EQUIVALENCES OF TWISTED K3 SURFACES

DANIEL HUYBRECHTS AND PAOLO STELLARI

Abstract. We prove that two derived equivalent twisted K3 surfaces have isomorphic periods. The converse is shown for K3 surfaces with large Picard number. It is also shown that all possible twisted derived equivalences between arbitrary twisted K3 surfaces form a subgroup of the group of all orthogonal transformations of the cohomology of a K3 surface.

The passage from twisted derived equivalences to an action on the cohomology is made possible by twisted Chern characters that will be introduced for arbitrary smooth projective varieties.

By definition a K3 surface is a compact complex surface X with trivial canonical bundle and vanishing $H^1(X,\mathcal{O}_X)$. As was shown by Kodaira in [23] all K3 surfaces are deformation equivalent. In particular, any K3 surface is diffeomorphic to the four-dimensional manifold M underlying the Fermat quartic in $\mathbb{P}^3$ defined by $x_0^4+x_1^4+x_2^4+x_3^4=0$. Thus, we may think of a K3 surface X as a complex structure I on M. (As it turns out, every complex structure on M does indeed define a K3 surface, see [14].)

In the following, we shall fix the orientation on M that is induced by a complex structure and denote by Λ the cohomology $H^2(M,\mathbb{Z})$ endowed with the intersection pairing. This is an even unimodular lattice of signature $(3,19)$ and hence isomorphic to $(-E_8)^{\oplus 2}\oplus U^{\oplus 3}$ with E_8 the unique even positive definite unimodular lattice of rank eight and U the hyperbolic plane.

We shall denote by $\widetilde{\Lambda}$ the lattice given by the full integral cohomology $H^*(M,\mathbb{Z})$ (which is concentrated in even degree) endowed with the Mukai pairing $\langle\varphi_0+\varphi_2+\varphi_4,\psi_0+\psi_2+\psi_4\rangle=\varphi_2\wedge\psi_2-\varphi_0\wedge\psi_4-\varphi_4\wedge\psi_0$. In other words, $\widetilde{\Lambda}$ is the direct sum of $(H^0\oplus H^4)(M,\mathbb{Z})$ endowed with the negative intersection pairing and Λ. Hence, $\widetilde{\Lambda}\cong\Lambda\oplus U$.

An *isomorphism* between two K3 surfaces X and X' given by two complex structures I respectively I' on M is a diffeomorphism $f\in\mathrm{Diff}(M)$ such that $I=f^*(I')$. Any such diffeomorphism f acts on the cohomology of M and, therefore, induces a lattice automorphism $f_*:\Lambda\cong\Lambda$.

Conversely, one might wonder whether any element $g\in\mathrm{O}(\Lambda)$ is of this form. This is essentially true and has been proved by Borcea in [4]. The precise statement is:

For any $g\in\mathrm{O}(\Lambda)$ there exist two K3 surfaces $X=(M,I)$ and $X'=(M,I')$ and an isomorphism $f:X\cong X'$ with $f_*=\pm g$.

(In fact, we can even prescribe the K3 surface $X=(M,I)$, but stated like this the result compares nicely with Theorem 0.1.) The proof of this fact uses the full theory of K3 surfaces, i.e. Global Torelli theorem, surjectivity of the period map, etc. Donaldson showed in [13] that the image of $\mathrm{Diff}(M) \to \mathrm{O}(\Lambda)$ is in fact the index two subgroup $\mathrm{O}_+(\Lambda)$ of orthogonal transformation preserving the orientation of the positive directions. (Note that $\mathrm{O}(\Lambda)$ is generated by $\mathrm{O}_+(\Lambda)$ and $\pm\mathrm{id}$.)

In a next step, we consider a more flexible notion of isomorphisms of K3 surfaces: One says that two K3 surfaces X and X' are *derived equivalent* if there exists a Fourier–Mukai equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$. Here, $\mathrm{D}^{\mathrm{b}}(X)$ is the bounded derived category of the abelian category $\mathbf{Coh}(X)$ of coherent sheaves on X. (Usually, derived equivalence is only considered for algebraic K3 surfaces.)

Clearly, any isomorphism between X and X' given by $f \in \mathrm{Diff}(M)$ induces a Fourier–Mukai equivalence $\Phi := Rf_*$. By results of Mukai and Orlov one knows how to associate to any Fourier–Mukai equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$ an isomorphism $\Phi_* : H^*(X, \mathbb{Z}) \cong H^*(X', \mathbb{Z})$ or, thinking of X and X' as complex structures I respectively I' on M, an orthogonal transformation $\Phi_* \in \mathrm{O}(\widetilde{\Lambda})$. For $\Phi = Rf_*$ as above, this gives back the standard action of the diffeomorphism f on $H^*(M, \mathbb{Z})$. However, for more general derived equivalences Φ the induced $\Phi_* \in \mathrm{O}(\widetilde{\Lambda})$ does not respect the decomposition $\widetilde{\Lambda} = \Lambda \oplus U$. So it makes perfect sense to generalize the above question on the cohomological action of isomorphisms between K3 surfaces to the derived setting:

Is any element in $\mathrm{O}_+(\widetilde{\Lambda})$ *of the form* Φ_* *for some derived equivalence* Φ*? Do elements of the form* Φ_* *form a subgroup of* $\mathrm{O}_+(\widetilde{\Lambda})$*?*

As will be shown in Section 6, the answer to both questions is negative. So generalizing isomorphisms of K3 surfaces to derived equivalence seems not very natural from the cohomological point of view. In fact, one has to go one step further in order to get a nice cohomological behaviour.

