty\colon \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}} \to \mathbf{dgcoCat}^{\mathbf{n}}$$

is a functor, which actually turns out to be fully faithful.

Finally, given $\mathsf{F}, \mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ (more generally, F and G could be given by arbitrary maps $\mathsf{F}^0, \mathsf{G}^0\colon \mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and $\mathsf{F}^i, \mathsf{G}^i$ for $i > 0$ as in (1.4)), a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{G}$ of degree p determines $\Bbbk$-linear maps $\mathrm{T}^c(\mathbf{A}[1])(A, A') \to \mathbf{B}[1]\big(\mathsf{F}^0(A), \mathsf{G}^0(A')\big)$ of degree $p-1$ for every $A, A' \in \mathbf{A}$, which extend uniquely to a $(\widehat{\mathsf{F}}, \widehat{\mathsf{G}})$-coderivation $\widehat{\theta}\colon \mathrm{T}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{B}[1])$ of degree $p-1$. Here we still denote by $\widehat{\mathsf{F}}, \widehat{\mathsf{G}}\colon \mathrm{T}^c(\mathbf{A}[1]) \to \overline{\mathrm{T}}^c(\mathbf{B}[1])$ the extensions by 0 of $\widehat{\mathsf{F}}$ and $\widehat{\mathsf{G}}$. Again, it can be easily proved that $d_{\mathbf{B}} \circ \widehat{\theta} + (-1)^p \widehat{\theta} \circ d_{\mathbf{A}} = 0$ (where we still denote by $d_{\mathbf{A}}\colon \mathrm{T}^c(\mathbf{A}[1]) \to \mathrm{T}^c(\mathbf{A}[1])$ the extensions by 0 of $d_{\mathbf{A}}$) if and only if θ is a natural transformation.

Remark 1.5. Given $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ and a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{F}$ of degree 1 such that $\theta^0 = 0$, we can define $\mathsf{G}^0 := \mathsf{F}^0$ and $\mathsf{G}^i := \mathsf{F}^i + \theta^i$ for $i > 0$. Then we can regard θ as a prenatural transformation $\mathsf{F} \to \mathsf{G}$, and it is not difficult to show that in this way $\widehat{\mathsf{G}} = \widehat{\mathsf{F}} + \widehat{\theta}$. This clearly implies that G is a non-unital A_∞ functor if and only if $\theta\colon \mathsf{F} \to \mathsf{G}$ is a natural transformation.

As a matter of notation, we set

$$\mathrm{B} := \mathrm{B}_\infty|_{\mathbf{dgCat}^{\mathbf{n}}}\colon \mathbf{dgCat}^{\mathbf{n}} \to \mathbf{dgcoCat}^{\mathbf{n}},$$

which is a faithful (but not full) functor. Dually, the *cobar construction* yields a faithful (but not full) functor

$$\Omega\colon \mathbf{dgcoCat}^{\mathbf{n}} \to \mathbf{dgCat}^{\mathbf{n}}.$$

In particular, for $\mathbf{C} \in \mathbf{dgcoCat}^{\mathbf{n}}$, $\Omega(\mathbf{C})$ is simply defined to be $\overline{\mathrm{T}}(\mathbf{C}[-1])$ as a non-unital graded category, with differential induced from the differential and the cocomposition in $\mathbf{C}$.

By [5, Proposition 1.21] there is an adjunction

$$\Omega\colon \mathbf{dgcoCat}^{\mathbf{n}} \rightleftarrows \mathbf{dgCat}^{\mathbf{n}} :\mathrm{B},$$

with counit denoted by $\alpha\colon \Omega \circ \mathrm{B} \to \mathrm{id}_{\mathbf{dgCat}^{\mathbf{n}}}$ and unit denoted by $\beta\colon \mathrm{id}_{\mathbf{dgcoCat}^{\mathbf{n}}} \to \mathrm{B} \circ \Omega$. Since B_∞ is fully faithful, for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ there exists unique $\gamma_{\mathbf{A}} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \Omega(\mathrm{B}_\infty(\mathbf{A})))$ such that

$$\beta_{\mathrm{B}_\infty(\mathbf{A})} = \mathrm{B}_\infty(\gamma_{\mathbf{A}})\colon \mathrm{B}_\infty(\mathbf{A}) \to \mathrm{B}_\infty(\Omega(\mathrm{B}_\infty(\mathbf{A}))) = \mathrm{B}(\Omega(\mathrm{B}_\infty(\mathbf{A}))).$$

Denoting by $\mathsf{I}^{\mathsf{n}}\colon \mathbf{dgCat}^{\mathbf{n}} \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ the inclusion functor and setting

$$\mathsf{U}^{\mathsf{n}} := \Omega \circ \mathrm{B}_\infty\colon \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}} \to \mathbf{dgCat}^{\mathbf{n}},$$

it is clear that the A_∞ functors $\gamma_{\mathbf{A}}\colon \mathbf{A} \to \Omega(\mathrm{B}_\infty(\mathbf{A})) = \mathsf{U}^{\mathsf{n}}(\mathbf{A})$ (for $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$) define a natural transformation $\gamma\colon \mathrm{id}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}} \to \mathsf{I}^{\mathsf{n}} \circ \mathsf{U}^{\mathsf{n}}$.

1.3. **Notions of unit.** Now we need to discuss the various notions of unit which will be used in the rest of the paper.

Definition 1.6. A *cohomologically unital A_∞ category* is a non-unital A_∞ category $\mathbf{A}$ such that $H(\mathbf{A})$ is a category (i.e. $H(\mathbf{A})$ is unital).

Remark 1.7. If $\mathbf{A}$ is a cohomologically unital A_∞ category, then for every $A \in \mathbf{A}$ there exists (unique up to coboundaries) a closed degree 0 element $e_A \in \mathbf{A}(A,A)$ representing $\mathrm{id}_A \in H(\mathbf{A})(A,A)$. We will say that e_A is a *cohomological unit* of A.

Definition 1.8 ([20, Definition 7.3 and Lemma 7.4])**.** An A_∞ category $\mathbf{A}$ is *unital* if it is cohomologically unital and the following morphisms of complexes

$$\mathrm{m}^2(-\otimes e_A)\colon \mathbf{A}(A,A') \to \mathbf{A}(A,A') \qquad \mathrm{m}^2(e_A\otimes -)\colon \mathbf{A}(A',A) \to \mathbf{A}(A',A)$$

are homotopic to the identity for every $A, A' \in \mathbf{A}$, and for some (hence for every) cohomological unit e_A of A. In this case the cohomological units of $\mathbf{A}$ can be simply called *units*.

Definition 1.9. A *strictly unital A_∞ category* is a non-unital A_∞ category $\mathbf{A}$ such that for every $A \in \mathbf{A}$ there exists (unique) a degree 0 morphisms $\mathrm{id}_A \in \mathbf{A}(A,A)$ (called a *strict unit* of A) satisfying the following properties:

(1) $\mathrm{m}^2(-\otimes \mathrm{id}_A) = \mathrm{id}_{\mathbf{A}(A,A')}$ and $\mathrm{m}^2(\mathrm{id}_A\otimes -) = \mathrm{id}_{\mathbf{A}(A',A)}$ for every $A, A' \in \mathbf{A}$;
(2) $\mathrm{m}^i(f_i\otimes\cdots\otimes f_1) = 0$ if $i \neq 2$ and $f_j = \mathrm{id}_A$ for some $j \in \{1,\dots,i\}$ and some $A \in \mathbf{A}$.

For a non-unital A_∞ category $\mathbf{A}$, its *augmentation* is the strictly unital A_∞ category $\mathbf{A}^+$ such that $\mathrm{Ob}(\mathbf{A}^+) = \mathrm{Ob}(\mathbf{A})$ and

$$\mathbf{A}^+(A,A') = \begin{cases} \mathbf{A}(A,A') & \text{if } A \neq A' \\ \mathbf{A}(A,A') \oplus \Bbbk\, 1_A & \text{if } A = A', \end{cases}$$

with $\mathrm{m}^i_{\mathbf{A}^+}$ the unique extension of $\mathrm{m}^i_{\mathbf{A}}$ such that the additional morphisms 1_A is the unit of A in $\mathbf{A}^+$, for every $A \in \mathbf{A}$ and every $i > 0$. To avoid confusion, when $\mathbf{A}$ is strictly unital, the strict unit in $\mathbf{A}$ is denoted by id_A while the one in $\mathbf{A}^+$ is 1_A, for every $A \in \mathbf{A}$.

Similarly, we get *cohomologically unital dg categories*, *unital dg categories* and *strictly unital dg categories*. In accordance to the existing literature, strictly unital dg categories (respectively functors) will be simply referred to as dg categories (respectively functors).

Example 1.10. (i) In the special case of (dg or A_∞) categories with only one object, then, for obvious reasons, we will talk about (dg or A_∞) algebras.

(ii) Given two (non-unital, cohomologically unital, unital or strictly unital) dg categories $\mathbf{A}$ and $\mathbf{B}$, we can define a (non-unital, cohomologically unital, unital or strictly unital) dg category $\mathbf{A}\otimes\mathbf{B}$, which is the tensor product of $\mathbf{A}$ and $\mathbf{B}$. Its objects are the pairs (A,B) with $A \in \mathbf{A}$ and $B \in \mathbf{B}$, while $(\mathbf{A}\otimes\mathbf{B})\big((A,B),(A',B')\big) = \mathbf{A}(A,A')\otimes\mathbf{B}(B,B')$. If $\mathbf{A}$ and $\mathbf{B}$ are A_∞ categories, then defining an appropriate tensor product is a more delicate issue which will be discussed in [24].

There is also a notion of homotopy unital A_∞ category (which will not be used in this paper), such that the following implications hold for A_∞ categories (see [20, Section 8.12.]):

$$\text{strictly unital} \implies \text{homotopy unital} \implies \text{unital} \implies \text{cohomologically unital}$$

Remark 1.11. If $\Bbbk$ is a field, then an A_∞ category is unital if and only if it is cohomologically unital. This is simply due to the fact that, over a field, two morphisms of complexes are homotopic if they induce the same map in cohomology.

Of course, there are also the corresponding notions of strictly unital, unital and cohomologically unital A_∞ functors (see [27, pp 23] and [20, Definition 8.1. and Proposition 8.2.]).

Definition 1.12. Let $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ be a non-unital A_∞ functor.

(i) F is *(cohomologically) unital* if $\mathbf{A}$ and $\mathbf{B}$ are (cohomologically) unital and $H(\mathsf{F})$ is unital.

(ii) F is *strictly unital* if $\mathbf{A}$ and $\mathbf{B}$ are strictly unital and the following properties are satisfied:

(1) $\mathsf{F}^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)}$ for every $A \in \mathbf{A}$;

(2) $\mathsf{F}^i(f_i \otimes \cdots \otimes f_1) = 0$ if $i > 1$ and $f_j = \mathrm{id}_A$ for some $j \in \{1, \dots, i\}$ and some $A \in \mathbf{A}$.

The following definition is only partially standard.

Definition 1.13. A non-unital A_∞ functor $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ is a *quasi-isomorphism* (respectively a *homotopy isomorphism*) if F^0 is bijective and $\mathsf{F}^1\colon \mathbf{A}(A, A') \to \mathbf{B}(\mathsf{F}^0(A), \mathsf{F}^0(A'))$ is a quasi-isomorphism (respectively a homotopy equivalence) of complexes for every $A, A' \in \mathbf{A}$.

When $\mathbf{B}$ is cohomologically unital (respectively unital), $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ is a *quasi-equivalence* (respectively a *homotopy equivalence*) if $H(\mathsf{F})$ is essentially surjective and $\mathsf{F}^1\colon \mathbf{A}(A, A') \to \mathbf{B}(\mathsf{F}^0(A), \mathsf{F}^0(A'))$ is a quasi-isomorphism (respectively a homotopy equivalence) of complexes for every $A, A' \in \mathbf{A}$.

Remark 1.14. Clearly every homotopy isomorphism (respectively homotopy equivalence) is a quasi-isomorphism (respectively quasi-equivalence), and the converse holds if $\Bbbk$ is a field. It is also easy to see that, if a non-unital A_∞ functor $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ is a quasi-isomorphism (respectively a homotopy isomorphism), then $\mathbf{A}$ is cohomologically unital (respectively unital) if and only if $\mathbf{B}$ is cohomologically unital (respectively unital), and in this case F is cohomologically unital (respectively unital).

We will denote by $\mathbf{A_\infty Cat}$ (respectively $\mathbf{A_\infty Cat^u}$, respectively $\mathbf{A_\infty Cat^c}$) the subcategory of $\mathbf{A_\infty Cat^n}$ whose objects are strictly unital (respectively unital, respectively cohomologically unital) A_∞ categories and whose morphisms are strictly unital (respectively cohomologically unital) A_∞ functors. Similarly, $\mathbf{dgCat}$ (respectively $\mathbf{dgCat^u}$, respectively $\mathbf{dgCat^c}$) denotes the subcategory of $\mathbf{dgCat^n}$ whose objects are strictly unital (respectively unital, respectively cohomologically unital) dg categories and whose morphisms are strictly unital (respectively cohomologically unital) dg functors. Moreover, $\mathbf{A_\infty Cat_{dg}}$ will be the full subcategory of $\mathbf{A_\infty Cat}$ whose objects are dg categories.

Remark 1.15. In any case $\mathbf{A_\infty Cat^u}$ (respectively $\mathbf{dgCat^u}$) is a full subcategory of $\mathbf{A_\infty Cat^c}$ (respectively $\mathbf{dgCat^c}$). Moreover, if $\Bbbk$ is a field, it follows from Remark 1.11 that $\mathbf{A_\infty Cat^u} = \mathbf{A_\infty Cat^c}$ and $\mathbf{dgCat^u} = \mathbf{dgCat^c}$.

In order to study the relation between $\mathbf{A_\infty Cat^u}$ and $\mathbf{dgCat}$, later we will need the following result (which comes from an A_∞ categorical version of Yoneda's lemma).

Lemma 1.16. *Given $\mathbf{A} \in \mathbf{A_\infty Cat^u}$, there exists a homotopy isomorphism $\mathsf{Y}_\mathbf{A}\colon \mathbf{A} \to \mathbf{R_A}$ with $\mathbf{R_A} \in \mathbf{dgCat}$.*

Proof. See [2, Corollary 1.4]. □

Coming to prenatural transformations, we will consider the following notion.

Definition 1.17. Given $\mathsf{F}, \mathsf{G}\colon \mathbf{A} \to \mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ with $\mathbf{A}$ strictly unital, a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{G}$ is *strictly unital* if $\theta^i(f_i \otimes \cdots \otimes f_1) = 0$ whenever $i > 0$ and there exists $j \in \{1, \dots, i\}$ such that $f_j = \mathrm{id}_A$ for some $A \in \mathbf{A}$.

1.4. **Category of functors and equivalence relations.** Given $\mathbf{A}, \mathbf{B} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$, there is a natural non-unital A_∞ category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}}(\mathbf{A}, \mathbf{B})$, whose set of objects is $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ and whose morphisms are prenatural transformations (see [5, Section 1.4]). As for the maps $\mathrm{m}^i = \mathrm{m}^i_{\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}}(\mathbf{A},\mathbf{B})}$, for our aims it is enough to know that $\mathrm{m}^1(\theta)^n$ is given by the left-hand-side of (1.7), for every $\mathsf{F}, \mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ and every prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{G}$ of degree p.

Observe that, if $\mathbf{B}$ is (strictly) unital or is a dg category, then $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}}(\mathbf{A}, \mathbf{B})$ has the same property. When $\mathbf{A}$ and $\mathbf{B}$ are strictly unital (respectively unital), the full A_∞ subcategory of $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}}(\mathbf{A}, \mathbf{B})$ whose set of objects is $\mathbf{A}_\infty\mathbf{Cat}(\mathbf{A}, \mathbf{B})$ (respectively $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}(\mathbf{A}, \mathbf{B})$) will be denoted by $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}, \mathbf{B})$ (respectively $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}, \mathbf{B})$).

Definition 1.18. Let $\mathsf{F}, \mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$.

(i) F and G are *weakly equivalent* (denoted by $\mathsf{F} \approx \mathsf{G}$) if $\mathbf{B}$ is unital and $\mathsf{F} \cong \mathsf{G}$ in the (unital) category $H^0\big(\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}}(\mathbf{A}, \mathbf{B})\big)$.

(ii) F and G are *homotopic* (denoted by $\mathsf{F} \sim \mathsf{G}$) if $\mathsf{F}^0 = \mathsf{G}^0$ and there exists a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{G}$ of degree 0 such that $\theta^0 = 0$ and $\mathsf{G}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for every $i > 0$.

Remark 1.19. As it is proved in [27, Section (1h)], homotopy is an equivalence relation. Since it will be useful later, we also point out the following property, which can be directly deduced from the proof. Let $\mathsf{F}, \mathsf{G}, \mathsf{H} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ with $\mathsf{F} \sim \mathsf{G}$ and $\mathsf{G} \sim \mathsf{H}$ through homotopies θ_1 and θ_2, respectively. Assuming that there exists $n > 0$ such that $\theta_2^i = 0$ for $i < n$, then $\mathsf{F} \sim \mathsf{H}$ through a homotopy θ such that $\theta^i = \theta_1^i$ for $i < n$.