Instead of allowing only equivalences $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$ one considers more generally Fourier–Mukai equivalences $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$ of twisted derived categories. Here, $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ is the bounded derived category of α-twisted coherent sheaves on X, where α is a Brauer class, i.e. a torsion class in $H^2(X, \mathcal{O}_X^*)$. (More details are recalled in Section 1.)

By means of twisted Chern characters and twisted Mukai vector, that will be introduced in full generality for arbitrary smooth projective varieties in Section 1, we will show how to associate to any twisted Fourier–Mukai equivalence Φ an isometry $\Phi_* \in \mathrm{O}(\widetilde{\Lambda})$ (see also Theorem 0.4, i)). In fact, Φ_* depends on the additional choices of B-field lifts of α and α'. This will be spelled out in detail in Sections 1 and 2, but for the relation between B-fields and Brauer classes see the discussion further below.

In the twisted context, the above question has an affirmative answer (see Proposition 6.6):

Theorem 0.1. *For any $g \in \mathrm{O}_+(\widetilde{\Lambda})$ there exists a twisted Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$ of algebraic K3 surfaces such that $\Phi_* = g$.*

Note that in contrast to the above result of Donaldson, the K3 surface X has to be chosen carefully here. The choice of B-field lifts of α and α' are tacitly assumed. A B-field is by definition a class in $H^2(X,\mathbb{Q})$. As it will be shown, it suffices actually to consider derived equivalences between untwisted derived categories, but with a non-trivial B-field B' turned on. So, studying twisted derived categories teaches us that even in the untwisted situation a non-trivial lift, e.g. $0 \neq B \in H^2(X,\mathbb{Z})$, might be necessary to obtain a clean picture.

Note that at least conjecturally, $\Phi_* \in \mathrm{O}_+(\widetilde{\Lambda})$ for any Fourier–Mukai equivalence Φ.

It will turn out useful to look at these purely algebraic questions from a more differential geometric angle. For this, one uses the generalization of the notion of K3 surfaces, i.e. complex structures on M, provided by generalized Calabi–Yau structures on M. The relation between twisted derived equivalence introduced above and generalized Calabi–Yau structures works perfectly well on the level of cohomology and is used to formulate our results on twisted derived equivalences. A deeper understanding of the interplay between these quite different mathematical structures needs still to be developed.

A generalized Calabi–Yau structure on M consists of an even closed complex form φ such that $\varphi_2^2 - 2\varphi_0 \wedge \varphi_4$ is a zero four-form and $\varphi_2 \wedge \bar{\varphi}_2 - \varphi_0 \wedge \bar{\varphi}_4 - \bar{\varphi}_0 \wedge \varphi_4$ is a volume form. This notion was introduced by Hitchin in [16], further discussed in Gualtieri's theses [15] and in the case of K3 surfaces in [20].

There are two types of generalized Calabi–Yau structures on M: Either $\varphi = \exp(B)\cdot\sigma = \sigma + B\wedge\sigma$, where σ is a holomorphic volume form on a K3 surface $X = (M,I)$, or $\varphi = \exp(B + i\omega)$ with ω a symplectic structure on M. In both cases B is a real closed two-form.

To any generalized Calabi–Yau structure φ one naturally associates a weight two-Hodge structure on $H^*(M,\mathbb{Z})$ be declaring $[\varphi]\mathbb{C}$ to be the $(2,0)$-part of it. The $(1,1)$-part is then given as the orthogonal complement of it with respect to the Mukai pairing. If $\varphi = \exp(B)\cdot\sigma$ with σ a holomorphic volume form on $X = (M,I)$ then we write $\widetilde{H}(X,B,\mathbb{Z})$ for this Hodge structure. For details see [20] or Section 2.

Twisted derived categories and generalized Calabi–Yau structures of the form $\exp(B)\sigma$ are related by the following construction: If $X = (M,I)$ is a K3 surface, then the cohomology class of B has its $(0,2)$-part in $H^2(X,\mathcal{O}_X)$. Via the exponential map $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ this yields an element $\alpha_B \in H^2(X,\mathcal{O}_X^*)$. If σ is a holomorphic volume form on X, then α_B only depends on the cohomology class of $B\wedge\sigma$. In other words, α_B is naturally associated to the cohomology class of $\varphi = \exp(B) \cdot \sigma$. Conversely, as $H^3(M, \mathbb{Z}) = 0$, any class $\alpha \in H^2(X, \mathcal{O}_X^*)$ is of the form α_B for some B. Moreover, α is a torsion class if and only if $\alpha = \alpha_B$ for some $B \in H^2(M, \mathbb{Q})$.

There are two problems naturally arising in this discussion:

I) *Let $X = (M, I)$ be an algebraic K3 surface with a Brauer class α. Describe the image of the following three homomorphisms*

i) $\mathrm{Aut}(X) \to \mathrm{O}(\Lambda)$, ii) $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\widetilde{\Lambda})$ and iii) $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha)) \to \mathrm{O}(\widetilde{\Lambda})$.

As a consequence of the Global Torelli theorem, one can describe the image of $\mathrm{Aut}(X) \to \mathrm{O}(\Lambda)$ as a certain subgroup of the group of Hodge isometries of the Hodge structure $H^2(X, \mathbb{Z})$. A complete answer to ii) is not yet known, but using results of Mukai and Orlov it was observed in [17, 31] that the image of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\widetilde{\Lambda})$ is a subgroup of index at most two inside the subgroup of all Hodge isometries of the lattice $\widetilde{H}(X, \mathbb{Z})$. In fact, it can be shown that the image contains the subgroup of all Hodge isometries that preserve the natural orientation of the four positive directions (see Section 5 for more details). Already in [33] Szendrői argues that every derived equivalence should preserve the orientation of the positive directions. In particular, one indeed expects that the image of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\widetilde{\Lambda})$ is the group of all orientation preserving Hodge isometries.