Remark 1.20. If $\mathsf{F} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ and $\theta\colon \mathsf{F} \to \mathsf{F}$ is a prenatural transformation of degree 0 such that $\theta^0 = 0$, then there exists $\mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ such that $\mathsf{F} \sim \mathsf{G}$ with $\mathsf{G}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for every $i > 0$, where θ is regarded as a prenatural transformation $\mathsf{F} \to \mathsf{G}$. Indeed, we set $\mathsf{G}^0 := \mathsf{F}^0$ and, for $n > 0$, we define inductively G^n as the sum of F^n and of the left-hand-side of (1.7) (which involves G^i only with $i < n$, since $\theta^0 = 0$). Then $\mathsf{G}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for $i > 0$ by construction, while the fact that $\mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ follows from Remark 1.5, taking into account that $\mathrm{m}^1(\theta)$ is a natural transformation (because $\mathrm{m}^1 \circ \mathrm{m}^1 = 0$) of degree 1 and clearly $\mathrm{m}^1(\theta)^0 = 0$.

Since $\approx$ is compatible with compositions, from the category $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}$ one can obtain a quotient category $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx$ with the same objects and whose morphisms are given by

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx(\mathbf{A}, \mathbf{B}) := \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}(\mathbf{A}, \mathbf{B})/\!\approx .$$

Similarly one can construct $\mathbf{A}_\infty\mathbf{Cat}/\!\approx$ from $\mathbf{A}_\infty\mathbf{Cat}$.

Later we will need the following results.

Lemma 1.21. *Let $\mathsf{F}, \mathsf{F}' \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ with $\mathbf{B}$ unital. If there exists a natural transformation $\theta\colon \mathsf{F} \to \mathsf{F}'$ of degree 0 such that $H(\theta)\colon H(\mathsf{F}) \to H(\mathsf{F}')$ is an isomorphism, then $\mathsf{F} \approx \mathsf{F}'$.*

Proof. It follows from [20, Proposition 7.15]. □

Lemma 1.22. *Let* $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ *be a homotopy equivalence (in particular,* $\mathbf{B}$ *is unital). Then* $\mathbf{A}$ *and* F *are also unital and there exists* $\mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}(\mathbf{B}, \mathbf{A})$ *such that* $\mathsf{G} \circ \mathsf{F} \approx \mathrm{id}_{\mathbf{A}}$ *and* $\mathsf{F} \circ \mathsf{G} \approx \mathrm{id}_{\mathbf{B}}$ *(hence the image of* F *is an isomorphism in* $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx$*).*

Proof. It follows from [20, Theorem 8.8]. □

Corollary 1.23. *Let* $\mathsf{F}, \mathsf{F}' \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A}, \mathbf{B})$ *with* $\mathbf{B}$ *unital. Then* $\mathsf{F} \approx \mathsf{F}'$ *in each of the following cases.*

(1) $\mathbf{B}$ *is strictly unital and* $\mathsf{F} \sim \mathsf{F}'$*.*
(2) *There exists a homotopy equivalence* $\mathsf{G}\colon \mathbf{B} \to \mathbf{B}'$ *such that* $\mathsf{G} \circ \mathsf{F} \approx \mathsf{G} \circ \mathsf{F}'$*.*
(3) *There exists a homotopy equivalence* $\mathsf{H}\colon \mathbf{A}' \to \mathbf{A}$ *such that* $\mathsf{F} \circ \mathsf{H} \approx \mathsf{F}' \circ \mathsf{H}$*.*

Proof. If $\mathbf{B}$ is strictly unital and $\theta\colon \mathsf{F} \to \mathsf{F}'$ is a homotopy, it is straightforward to check that the prenatural transformation $\widetilde{\theta}\colon \mathsf{F} \to \mathsf{F}'$ defined by $\widetilde{\theta}^i := \theta^i$ for $i > 0$ and $\widetilde{\theta}^0(A) := \mathrm{id}_{\mathsf{F}^0(A)}$ for every $A \in \mathbf{A}$ is a natural transformation (see also the paragraph before [27, Lemma 2.5]). Thus part (1) follows from Lemma 1.21, whereas parts (2) and (3) are easy consequences of Lemma 1.22. □

2. The non-unital case

Using the notation introduced in Section 1.2, we first state the following result.

Proposition 2.1. *There is an adjunction*

$$\mathsf{U}^{\mathsf{n}}\colon \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}} \rightleftarrows \mathbf{dgCat}^{\mathbf{n}} :\mathsf{I}^{\mathsf{n}},$$

whose unit is $\gamma\colon \mathrm{id}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}} \to \mathsf{I}^{\mathsf{n}} \circ \mathsf{U}^{\mathsf{n}}$ *and whose counit is* $\alpha\colon \mathsf{U}^{\mathsf{n}} \circ \mathsf{I}^{\mathsf{n}} = \Omega \circ \mathrm{B} \to \mathrm{id}_{\mathbf{dgCat}^{\mathbf{n}}}$*. Moreover,* $\gamma_{\mathbf{A}}$ *(for every* $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$*) and* $\alpha_{\mathbf{B}}$ *(for every* $\mathbf{B} \in \mathbf{dgCat}^{\mathbf{n}}$*) are homotopy isomorphisms.*

It is easy to see that the same proof of [5, Proposition 1.22] can be adapted to work when $\Bbbk$ is an arbitrary commutative ring and with homotopy isomorphism in place of quasi-isomorphism. However, when dealing with the strictly unital case in Section 3, it will be useful to know the crucial proof of the fact that, for every $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}$ and every $A, A' \in \mathbf{A}$,

$$\gamma^1_{\mathbf{A}}\colon \mathbf{A}(A, A') \to \mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A') = \Omega(\mathrm{B}_\infty(\mathbf{A}))(A, A') \tag{2.1}$$

is a homotopy equivalence of complexes. Actually we will prove a more general statement in Section 2.3, from which we will also deduce a result that will be needed in Section 5. To this aim, we first prove a technical result in Section 2.1 and then introduce the morphism which replaces (2.1) in a more general setting in Section 2.2.

Remark 2.2. In view of Remark 1.14, the adjunction of Proposition 2.1 restricts to adjunctions

$$\mathsf{U}^{\mathsf{n}}\colon \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{c}} \rightleftarrows \mathbf{dgCat}^{\mathbf{c}} :\mathsf{I}^{\mathsf{n}}, \qquad \mathsf{U}^{\mathsf{n}}\colon \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}} \rightleftarrows \mathbf{dgCat}^{\mathbf{u}} :\mathsf{I}^{\mathsf{n}}.$$

2.1. **A criterion for homotopy equivalence.** We will need the following general and possibly known result about filtered complexes. We include the proof since we could not find a suitable reference.

Lemma 2.3. *Let C be a complex of $\Bbbk$-modules endowed with an ascending and exhaustive filtration F^nC (with $n \in \mathbb{N}$) such that $\mathrm{F}^0C = 0$. Assume that for every $n > 1$ the exact sequence of complexes*

$$0 \to \mathrm{F}^{n-1}C \to \mathrm{F}^nC \to \mathrm{gr}^nC := (\mathrm{F}^nC)/(\mathrm{F}^{n-1}C) \to 0$$

splits as a sequence of graded modules and the complex gr^nC is null-homotopic. Then the inclusion $\mathrm{F}^1C \hookrightarrow C$ is a homotopy equivalence of complexes.

Proof. We can assume that as a graded module $C = \bigoplus_{n>0} C_n$ with $\mathrm{F}^nC = \bigoplus_{0<m\leq n} C_m$ and $\mathrm{gr}^nC = C_n$. We will write $C_{\leq n}$ or $C_{<n+1}$ instead of F^nC. For $n > 0$ we will denote by $d_{\leq n}$ the differential of $C_{\leq n}$ (which is the restriction of the differential d of C) and by d_n the induced differential on C_n. Observe that $d_1 = d_{\leq 1}$ and

$$d_{\leq n} = \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} : C_{\leq n} = C_{<n} \oplus C_n \to C_{\leq n} = C_{<n} \oplus C_n$$

for $n > 1$, where $e_n \colon C_n \to C_{<n}$ is a degree 1 map such that

$$d_{<n} \circ e_n = -e_n \circ d_n. \tag{2.2}$$

Denoting by $i_n \colon C_1 \hookrightarrow C_{\leq n}$ the inclusion, we claim that there exist morphisms of complexes $p_n \colon C_{\leq n} \to C_1$ and degree -1 maps $h_{\leq n} \colon C_{\leq n} \to C_{\leq n}$ such that

$$\mathrm{id}_{C_1} = p_n \circ i_n, \tag{2.3}$$

$$\mathrm{id}_{C_{\leq n}} = i_n \circ p_n + d_{\leq n} \circ h_{\leq n} + h_{\leq n} \circ d_{\leq n} \tag{2.4}$$

for every $n > 0$, and satisfying the compatibility conditions

$$p_n|_{C_{<n}} = p_{n-1}, \qquad h_{\leq n}|_{C_{<n}} = h_{<n} \tag{2.5}$$

for every $n > 1$. Assuming this, one can conclude the proof very easily. Indeed, the maps $p \colon C \to C_1$ and $h \colon C \to C$ such that $p|_{C_{\leq n}} = p_n$ and $h|_{C_{\leq n}} = h_{\leq n}$ for every $n > 0$ are well defined (and unique), thanks to (2.5). Moreover, since each p_n is a morphism of complexes and (2.3) and (2.4) hold, also p is a morphism of complexes and (denoting by $i \colon C_1 \hookrightarrow C$ the inclusion) we obtain

$$\mathrm{id}_{C_1} = p \circ i, \qquad \mathrm{id}_C = i \circ p + d \circ h + h \circ d,$$

thus proving that i is a homotopy equivalence.

So it remains to prove the claim, and to this purpose we proceed by induction on n. As we can obviously take $p_1 = i_1 = \mathrm{id}_{C_1}$ and $h_{\leq 1} = 0$, we assume that $n > 1$ and that the maps p_m and h_m with all the required properties have already been chosen for $0 < m < n$. In particular, p_{n-1} is a morphism of complexes and

$$\mathrm{id}_{C_1} = p_{n-1} \circ i_{n-1}, \tag{2.6}$$

$$\mathrm{id}_{C_{<n}} = i_{n-1} \circ p_{n-1} + d_{<n} \circ h_{<n} + h_{<n} \circ d_{<n}. \tag{2.7}$$

Moreover, since C_n is null-homotopic, there exists a degree -1 map $h_n\colon C_n \to C_n$ such that

$$\mathrm{id}_{C_n} = d_n \circ h_n + h_n \circ d_n. \tag{2.8}$$

Setting

$$p_n := \begin{pmatrix} p_{n-1} & -p_{n-1}\circ e_n \circ h_n \end{pmatrix} : C_{\leq n} = C_{<n} \oplus C_n \to C_1,$$
$$h_{\leq n} := \begin{pmatrix} h_{<n} & -h_{<n}\circ e_n \circ h_n \\ 0 & h_n \end{pmatrix} : C_{\leq n} = C_{<n} \oplus C_n \to C_{\leq n} = C_{<n} \oplus C_n,$$

(2.5) is certainly satisfied. Using, beyond the fact that p_{n-1} is a morphism of complexes, (2.2) and (2.8), we obtain also

$$\begin{aligned} d_1 \circ p_n &= d_1 \circ \begin{pmatrix} p_{n-1} & -p_{n-1}\circ e_n \circ h_n \end{pmatrix} = \begin{pmatrix} d_1 \circ p_{n-1} & -d_1 \circ p_{n-1}\circ e_n \circ h_n \end{pmatrix} \\ &= \begin{pmatrix} p_{n-1}\circ d_{<n} & -p_{n-1}\circ d_{<n}\circ e_n \circ h_n \end{pmatrix} = \begin{pmatrix} p_{n-1}\circ d_{<n} & p_{n-1}\circ e_n \circ d_n \circ h_n \end{pmatrix} \\ &= \begin{pmatrix} p_{n-1}\circ d_{<n} & p_{n-1}\circ e_n \circ (\mathrm{id}_{C_n} - h_n \circ d_n) \end{pmatrix} = \begin{pmatrix} p_{n-1} & -p_{n-1}\circ e_n \circ h_n \end{pmatrix} \circ \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} = p_n \circ d_{\leq n}, \end{aligned}$$

which shows that p_n is a morphism of complexes. Taking into account that $p_n \circ i_n = p_{n-1} \circ i_{n-1}$, (2.3) follows directly from (2.6). Finally, by (2.7), (2.8) and (2.2),

$$\begin{aligned} d_{\leq n}\circ h_{\leq n} + h_{\leq n}\circ d_{\leq n} &= \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} \circ \begin{pmatrix} h_{<n} & -h_{<n}\circ e_n \circ h_n \\ 0 & h_n \end{pmatrix} + \begin{pmatrix} h_{<n} & -h_{<n}\circ e_n \circ h_n \\ 0 & h_n \end{pmatrix} \circ \begin{pmatrix} d_{<n} & e_n \\ 0 & d_n \end{pmatrix} \\ &= \begin{pmatrix} d_{<n}\circ h_{<n} + h_{<n}\circ d_{<n} & -d_{<n}\circ h_{<n}\circ e_n \circ h_n + e_n \circ h_n + h_{<n}\circ e_n - h_{<n}\circ e_n \circ h_n \circ d_n \\ 0 & d_n \circ h_n + h_n \circ d_n \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_{C_{<n}} - i_{n-1}\circ p_{n-1} & (h_{<n}\circ d_{<n} + i_{n-1}\circ p_{n-1})\circ e_n \circ h_n + h_{<n}\circ e_n \circ d_n \circ h_n \\ 0 & \mathrm{id}_{C_n} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_{C_{<n}} - i_{n-1}\circ p_{n-1} & i_{n-1}\circ p_{n-1}\circ e_n \circ h_n \\ 0 & \mathrm{id}_{C_n} \end{pmatrix} = \mathrm{id}_{C_{\leq n}} - i_n \circ p_n, \end{aligned}$$

which proves (2.4). □

Remark 2.4. The assumption that the sequence splits in Lemma 2.3 is essential. To see this, just consider the case in which C is given by a non-split short exact sequence $0 \to M \xrightarrow{i} N \to P \to 0$ and $\mathrm{F}^1 C$ is the subcomplex $0 \to M \xrightarrow{\sim} i(M) \to 0$, while $\mathrm{F}^n C = C$ for $n > 1$. On the other hand, one can very easily prove that the inclusion $\mathrm{F}^1 C \hookrightarrow C$ is a quasi-isomorphism of complexes, even without that assumption.

2.2. **The relevant morphism.** In this section, we fix two non-unital A_∞ categories $\mathbf{A}$ and $\mathbf{B}$, and we denote by $\mathbf{C}$ the non-unital dg cocategory $\overline{\mathrm{B}_\infty(\mathbf{A})^+ \otimes \mathrm{B}_\infty(\mathbf{B})^+}$. We will also consider the dg quiver $\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}$.

Our aim here is to construct a suitable morphism of complexes

$$\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}\big((A,B),(A',B')\big) \to \Omega(\mathbf{C})\big((A,B),(A',B')\big)$$

where $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$. Its precise definition is in (2.15) below.

Moving in this direction, note that, given $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$, we have

$$\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}\big((A,B),(A',B')\big) = \mathbf{A}(A,A') \otimes \mathbf{B}(B,B') \oplus \mathbf{A}(A,A')^{\delta_{B,B'}} \oplus \mathbf{B}(B,B')^{\delta_{A,A'}} \tag{2.9}$$

as a dg $\Bbbk$-module. In order to explicitly describe also $\Omega(\mathbf{C})\big((A,B),(A',B')\big)$, we first introduce some notation.