The last problem iii) is related to Căldăraru's conjecture, see Remark 0.3.

II) *Let $X = (M, I)$ and $X' = (M, I')$ be two algebraic K3 surfaces endowed with Brauer classes α respectively α'. Find cohomological criteria which determine when*

i) $X \cong X'$, ii) $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$, *or* iii) $\mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$.

The answer to i) is provided by the Global Torelli theorem: $X \cong X'$ if and only if there exists a Hodge isometry $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$ (see [1]). (But not any Hodge isometry can be lifted to an isomorphism.)

The derived version of it yields an answer to ii): Let X and X' be two algebraic K3 surfaces. Then $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$ if and only if $\widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(X', \mathbb{Z})$. This result is due to Orlov [29] and relies on techniques introduced by Mukai [25].

The answer to iii) is supposed to be provided by the following conjecture formulated, though in a slightly different form, by Căldăraru in his thesis [7].

Conjecture 0.2. *Let X and X' be two algebraic K3 surfaces with rational B-fields B respectively B' inducing Brauer classes α respectively α'. Then there exists a Fourier–Mukai equivalence* $\mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$ *if and only if there exists a Hodge isometry $\widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$ that respects the natural orientation of the four positive directions.*

Remark 0.3. A slightly refined version of this conjectures predicts that actually any Hodge isometry in O_+ can be lifted to a twisted derived equivalence. This would in particular answer I, iii).

Căldăraru actually conjectured that $\mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$ if and only if the transcendental lattices of the twisted Hodge structures are Hodge isometric. In fact, in the untwisted situation it is easy to see that any Hodge isometry $T(X) \cong T(X')$ of the transcendental lattices lifts to a Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(X',\mathbb{Z})$. This does not hold any longer in the twisted case. Thus, as will be explained in detail in Section 4, one has to modify the original conjecture of Căldăraru's and use the full twisted Hodge structure $\widetilde{H}(X,B,\mathbb{Z})$ instead of just its transcendental lattice.

Using the twisted Chern character introduced in Section 1 one can at least prove parts of this conjecture (cf. Propositions 4.3 and 7.7).

Theorem 0.4. *Let $X, B, \alpha, X', B', \alpha'$ as before.*

i) *If $\Phi : \mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$ is a Fourier–Mukai equivalence, then there exists a naturally defined Hodge isometry $\Phi_*^{B,B'} : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z})$.*

ii) *If the Picard number $\rho(X)$ satisfies $\rho(X) \geq 12$, then for any orientation preserving Hodge isometry $g : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z})$ there exists a Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha_{B'})$ such that $\Phi_* = g$.*

The first part will be shown (cf. Corollary 4.6) to imply

Corollary 0.5. *Any twisted algebraic K3 surface (X,α) admits only finitely many Fourier–Mukai partners, i.e. there exists only a finite number of isomorphism classes of twisted K3 surfaces (X',α') such that one can find a Fourier–Mukai equivalence $\mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$.*

The untwisted case of this corollary had been proved in [5].

Remark 0.6. Orlov has proved that in fact any equivalence between untwisted derived categories is of Fourier–Mukai type. So in the untwisted case one could just consider equivalences of derived categories. An analogous result is expected (and in fact proved in Section 7 for large Picard number) also in the twisted situation, but for the time being we have to restrict to the geometrically relevant case of Fourier–Mukai equivalences.

Here is an outline of the paper. In Section 1 we introduce twisted Chern characters on arbitrary smooth projective varieties as an additive map from the K-group of twisted coherent sheaves to rational cohomology and prove a few basic facts about them. In particular, we will see that the standard Hodge conjecture in the twisted context is equivalent to the standard Hodge conjecture.

Section 2 explains the relation between generalized Calabi–Yau structures and twisted K3 surfaces, i.e. K3 surfaces endowed with a Brauer class.

In Section 3 we will study the Brauer group of a K3 surface by introducing three equivalence relations on it. The main result asserts the finiteness of each equivalence class modulo the action of the group of automorphisms.

The general framework associating a natural map on the cohomology to a twisted Fourier–Mukai transform is explained in Section 4. For K3 surfaces we will find that a Fourier–Mukai equivalence yields a Hodge isometry between Hodge structures that are defined by means of related generalized Calabi–Yau structures. This will in particular lead to the finiteness result for twisted Fourier–Mukai partners. The section also contains a detailed discussion of Căldăraru's conjecture and explains why and how it has to be modified.

Section 6 contains the proof of Theorem 0.1. This part is based on purely lattice theoretical considerations and a proof of Căldăraru's conjecture for large Picard number given in Section 7.

The last section shows that despite the finiteness of Fourier–Mukai partners, one always finds arbitrarily many twisted Fourier–Mukai partners and, unlike the untwisted case, the Picard group may even be chosen large. The examples in this section illustrate the difference between the twisted and the untwisted world alluded to in the preceding sections.

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Note added in Proof: Using results of Yoshioka [35] on the existence and non-triviality of moduli spaces of twisted sheaves on K3 surfaces we can now prove Căldăraru's conjecture, see [21].