For every $i, j \in \mathbb{N}$, we denote by $C_{i,j}\big((A,B),(A',B')\big)$ the graded $\Bbbk$-module

$$\bigoplus_{\substack{A=A_0,A_1,\dots,A_{i-1},A_i=A'\in\mathbf{A}\\ B=B_0,B_1,\dots,B_{j-1},B_j=B'\in\mathbf{B}}} \mathbf{A}(A_{i-1},A_i)[1] \otimes \cdots \otimes \mathbf{A}(A_0,A_1)[1] \otimes \mathbf{B}(B_{j-1},B_j)[1] \otimes \cdots \otimes \mathbf{B}(B_0,B_1)[1],$$

which is meant to be 0 when $i = 0$ and $A \neq A'$ or $j = 0$ and $B \neq B'$ or $i = j = 0$. Given $m_1, n_1, \dots, m_l, n_l \in \mathbb{N}$, we denote by $C_{(m_1,\dots,m_l),(n_1,\dots,n_l)}\big((A,B),(A',B')\big)$ the graded $\Bbbk$-module

$$\bigoplus_{\substack{A=A_0,A_1,\dots,A_{l-1},A_l=A'\in\mathbf{A}\\ B=B_0,B_1,\dots,B_{l-1},B_l=B'\in\mathbf{B}}} C_{m_l,n_l}\big((A_{l-1},B_{l-1}),(A_l,B_l)\big)[-1] \otimes \cdots \otimes C_{m_1,n_1}\big((A_0,B_0),(A_1,B_1)\big)[-1]$$

(in particular, $C_{(i),(j)}\big((A,B),(A',B')\big) = C_{i,j}\big((A,B),(A',B')\big)[-1]$). For every $m, n \geq 0$ we define moreover

$$L_{m,n}\big((A,B),(A',B')\big) := \bigoplus_{\substack{m_1+\cdots+m_l=m\\ n_1+\cdots+n_l=n}} C_{(m_1,\dots,m_l),(n_1,\dots,n_l)}\big((A,B),(A',B')\big).$$

Note that, in particular, $L_{0,0}\big((A,B),(A',B')\big) = 0$ and

$$L_{1,0}\big((A,B),(A',B')\big) = \mathbf{A}(A,A')^{\delta_{B,B'}} \tag{2.10}$$

$$L_{0,1}\big((A,B),(A',B')\big) = \mathbf{B}(B,B')^{\delta_{A,A'}} \tag{2.11}$$

$$\begin{aligned} L_{1,1}\big((A,B),(A',B')\big) =& \mathbf{A}(A,A') \otimes \mathbf{B}(B,B') \oplus \mathbf{B}(B,B') \otimes \mathbf{A}(A,A') \\ &\oplus (\mathbf{A}(A,A')[1] \otimes \mathbf{B}(B,B')[1])[-1] \end{aligned} \tag{2.12}$$

Then the non-unital dg category $\Omega(\mathbf{C})$ has the same objects as $\overline{\mathbf{A}^+ \otimes \mathbf{B}^+}$, and

$$\Omega(\mathbf{C})\big((A,B),(A',B')\big) = \bigoplus_{m,n\geq 0} L_{m,n}\big((A,B),(A',B')\big)$$

as a graded $\Bbbk$-module for every $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$. While the composition in $\Omega(\mathbf{C})$ is the natural one given by the tensor product of the cobar construction, the differential on $\Omega(\mathbf{C})\big((A,B),(A',B')\big)$ extends (in such a way that the graded Leibnitz rule holds) $\mu + \Delta$, where μ and Δ are determined, respectively, by the differential and the comultiplication on the dg cocategory $\mathbf{C}$. More precisely, given

$$c = (f_m[1] \otimes \cdots \otimes f_1[1] \otimes g_n[1] \otimes \cdots \otimes g_1[1])[-1] \in C_{(m),(n)}\big((A,B),(A',B')\big)$$

with the f_i and the g_j homogeneous, we have

$$\Delta(c) = \sum_{(i,j)\in I_{m,n}} (-1)^{\deg(c_{\leq i,\emptyset})\deg'(c_{\emptyset,>j})+\deg(c_{>i,\emptyset})} c_{>i,>j} \otimes c_{\leq i,\leq j},$$

where $I_{m,n} := \{0,\dots,m\}\times\{0,\dots,n\}\setminus\{(0,0),(m,n)\}$. Here $>i$ and $\leq i$ (respectively $>j$ and $\leq j$) denote the (descending) intervals $[m,i) = [m.i+1]$ and $[i,1]$ (respectively $[n,j)$ and $[j,1]$), and in general

$$c_{[i',i),[j',j)} := (f_{i'}[1]\otimes\cdots\otimes f_{i+1}[1]\otimes g_{j'}[1]\otimes\cdots\otimes g_{j+1}[1])[-1]$$

for $1\leq i\leq i'\leq m$ and $1\leq j\leq j'\leq n$. Obviously the empty interval is denoted also by $\emptyset$, while the full interval $[m,1]$ or $[n,1]$ will be denoted by $*$. Clearly

$$\Delta\Big(L_{m,n}\big((A,B),(A',B')\big)\Big)\subseteq L_{m,n}\big((A,B),(A',B')\big)$$

for every $m,n\geq 0$. On the other hand, the component μ^1 of μ induced from $\mathrm{m}^1_{\mathbf{A}}$ and $\mathrm{m}^1_{\mathbf{B}}$ is given on c as above by

$$\mu^1(c) = \sum_{i=1}^{m}(-1)^{\deg'(c_{>i,\emptyset})}\mu^1_{i,0}(c) + \sum_{j=1}^{n}(-1)^{\deg'(c_{*,>j})}\mu^1_{0,j}(c),$$

where

$$\mu^1_{i,0}(c) := (f_m[1]\otimes\cdots\otimes f_{i+1}[1]\otimes \mathrm{m}^1_{\mathbf{A}}(f_i)[1]\otimes f_{i-1}[1]\otimes\cdots\otimes f_1[1]\otimes g_n[1]\otimes\cdots\otimes g_1[1])[-1],$$
$$\mu^1_{0,j}(c) := (f_m[1]\otimes\cdots\otimes f_1[1]\otimes g_n[1]\otimes\cdots\otimes g_{j+1}[1]\otimes \mathrm{m}^1_{\mathbf{B}}(g_j)[1]\otimes g_{j-1}[1]\otimes\cdots\otimes g_1[1])[-1].$$

Hence also in this case

$$\mu^1\Big(L_{m,n}\big((A,B),(A',B')\big)\Big)\subseteq L_{m,n}\big((A,B),(A',B')\big)$$

for every $m,n\geq 0$. As for the other components μ^i of μ induced from $\mathrm{m}^i_{\mathbf{A}}$ and $\mathrm{m}^i_{\mathbf{B}}$ with $i>1$, for our purposes it is enough to observe that

$$\mu^i\Big(L_{m,n}\big((A,B),(A',B')\big)\Big)\subseteq \bigoplus_{0<m'<m} L_{m',n}\big((A,B),(A',B')\big)\bigoplus_{0<n'<n} L_{m,n'}\big((A,B),(A',B')\big)$$

for every $m,n\geq 0$. This implies that $\mu^1+\Delta$ is a differential (often denoted simply by d) on each $L_{m,n}\big((A,B),(A',B')\big)$, which will be regarded as a complex in this way. Moreover,

$$L_{*,0}\big((A,B),(A',B')\big) := \bigoplus_{m\geq 0} L_{m,0}\big((A,B),(A',B')\big)$$
$$L_{0,*}\big((A,B),(A',B')\big) := \bigoplus_{n\geq 0} L_{0,n}\big((A,B),(A',B')\big)$$
$$L_{>0}\big((A,B),(A',B')\big) := \bigoplus_{m,n>0} L_{m,n}\big((A,B),(A',B')\big)$$

are subcomplexes of $\Omega(\mathbf{C})\big((A,B),(A',B')\big)$, and obviously there is a decomposition

$$\Omega(\mathbf{C})\big((A,B),(A',B')\big) = L_{*,0}\big((A,B),(A',B')\big)\oplus L_{0,*}\big((A,B),(A',B')\big)\oplus L_{>0}\big((A,B),(A',B')\big).$$

Now, for every $A,A'\in\mathbf{A}$ and $B,B'\in\mathbf{B}$, we will consider the maps (see (2.12))

$$\begin{aligned}\mathbf{A}(A,A')\otimes\mathbf{B}(B,B') &\to L_{1,1}\big((A,B),(A',B')\big)\\ f\otimes g &\mapsto (0,(-1)^{\deg(f)\deg(g)}g\otimes f,0)\end{aligned}\tag{2.13}$$

and

$$\begin{aligned} L_{1,1}\big((A,B),(A',B')\big) &\to \mathbf{A}(A,A')\otimes\mathbf{B}(B,B') \\ \big(f\otimes g, g'\otimes f', (f''[1]\otimes g''[1])[-1]\big) &\mapsto f\otimes g + (-1)^{\deg(f')\deg(g')} f'\otimes g' \end{aligned} \tag{2.14}$$

It is easy to check that (2.13) and (2.14) are morphisms of complexes. Remembering (2.9), it is then clear that (2.10), (2.11) and (2.13) define the morphism of complexes

$$\overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big) \to \Omega(\mathbf{C})\big((A,B),(A',B')\big) = \bigoplus_{m,n\geq 0} L_{m,n}\big((A,B),(A',B')\big) \tag{2.15}$$

we are interested in.

2.3. **A general result.** In order to prove the properties of (2.15) we first need the following key result, whose proof is rather technical.

Lemma 2.5. *Let $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$.*

(1) *The maps* (2.13) *and* (2.14) *are mutually inverse homotopy equivalences of complexes.*
(2) *Given $m, n \in \mathbb{N}$ with $m > 1$ or $n > 1$, the complex $L_{m,n}\big((A,B),(A',B')\big)$ is null-homotopic.*

Proof. We start by observing that the composition of (2.14) with (2.13) is $\mathrm{id}_{\mathbf{A}(A,A')\otimes\mathbf{B}(B,B')}$, while the composition of (2.13) with (2.14) is the map

$$\begin{aligned} \xi\colon L_{1,1}\big((A,B),(A',B')\big) &\to L_{1,1}\big((A,B),(A',B')\big) \\ \big(f\otimes g, g'\otimes f', (f''[1]\otimes g''[1])[-1]\big) &\mapsto \big(0, (-1)^{\deg(f)\deg(g)} g\otimes f + g'\otimes f', 0\big) \end{aligned}$$

We define also $\xi\colon L_{m,n}\big((A,B),(A',B')\big) \to L_{m,n}\big((A,B),(A',B')\big)$ to be 0 for $m > 1$, and to be the identity for $m = 1$ and $n = 0$.

Therefore, in order to prove both (1) and (2) (where, by symmetry, we can assume $m > 1$), we just need to find, when $m > 1$ or $m = n = 1$, a $\Bbbk$-linear map

$$r\colon L_{m,n}\big((A,B),(A',B')\big) \to L_{m,n}\big((A,B),(A',B')\big)$$

of degree -1 such that

$$d\circ r + r\circ d = \mathrm{id} - \xi.$$

More generally, we define r for $m > 0$ as follows. By linearity an element of $L_{m,n}\big((A,B),(A',B')\big)$ can be assumed to be of the form

$$c = c^l\otimes\cdots\otimes c^1 \in C_{(m_1,\dots,m_l),(n_1,\dots,n_l)}\big((A,B),(A',B')\big),$$

where $m_1+\cdots+m_l = m$, $n_1+\cdots+n_l = n$ and $c^k \in C_{(m_k),(n_k)}\big((A_{k-1},B_{k-1}),(A_k,B_k)\big)$ homogeneous (for $k = 1,\dots,l$), with $A_0 = A$, $A_l = A'$, $B_0 = B$ and $B_l = B'$. Given $1 \leq i \leq j \leq l$, we will often use the shorthand $c^{[j,i]} := c^j\otimes\cdots\otimes c^i$, as well as its variants $c^{[j,i)}$, $c^{(j,i]}$ and $c^{(j,i)}$ (with obvious meanings). Setting

$$\begin{aligned} t &:= \max\big\{k\in\{1,\dots,l\} \mid m_k > 0\big\} \\ s' &:= \min\big\{k\in\{1,\dots,t\} \mid m_i = 0 \text{ for } k < i < t\big\} \\ s &:= \begin{cases} s' & \text{if } (m_t,n_t) = (1,0) \\ t & \text{otherwise} \end{cases} \end{aligned}$$

(note that they are well defined because $m > 0$), we define recursively

$$r(c) := \sum_{k=s}^{t-1} (-1)^{\deg(c^{[l,t)})+\deg'(c^t)\deg(c^{(t,k)})} c^{[l,t)} \otimes c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]}$$

(so $r(c) = 0$ if $s = t$, in particular if $(m_t, n_t) \neq (1,0)$). If $(m_t, n_t) = (1,0)$, $c^t = f \in \mathbf{A}(A_{t-1}, A_t)$ and $c^k = (f_{m_k}[1] \otimes \cdots \otimes f_1[1] \otimes g_{n_k}[1] \otimes \cdots \otimes g_1[1])[-1]$ (with $s \leq k < t$), then

$$r(c^t \otimes c^k) := (-1)^{\deg(f)}(f[1] \otimes f_{m_k}[1] \otimes \cdots \otimes f_1[1] \otimes g_{n_k}[1] \otimes \cdots \otimes g_1[1])[-1].$$

For the rest of the proof we assume $m > 1$ or $m = n = 1$. First we note that $\xi(c) = c^{[l,t)} \otimes \xi(c^{[t,1]})$. Indeed, we can assume $t < l$, and then $\xi(c) = \xi(c^{[t,1]}) = 0$ if $m > 1$, whereas $\xi(c) = c$ and $\xi(c^{[t,1]}) = c^{[t,1]}$ if $m = n = 1$ (in which case $l = 2$, $(m_2, n_2) = (0,1)$ and $(m_1, n_1) = (1,0)$). Since moreover

$$\begin{aligned}(d \circ r + r \circ d)(c) &= d\big((-1)^{\deg(c^{[l,t)})} c^{[l,t)} \otimes r(c^{[t,1]})\big) + r\big(d(c^{[l,t)}) \otimes c^{[t,1]} + (-1)^{\deg(c^{[l,t)})} c^{[l,t)} \otimes d(c^{[t,1]})\big) \\ &= (-1)^{\deg(c^{[l,t)})} d(c^{[l,t)}) \otimes r(c^{[t,1]}) + c^{[l,t)} \otimes d\big(r(c^{[t,1]})\big) + (-1)^{\deg'(c^{[l,t)})} d(c^{[l,t)}) \otimes r(c^{[t,1]}) + c^{[l,t)} \otimes r\big(d(c^{[t,1]})\big) \\ &= c^{[l,t)} \otimes (d \circ r + r \circ d)(c^{[t,1]}),\end{aligned}$$

it is enough to prove that

$$(d \circ r + r \circ d)(c^{[t,1]}) = c^{[t,1]} - \xi(c^{[t,1]}). \tag{2.16}$$

We have

$$\begin{aligned} d\big(r(c^{[t,1]})\big) &= d\Big(\sum_{k=s}^{t-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})} c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]}\Big) \\ &= \sum_{k=s}^{t-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})+\deg(c^{(t,k)})} c^{(t,k)} \otimes d\big(r(c^t \otimes c^k)\big) \otimes c^{(k,1]} \\ &\quad + \sum_{k=s}^{t-1} \sum_{i=1}^{k-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})+\deg'(c^{[t,i)})} c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,i)} \otimes d(c^i) \otimes c^{(i,1]} \\ &\quad + \sum_{k=s}^{t-1} \sum_{i=k+1}^{t-1} (-1)^{\deg'(c^t)\deg(c^{(t,k)})+\deg(c^{(t,i)})} c^{(t,i)} \otimes d(c^i) \otimes c^{(i,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]} \end{aligned}$$

and

$$\begin{aligned} r\big(d(c^{[t,1]})\big) &= r\Big(\sum_{i=1}^{t} (-1)^{\deg(c^{[t,i)})} c^{[t,i)} \otimes d(c^i) \otimes c^{(i,1]}\Big) \\ &= r\big(d(c^t) \otimes c^{(t,1]}\big) + \sum_{i=s}^{t-1} (-1)^{\deg(c^{[t,i)})+\deg'(c^t)\deg(c^{(t,i)})} c^{(t,i)} \otimes r\big(c^t \otimes d(c^i)\big) \otimes c^{(i,1]} \\ &\quad + \sum_{i=1}^{t-1} \sum_{k=\max\{i+1,s\}}^{t-1} (-1)^{\deg(c^{[t,i)})+\deg'(c^t)\deg(c^{(t,k)})} c^{(t,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,i)} \otimes d(c^i) \otimes c^{(i,1]} \\ &\quad + \sum_{i=1}^{t-1} \sum_{k=s}^{i-1} (-1)^{\deg(c^{[t,i)})+\deg'(c^t)\deg'(c^{(t,k)})} c^{(t,i)} \otimes d(c^i) \otimes c^{(i,k)} \otimes r(c^t \otimes c^k) \otimes c^{(k,1]}, \end{aligned}$$

whence

$$(d\circ r + r\circ d)(c^{[t,1]}) \tag{2.17}$$
$$= r\big(d(c^t)\otimes c^{(t,1]}\big) + \sum_{k=s}^{t-1}(-1)^{\deg(c^t)\deg(c^{(t,k)})}c^{(t,k)}\otimes\Big(d\big(r(c^t\otimes c^k)\big)+(-1)^{\deg(c^t)}r\big(c^t\otimes d(c^k)\big)\Big)\otimes c^{(k,1]}.$$

First we assume $(m_t,n_t)\neq(1,0)$, in which case the right-hand-side of (2.17) is just $r\big(\Delta(c^t)\otimes c^{(t,1]}\big)$. If $m_t>1$ then

$$(d\circ r + r\circ d)(c^{[t,1]}) = r\big((-1)^{\deg(c^t_{m_t,\emptyset})}c^t_{m_t,\emptyset}\otimes c^t_{<m_t,*}\otimes c^{(t,1]}\big)$$
$$= (-1)^{\deg(c^t_{m_t,\emptyset})}r(c^t_{m_t,\emptyset}\otimes c^t_{<m_t,*})\otimes c^{(t,1]} = c^t\otimes c^{(t,1]} = c^{[t,1]},$$

hence (2.16) holds in this case. If $m_t=1$ and $n_t>0$ then

$$(d\circ r + r\circ d)(c^{[t,1]}) = r\Big(\big((-1)^{\deg(c^t_{1,\emptyset})}c^t_{1,\emptyset}\otimes c^t_{\emptyset,*} - (-1)^{\deg(c^t_{1,\emptyset})\deg'(c^t_{\emptyset,*})}c^t_{\emptyset,*}\otimes c^t_{1,\emptyset}\big)\otimes c^{(t,1]}\Big)$$
$$= (-1)^{\deg(c^t_{1,\emptyset})}r(c^t_{1,\emptyset}\otimes c^t_{\emptyset,*})\otimes c^{(t,1]}$$
$$+ (-1)^{\deg(c^t_{1,\emptyset})}\sum_{k=s'}^{t-1}(-1)^{\deg'(c^t_{1,\emptyset})\left(\deg(c^t_{\emptyset,*})+\deg(c^{(t,k)})\right)}c^t_{\emptyset,*}\otimes c^{(t,k)}\otimes r(c^t_{1,\emptyset}\otimes c^k)\otimes c^{(k,1]}$$
$$- (-1)^{\deg(c^t_{1,\emptyset})\deg'(c^t_{\emptyset,*})}\sum_{k=s'}^{t-1}(-1)^{\deg(c^t_{\emptyset,*})+\deg'(c^t_{1,\emptyset})\deg(c^{(t,k)})}c^t_{\emptyset,*}\otimes c^{(t,k)}\otimes r(c^t_{1,\emptyset}\otimes c^k)\otimes c^{(k,1]} = c^{[t,1]},$$

thus proving (2.16) also in this case.