1. Twisted Chern characters

In the following, we let X be a smooth projective variety over $\mathbb{C}$ and $\alpha \in H^2(X, \mathcal{O}_X^*)$ be a torsion class, i.e. an element in the Brauer group $\mathrm{Br}(X)$. The pair (X, α) will sometimes be called a twisted variety (see Definition 2.1). We may represent α by a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$ with $X = \bigcup_{i\in I} U_i$ an appropriate open analytic cover. An *α-twisted (coherent) sheaf* E consists of pairs $(\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ such that the E_i are (coherent) sheaves on U_i and $\varphi_{ij} : E_j|_{U_i\cap U_j} \to E_i|_{U_i\cap U_j}$ are isomorphisms satisfying the following conditions:

i) $\varphi_{ii} = \mathrm{id}$,
ii) $\varphi_{ji} = \varphi_{ij}^{-1}$, and
iii) $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

Definition 1.1. *By* $\mathbf{Coh}(X, \alpha)$ *we denote the abelian category of α-twisted coherent sheaves. Its K-group is denoted $K(X, \alpha)$.*

A priori, the above definition of $\mathbf{Coh}(X, \alpha)$ depends on the chosen Čech representative of α. However, it is not difficult to see that for two different choices the two abelian categories are equivalent (see [7]). Note however that the equivalence depends on the additional choice of $\{\beta_{ij} \in \mathcal{O}^*(U_i \cap U_j)\}$ satisfying $\alpha'_{ijk} \cdot \alpha_{ijk}^{-1} = \beta_{ij} \cdot \beta_{jk} \cdot \beta_{ki}$, where $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ are two Čech cocycles representing the same Brauer class. Let us also introduce the notation $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ for the bounded derived category of $\mathbf{Coh}(X, \alpha)$, although we won't say anything about it in this section.

As the tensor product $\mathcal{F} \otimes \mathcal{E}$ of an α-twisted sheaf $\mathcal{F}$ with a β-twisted sheaf $\mathcal{E}$ is an $\alpha \cdot \beta$-twisted sheaf, the abelian category $\mathbf{Coh}(X, \alpha)$ has no natural tensor structure (except when α is trivial). Hence, its K-group $K(X, \alpha)$ is indeed just an additive group and not a ring.

The aim of this section is to construct a twisted Chern character

$$K(X, \alpha) \longrightarrow H^*(X, \mathbb{Q})$$

with a number of specific properties which are all twisted versions of the standard results on Chern characters. As it will turn out, however, the definition depends on the additional choice of a B-field, i.e. a cohomology class in $H^2(X, \mathbb{Q})$, and only works for topologically trivial Brauer classes, i.e. for those $\alpha \in H^2(X, \mathcal{O}_X^*)$ with trivial boundary in $H^3(X, \mathbb{Z})$ under the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0.$$

Thus, our construction works for arbitrary Brauer classes on smooth projective varieties X with $H^3(X, \mathbb{Z})_{\mathrm{tor}} = 0$, e.g. for K3 surfaces.

Proposition 1.2. *Suppose $B \in H^2(X, \mathbb{Q})$ is a rational B-field such that its $(0,2)$-part $B^{0,2} \in H^2(X, \mathcal{O}_X)$ maps to α, i.e. $\exp(B^{0,2}) = \alpha$.*

Then there exists a map

$$\mathrm{ch}^B : K(X, \alpha) \longrightarrow H^*(X, \mathbb{Q})$$

such that:

i) ch^B *is additive, i.e.* $\mathrm{ch}^B(E_1 \oplus E_2) = \mathrm{ch}^B(E_1) + \mathrm{ch}^B(E_2)$.

ii) *If* $B = \mathrm{c}_1(L) \in H^2(X, \mathbb{Z})$, *then* $\mathrm{ch}^B(E) = \exp(\mathrm{c}_1(L)) \cdot \mathrm{ch}(E)$. *(Note that with this assumption α is trivial and an α-twisted sheaf is just an ordinary sheaf.)*

iii) *For two choices* $(B_1, \alpha_1 := \exp(B_1^{0,2}))$, $(B_2, \alpha_2 := \exp(B_2^{0,2}))$ *and* $E_i \in K(X, \alpha_i)$ *one has*

$$\mathrm{ch}^{B_1}(E_1) \cdot \mathrm{ch}^{B_2}(E_2) = \mathrm{ch}^{B_1 + B_2}(E_1 \otimes E_2).$$

iv) *For any* $E \in K(X, \alpha)$ *one has* $\mathrm{ch}^B(E) \in \exp(B) \left(\bigoplus H^{p,p}(X)\right)$.

Proof. For the basic facts on twisted sheaves we refer to [7].

Any coherent α-twisted sheaf admits a finite resolution by locally free sheaves. (Here, one definitely needs α be torsion. For the argument see [7, Lemma 2.1.4].) Hence, $K(X, \alpha)$ can also be regarded as the Grothendieck group of locally free α-twisted coherent sheaves. Therefore, it suffices to define ch^B for locally free α-twisted sheaves in a way that it becomes additive for short exact sequences.

Let us fix a Čech representative $\{\alpha_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}^*)\}$ of $\alpha \in H^2(X, \mathcal{O}^*)$.

In fact, in our situation we may start with a Čech representative $B_{ijk} \in \Gamma(U_{ijk}, \mathbb{Q})$ of the B-field B and use the cocycle given by $\alpha_{ijk} := \exp(B_{ijk})$ regarded as local sections of $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1) \subset \mathcal{O}^*$ that represents α.

Since $\mathcal{C}^\infty$ is acyclic, there exist functions $a_{ij} \in \Gamma(U_{ij}, \mathcal{C}^\infty)$ with $-a_{ij} + a_{ik} - a_{jk} = B_{ijk}$. (We assume that the cover is sufficiently fine.)

Let now E be an α-twisted sheaf (it does not need to be coherent) given by $\{E_i, \varphi_{ij}\}$. Then consider $\{E_i, \varphi'_{ij} := \varphi_{ij} \cdot \exp(a_{ij})\}$. It is easy to check that $\{\varphi'_{ij}\}$ is an honest cocycle, i.e. $\varphi'_{ij} \circ \varphi'_{jk} \circ \varphi'_{ki} = \mathrm{id}$. Hence, we have constructed a(n untwisted) sheaf $E_B := \{E_i, \varphi'_{ij}\}$.

Then define

$$\mathrm{ch}^B(E) := \mathrm{ch}(E_B).$$

We have to check that with this definition $\mathrm{ch}^B(E)$ does not depend on any of the choices. This is straightforward and left to the reader.

i) is also an immediate consequence of the construction and ii) follows from $E_B = E \otimes L$ under the assumptions. For iii) one observes that $E_{1B_1} \otimes E_{2B_2} \cong (E_1 \otimes E_2)_{B_1 + B_2}$.

Suppose $B_0 := k \cdot B \in H^2(X, \mathbb{Z})$ for some non-trivial $k \in \mathbb{Z}$. Due to ii) and iii) one has for any $E \in K(X, \alpha)$

$$\mathrm{ch}^B(E)^k = \mathrm{ch}^{B_0}(E^{\otimes k}) = \exp(B_0) \cdot \mathrm{ch}(E^{\otimes k}).$$

Hence,

$$\begin{aligned} \left(\exp(-B) \cdot \mathrm{ch}^B(E)\right)^k &= \exp(-B_0) \cdot \mathrm{ch}^B(E)^k \\ &= \mathrm{ch}(E^{\otimes k}) \in \bigoplus H^{p,p}(X), \end{aligned}$$

for $E^{\otimes k}$ is an algebraic vector bundle and has thus Chern classes of pure type.

The assertion iv) now follows from the following easy observation:

If $(v_0, v_1, \ldots, v_n) \in H^0 \oplus H^2 \oplus \ldots \oplus H^{2n}$ with $v_0 \neq 0$ and such that $v^k \in \bigoplus H^{p,p}$, then also $v \in \bigoplus H^{p,p}$, i.e. $v_p \in H^{p,p}$ for all p.

Suppose we have shown that $v_i \in H^{i,i}$ for all $i < j$. Write $(v^k)_j = k \cdot (v_j v_0) + P(v_0, \ldots, v_{j-1})$ with P a certain polynomial. By assumption $(v^k)_j$ is of pure type and by induction hypotheses the same holds for $P(v_0, \ldots, v_{j-1})$. Since $v_0 \neq 0$, this yields $v_j \in H^{j,j}$. □

Remark 1.3. (Mixed Hodge structures by twisting)

i) The cohomology $H^*(X, \mathbb{Z})$ can be seen as a direct sum of Hodge structures (over $\mathbb{Z}$) and, therefore, as a mixed Hodge structure with ascending filtration W_i given by $W_i = \bigoplus_{j \leq i} H^j(X, \mathbb{Q})$. For any $B \in H^2(X, \mathbb{Q})$ the isomorphism

$$\exp(B) : H^*(X, \mathbb{Q}) \cong H^*(X, \mathbb{Q})$$

induces a natural mixed Hodge structure given by

$$W_i := \exp(B)\left(\bigoplus_{j\leq i} H^j(X,\mathbb{Q})\right).$$

We denote the cohomology endowed with this mixed Hodge structure by $H^*(X,B,\mathbb{Z})$. Clearly, the induced rational(!) mixed Hodge structure is by definition isomorphic to the standard one, but as soon as B is not integral the decomposition $W_{i+1} = W_i \oplus \exp(B)H^{i+1}(X,\mathbb{Q})$ is not defined over $\mathbb{Z}$. Also note that the definition depends on B and not just on α, e.g. for $B \in H^{1,1}(X,\mathbb{Q})$, which induces the trivial Brauer class, the mixed Hodge structure will nevertheless be non-split in general.

ii) In the sequel we shall denote by $H^{*,*}(X,B,\mathbb{Z})$ (respectively $H^{*,*}(X,B,\mathbb{Q})$) the integral (resp. rational) part of $\exp(B)\,(\bigoplus H^{*,*}(X))$. More precisely, $H^{p,p}(X,B,\mathbb{Z}) = \exp(B)(H^{p,p}(X,\mathbb{Q})) \cap H^*(X,\mathbb{Z})$. Proposition 1.2, iv) then says that

$$\mathrm{Im}\left(\mathrm{ch}^B : K(X,\alpha) \longrightarrow H^*(X,\mathbb{Q})\right) \subset H^{*,*}(X,B,\mathbb{Q}) .$$

iii) Note that the natural product defines homomorphisms of mixed Hodge structures

$$H^*(X,B_1,\mathbb{Z}) \otimes H^*(X,B_2,\mathbb{Z}) \longrightarrow H^*(X,B_1+B_2,\mathbb{Z}) .$$

All mixed Hodge structures together yield a mixed Hodge structure on $H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^2(X,\mathbb{Q})$ that is endowed with an inner product.