Finally we assume $(m_t,n_t)=(1,0)$. Then we have

$$r\big(d(c^t)\otimes c^{(t,1]}\big) = \sum_{k=s}^{t-1}(-1)^{\deg(c^t)\deg(c^{(t,k)})}c^{(t,k)}\otimes r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)\otimes c^{(k,1]},$$

and so from (2.17) we obtain

$$(d\circ r + r\circ d)(c^{[t,1]}) \tag{2.18}$$
$$= \sum_{k=s}^{t-1}(-1)^{\deg(c^t)\deg(c^{(t,k)})}c^{(t,k)}\otimes\Big(r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)+d\big(r(c^t\otimes c^k)\big)+(-1)^{\deg(c^t)}r\big(c^t\otimes d(c^k)\big)\Big)\otimes c^{(k,1]}.$$

Since

$$\mu^1\big(r(c^t\otimes c^k)\big) = \sum_{i=1}^{m_k+1}(-1)^{\deg'\left(r(c^t\otimes c^k)_{>i,\emptyset}\right)}\mu^1_{i,0}\big(r(c^t\otimes c^k)\big)+\sum_{j=1}^{n_k}(-1)^{\deg'\left(r(c^t\otimes c^k)_{*,>j}\right)}\mu^1_{0,j}\big(r(c^t\otimes c^k)\big)$$
$$= \sum_{i=1}^{m_k}(-1)^{\deg(c^t)+\deg(c^k_{>i,\emptyset})}r\big(c^t\otimes\mu^1_{i,0}(c^k)\big) - r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)+\sum_{j=1}^{n_k}(-1)^{\deg(c^t)+\deg(c^k_{*,>j})}r\big(c^t\otimes\mu^1_{0,j}(c^k)\big)$$
$$= -r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big) - (-1)^{\deg(c^t)}r\Big(c^t\otimes\big(\sum_{i=1}^{m_k}(-1)^{\deg'(c^k_{>i,\emptyset})}\mu^1_{i,0}(c^k)+\sum_{j=1}^{n_k}(-1)^{\deg'(c^k_{*,>j})}\mu^1_{0,j}(c^k)\big)\Big)$$
$$= -r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big) - (-1)^{\deg(c^t)}r\big(c^t\otimes\mu^1(c^k)\big)$$

and

$$\begin{aligned}
&\Delta\big(r(c^t\otimes c^k)\big)\\
&=\sum_{(i,j)\in I_{m_k+1,n_k}}(-1)^{\deg\left(r(c^t\otimes c^k)_{\leq i,\emptyset}\right)\deg'\left(r(c^t\otimes c^k)_{\emptyset,>j}\right)+\deg\left(r(c^t\otimes c^k)_{>i,\emptyset}\right)}r(c^t\otimes c^k)_{>i,>j}\otimes r(c^t\otimes c^k)_{\leq i,\leq j}\\
&\qquad=\sum_{(i,j)\in I_{m_k,n_k}}(-1)^{\deg(c^k_{\leq i,\emptyset})\deg'(c^k_{\emptyset,>j})+\deg(c^k_{>i,\emptyset})+\deg'(c^t)}r\big(c^t\otimes c^k_{>i,>j}\big)\otimes c^k_{\leq i,\leq j}+c^t\otimes c^k\\
&-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}+\sum_{j=\delta_{m_k,0}}^{n_k-1}(-1)^{\left(\deg(c^k_{*,\emptyset})+\deg'(c^t)\right)\deg'(c^k_{\emptyset,>j})+1}c^k_{\emptyset,>j}\otimes r\big(c^t\otimes c^k_{*,\leq j}\big)\\
&\qquad=c^t\otimes c^k-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}\\
&\qquad-(-1)^{\deg(c^t)}r\big(c^t\otimes\sum_{(i,j)\in I_{m_k,n_k}}(-1)^{\deg(c^k_{\leq i,\emptyset})\deg'(c^k_{\emptyset,>j})+\deg(c^k_{>i,\emptyset})}c^k_{>i,>j}\otimes c^k_{\leq i,\leq j}\big)\\
&\qquad=c^t\otimes c^k-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}-(-1)^{\deg(c^t)}r\big(c^t\otimes\Delta(c^k)\big),
\end{aligned}$$

we see that

$$r\big(\mathrm{m}^1_{\mathbf{A}}(c^t)\otimes c^k\big)+d\big(r(c^t\otimes c^k)\big)+(-1)^{\deg(c^t)}r\big(c^t\otimes d(c^k)\big)=c^t\otimes c^k-(-1)^{\deg(c^t)\deg(c^k)}(c^k\otimes c^t)^{\delta_{m_k,0}}.$$

Substituting the last equality in (2.18) and remembering that $m_k=0$ for $s<k<t$, while $m_s=0$ if and only if $m=1$ (in which case $s=1$), we get

$$(d\circ r+r\circ d)(c^{[t,1]})=\begin{cases}c^{[t,1]} & \text{if } m>1\\ c^{[t,1]}-(-1)^{\deg(c^t)\deg(c^{(t,1]})}c^{(t,1]}\otimes c^t & \text{if } m=1,\end{cases}$$

from which we conclude that (2.16) is satisfied also in this case. □

We can finally prove the main result of this section. Note that, when $\mathbf{B}$ is the 0 dg algebra, (2.15) boils down to (2.1). Hence the first part of the following result shows, in particular, that (2.1) is a homotopy equivalence, as wanted.

Proposition 2.6. *For every $A,A'\in\mathbf{A}$ and $B,B'\in\mathbf{B}$ the map* (2.15) *is a homotopy equivalence. Moreover, a morphism of complexes*

$$\Omega(\mathbf{C})\big((A,B),(A',B')\big)\to\overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big)$$

is a homotopy equivalence if and only if its restriction to $\bigoplus_{0\leq m,n\leq 1}L_{m,n}\big((A,B),(A',B')\big)$ is a homotopy equivalence.

Proof. By construction (2.15) is the composition of a map

$$\overline{\mathbf{A}^+\otimes\mathbf{B}^+}\big((A,B),(A',B')\big)\to\bigoplus_{0\leq m,n\leq 1}L_{m,n}\big((A,B),(A',B')\big)\tag{2.19}$$

with the inclusion

$$\bigoplus_{0\leq m,n\leq 1}L_{m,n}\big((A,B),(A',B')\big)\hookrightarrow\Omega(\mathbf{C})\big((A,B),(A',B')\big).\tag{2.20}$$

Now, part (1) of Lemma 2.5 immediately implies that (2.19) is a homotopy equivalence. Therefore it is enough to prove that (2.20) is a homotopy equivalence, as well. Clearly this is true if (and only if) each of the three inclusions

$$L_{1,0}\big((A,B),(A',B')\big) \hookrightarrow L_{*,0}\big((A,B),(A',B')\big)$$
$$L_{0,1}\big((A,B),(A',B')\big) \hookrightarrow L_{0,*}\big((A,B),(A',B')\big)$$
$$L_{1,1}\big((A,B),(A',B')\big) \hookrightarrow L_{>0}\big((A,B),(A',B')\big)$$

is a homotopy equivalence. To this aim we apply Lemma 2.3 to the complexes on the right-hand-sides of the above inclusions, endowed with the filtrations

$$\mathrm{F}^n L_{*,0}\big((A,B),(A',B')\big) := \bigoplus_{0\le m\le n} L_{m,0}\big((A,B),(A',B')\big)$$
$$\mathrm{F}^n L_{0,*}\big((A,B),(A',B')\big) := \bigoplus_{0\le m\le n} L_{0,m}\big((A,B),(A',B')\big)$$
$$\mathrm{F}^n L_{>0}\big((A,B),(A',B')\big) := \bigoplus_{m,m'>0,\, m+m'\le n+1} L_{m,m'}\big((A,B),(A',B')\big)$$

Note that the assumptions of Lemma 2.3 are satisfied because

$$\mathrm{gr}^n L_{*,0}\big((A,B),(A',B')\big) = L_{n,0}\big((A,B),(A',B')\big)$$
$$\mathrm{gr}^n L_{0,*}\big((A,B),(A',B')\big) = L_{0,n}\big((A,B),(A',B')\big)$$
$$\mathrm{gr}^n L_{>0}\big((A,B),(A',B')\big) = \bigoplus_{0<m\le n} L_{m,n+1-m}\big((A,B),(A',B')\big)$$

are null-homotopic for $n>1$ by part (2) of Lemma 2.5. □

Let us now single out the following direct consequence.

Corollary 2.7. *Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are non-unital dg categories. Then there is a natural non-unital dg functor* $\tilde{\mathsf{N}}\colon \Omega(\mathbf{C}) \to \overline{\mathbf{A}^+ \otimes \mathbf{B}^+}$*, which is a homotopy isomorphism.*

Proof. The definition of $\tilde{\mathsf{N}}$ can be found in [5, Section 3.1], where it is also proved that it is a quasi-isomorphism (hence a homotopy isomorphism) when $\Bbbk$ is a field. Over an arbitrary commutative ring we can apply the second part of Proposition 2.6. Indeed, it can be readily checked that, for every $A, A' \in \mathbf{A}$ and $B, B' \in \mathbf{B}$, the restriction of

$$\tilde{\mathsf{N}}\colon \Omega(\mathbf{C})\big((A,B),(A',B')\big) \to \overline{\mathbf{A}^+ \otimes \mathbf{B}^+}\big((A,B),(A',B')\big)$$

to $\bigoplus_{0\le m,n\le 1} L_{m,n}\big((A,B),(A',B')\big)$ is given by the natural maps (2.10), (2.11) and (2.14). Such restriction is then a homotopy equivalence by part (1) of Lemma 2.5. □

3. Proof of Theorem A

We refer to [29] and [26] for the (few) basic notions about ∞-categories which are needed in this section.

Recall that a *category with weak equivalences* is a pair $(\mathbf{C}, \mathcal{W})$ (often denoted simply by $\mathbf{C}$), where $\mathbf{C}$ is an ordinary category and $\mathcal{W}$ is a class of morphisms of $\mathbf{C}$ containing the isomorphisms and satisfying the 2-out-of-3 property. We can then consider both the localization $\mathrm{Ho}(\mathbf{C})$ of $\mathbf{C}$

with respect to $\mathcal{W}$, which is an ordinary category,[1] and the ∞-category $\mathrm{Ho}_\infty(\mathbf{C})$ obtained as the localization of the nerve of $\mathbf{C}$ with respect to the image under the nerve functor of $\mathcal{W}$.

Given another category with weak equivalences $\mathbf{D}$, a functor $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$ preserving weak equivalences induces a functor $\mathrm{Ho}(\mathsf{F})\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ and an ∞-functor $\mathrm{Ho}_\infty(\mathsf{F})\colon \mathrm{Ho}_\infty(\mathbf{C}) \to \mathrm{Ho}_\infty(\mathbf{D})$. Now, let $\mathsf{F}'\colon \mathbf{C} \to \mathbf{D}$ be another functor such that there exists a natural transformation $\mathsf{F} \to \mathsf{F}'$ or $\mathsf{F}' \to \mathsf{F}$ whose components are all weak equivalences. Then it is easy to see (using the 2-out-of-3 property) that also F' preserves weak equivalences and that $\mathrm{Ho}(\mathsf{F})$ is isomorphic to $\mathrm{Ho}(\mathsf{F}')$. Moreover, by [29, Proposition 2.3], $\mathrm{Ho}_\infty(\mathsf{F})$ is homotopic to $\mathrm{Ho}_\infty(\mathsf{F}')$. Clearly this implies the following result.

Proposition 3.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories with weak equivalences and let* $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$ *and* $\mathsf{G}\colon \mathbf{D} \to \mathbf{C}$ *be functors preserving weak equivalences. Assume moreover that* $\mathrm{id}_{\mathbf{C}}$ *(respectively* $\mathrm{id}_{\mathbf{D}}$*) and* $\mathsf{G} \circ \mathsf{F}$ *(respectively* $\mathsf{F} \circ \mathsf{G}$*) are related by a zigzag of natural transformations whose components are all weak equivalences. Then* $\mathrm{Ho}(\mathsf{F})$ *and* $\mathrm{Ho}(\mathsf{G})$ *are quasi-inverse equivalences of categories and* $\mathrm{Ho}_\infty(\mathsf{F})$ *and* $\mathrm{Ho}_\infty(\mathsf{G})$ *are quasi-inverse equivalences of* ∞*-categories.*

Remark 3.2. The assumptions of Proposition 3.1 are certainly satisfied when F and G form a *Dwyer-Kan adjunction*, meaning that they are adjoint functors which preserve weak equivalences and such that the components of both the unit and the counit of the adjunction are all weak equivalences. The fact that in this case $\mathrm{Ho}_\infty(\mathsf{F})$ and $\mathrm{Ho}_\infty(\mathsf{G})$ are quasi-inverse equivalences of ∞-categories was already observed in [26].

This discussion is crucial for our proof of Theorem A which is divided into several sections each of which compares the various ∞-categorical localizations mentioned in the statement.

Remark 3.3. The reader should approach the next few sections by paying attention to the fact that all functors involved in our constructions induce bijections of objects. Thus our argument provides a proof of the variant of Theorem A where categories are replaced by algebras (namely, categories with just one object). Clearly this will prove all the equivalences in Theorem E, except the last one, which can however be very easily deduced from Proposition 2.1 (see also Remark 3.6).

3.1. **Dg vs. strictly unital A_∞ categories.** In this subsection we prove the equivalence between $\mathrm{Ho}_\infty(\mathbf{dgCat})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$. As a preliminary step, let us consider the following.

Definition 3.4. A strictly unital A_∞ category A *has split units* if $\Bbbk \cong \Bbbk\mathrm{id}_A$ and the inclusion $\Bbbk\mathrm{id}_A \hookrightarrow \mathbf{A}(A,A)^0$ of $\Bbbk$-modules splits, for every $A \in \mathbf{A}$.

In view of Remark 3.2, the result we want to prove is a straightforward consequence of the following one, which is the strictly unital analogue of Proposition 2.1 in the non-unital setting.

Proposition 3.5. *There is an adjunction*

$$\mathsf{U}\colon \mathbf{A}_\infty\mathbf{Cat} \rightleftarrows \mathbf{dgCat} :\mathsf{I},$$

[1] In general it can be necessary to enlarge the universe of definition in order to get a category, because the morphisms in $\mathrm{Ho}(\mathbf{C})$ between two objects need not be a set in the original universe. However a posteriori this does not happen in the cases we are interested in, namely when $\mathbf{C}$ is one of the four categories in (1) and $\mathcal{W}$ is the corresponding class of quasi-equivalences. Indeed, this follows from Corollary B and from the fact that $\mathrm{Ho}(\mathbf{dgCat})$ can be alternatively described as the homotopy category of $\mathbf{dgCat}$ endowed with Tabuada's model structure [28].

where I *is the inclusion functor. Moreover, the unit* $\rho\colon \mathrm{id}_{\mathbf{A}_\infty\mathbf{Cat}} \to \mathsf{I}\circ\mathsf{U}$ *and the counit* $\sigma\colon \mathsf{U}\circ\mathsf{I} \to \mathrm{id}_{\mathbf{dgCat}}$ *are such that* $\rho_{\mathbf{A}}$ *(for every* $\mathbf{A}\in\mathbf{A}_\infty\mathbf{Cat}$*) and* $\sigma_{\mathbf{B}}$ *(for every* $\mathbf{B}\in\mathbf{dgCat}$*) are quasi-isomorphisms. Finally,* $\rho_{\mathbf{A}}$ *is even a homotopy isomorphism if* $\mathbf{A}$ *has split units.*

Before turning to the proof of Proposition 3.5, a few observations are in order.

Remark 3.6. In a completely similar way, replacing Proposition 3.5 with Proposition 2.1 and recalling Remark 2.2, the adjunction between U^{n} and I^{n} induces a natural equivalence of ∞-categories between $\mathrm{Ho}_\infty(\mathbf{dgCat}^{\mathbf{u}})$ (respectively $\mathrm{Ho}_\infty(\mathbf{dgCat}^{\mathbf{c}})$) and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}})$ (respectively $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{c}})$).

Remark 3.7. When $\Bbbk$ is a field, the equivalence between $\mathrm{Ho}_\infty(\mathbf{dgCat})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$ was already proved in [26, Corollary 5.2] (using the results of [5]). As it is pointed out in [26], one can consider different models for ∞-categories and for all of them there is an analogue of [26, Corollary 5.2]. Namely, one gets [26, Corollaries 2.5, 3.2, 4.3, 4.5, 5.1]. Using Proposition 3.5 one sees that they remain valid when $\Bbbk$ is an arbitrary commutative ring.

Remark 3.8. Actually we can say that

$$\mathsf{U}\colon \mathbf{A}_\infty\mathbf{Cat} \rightleftarrows \mathbf{dgCat} :\mathsf{I}$$

is a Dwyer-Kan adjunction even if we take as weak equivalences, instead of the quasi-equivalences, the pretriangulated (or Morita) equivalences (see [28, §1.4 and Definition 1.36] or [23, Definition 1.4.7.]). Indeed, every quasi-equivalence is a pretriangulated equivalence, so it is still true that all the components of both the unit and the counit of the adjunction are pretriangulated equivalences. To prove that I preserves pretriangulated equivalences it suffices to notice that $\mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{A}) = \mathrm{pretr}_{\mathrm{dg}}(\mathbf{A})$ if $\mathbf{A}\in\mathbf{dgCat}$ (see [23, Remark 1.7]). As for the fact that U preserve pretriangulated equivalences, suppose that $\mathsf{F}:\mathbf{A}\to\mathbf{B}$ in $\mathbf{A}_\infty\mathbf{Cat}$ induces a quasi-equivalence $\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{F})\colon \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{A}) \to \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{B})$. Then

$$\begin{array}{ccc} \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{A}) & \xrightarrow{\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{F})} & \mathrm{pretr}_{\mathrm{A}_\infty}(\mathbf{B}) \\ {\scriptstyle \mathrm{pretr}_{\mathrm{A}_\infty}(\rho_{\mathbf{A}})}\downarrow & & \downarrow{\scriptstyle \mathrm{pretr}_{\mathrm{A}_\infty}(\rho_{\mathbf{B}})} \\ \mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathbf{A})) & \xrightarrow[\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathsf{F}))]{} & \mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathbf{B})) \end{array}$$

is a commutative diagram in $\mathbf{A}_\infty\mathbf{Cat}$ in which the upper and the vertical arrows are quasi-equivalences. Hence $\mathrm{pretr}_{\mathrm{A}_\infty}(\mathsf{U}(\mathsf{F})) = \mathrm{pretr}_{\mathrm{dg}}(\mathsf{U}(\mathsf{F}))$ is a quasi-equivalence, as well. One can argue in a similar way for Morita equivalences.

Coming to the proof of Proposition 3.5, this result is proved in [5, Proposition 2.1] assuming that $\Bbbk$ is a field. Actually the first part of the proof works without changes over an arbitrary commutative ring. Only the argument showing that $\rho_{\mathbf{A}}\colon \mathbf{A}\to\mathsf{U}(\mathbf{A})$ is a quasi-isomorphism for every $\mathbf{A}\in\mathbf{A}_\infty\mathbf{Cat}$ (respectively, a homotopy isomorphism when $\mathbf{A}$ has split units needs to be modified. We now give a different proof, valid over every commutative ring, which occupies the rest of this subsection.

To this aim, we fix a strictly unital A_∞ category $\mathbf{A}$. As it is explained at the beginning of the proof of [5, Proposition 2.1], $\rho_{\mathbf{A}} = \pi_{\mathbf{A}}\circ\gamma_{\mathbf{A}}$, where the non-unital A_∞ functor $\gamma_{\mathbf{A}}\colon \mathbf{A}\to\mathsf{U}^{\mathsf{n}}(\mathbf{A})$ is a

homotopy isomorphism by Proposition 2.1, and the non-unital dg functor $\pi_{\mathbf{A}}$ is defined to be the composition

$$\pi_{\mathbf{A}} \colon \mathsf{U}^{\mathsf{n}}(\mathbf{A}) \hookrightarrow \mathsf{U}^{\mathsf{n}}(\mathbf{A})^{+} \twoheadrightarrow \mathsf{U}^{\mathsf{n}}(\mathbf{A})^{+}/J = \mathsf{U}(\mathbf{A}),$$

with $J = J_{\mathbf{A}}$ the smallest dg ideal of $\mathsf{U}^{\mathsf{n}}(\mathbf{A})^{+}$ such that $\rho_{\mathbf{A}}$ is a strictly unital A_∞ functor.

First we recall from Section 2.2 (whose notation we adapt and simplify in an obvious way to the setting where $\mathbf{B}$ is the 0 dg algebra) that, for every $A, A' \in \mathbf{A}$,

$$\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A') = \bigoplus_{n \geq 0} L_n(A, A')$$

as a graded $\Bbbk$-module, where

$$L_n(A, A') := \bigoplus_{n_1 + \cdots + n_l = n} C_{(n_1, \ldots, n_l)}(A, A'),$$

with

$$C_{(n_1,\ldots,n_l)}(A, A') := \bigoplus_{A = A_0, A_1, \ldots, A_{l-1}, A_l = A' \in \mathbf{A}} C_{n_l}(A_{l-1} A_l)[-1] \otimes \cdots \otimes C_{n_1}(A_0, A_1)[-1]$$

and, for every $i \geq 0$,

$$C_i(A, A') := \bigoplus_{A = A_0, A_1, \ldots, A_{i-1}, A_i = A' \in \mathbf{A}} \mathbf{A}(A_{i-1}, A_i)[1] \otimes \cdots \otimes \mathbf{A}(A_0, A_1)[1].$$

The differential d on $\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')$ extends $\mu + \Delta$, where μ and Δ are determined, respectively, by the differential and the comultiplication on the dg cocategory $\mathrm{B}_\infty(\mathbf{A})$. Explicitly, given

$$c = (f_n[1] \otimes \cdots \otimes f_1[1])[-1] \in C_{(n)}(A, A') \tag{3.1}$$

with the f_i homogeneous, we have

$$\Delta(c) = \sum_{i=1}^{n-1} (-1)^{\deg(c_{>i})} c_{>i} \otimes c_{\leq i}.$$

The components μ^k of μ induced from $\mathrm{m}^k = \mathrm{m}^k_{\mathbf{A}}$ are given by

$$\mu^k(c) = \sum_{i=1}^{n+1-k} \pm \mu^k_i(c), \tag{3.2}$$

(with $1 \leq k \leq n$), where

$$\mu^k_i(c) := \big(f_n[1] \otimes \cdots \otimes f_{i+k}[1] \otimes \mathrm{m}^k_{\mathbf{A}}(f_{i+k-1} \otimes \cdots \otimes f_i)[1] \otimes f_{i-1}[1] \otimes \cdots \otimes f_1[1]\big)[-1].$$

As for the signs in (3.2), we just need to know that they are $(-1)^{\deg'(c_{>i})}$ for $k = 2$.

Now we can give the following more explicit description of J.

Lemma 3.9. *The dg ideal J coincides with the (a priori not necessarily dg) ideal J' of $\mathsf{U}^{\mathsf{n}}(\mathbf{A})^{+}$ generated by all the elements of one of the following two forms:*

1. *$1_A - \mathrm{id}_A$, where $A \in \mathbf{A}$;*
2. *c as in (3.1) with $n > 1$ and such that $f_j = \mathrm{id}_{\tilde{A}}$ for some $j \in \{1, \ldots, n\}$ and some $\tilde{A} \in \mathbf{A}$.*

Proof. If c is as in (3.1), we have

$$\rho^n_{\mathbf{A}}(f_n \otimes \cdots \otimes f_1) = \pi_{\mathbf{A}}\big(\gamma^n_{\mathbf{A}}(f_n \otimes \cdots \otimes f_1)\big) = \pm\pi_{\mathbf{A}}(c).$$

Since $\rho_{\mathbf{A}}$ is strictly unital, it follows that $\pi_{\mathbf{A}}(c) = 0$ if c is a generator of J' of the form (2). On the other hand, $\pi_{\mathbf{A}}(\mathrm{id}_A) = \mathrm{id}_A$ coincides with the image of 1_A through the projection $\mathsf{U}^{\mathsf{n}}(\mathbf{A})^+ \twoheadrightarrow \mathsf{U}(\mathbf{A})$, for every $A \in \mathbf{A}$. Therefore J contains also the generators of J' of the form (1), hence $J' \subseteq J$. To prove the other inclusion it is clearly enough to show that $d(c) \in J'$ for every generator c of J'. As this is obviously true when c is of the form (1), we can assume that c is of the form (2). Then it is clear from the definition that $\mu^k(c) \in J'$ if $k \neq 2$, and so it remains to prove that J' contains

$$\mu^2(c)+\Delta(c) = \sum_{i=1}^{n-1}(-1)^{\deg'(c_{>i})}\mu^2_i(c)+\sum_{i=1}^{n-1}(-1)^{\deg(c_{>i})}c_{>i}\otimes c_{\leq i} = \sum_{i=1}^{n-1}(-1)^{\deg'(c_{>i})}\big(\mu^2_i(c)-c_{>i}\otimes c_{\leq i}\big).$$

Now, it is immediate to see that (for $0 < i < n$) $\mu^2_i(c) \in J'$ if $i \neq j, j-1$ and $c_{>i} \otimes c_{\leq i} \in J'$ if $1 < i < n-1$ or $i = 1 \neq j$ or $i = n-1 \neq j-1$. Moreover, if $j = 1$ then

$$\mu^2_1(c) - c_{>1} \otimes c_{\leq 1} = c_{>1} \otimes (1_{\tilde{A}} - \mathrm{id}_{\tilde{A}}) \in J'.$$

Similarly, if $j = n$ then

$$\mu^2_{n-1}(c) - c_{\geq n} \otimes c_{<n} = (1_{\tilde{A}} - \mathrm{id}_{\tilde{A}}) \otimes c_{<n} \in J'.$$

Finally, if $1 < j < n$ then

$$\mu^2_{j-1}(c) = \mu^2_j(c) = (f_n[1] \otimes \cdots f_{j+1}[1] \otimes f_{j-1}[1] \otimes \cdots \otimes f_1[1])[-1],$$

whence $(-1)^{\deg'(c_{\geq j})}\mu^2_{j-1}(c) + (-1)^{\deg'(c_{>j})}\mu^2_j(c) = 0$. □

From Lemma 3.9 we immediately deduce the following result.

Corollary 3.10. *The non-unital dg functor $\pi_{\mathbf{A}} \colon \mathsf{U}^{\mathsf{n}}(\mathbf{A}) \to \mathsf{U}(\mathbf{A})$ is full and its kernel $I = I_{\mathbf{A}}$ is a dg ideal of $\mathsf{U}^{\mathsf{n}}(\mathbf{A})$ such that $I(A, A')$ (for every $A, A' \in \mathbf{A}$) is the $\Bbbk$-subspace of $\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')$ generated by all the elements of one of the following two forms, where $c^l_{n_l} \otimes \cdots \otimes c^1_{n_1} \in C_{(n_1,\ldots,n_l)}(A, A')$:*

1. *$c^l_{n_l} \otimes \cdots \otimes c^1_{n_1} - c^l_{n_l} \otimes \cdots \otimes c^{i+1}_{n_{i+1}} \otimes \mathrm{id}_{\tilde{A}} \otimes c^i_{n_i} \otimes \cdots \otimes c^1_{n_1}$ (for suitable $\tilde{A} \in \mathbf{A}$), with $n_1 + \cdots + n_l > 0$ and $i \in \{0, \ldots, l\}$;*
2. *$c^l_{n_l} \otimes \cdots \otimes c^1_{n_1}$, with $c^i_{n_i}$ of the form (2) in Lemma 3.9 for some $i \in \{1, \ldots, l\}$.*

For every $A, A' \in \mathbf{A}$ the filtration $L_{\leq n}(A, A') := \bigoplus_{m \leq n} L_m(A, A')$ on $\mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')$ (where $n \geq 0$) induces a filtration $I_{\leq n}(A, A') := L_{\leq n}(A, A') \cap I(A, A')$ on $I(A, A')$ and a filtration

$$\mathrm{F}^n\mathsf{U}(\mathbf{A})(A, A') := \big(L_{\leq n}(A, A') + I(A, A')\big)/I(A, A') \cong L_{\leq n}(A, A')/I_{\leq n}(A, A')$$

on $\mathsf{U}(\mathbf{A})(A, A') \cong \mathsf{U}^{\mathsf{n}}(\mathbf{A})(A, A')/I(A, A')$.

Since $\gamma^1_{\mathbf{A}} \colon \mathbf{A}(A, A') \to L_{\leq 1}(A, A')$ is an isomorphism of complexes and $I_{\leq 1}(A, A') = 0$, we see that $\rho^1_{\mathbf{A}} \colon \mathbf{A}(A, A') \to \mathrm{F}^1\mathsf{U}(\mathbf{A})(A, A')$ is an isomorphism, as well. Therefore we just need to show that the inclusion $\mathrm{F}^1\mathsf{U}(\mathbf{A})(A, A') \hookrightarrow \mathsf{U}(\mathbf{A})(A, A')$ is a quasi-isomorphism, and even a homotopy equivalence if $\mathbf{A}$ has split units. By Lemma 2.3 and Remark 2.4 it is enough to prove that for every $n > 1$ the complex $\mathrm{gr}^n\mathsf{U}(\mathbf{A})(A, A')$ is null-homotopic, and also that the inclusion $\mathrm{F}^{n-1}\mathsf{U}(\mathbf{A})(A, A') \hookrightarrow \mathrm{F}^n\mathsf{U}(\mathbf{A})(A, A')$ splits as a morphism of graded $\Bbbk$-modules if $\mathbf{A}$ has split units.

Now, recall from Lemma 2.5 and its proof that, for $n > 1$, the complex $L_n(A, A')$ (endowed with the differential d extending $\mu^1 + \Delta$) is null-homotopic, and a map $r\colon L_n(A, A') \to L_n(A, A')$ of degree -1 satisfying $d \circ r + r \circ d = \mathrm{id}$ can be defined (also for $n = 1$) as follows. By linearity an element of $L_n(A, A')$ can be assumed to be of the form

$$c = c^l \otimes \cdots \otimes c^1 \in C_{(n_1,\dots,n_l)}(A, A'), \tag{3.3}$$

where $n_1 + \cdots + n_l = n$ and $c^k \in C_{(n_k)}(A_{k-1}, A_k)$ homogeneous (for $k = 1, \dots, l$), with $A_0 = A$ and $A_l = A'$. Then

$$r(c) := \begin{cases} 0 & \text{if } n_l > 1 \text{ or } n = 1 \\ r(c^l \otimes c^{l-1}) \otimes c^{l-2} \otimes \cdots \otimes c^1 & \text{if } n_l = 1 < n, \end{cases}$$

where, if $n_l = 1 < n$, $c^t = f \in \mathbf{A}(A_{l-1}, A_l)$ and $c^{l-1} = (f_{n_{l-1}}[1] \otimes \cdots \otimes f_1[1])[-1]$, then

$$r(c^l \otimes c^{l-1}) := (-1)^{\deg(f)}(f[1] \otimes f_{n_{l-1}}[1] \otimes \cdots \otimes f_1[1])[-1].$$

Since

$$\mathrm{gr}^n \mathsf{U}(\mathbf{A})(A, A') \cong \big(L_{\leq n}(A, A') + I(A, A')\big)/\big(L_{<n}(A, A') + I(A, A')\big) \cong L_n(A, A')/I_n(A, A'),$$

where

$$I_n(A, A') := L_n(A, A') \cap \big(L_{<n}(A, A') + I_{\leq n}(A, A')\big),$$

from Lemma 3.11 we deduce that $\mathrm{gr}^n \mathsf{U}(\mathbf{A})(A, A')$ is null-homotopic for $n > 1$.

Lemma 3.11. *The map $r\colon L_n(A, A') \to L_n(A, A')$ preserves the subcomplex $I_n(A, A')$ for every $n > 1$ and every $A, A' \in \mathbf{A}$.*

Proof. As r preserves both $L_n(A, A')$ and $L_{<n}(A, A')$, it is enough to prove that, if $c \in I_{\leq n}(A, A')$, then $r(c) \in L_{<n}(A, A') + I_{\leq n}(A, A')$. We can clearly assume that c is as in part (1) or (2) of Corollary 3.10. In the latter case it is obvious from the definition that $r(c)$ is either 0 or a generator of the same form in $I_{\leq n}(A, A')$. So we can assume c to be of the form (1) with $n_1 + \cdots + n_l = n - 1$, and it is enough to show that $r(c') \in L_{<n}(A, A') + I_{\leq n}(A, A')$, where

$$c' := c^l_{n_l} \otimes \cdots \otimes c^{i+1}_{n_{i+1}} \otimes \mathrm{id}_{\bar{A}} \otimes c^i_{n_i} \otimes \cdots \otimes c^1_{n_1}.$$

Now, if $i \geq l - 1$, then $r(c')$ is either 0 or a generator of the form (2) in $I_{\leq n}(A, A')$. On the other hand, if $i < l - 1$, then $r(c') \in L_{<n}(A, A') + I_{\leq n}(A, A')$ because $r(c^l_{n_l} \otimes \cdots \otimes c^1_{n_1}) \in L_{<n}(A, A')$ and $r(c^l_{n_l} \otimes \cdots \otimes c^1_{n_1}) - r(c')$ is either 0 or a generator of the form (1) in $I_{\leq n}(A, A')$. □

Finally, Lemma 3.12 easily implies that the inclusion $\mathrm{F}^{n-1}\mathsf{U}(\mathbf{A})(A, A') \hookrightarrow \mathrm{F}^n \mathsf{U}(\mathbf{A})(A, A')$ splits as a morphism of graded $\Bbbk$-modules if $\mathbf{A}$ has split units and $n > 1$.

Lemma 3.12. *If $\mathbf{A}$ has split units, then for every $n > 1$ and every $A, A' \in \mathbf{A}$ there exists a morphism of graded $\Bbbk$-modules $u\colon L_n(A, A') \to L_{<n}(A, A')$ such that the map*

$$\tilde{u} := \begin{pmatrix} \mathrm{id} & u \end{pmatrix} : L_{<n}(A, A') \oplus L_n(A, A') = L_{\leq n}(A, A') \to L_{<n}(A, A')$$

sends $I_{\leq n}(A, A')$ to $I_{<n}(A, A')$.