It seems plausible to generalize the standard Hodge conjecture to the question whether the map $\mathrm{ch}^B : K(X,\alpha)\otimes\mathbb{Q} \to H^{*,*}(X,B,\mathbb{Q})$ is surjective. As it turns out, the twisted Hodge conjecture is actually equivalent to the untwisted one. This is the following

Proposition 1.4. *Let X be a smooth complex projective variety and $B \in H^2(X,\mathbb{Q})$ a rational B-field inducing the Brauer class α. Then the following two assertions are equivalent:*

i) *The map* $\mathrm{ch} : K(X)_{\mathbb{Q}} \to \bigoplus H^{p,p}(X,\mathbb{Q})$ *is surjective.*

ii) *The map* $\mathrm{ch}^B : K(X,\alpha)_{\mathbb{Q}} \to \bigoplus H^{p,p}(X,B,\mathbb{Q})$ *is surjective.*

Proof. Let us denote the image of ch and ch^B by A respectively A_B. If $E \in K(X,\alpha)$ and $F \in K(X)$, then $E\otimes F \in K(X,\alpha)$ and $\mathrm{ch}^B(E\otimes F) = \mathrm{ch}^B(E)\cdot\mathrm{ch}(F)$. Hence, A_B is invariant under multiplication with A.

Assume that $A = \bigoplus H^{p,p}(X,\mathbb{Q})$, then $A_B \subset \exp(B)(\bigoplus H^{p,p}(X,\mathbb{Q}))$ is invariant under multiplication with $\bigoplus H^{p,p}(X,\mathbb{Q})$. As $\mathbf{Coh}(X,\alpha)$ contains a locally free sheaf of finite rank, there exists an element $c \in A_B$ of the form $c = 1 +$ higher order terms. From these two statements one easily deduces that $\mathrm{ch}^B : K(X,\alpha)_{\mathbb{Q}} \to \bigoplus H^{p,p}(X,B,\mathbb{Q})$ is surjective, i.e. $A_B = \bigoplus H^{p,p}(X,B,\mathbb{Q})$.

Let us conversely assume that $A_B = \bigoplus H^{p,p}(X,B,\mathbb{Q})$. The idea of the proof is to use the commutativity of the following diagram

$$\begin{tikzcd} K(X,\alpha)_{\mathbb{Q}} \arrow[r, two heads, "\mathrm{ch}^B"] \arrow[d, "(\ )^k"'] & \bigoplus H^{p,p}(X,B,\mathbb{Q}) \arrow[d, "(\ )^k"] \\ K(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}^{B_0}"] & \bigoplus H^{p,p}(X,B_0,\mathbb{Q}) \end{tikzcd}$$

where $B_0 := k \cdot B \in H^2(X,\mathbb{Z})$ for some $k \in \mathbb{Z}$. As the image of the vertical map on the right hand side spans, this suffices to conclude that ch^{B_0} is surjective as well.

More explicitly, consider $\beta \in H^{p,p}(X,\mathbb{Q})$. Then $\exp(B)\cdot(1+\beta) \in A_B$ and, therefore, (up to a scaling factor) $\exp(B)\cdot(1+\beta) = \mathrm{ch}^B(E) - \mathrm{ch}^B(F)$ for certain α-twisted vector bundles E and F. Passing to the k-th power yields $\exp(B_0)\cdot(1+\beta)^k = \sum(-1)^i\binom{k}{i}\mathrm{ch}^{B_0}(E^{\otimes i}\otimes F^{\otimes k-i}) = \exp(B_0)\cdot\sum(-1)^i\binom{k}{i}\mathrm{ch}(E^{\otimes i}\otimes F^{\otimes k-i})$, because $E^{\otimes i}\otimes F^{\otimes k-i}$ is an untwisted vector bundle. Thus, $(1+\beta)^k \in A$ and hence $\beta \in A$. $\square$

Remark 1.5. i) There is a simple way to construct a Chern character for twisted sheaves which only depends on the B-field as an element in $H^2(X,\mathbb{Q})/H^2(X,\mathbb{Z})$. Indeed, one may consider $\exp(-B)\mathrm{ch}^B(\ )$. Changing B by an integral B-field does not affect this expression. However, the above approach has the advantage that the denominators that occur in $\mathrm{ch}^B \in H^*(X,\mathbb{Q})$ are universal. This fact will be important for the construction of certain Hodge isometries over $\mathbb{Z}$ (and not only over $\mathbb{Q}$), in Section 4.

ii) Twisted Chern characters are alluded to at different places in the literature, but we couldn't find the above explicit construction. However, Eyal Markman informed us that Jun Li has developed a theory of of connections on twisted holomorphic bundles and a twisted analogue of the Hermite–Einstein equation.

2. Generalized CY structures versus twisted K3 surfaces

In this section we shall compare the notions of twisted K3 surfaces and generalized Calabi–Yau structures on M.

Definition 2.1. *A* twisted K3 surface (X,α) *consists of a K3 surface X together with a Brauer class $\alpha \in \mathrm{Br}(X)$. We say that $(X,\alpha)\cong(X',\alpha')$ if there exists an isomorphism $f : X \cong X'$ with $f^*\alpha' = \alpha$.*

Definition 2.2. *A* generalized Calabi–Yau *structure on M is an even closed complex form $\varphi = \varphi_0+\varphi_2+\varphi_4$ such that $\langle\varphi,\varphi\rangle = 0$ and $\langle\varphi,\overline{\varphi}\rangle > 0$.*

Here, $\langle\ ,\ \rangle$ is the Mukai pairing on the level of forms and the positivity of $\langle\varphi,\overline{\varphi}\rangle$ is meant with respect to a fixed volume form. For details see [20]. In this paper we are only interested in generalized Calabi–Yau structures of the form $\varphi = \exp(B)\cdot\sigma = \sigma + B\wedge\sigma$, where σ is a holomorphic two-form on a K3 surface $X = (M, I)$ and B a real closed two-form. If B can be chosen such that $[B]$ is a rational class, then φ is called a *rational generalized Calabi–Yau structure*. In the sequel, we often just work with the cohomology classes of φ and B, which for simplicity will be denoted by the same symbols.