Proof. By hypothesis for every $\tilde{A} \in \mathbf{A}$ there exists a morphism of graded $\Bbbk$-modules $p\colon \mathbf{A}(\tilde{A},\tilde{A}) \to \Bbbk$ such that $p(\mathrm{id}_{\tilde{A}}) = 1$. First, by linearity, every $c \in L_n(A,A')$ can be assumed to be as in (3.3). Setting

$$S(c) := \{i = 1,\dots,l \mid n_i = 1 \text{ and } A_{i-1} = A_i\},$$

we denote, for every subset S of $S(c)$, by $u_S(c)$ the expression obtained from c by deleting the terms c^i with $i \in S$. In case $S = S(c) = \{1,\dots,l\}$ (which implies $A = A'$), we mean $u_S(c) = \mathrm{id}_A$. Now we can define

$$u(c) := \sum_{\emptyset \neq S \subseteq S(c)} (-1)^{|S|-1} \prod_{i\in S} p(c^i) u_S(c).$$

It is immediate from the definition that $\tilde{u}$ sends a generator of the form (2) in Corollary 3.10 to a linear combination of generators of the same form. Hence, given c as above with the additional assumption that there exists $j \in \{1,\dots,l\}$ such that $c^j = \mathrm{id}_{A_j}$ (in particular, $j \in S(c)$), we just need to show that $\tilde{u}(\tilde{c}) \in I(A,A')$, where $\tilde{c} := u_{\{j\}}(c) - c \in I_{\leq n}(A,A')$ is a generator of the form (1). Equivalently, we must prove that $-\tilde{c} + \tilde{u}(\tilde{c}) \in I(A,A')$. In fact we have

$$\begin{aligned} -\tilde{c} + \tilde{u}(\tilde{c}) &= -u_{\{j\}}(c) + c + u_{\{j\}}(c) - u(c) = c - \sum_{\emptyset \neq S \subseteq S(c)} (-1)^{|S|-1} \prod_{i\in S} p(c^i) u_S(c) \\ &= \sum_{S \subseteq S(c)} (-1)^{|S|} \prod_{i\in S} p(c^i) u_S(c) = \sum_{S \subseteq S(c)\setminus\{j\}} (-1)^{|S|} \prod_{i\in S} p(c^i)(u_{S\cup\{j\}}(c) - u_S(c)), \end{aligned}$$

and each $u_{S\cup\{j\}}(c) - u_S(c)$ is a generator of the form (1) (or 0 if $S \cup \{j\} = S(c) = \{1,\dots,l\}$). □

This concludes the proof of Proposition 3.5.

Remark 3.13. The proof of the equivalence between $\mathrm{Ho}_\infty(\mathbf{dgCat})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$ can be modified in an obvious way to prove that there is an equivalence between $\mathrm{Ho}_\infty(\mathbf{dgCat})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat_{dg}})$. Hence $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$ and $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat_{dg}})$ are equivalent, as well.

3.2. **Strictly unital vs. unital A_∞ categories.** A proof of the natural equivalence of ∞-categories $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat}) \to \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat^u})$ has already appeared in [29] (see Theorem 1.1 therein). Here we provide a different and simpler proof (the last part of which has already been included in [29]). A first simplification consists in the fact that, due to the equivalences $\mathrm{Ho}_\infty(\mathbf{dgCat}) \to \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat})$ and $\mathrm{Ho}_\infty(\mathbf{dgCat^u}) \to \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat^u})$, already obtained in Section 3.1, it is enough to prove that the ∞-functor $\mathrm{Ho}_\infty(\mathsf{J})\colon \mathrm{Ho}_\infty(\mathbf{dgCat}) \to \mathrm{Ho}_\infty(\mathbf{dgCat^u})$ is an equivalence of ∞-categories. Here $\mathsf{J}\colon \mathbf{dgCat} \to \mathbf{dgCat^u}$ denotes the natural inclusion functor, which obviously preserves quasi-equivalences.

First of all, given $\mathbf{A} \in \mathbf{dgCat^u}$ and denoting by $\mathrm{Id}_{\mathbf{A}}$ the family of all units in $\mathbf{A}$, we consider the full dg subcategory $\mathbf{C}$ of $\mathrm{pretr}(\mathbf{A}^+)$ consisting of the cones (in $\mathrm{pretr}(\mathbf{A}^+)$) of all the morphisms in $\mathrm{Id}_{\mathbf{A}}$. Then we define $\mathsf{T}(\mathbf{A})$ to be the full dg subcategory of the Drinfeld quotient $\mathrm{pretr}(\mathbf{A}^+)/\mathbf{C}$ whose objects are those in the image of the natural non-unital dg functor

$$\tilde{\eta}_{\mathbf{A}}\colon \mathbf{A} \hookrightarrow \mathbf{A}^+ \hookrightarrow \mathrm{pretr}(\mathbf{A}^+) \to \mathrm{pretr}(\mathbf{A}^+)/\mathbf{C}. \tag{3.4}$$

We will soon see that $\tilde{\eta}_{\mathbf{A}}$ is actually unital (but not strictly unital). For now it is useful to observe that, while $\mathbf{A} \hookrightarrow \mathbf{A}^+$ is really not unital, the other two maps in (3.4) are (strictly unital) dg functors. The crucial result is then the following.

Lemma 3.14. *The non-unital dg functor* $\eta_{\mathbf{A}}\colon \mathbf{A} \to \mathsf{T}(\mathbf{A})$ *induced by* $\tilde{\eta}_{\mathbf{A}}$ *is a quasi-isomorphism for every* $\mathbf{A} \in \mathbf{dgCat}^{\mathbf{u}}$.

Proof. Instead of the argument in [29, Section 2.3] we propose the following much shorter one.

Since $\tilde{\eta}_{\mathbf{A}}$ is injective on objects, $\eta_{\mathbf{A}}$ is bijective on objects by definition of $\mathsf{T}(\mathbf{A})$. Thus we just need to show that the $\tilde{\eta}_{\mathbf{A}}$ is quasi-fully faithful, or, equivalently that the non-unital graded functor

$$H^*(\tilde{\eta}_{\mathbf{A}})\colon H^*(\mathbf{A}) \hookrightarrow H^*(\mathbf{A})^+ = H^*(\mathbf{A}^+) \hookrightarrow H^*\big(\mathrm{pretr}(\mathbf{A}^+)\big) \to H^*\big(\mathrm{pretr}(\mathbf{A}^+)/\mathbf{C}\big)$$

is fully faithful.

It is an easy exercise (using the fact that the elements of $\mathrm{Id}_{\mathbf{A}}$ are units) to prove that the complex of $\Bbbk$-modules $\mathrm{pretr}(\mathbf{A}^+)(X,Y)$ is h-flat (even h-projective) for every $X \in \mathrm{pretr}(\mathbf{A}^+)$ and every $Y \in \mathbf{C}$. Hence [8, Theorem 3.4] shows that the natural functor

$$H^0\Big(\mathrm{pretr}\big(\mathrm{pretr}(\mathbf{A}^+)\big)\Big)/H^0\big((\mathrm{pretr}(\mathbf{C})\big) \to H^0\Big(\mathrm{pretr}\big(\mathrm{pretr}(\mathbf{A}^+)/\mathbf{C}\big)\Big) \tag{3.5}$$

is a triangulated equivalence. Since the natural dg functor $\mathrm{pretr}(\mathbf{A}^+) \hookrightarrow \mathrm{pretr}\big(\mathrm{pretr}(\mathbf{A}^+)\big)$ is a quasi-equivalence, the left-hand-side of (3.5) can be identified with the Verdier quotient $\mathbf{T}/\mathbf{S}$, where $\mathbf{T} := H^0\big(\mathrm{pretr}(\mathbf{A}^+)\big)$ and $\mathbf{S}$ denotes the smallest strictly full triangulated subcategory of $\mathbf{T}$ containing the objects of $\mathbf{C}$. Note that $\mathbf{S}$ is the smallest strictly full triangulated subcategory of $\mathbf{T}$ containing the cones (in $\mathbf{T}$) C_A of id_A, for every $A \in \mathbf{A}$. Here id_A is the identity of A in $H^0(\mathbf{A})$, while 1_A will be the identity of A in $H^0(\mathbf{A})^+$. It should then be clear that $H^*(\tilde{\eta}_{\mathbf{A}})$ can be identified with the natural non-unital graded functor

$$\mathsf{H}\colon H^*(\mathbf{A}) \hookrightarrow H^*(\mathbf{A})^+ \hookrightarrow \mathbf{T}^{\mathrm{gr}} \to (\mathbf{T}/\mathbf{S})^{\mathrm{gr}},$$

where $\mathbf{T}^{\mathrm{gr}}$ is the graded category with the same objects as $\mathbf{T}$ and $\mathbf{T}^{\mathrm{gr}}(X,Y)^n := \mathbf{T}(X,Y[n])$ for all $X,Y \in \mathbf{T}$ and all $n \in \mathbb{Z}$ (hence $\mathbf{T}^{\mathrm{gr}}$ can be identified with $H^*\big(\mathrm{pretr}(\mathbf{A}^+)\big)$), and similarly for $(\mathbf{T}/\mathbf{S})^{\mathrm{gr}}$. We have to show that H is fully faithful, and to this purpose, up to passing to the idempotent completion, we can assume $\mathbf{T}$ to be idempotent complete.

Under this assumption, for every $A \in \mathbf{A}$, the idempotent morphism id_A splits in $\mathbf{T}$. Therefore we can write $A = X_A \oplus Y_A$ for suitable $X_A, Y_A \in \mathbf{T}$, and id_A (respectively $1_A - \mathrm{id}_A$) gets identified with $A \twoheadrightarrow X_A \hookrightarrow A$ (respectively $A \twoheadrightarrow Y_A \hookrightarrow A$). Note that this implies $C_A \cong Y_A \oplus Y_A[1]$.

On the other hand, using the fact that the composition of two morphisms in $H^*(\mathbf{A})^+$ is trivial if one of the two is in $H^*(\mathbf{A})$ and the other one is of the form $1_A - \mathrm{id}_A$, it is easy to see that, for every $A, A' \in \mathbf{A}$, in the decomposition

$$H^*(\mathbf{A})^+(A,A') = \mathbf{T}^{\mathrm{gr}}(A,A') = \mathbf{T}^{\mathrm{gr}}(X_A,X_{A'}) \oplus \mathbf{T}^{\mathrm{gr}}(X_A,Y_{A'}) \oplus \mathbf{T}^{\mathrm{gr}}(Y_A,X_{A'}) \oplus \mathbf{T}^{\mathrm{gr}}(Y_A,Y_{A'})$$

the first summand gets identified with $H^*(\mathbf{A})(A,A')$, the second and the third one are always 0, and the fourth one is a copy of $\Bbbk$ in degree 0 (with generator corresponding to $1_A - \mathrm{id}_A$) if $A = A'$ and 0 otherwise.

Furthermore $\mathbf{T}^{\mathrm{gr}}(X_A,X_{A'}) \to (\mathbf{T}/\mathbf{S})^{\mathrm{gr}}(X_A,X_{A'})$ is an isomorphism, thanks to the implication III) $\Rightarrow$ v) in [31, Chapt. II, Prop. 2.3.3a)] (which applies since $\mathbf{S}$ is orthogonal to all the objects of the form X_A). To conclude, it is then enough to note that $(\mathbf{T}/\mathbf{S})^{\mathrm{gr}}(A,A') = (\mathbf{T}/\mathbf{S})^{\mathrm{gr}}(X_A,X_{A'})$ because Y_A and $Y_{A'}$ are direct summands of objects in $\mathbf{S}$, and thus they are trivial in $\mathbf{T}/\mathbf{S}$. □

Observe that, as a consequence of Lemma 3.14, $\eta_{\mathbf{A}}$ is a unital dg functor (see Remark 1.14). It is then very easy to deduce that T extends to a functor $\mathsf{T}\colon \mathbf{dgCat^u} \to \mathbf{dgCat}$ and that we obtain a natural transformation $\eta\colon \mathrm{id}_{\mathbf{dgCat^u}} \to \mathsf{J}\circ\mathsf{T}$, whose components are all quasi-equivalences. Moreover $\mathsf{J}\circ\mathsf{T}$, hence also T, preserves quasi-equivalences.

On the other hand, since $\eta_{\mathbf{A}}$ is not strictly unital, η does not define a natural transformation $\mathrm{id}_{\mathbf{dgCat}} \to \mathsf{T}\circ\mathsf{J}$. However, we are going to see that there exist another functor $\mathsf{T}'\colon \mathbf{dgCat} \to \mathbf{dgCat}$ and two natural transformations $\eta'\colon \mathrm{id}_{\mathbf{dgCat}} \to \mathsf{T}'$ and $\eta''\colon \mathsf{T}\circ\mathsf{J} \to \mathsf{T}'$, whose components are all quasi-equivalences. By Proposition 3.1 this will allow to conclude that $\mathrm{Ho}_\infty(\mathsf{J})$ and $\mathrm{Ho}_\infty(\mathsf{T})$ are quasi-inverse equivalences of ∞-categories.

The definitions of T' and η' are similar to those of T and η. More precisely, given $\mathbf{A} \in \mathbf{dgCat}$ and still denoting by $\mathrm{Id}_{\mathbf{A}}$ the family of all units in $\mathbf{A}$, we consider the full dg subcategory $\mathbf{C}'$ of $\mathrm{pretr}(\mathbf{A})$ consisting of the cones (in $\mathrm{pretr}(\mathbf{A})$) of all the morphisms in $\mathrm{Id}_{\mathbf{A}}$. Then we define $\mathsf{T}'(\mathbf{A})$ to be the full dg subcategory of the Drinfeld quotient $\mathrm{pretr}(\mathbf{A})/\mathbf{C}'$ whose objects are those in the image of the natural (strictly unital) dg functor $\mathbf{A} \hookrightarrow \mathrm{pretr}(\mathbf{A}) \to \mathrm{pretr}(\mathbf{A})/\mathbf{C}'$. Thus we obtain a dg functor $\eta'_{\mathbf{A}}\colon \mathbf{A} \to \mathsf{T}'(\mathbf{A})$. Using the fact that the objects of $\mathbf{C}'$ are trivial in cohomology, a much simpler argument than that in the proof of Lemma 3.14 shows that $\eta'_{\mathbf{A}}$ is a quasi-isomorphism. It is then clear that T' extends to a functor $\mathsf{T}'\colon \mathbf{dgCat} \to \mathbf{dgCat}$ and that $\eta'\colon \mathrm{id}_{\mathbf{dgCat}} \to \mathsf{T}'$ is a natural transformation, whose components are all quasi-equivalences.

Finally, for every $\mathbf{A} \in \mathbf{dgCat}$, the unique dg functor $\mathbf{A}^+ \to \mathbf{A}$ such that $\mathbf{A} \hookrightarrow \mathbf{A}^+ \to \mathbf{A}$ is $\mathrm{id}_{\mathbf{A}}$ extends to a dg functor $\mathrm{pretr}(\mathbf{A}^+) \to \mathrm{pretr}(\mathbf{A})$ mapping $\mathbf{C}$ to $\mathbf{C}'$, whence it induces a dg functor $\eta''_{\mathbf{A}}\colon \mathsf{T}(\mathbf{A}) \to \mathsf{T}'(\mathbf{A})$. It is easy to see, by construction, that $\eta'_{\mathbf{A}} = \eta''_{\mathbf{A}} \circ \eta_{\mathbf{A}}$. Since $\eta_{\mathbf{A}}$ and $\eta'_{\mathbf{A}}$ are quasi-isomorphism, $\eta''_{\mathbf{A}}$ is a quasi-isomorphism, as well. Clearly we conclude that $\eta''\colon \mathsf{T}\circ\mathsf{J} \to \mathsf{T}'$ is a natural transformation, whose components are all quasi-equivalences.

3.3. **Unital vs. cohomologically unital A_∞ categories.** In this subsection we prove the last equivalence in Theorem A, namely $\mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat^u}) \to \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat^c})$. As we already know the natural equivalences $\mathrm{Ho}_\infty(\mathbf{dgCat^u}) \to \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat^u})$ and $\mathrm{Ho}_\infty(\mathbf{dgCat^c}) \to \mathrm{Ho}_\infty(\mathbf{A}_\infty\mathbf{Cat^c})$ (see Remark 3.6), it is enough to prove that $\mathrm{Ho}_\infty(\mathsf{K})\colon \mathrm{Ho}_\infty(\mathbf{dgCat^u}) \to \mathrm{Ho}_\infty(\mathbf{dgCat^c})$ is an equivalence of ∞-categories. Here $\mathsf{K}\colon \mathbf{dgCat^u} \to \mathbf{dgCat^c}$ denotes the inclusion functor, which obviously preserves quasi-equivalences.

We start by recalling that, in general, $\mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat^n}$ is *h-projective* if $\mathbf{A}(A, A')$ is a h-projective complex[2] of $\Bbbk$-modules for every $A, A' \in \mathbf{A}$. We will need the following easy result.