Suppose $X = (M, I)$ is a given K3 surface. To any rational B-field $B \in H^2(M, \mathbb{Q})$ one can associate the twisted K3 surface (X, α_B) (for the definition of α_B see the introduction) and a generalized Calabi–Yau structure $\varphi = \exp(B)\cdot\sigma$.

In fact, (X, α_B) depends only on the cohomology class of φ and we therefore get a natural map

$$\Big\{\text{rational gen. CYs } [\exp(B)\cdot\sigma]\Big\} \longrightarrow \Big\{\text{twisted K3s } (X,\alpha)\Big\},$$

which is surjective due to $H^3(X, \mathbb{Z}) = 0$.

Definition 2.3. *Let X be a K3 surface with a (rational) B-field $B \in H^2(X, \mathbb{Q})$. Then we denote by $\widetilde{H}(X, B, \mathbb{Z})$ the weight-two Hodge structure on $H^*(X, \mathbb{Z})$ with*

$$\widetilde{H}^{2,0}(X, B) := \exp(B)\left(H^{2,0}(X)\right)$$

and $\widetilde{H}^{1,1}(X, B)$ its orthogonal (with respect to the Mukai pairing) complement.

Clearly, $\widetilde{H}(X, B, \mathbb{Z})$ only depends on the generalized Calabi–Yau structure $\varphi = \sigma + B\wedge\sigma$. In other words, B and B' define the same weight-two Hodge structure on $H^*(M, \mathbb{Z})$ if and only if $B^{0,2} = B'^{0,2} \in H^2(X, \mathcal{O}_X)$. If B and B' differ by an integral class $B_0 \in H^2(M, \mathbb{Z})$ then

$$\exp(B_0) : \widetilde{H}(X, B, \mathbb{Z}) \longrightarrow \widetilde{H}(X, B', \mathbb{Z})$$

is a Hodge isometry. This yields the diagram

$$\begin{tikzcd}
\Big\{\text{rational gen. CYs } [\exp(B)\cdot\sigma]\Big\} \arrow[r] \arrow[d] & \Big\{\text{twisted K3s } (X,\alpha)\Big\} \arrow[d] \\
\Big\{\text{weight} - 2 \text{ HS on } H^*(M,\mathbb{Z})\Big\} \arrow[r] & \Big\{\text{weight} - 2 \text{ HS on } H^*(M,\mathbb{Z})\Big\}/_{\cong}
\end{tikzcd}$$

This new Hodge structure comes along with a natural orientation of its positive directions. We shall briefly explain what this means. If X is a K3 surface with σ a generator of $H^{2,0}(X)$ and ω a Kähler class (e.g. an ample class if X is algebraic), then $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1 - \omega^2/2, \omega\rangle$ is a positive four-space in $\widetilde{H}(X, \mathbb{R})$ which comes, by the choice of the basis, with a natural orientation. It is easy to see that this orientation is independent of the choice of σ and ω. Let $g : \Gamma \to \Gamma'$ be an isometry of lattices with signature $(4, t)$. Suppose positive four-spaces $V \subset \Gamma_{\mathbb{R}}$ and $V' \subset \Gamma'_{\mathbb{R}}$ and orientations for both of them have been chosen. Then one says that g preserves the given orientation of the positive directions (or, simply, that g is orientation preserving) if the composition of $g_{\mathbb{R}} : V \to \Gamma'_{\mathbb{R}}$ and the orthogonal projection $\Gamma'_{\mathbb{R}} \to V'$ is compatible with the given orientations of V and V'. By $\mathrm{O}_+(\Gamma)$ one denotes the group of all orientation preserving orthogonal transformations. We shall also use the analogous notation $\mathrm{O}_+(U)$ for the hyperbolic plane $U = H^0 \oplus H^4$.

In [20] we explained how to associate to any generalized Calabi–Yau structure $\varphi \in H^*(M, \mathbb{C})$ the *generalized (or twisted) Picard group* and the *generalized (or twisted) transcendental lattice*. Let us discuss these two lattices a bit further in the case of a generalized Calabi–Yau structure $\varphi = \exp(B) \cdot \sigma$ with σ the holomorphic two-form on the K3 surface $X = (M, I)$. By definition the generalized Picard group is

$$\mathrm{Pic}(X, \varphi) := \{\beta \in H^*(M, \mathbb{Z}) \mid \langle \beta, \varphi \rangle = 0\}$$

and the *generalized transcendental lattice*

$$T(X, \varphi) := \mathrm{Pic}(X, \varphi)^{\perp} \subset H^*(M, \mathbb{Z}),$$

where the orthogonal complement is taken with respect to the Mukai pairing. The latter comes with a natural weight-two Hodge structure which will always be understood. Note that both lattices depend only on the Hodge structure $\widetilde{H}(X, B, \mathbb{Z})$ and, therefore, their isomorphism classes only on the induced Brauer class α_B. Later we often write $\mathrm{Pic}(X, B)$ and $T(X, B)$ or, if only the isomorphism type is relevant, $\mathrm{Pic}(X, \alpha_B)$ and $T(X, \alpha_B)$.