Lemma 3.15. *Let $\mathbf{A}$ be a h-projective and cohomologically unital dg category. Then $\mathbf{A}$ is unital.*

Proof. Given a cohomological unit e_A of some $A \in \mathbf{A}$, the difference $e_A - e_A \circ e_A$ is a coboundary. Since composition with a coboundary is always null-homotopic, it follows that both maps

$$-\circ e_A\colon \mathbf{A}(A, A') \to \mathbf{A}(A, A') \qquad e_A \circ -\colon \mathbf{A}(A', A) \to \mathbf{A}(A', A) \tag{3.6}$$

are idempotent endomorphisms in the homotopy category of complexes, for every $A' \in \mathbf{A}$. On the other hand (3.6) induce isomorphisms (in fact the identity) in cohomology. As $\mathbf{A}$ is h-projective, this implies that (3.6) are isomorphisms in the homotopy category of complexes. Taking into

[2]A complex in an abelian category is *h-projective* if every morphism from it to an acyclic complex is null-homotopic.

account that an idempotent isomorphism is necessarily the identity, we conclude that (3.6) are homotopic to the identity. □

Then it is enough to show that non-unital dg categories admit functorial h-projective resolutions, meaning that there exist a functor $\tilde{\mathsf{S}}\colon \mathbf{dgCat^n} \to \mathbf{dgCat^n}$ together with a natural transformation $\varepsilon\colon \tilde{\mathsf{S}} \to \mathrm{id}_{\mathbf{dgCat^n}}$ such that $\tilde{\mathsf{S}}(\mathbf{A})$ is h-projective and $\varepsilon_{\mathbf{A}}\colon \tilde{\mathsf{S}}(\mathbf{A}) \to \mathbf{A}$ is a quasi-isomorphism, for every $\mathbf{A} \in \mathbf{dgCat^n}$. Indeed, assuming this, by Lemma 3.15 (and recalling Remark 1.14) $\tilde{\mathsf{S}}$ restricts to a functor $\mathsf{S}\colon \mathbf{dgCat^c} \to \mathbf{dgCat^u}$ and ε to natural transformations $\mathsf{K} \circ \mathsf{S} \to \mathrm{id}_{\mathbf{dgCat^c}}$ and $\mathsf{S} \circ \mathsf{K} \to \mathrm{id}_{\mathbf{dgCat^u}}$, whose components are all quasi-equivalences. This also implies that S preserves quasi-equivalences, and we conclude from Proposition 3.1 that $\mathrm{Ho}_\infty(\mathsf{K})$ and $\mathrm{Ho}_\infty(\mathsf{S})$ are quasi-inverse equivalences of ∞-categories.

As for the existence of functorial h-projective resolutions for non-unital dg categories, we just very briefly sketch the argument, as one can proceed similarly to the strictly unital case, for which we refer to [4, Section 3.2]. Indeed, in [4] the h-projective resolution of $\mathbf{A} \in \mathbf{dgCat}$ is obtained as the colimit of a sequence of dg categories $\mathbf{A}_n$ ($n \in \mathbb{N}$), all with the same objects as $\mathbf{A}$. In the non-unital case we can keep the same definition of $\mathbf{A}_1$ and change the inductive step from $\mathbf{A}_{n-1}$ to $\mathbf{A}_n$ as follows. While in [4] $\mathbf{A}_n$ is the graded $\Bbbk$-linear *category* freely generated over $\mathbf{A}_{n-1}$ by a suitable set of generators, in the non-unital case we consider the graded $\Bbbk$-linear *non-unital category* freely generated over $\mathbf{A}_{n-1}$ by the same set of generators. In [25] it is explained how the same result holds more generally for (non-unital) A_∞ categories.

4. Ho(**dgCat**) and categories of functors

In this section, we show that $\mathrm{Ho}(\mathbf{dgCat})$ is naturally equivalent to the categories $\mathbf{A}_\infty\mathbf{Cat^u_{hp}}/\!\approx$ and $\mathbf{A}_\infty\mathbf{Cat_{hps}}/\!\approx$, thus proving the second part of Theorem C.

To clarify the notation, we set $\mathbf{A}_\infty\mathbf{Cat^u_{hp}}$ (respectively $\mathbf{A}_\infty\mathbf{Cat_{hps}}$) to be the full subcategory of $\mathbf{A}_\infty\mathbf{Cat^u}$ (respectively $\mathbf{A}_\infty\mathbf{Cat}$) whose objects are h-projective (respectively h-projective and with split units, see Definition 3.4). Such equivalences are established in Section 4.2 (see, in particular, Proposition 4.12), after a good amount of technical preliminaries in Section 4.1.

Actually the effort to prove Proposition 4.12 will receive an extra reward in Section 5, where we will use Proposition 4.11 (the key technical result of this section) to prove the first part of Theorem C.

4.1. Preliminary results.

We begin with the following result about A_∞ functors which corrects a similar statement in [17].

Lemma 4.1. *Let $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ be a unital A_∞ functor between two strictly unital A_∞ categories. If $\mathbf{A}$ has split units, then F is homotopic to a strictly unital A_∞ functor.*

Proof. Since F is unital, for every $A \in \mathbf{A}$ there exists $h_A \in \mathbf{B}(\mathsf{F}^0(A), \mathsf{F}^0(A))^{-1}$ such that $\mathsf{F}^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)} - \mathrm{m}^1_{\mathbf{B}}(h_A)$. Then we define a prenatural transformation $\theta\colon \mathsf{F} \to \mathsf{F}$ of degree 0 as follows: for every $f \in \mathbf{A}(A, A')$ we set

$$\theta^1(f) := \begin{cases} p(f)h_A & \text{if } A = A' \\ 0 & \text{if } A \neq A' \end{cases}$$

(where $p\colon \mathbf{A}(A,A) \to \Bbbk$ is as in the proof of Lemma 3.12) and $\theta^i := 0$ for $i \neq 1$. By Remark 1.20 we can find $\widetilde{\mathsf{F}} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A},\mathbf{B})$ such that $\mathsf{F} \sim \widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{F}}^i = \mathsf{F}^i + \mathrm{m}^1(\theta)^i$ for $i > 0$. By definition for every $A \in \mathbf{A}$ we have

$$\widetilde{\mathsf{F}}^1(\mathrm{id}_A) = \mathsf{F}^1(\mathrm{id}_A) + \mathrm{m}^1(\theta)^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)} - \mathrm{m}^1_{\mathbf{B}}(h_A) + \theta^1\big(\mathrm{m}^1_{\mathbf{A}}(\mathrm{id}_A)\big) + \mathrm{m}^1_{\mathbf{B}}\big(\theta^1(\mathrm{id}_A)\big) = \mathrm{id}_{\mathsf{F}^0(A)}.$$

To conclude, using Remark 1.19 and an easy recursive argument, it should be clear that it is enough to prove the following statement. Assume that $\mathsf{F}^1(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}^0(A)}$ for every $A \in \mathbf{A}$ and that there exist $n > 1$ and $1 \leq m \leq n$ such that $\mathsf{F}^i(f_i \otimes \cdots \otimes f_1) = 0$ if there exists $j \in \{1,\dots,i\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and either $1 < i < n$ or $i = n$ and $j < m$. Then we can find $\mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{n}}(\mathbf{A},\mathbf{B})$ such that $\mathsf{F} \sim \mathsf{G}$ through a homotopy θ with $\theta^i = 0$ for $i < n-1$, $\mathsf{G}^i = \mathsf{F}^i$ for $i < n$ and $\mathsf{G}^n(f_n \otimes \cdots \otimes f_1) = 0$ if there exists $j \in \{1,\dots,n\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and $j \leq m$.

To this aim, a direct but tedious check shows that we can define θ by

$$\theta^{n-1}(f_{n-1} \otimes \cdots \otimes f_1) := (-1)^m \mathsf{F}^n(f_{n-1} \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$
$$\theta^n(f_n \otimes \cdots \otimes f_1) := (-1)^m \mathsf{F}^{n+1}(f_n \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$

and $\theta^i := 0$ for $i \neq n-1, n$. See also [23, Lemma 3.7] (where the assumption that the category has split units is erroneously missing) for more details of the computation. □

The following is then a straightforward consequence using Corollary 1.23 (1).

Corollary 4.2. *If $\mathbf{A},\mathbf{B} \in \mathbf{A}_\infty\mathbf{Cat}$ and $\mathbf{A}$ has split units, then the natural injective map*

$$\mathbf{A}_\infty\mathbf{Cat}/\approx(\mathbf{A},\mathbf{B}) \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A},\mathbf{B})$$

is also surjective.

With an argument similar to the one in the proof of Lemma 4.1, we get the following result about natural transformations.

Lemma 4.3. *Let $\mathsf{F},\mathsf{G} \in \mathbf{A}_\infty\mathbf{Cat}(\mathbf{A},\mathbf{B})$ and let $\theta\colon \mathsf{F} \to \mathsf{G}$ be a natural transformation of degree p. Then there exists a prenatural transformation $\widetilde{\theta}\colon \mathsf{F} \to \mathsf{G}$ of degree $p-1$ such that $\theta - \mathrm{m}^1(\widetilde{\theta})\colon \mathsf{F} \to \mathsf{G}$ is a strictly unital natural transformation.*

Proof. The argument is similar (and a bit simpler) to the one of Lemma 4.1. In this case the only key step consists in the proof of the following statement. Assume that there exist $n > 0$ and $1 \leq m \leq n$ such that $\theta^i(f_i \otimes \cdots \otimes f_1) = 0$ if there exists $j \in \{1,\dots,i\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and either $0 < i < n$ or $i = n$ and $j < m$. Then we can find a prenatural transformation $\overline{\theta}\colon \mathsf{F} \to \mathsf{G}$ of degree $p-1$ such that $\overline{\theta}^i = 0$ for $i < n-1$, $\mathrm{m}^1(\overline{\theta})^i = 0$ for $i < n$ and $\mathrm{m}^1(\overline{\theta})^n(f_n \otimes \cdots \otimes f_1) = \theta^n(f_n \otimes \cdots \otimes f_1)$ if there exists $j \in \{1,\dots,n\}$ such that $f_j = \mathrm{id}_A$ (for some $A \in \mathbf{A}$) and $j \leq m$. Here we can define $\overline{\theta}$ by

$$\overline{\theta}^{n-1}(f_{n-1} \otimes \cdots \otimes f_1) := (-1)^m \theta^n(f_{n-1} \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$
$$\overline{\theta}^n(f_n \otimes \cdots \otimes f_1) := (-1)^m \theta^{n+1}(f_n \otimes \cdots \otimes f_m \otimes \mathrm{id}_A \otimes f_{m-1} \otimes \cdots \otimes f_1)$$

and $\overline{\theta}^i := 0$ for $i \neq n-1, n$. See also [23, Lemma 3.8] for more details. □

We can then prove the following.

Lemma 4.4. *If* $\mathsf{F}, \mathsf{F}' \in \mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}}(\mathbf{A}, \mathbf{B})$ *are such that* $\mathsf{F} \approx \mathsf{F}'$*, then* F *and* F' *have the same image in* $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}})$.

Proof. This is [5, Lemma 2.10]. The only point of the proof that must be modified is the existence of a suitable strictly unital natural transformation $\mathsf{F} \to \mathsf{F}'$, for which we invoke Lemma 4.3. □

In the following we will need to use the fact that **dgCat** admits a model structure, where the weak equivalences are the quasi-equivalences and the fibrations are the full dg functors whose H^0 is an isofibration (see [28]). Recall that, in general, if $\mathbf{C}$ is a model category, $X \in \mathbf{C}$ is cofibrant and $Y \in \mathbf{C}$ is fibrant, then the natural map $\mathbf{C}(X, Y) \to \mathrm{Ho}(\mathbf{C})(X, Y)$ induces a bijection

$$\mathbf{C}(X, Y)/\asymp \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{C})(X, Y) \tag{4.1}$$

(see [14, Theorem 1.2.10]), where the equivalence relation $\asymp$ on $\mathbf{C}(X, Y)$ can be defined as follows.[3] First a *cylinder object* for X is given by morphisms $i_0, i_1 \colon X \to X'$ and a weak equivalence $s \colon X' \to X$ such that $s \circ i_0 = s \circ i_1 = \mathrm{id}_X$ and $(i_0, i_1) \colon X \coprod X \to X'$ is a cofibration. Then, given $f_0, f_1 \in \mathbf{C}(X, Y)$, we have $f_0 \asymp f_1$ if and only if there exist a cylinder object (X', i_0, i_1, s) for X and $h \in \mathbf{C}(X', Y)$ such that $f_k = h \circ i_k$, for $k = 0, 1$ (see [14, Definition 1.2.4 and Corollary 1.2.6]). Moreover, if $f \in \mathbf{C}(X, Y)$ is a weak equivalence between two fibrant and cofibrant objects, then (always by [14, Theorem 1.2.10]) there exists $g \in \mathbf{C}(Y, X)$ such that $g \circ f \asymp \mathrm{id}_X$ and $f \circ g \asymp \mathrm{id}_Y$.

Remark 4.5. One can easily see that, by construction, the bijection in (4.1) is indeed natural with respect to pre and post composition, if one restricts to fibrant and cofibrant objects of $\mathbf{C}$.

Remark 4.6. If (X', i_0, i_1, s) is a cylinder object for a cofibrant object X, then X' is cofibrant, as well: this follows immediately from the fact that cofibrations are stable under composition and pushouts (see [14, Corollary 1.1.11]).

Remark 4.7. It is clear from the definition that every dg category is fibrant. On the other hand, if $\mathbf{A} \in \mathbf{dgCat}$ is cofibrant, then $\mathbf{A}$ is also h-projective. Indeed, for every $A, B \in \mathbf{A}$ the complex of $\Bbbk$-modules $\mathbf{A}(A, B)$ is cofibrant by [30, Proposition 2.3], hence h-projective by [14, Lemma 2.3.8]. It follows that every dg category admits a h-projective resolution, namely a quasi-equivalence from a h-projective dg category. Actually, we can be more precise as, by [8, Lemma B.5], every $\mathbf{A} \in \mathbf{dgCat}$ has a semi-free resolution, which is cofibrant by [8, Lemma B.6] (hence h-projective) and has split units.

Remark 4.8. If $\mathbf{A}$ and $\mathbf{B}$ are h-projective dg-categories, then so is $\mathbf{A} \otimes \mathbf{B}$.

Lemma 4.9. *If* $\mathsf{F}_0, \mathsf{F}_1 \in \mathbf{dgCat}(\mathbf{A}, \mathbf{B})$ *are such that* $\mathbf{A}$ *is cofibrant and* $\mathsf{F}_0 \asymp \mathsf{F}_1$*, then* $\mathsf{F}_0 \approx \mathsf{F}_1$.

Proof. By definition there exist a cylinder object $(\mathbf{A}', \mathsf{I}_0, \mathsf{I}_1, \mathsf{S})$ for $\mathbf{A}$ and $\mathsf{H} \in \mathbf{dgCat}(\mathbf{A}', \mathbf{B})$ such that $\mathsf{F}_k = \mathsf{H} \circ \mathsf{I}_k$, for $k = 0, 1$. Note that both $\mathbf{A}'$ (by Remark 4.6) and $\mathbf{A}$ are cofibrant, hence h-projective by Remark 4.7. So the quasi-equivalence S is actually a homotopy equivalence. Since $\mathsf{S} \circ \mathsf{I}_0 = \mathsf{S} \circ \mathsf{I}_1$, part (2) of Corollary 1.23 implies $\mathsf{I}_0 \approx \mathsf{I}_1$. It follows that $\mathsf{F}_0 = \mathsf{H} \circ \mathsf{I}_0 \approx \mathsf{H} \circ \mathsf{I}_1 = \mathsf{F}_1$. □

[3] Usually this equivalence relation is called homotopy and is denoted by $\sim$, but we will not do that, in order to avoid confusion with the already defined notion of homotopy for A_∞ functors.

Lemma 4.10. *Given* $\mathsf{F} \in \mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}}(\mathbf{A}, \mathbf{B})$ *with* $\mathbf{A}$ *cofibrant and with split units, there exists* $\mathsf{F}' \in \mathbf{dgCat}(\mathbf{A}, \mathbf{B})$ *such that* $\mathsf{F} \approx \mathsf{F}'$.

Proof. By Proposition 3.5 the diagram in $\mathbf{A}_\infty\mathbf{Cat}$

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{\rho_{\mathbf{A}}} & \mathsf{U}(\mathbf{A}) & \xrightarrow{\sigma_{\mathbf{A}}} & \mathbf{A} \\ {\scriptstyle \mathsf{F}}\downarrow & & \downarrow{\scriptstyle \mathsf{U}(\mathsf{F})} & & \downarrow{\scriptstyle \mathsf{F}} \\ \mathbf{B} & \xrightarrow[\rho_{\mathbf{B}}]{} & \mathsf{U}(\mathbf{B}) & \xrightarrow[\sigma_{\mathbf{B}}]{} & \mathbf{B} \end{array}$$

is such that the square on the left commutes. Instead the square on the right commutes when F is a dg functor, but not in general. Taking into account that $\sigma_{\mathbf{A}} \circ \rho_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}}$ and $\sigma_{\mathbf{B}} \circ \rho_{\mathbf{B}} = \mathrm{id}_{\mathbf{B}}$, in any case we have

$$\mathsf{F} \circ \sigma_{\mathbf{A}} \circ \rho_{\mathbf{A}} = \mathsf{F} = \sigma_{\mathbf{B}} \circ \rho_{\mathbf{B}} \circ \mathsf{F} = \sigma_{\mathbf{B}} \circ \mathsf{U}(\mathsf{F}) \circ \rho_{\mathbf{A}}.$$

Since $\rho_{\mathbf{A}}$ is a homotopy equivalence, from part (3) of Corollary 1.23 we obtain

$$\mathsf{F} \circ \sigma_{\mathbf{A}} \approx \sigma_{\mathbf{B}} \circ \mathsf{U}(\mathsf{F}). \tag{4.2}$$

Now, let $\mathsf{S}\colon \mathbf{C} \to \mathsf{U}(\mathbf{A})$ be a quasi-equivalence in $\mathbf{dgCat}$ with $\mathbf{C}$ cofibrant (given, for instance, by a cofibrant replacement of $\mathsf{U}(\mathbf{A})$). Then $\sigma_{\mathbf{A}} \circ \mathsf{S}\colon \mathbf{C} \to \mathbf{A}$ is a quasi-equivalence in $\mathbf{dgCat}$ between two fibrant and cofibrant objects. Therefore there exists $\mathsf{G} \in \mathbf{dgCat}(\mathbf{A}, \mathbf{C})$ such that $\sigma_{\mathbf{A}} \circ \mathsf{S} \circ \mathsf{G} \asymp \mathrm{id}_{\mathbf{A}}$. By Lemma 4.9 this implies

$$\sigma_{\mathbf{A}} \circ \mathsf{S} \circ \mathsf{G} \approx \mathrm{id}_{\mathbf{A}}. \tag{4.3}$$

Using (4.3) and (4.2) we obtain

$$\mathsf{F} = \mathsf{F} \circ \mathrm{id}_{\mathbf{A}} \approx \mathsf{F} \circ \sigma_{\mathbf{A}} \circ \mathsf{S} \circ \mathsf{G} \approx \sigma_{\mathbf{B}} \circ \mathsf{U}(\mathsf{F}) \circ \mathsf{S} \circ \mathsf{G}.$$

To conclude, just observe that $\sigma_{\mathbf{B}}$, $\mathsf{U}(\mathsf{F})$, S and G are all dg functors, hence the same is true for their composition. □

4.2. **The missing equivalences.** The following is the crucial technical result of this section. It will also play an important role in the description of the internal Homs in Section 5.

Proposition 4.11. *For every* $\mathbf{A}, \mathbf{B} \in \mathbf{dgCat}$ *with* $\mathbf{A}$ *h-projective there is a natural bijection*

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}).$$

Proof. First we claim that we can assume that $\mathbf{A}$ is semi-free. Indeed, by Remark 4.7), there exists $\mathbf{A} \cong \tilde{\mathbf{A}}$ in $\mathrm{Ho}(\mathbf{dgCat})$, with $\tilde{\mathbf{A}}$ semi-free. Hence there is a natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\tilde{\mathbf{A}}, \mathbf{B}).$$

Taking into account that $\tilde{\mathbf{A}}$ (by Remark 4.7) and $\mathbf{A}$ are h-projective, Lemma 1.22 implies $\mathbf{A} \cong \tilde{\mathbf{A}}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx$. Thus there is a natural bijection

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\tilde{\mathbf{A}}, \mathbf{B}).$$

It follows that the existence of the required bijection is equivalent to the existence of a natural bijection

$$\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\tilde{\mathbf{A}},\mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\tilde{\mathbf{A}},\mathbf{B}).$$

This proves the claim, and so for the rest of the proof we assume that $\mathbf{A}$ is semi-free.

By Lemma 4.9 the inclusion $\mathbf{dgCat}(\mathbf{A},\mathbf{B}) \hookrightarrow \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}(\mathbf{A},\mathbf{B})$ induces a map

$$\varphi\colon \mathbf{dgCat}(\mathbf{A},\mathbf{B})/\asymp \;\to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A},\mathbf{B}). \tag{4.4}$$

By (4.1) it is enough to prove that φ is bijective. Indeed, given $\mathsf{F},\mathsf{F}' \in \mathbf{dgCat}(\mathbf{A},\mathbf{B})$ such that $\mathsf{F}\approx\mathsf{F}'$, by Lemma 4.4 F and F' have the same image in $\mathrm{Ho}(\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{dg}})$, hence also in $\mathrm{Ho}(\mathbf{dgCat})$ (see Remark 3.13). Therefore $\mathsf{F}\asymp\mathsf{F}'$, again by (4.1), and this proves that φ is injective. Finally, since $\mathbf{A}$ has split units, Corollary 4.2 shows that the natural map

$$\mathbf{A}_\infty\mathbf{Cat}/\approx(\mathbf{A},\mathbf{B}) \to \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx(\mathbf{A},\mathbf{B})$$

is bijective. We conclude that φ is surjective by Lemma 4.10. □

We can now apply the previous result to provide the equivalences in the last part of Theorem C.

Proposition 4.12. *The categories* $\mathrm{Ho}(\mathbf{dgCat})$, $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ *and* $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}/\approx$ *are equivalent. In particular,* $\mathrm{Ho}(\mathbf{dgCat})$ *and* $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\approx$ *are equivalent if* $\Bbbk$ *is a field.*

Proof. Let $\mathbf{C}$ be the full subcategory of $\mathrm{Ho}(\mathbf{dgCat})$ whose objects are h-projective, and let $\mathbf{C}'$ be the full subcategory of $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ whose objects are (strictly unital) dg categories. The inclusion $\mathbf{C}\hookrightarrow\mathrm{Ho}(\mathbf{dgCat})$ is clearly an equivalence, and we claim that the same is true for the inclusion $\mathbf{C}'\hookrightarrow\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$. Indeed, given $\mathbf{A}\in\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}$, by Lemma 1.16 and Lemma 1.22 we can find a homotopy equivalence $\mathsf{F}\colon\mathbf{R}_{\mathbf{A}}\to\mathbf{A}$ with $\mathbf{R}_{\mathbf{A}}\in\mathbf{dgCat}$. Denoting by $\mathsf{G}\colon\mathbf{B}\to\mathbf{R}_{\mathbf{A}}$ a quasi-equivalence with $\mathbf{B}$ h-projective, we obtain a quasi-equivalence $\mathsf{F}\circ\mathsf{G}\colon\mathbf{B}\to\mathbf{A}$ with $\mathbf{B}\in\mathbf{C}'$. Since both $\mathbf{A}$ and $\mathbf{B}$ are h-projective, $\mathsf{F}\circ\mathsf{G}$ is actually a homotopy equivalence, hence its image in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ is an isomorphism by Lemma 1.22. This proves the claim, and the equivalence between $\mathrm{Ho}(\mathbf{dgCat})$ and $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ follows from the fact that, as an easy consequence of Proposition 4.11, $\mathbf{C}$ and $\mathbf{C}'$ are isomorphic categories.

To conclude, it is obviously enough to prove that the inclusion functor

$$\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}/\approx\to\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$$

is an equivalence. It is clearly faithful and, by Corollary 4.2, it is full as well. It is also essentially surjective because, for any $\mathbf{A}\in\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}$, we can find $\mathbf{B}\in\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}$ which is isomorphic to $\mathbf{A}$ in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}_{\mathbf{hp}}/\approx$ (for instance, arguing similarly as above, we can take $\mathbf{B}$ to be a semi-free resolution of $\mathbf{R}_{\mathbf{A}}$). □

Remark 4.13. The fully faithful inclusion functor $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}/\approx\to\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}/\approx$ is not essentially surjective, even when $\Bbbk$ is a field (in which case $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}/\approx=\mathbf{A}_\infty\mathbf{Cat}/\approx$). Indeed, the 0 dg algebra is certainly in $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hp}}/\approx$, but it is not in the essential image of the functor, since there is no morphism in $\mathbf{A}_\infty\mathbf{Cat}$ from it to an object of $\mathbf{A}_\infty\mathbf{Cat}_{\mathbf{hps}}$.

5. Internal Homs via A_∞ functors

In this section we complete the proof of Theorem C in Section 5.2, after introducing some preliminary results about A_∞ multifunctors in Section 5.1.

5.1. **A_∞ multifunctors.** Let us briefly recall some constructions which are carefully described in Sections 1.2 and 1.4 in [5] and which were originally introduced in [2]. For this reason we will be concise in the presentation and we will refer to these original sources for more details.

More specifically, given $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A} \in \mathbf{A}_\infty\mathbf{Cat^u}$, an A_∞ *multifunctor* from $\mathbf{A}_1, \dots, \mathbf{A}_n$ to $\mathbf{A}$ is a morphism of graded quivers

$$\mathsf{F}\colon \overline{\mathrm{B}_\infty(\mathbf{A}_1)^+ \otimes \cdots \otimes \mathrm{B}_\infty(\mathbf{A}_n)^+} \to \mathbf{A}[1]$$

such that the natural extension

$$\overline{\mathrm{B}_\infty(\mathbf{A}_1)^+ \otimes \cdots \otimes \mathrm{B}_\infty(\mathbf{A}_n)^+} \to \mathrm{B}_\infty(\mathbf{A})$$

of F as graded cofunctor commutes with the differentials. Furthermore, an A_∞ multifunctor is *unital* if all its restrictions are unital. The set of all unital A_∞ multifunctors from $\mathbf{A}_1, \dots, \mathbf{A}_n$ to $\mathbf{A}$ will be denoted by $\mathbf{A}_\infty\mathbf{Cat^u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$. It is also important to know that, by [2, Proposition 8.15], there is a unital A_∞ category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat^u}}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$ whose set of objects is $\mathbf{A}_\infty\mathbf{Cat^u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$ (morphisms are suitably defined prenatural transformations). Note that, if $\mathbf{A}$ is a dg category, then $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat^u}}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$ is a dg category as well.

Proposition 5.1. *For every $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbf{A}_\infty\mathbf{Cat^u}$ there is an isomorphism in $\mathbf{A}_\infty\mathbf{Cat^u}$*

$$\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat^u}}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat^u}}(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat^u}}(\mathbf{A}_2, \mathbf{A}_3)).$$

Proof. It follows from [2, Proposition 9.18] together with [2, Proposition 4.12]. □

In complete analogy with the case of A_∞ functors (see Section 1.4), if F_1 and F_2 are in $\mathbf{A}_\infty\mathbf{Cat^u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})$, we say that F_1 and F_2 are *weakly equivalent* (denoted by $\mathsf{F}_1 \approx \mathsf{F}_2$) if they are isomorphic in the category $H^0(\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat^u}}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A}))$. The relation $\approx$ is clearly compatible with compositions and then we can define a quotient (multi)category $\mathbf{A}_\infty\mathbf{Cat^u}/\!\approx$ with the same objects and whose morphisms are given by

$$\mathbf{A}_\infty\mathbf{Cat^u}/\!\approx(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A}) := \mathbf{A}_\infty\mathbf{Cat^u}(\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{A})/\approx .$$

Proposition 5.2. *For every $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbf{dgCat}$ there is a natural bijection*

$$\mathbf{A}_\infty\mathbf{Cat^u}/\!\approx(\mathbf{A}_1 \otimes \mathbf{A}_2, \mathbf{A}_3) \overset{1:1}{\longleftrightarrow} \mathbf{A}_\infty\mathbf{Cat^u}/\!\approx(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3).$$

Proof. We just sketch the proof, which is essentially the same as that of [5, Proposition 3.8], with a few adjustments at some points. Keeping the same notation of [5, Section 3], $\tilde{\mathsf{N}} \in \mathbf{dgCat^n}(\Omega(\mathbf{C}), \overline{\mathbf{A}_1^+ \otimes \mathbf{A}_2^+})$ (where $\mathbf{C} := \overline{\mathrm{B}(\mathbf{A}_1)^+ \otimes \mathrm{B}(\mathbf{A}_2)^+}$) is actually a homotopy isomorphism by Corollary 2.7. Then, as $\overline{\mathbf{A}_1^+ \otimes \mathbf{A}_2^+} \in \mathbf{dgCat}$ when $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{dgCat}$ (see the proof of [5, Lemma 3.7]), by Lemma 1.22 $\tilde{\mathsf{N}}$ is unital and there exists $\mathsf{H} \in \mathbf{A}_\infty\mathbf{Cat^u}(\overline{\mathbf{A}_1^+ \otimes \mathbf{A}_2^+}, \Omega(\mathbf{C}))$ such that $\mathsf{H} \circ \tilde{\mathsf{N}} \approx \mathrm{id}_{\Omega(\mathbf{C})}$. No further change is needed in the rest of the proof, except that in the end we use Lemma 1.21 instead of [27, Lemma 1.6]. □

5.2. **Proof of the first part of Theorem C.** Given $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbf{dgCat}$, we want to prove that there are the following natural bijections (where $\mathbf{A}_i^{\mathrm{hp}}$ denotes a h-projective resolution of $\mathbf{A}_i$, for $i = 1, 2$)

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) & & \\
\Big\updownarrow{\scriptstyle 1:1}\ {\scriptstyle (\mathrm{A})} & & \\
\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3) & & \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)) \\
\Big\updownarrow{\scriptstyle 1:1}\ {\scriptstyle (\mathrm{B})} & & \Big\updownarrow{\scriptstyle 1:1}\ {\scriptstyle (\mathrm{F})} \\
\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3) & & \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1^{\mathrm{hp}}, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)) \\
\Big\updownarrow{\scriptstyle 1:1}\ {\scriptstyle (\mathrm{C})} & & \Big\updownarrow{\scriptstyle 1:1}\ {\scriptstyle (\mathrm{E})} \\
\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\mathbf{A}_1^{\mathrm{hp}}, \mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3) & \xleftrightarrow[(\mathrm{D})]{1:1} & \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\mathbf{A}_1^{\mathrm{hp}}, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)),
\end{array}
\tag{5.1}
$$

since this would imply the wanted natural bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2, \mathbf{A}_3) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}_1, \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}_2^{\mathrm{hp}}, \mathbf{A}_3)).$$

Now, the existence of (A), and (F) follows from the isomorphisms $\mathbf{A}_1 \otimes^{\mathbb{L}} \mathbf{A}_2 \cong \mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}$ and $\mathbf{A}_1^{\mathrm{hp}} \cong \mathbf{A}_1$ in $\mathrm{Ho}(\mathbf{dgCat})$. Taking into account that $\mathbf{A}_1^{\mathrm{hp}} \otimes \mathbf{A}_2^{\mathrm{hp}}$ is h-projective by Remark 4.8, (B) and (E) are due to Proposition 4.11. Finally, Proposition 5.2 implies (C), whereas (D) is a direct consequence of Proposition 5.1.

This clearly implies that in the symmetric monoidal category $\mathrm{Ho}(\mathbf{dgCat})$ the internal Hom $\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B})$ between two dg categories $\mathbf{A}$ and $\mathbf{B}$ is, up to isomorphism in $\mathrm{Ho}(\mathbf{dgCat})$, the dg category $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B})$.

Remark 5.3. Let us now show that our new construction of the internal Homs naturally answers Toën question about the compatibility with compositions in the introduction. Indeed, as $\mathbf{B}$ and its cofibrant replacements $\tilde{\mathbf{B}}$ are isomorphic in $\mathrm{Ho}(\mathbf{dgCat})$, the universal property of the internal Hom yields the isomorphism

$$\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B}) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \tilde{\mathbf{B}})$$

in $\mathrm{Ho}(\mathbf{dgCat})$. Finally, as we explained in the proof of Proposition 4.11, $\mathbf{A}^{\mathrm{hp}}$ and the cofibrant replacement $\tilde{\mathbf{A}}$ are isomorphic in $\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx$, thus we can simply set

$$\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B}) := \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$$

yielding a bijection

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Isom}(H^0(\mathbb{R}\underline{Hom}(\mathbf{A}, \mathbf{B}))) = \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\tilde{\mathbf{A}}, \tilde{\mathbf{B}}), \tag{5.2}$$

where the last equality is by definition.

As in the proof Proposition 4.11, the bijection (5.2) boils down to the composition of bijections

$$\mathrm{Ho}(\mathbf{dgCat})(\mathbf{A}, \mathbf{B}) \xleftrightarrow{1:1} \mathrm{Ho}(\mathbf{dgCat})(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \xleftrightarrow{1:1} \mathbf{dgCat}(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})/\asymp \xleftrightarrow{1:1} \mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}/\!\approx (\tilde{\mathbf{A}}, \tilde{\mathbf{B}}).$$

Now, the first and the last bijections are compatible with pre and post compositions by definition, while the second one is such in view of Remark 4.5.

In order to complete the proof of Theorem C, we have to show that $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B})$ and $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}^{\mathrm{hps}}, \mathbf{B})$ (where $\mathbf{A}^{\mathrm{hps}}$ denotes a h-projective resolution with split units of $\mathbf{A}$) are isomorphic in $\mathrm{Ho}(\mathbf{dgCat})$, for any dg categories $\mathbf{A}$ and $\mathbf{B}$. To this aim, note that, in general, if $\mathbf{C}$ is a h-projective dg category with split units, then the fully faithful embedding $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{C}, \mathbf{B}) \hookrightarrow \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{C}, \mathbf{B})$ is indeed a quasi-equivalence. The argument is the same as in [5, Corollary 2.6], where we replace (the erroneous) [5, Proposition 2.5] with Lemma 4.1. Thus $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hps}}, \mathbf{B}) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}^{\mathrm{hps}}, \mathbf{B})$ in $\mathrm{Ho}(\mathbf{dgCat})$. On the other hand, $\mathbf{A}^{\mathrm{hp}}$ and $\mathbf{A}^{\mathrm{hps}}$ are isomorphic in $\mathrm{Ho}(\mathbf{dgCat})$ as well. Thus the universal property of the internal Homs implies that $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hp}}, \mathbf{B}) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}^{\mathbf{u}}}(\mathbf{A}^{\mathrm{hps}}, \mathbf{B})$ in $\mathrm{Ho}(\mathbf{dgCat})$. This concludes the proof.

Remark 5.4. It is worth pointing out that Theorem C implies Kontsevich–Keller's Claim in the introduction. Indeed, if $\mathbf{A}$ is a dg category such that $\mathbf{A}(A, A')$ is cofibrant, for all $A, A' \in \mathbf{A}$, then $\mathbf{A}$ is h-projective by [14, Lemma 2.3.8]. Moreover, if $\mathbf{A}$ has the additional property that the unit map $\Bbbk \to \mathbf{A}(A, A)$ admits a retraction as a morphism of complexes, for all $A \in \mathbf{A}$, then $\mathbf{A}$ has split units. Thus $\mathbf{A} \cong \mathbf{A}^{\mathrm{hps}}$ in $\mathrm{Ho}(\mathbf{dgCat})$, whence $\mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}^{\mathrm{hps}}, \mathbf{B}) \cong \mathbf{Fun}_{\mathbf{A}_\infty\mathbf{Cat}}(\mathbf{A}, \mathbf{B})$ in $\mathrm{Ho}(\mathbf{dgCat})$.

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