By definition, $\mathrm{Pic}(X, \varphi)$ is the set of integral classes $(\delta_0, \delta_2, \delta_4)$ with $\delta_0 \int \sigma \wedge B = \int \sigma \wedge \delta_2$. In particular, no condition on δ_4 and, hence, $\mathrm{Pic}(X) \oplus H^4(X, \mathbb{Z}) \subset \mathrm{Pic}(X, \varphi)$. One also has the following description of the generalized Picard group

$$\mathrm{Pic}(X, B) = \mathrm{Pic}(X, \varphi) = H^{*,*}(X, B, \mathbb{Z}) = \widetilde{H}^{1,1}(X, B, \mathbb{Z}).$$

Indeed, if one writes $(\delta_0, \delta_2, \delta_4) = \exp(B)(\alpha_0, \alpha_2, \alpha_4)$, one finds that $\alpha_2 \in H^{1,1}(X)$, i.e. $\int \sigma \wedge \alpha_2 = 0$, is equivalent to $\delta_0 \int \sigma \wedge B = \int \sigma \wedge \delta_2$. Hence, $\mathrm{Pic}(X, \varphi) = H^{*,*}(X, B, \mathbb{Z})$.

The third equality follows from the fact that $\exp(B)$ is an orthogonal transformation with respect to the Mukai pairing, i.e. $\langle \exp(B)(\ ), \exp(B)(\ ) \rangle = \langle\ ,\ \rangle$.

The next result can be seen as a consequence of Proposition 1.4.

Corollary 2.4. *Let X be a K3 surface and $\alpha := \exp(B^{0,2}) \in H^2(X, \mathcal{O}_X^*)$ a given Brauer class induced by a rational B-field B. Then the twisted Chern character*

$$\mathrm{ch}^B : K(X, \alpha) \twoheadrightarrow \mathrm{Pic}(X, B)$$

is surjective. □

It is easy to see that there exists a finite index immersion

$$\mathrm{Pic}(X) \oplus (\lambda_B u_2 + \lambda_B B)\mathbb{Z} \oplus H^4(X, \mathbb{Z}) \hookrightarrow \mathrm{Pic}(X, B)$$

for a certain integer λ_B such that $\lambda_B B \in H^2(X, \mathbb{Z})$. In particular, $\mathrm{Pic}(X, B)$ and $\mathrm{Pic}(X) \oplus U$ are always of the same rank. Moreover, X is algebraic if and only if $\mathrm{Pic}(X, B)$ contains two positive directions.

We also need to clarify the relation between $T(X, B)$ and the transcendental lattice $T(X)$. In [20] we have shown that $\exp(-B)$ defines a Hodge isometry $T(X, B) \cong T(X, \alpha_B)$, where $T(X, \alpha_B)$ is the kernel of the natural map $(B, \ ) : T(X) \to \mathbb{Q}/\mathbb{Z}$ defined by the intersection product with B (it only depends on α_B). Once more, $T(X, B)$ and $T(X)$ are therefore lattices of the same rank.

3. Some remarks on Brauer groups of K3 surfaces

By definition, the Brauer group $\mathrm{Br}(X)$ of X is the set of torsion classes in $H^2(X, \mathcal{O}_X^*)$. If the order of a class $\alpha \in \mathrm{Br}(X)$ divides k then there exists a B-field lift B of α, i.e. a class $B \in H^2(X, \mathbb{Q})$ with $\alpha_B = \alpha$, such that $kB \in H^2(X, \mathbb{Z})$. Using the intersection product with B yields a linear map $(B, \ ) : T(X) \to \mathbb{Q}$ with image contained in $\frac{1}{k}\mathbb{Z}$ and, by dividing by $\mathbb{Z} \subset \frac{1}{k}\mathbb{Z}$, a linear map $\alpha : T(X) \to \frac{1}{k}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/k\mathbb{Z}$. It is straightforward to check that this construction yields an isomorphism

$$\mathrm{Br}_{k-\mathrm{tor}}(X) \cong \mathrm{Hom}(T(X), \mathbb{Z}/k\mathbb{Z}).$$

Moreover, if the order of α equals k then the induced map $\alpha : T(X) \to \mathbb{Z}/k\mathbb{Z}$ is surjective.

If one is not interested in any specific torsion, the construction yields an alternative description of the Brauer group as $\mathrm{Br}(X) = \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$.

Remark 3.1. In the last paragraph, we used the notation $T(X, \alpha)$ for the kernel of the map $T(X) \to \mathbb{Q}/\mathbb{Z}$. Thus, the order of a Brauer class α can be computed by using the standard formula

$$|\mathrm{disc}(T(X, \alpha))| = |\mathrm{disc}(T(X))| \cdot |\alpha|^2.$$

Moreover, the existence of the aforementioned Hodge isometry $\exp(-B) : T(X, B) \cong T(X, \alpha)$ proves that the order of a Brauer class is encoded by the twisted transcendental lattice $T(X, B)$, where B is an arbitrary rational B-field lift of α and, of course, the standard transcendental lattice $T(X)$.

But even with this description the Brauer group $\mathrm{Br}(X)$, which is abstractly isomorphic to $(\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}$, is otherwise a rather mysterious object. In order to get a better understanding of it, we shall introduce various equivalence relations.

Definition 3.2. *We define* derived equivalence *of two Brauer classes $\alpha, \alpha' \in \mathrm{Br}(X)$ by*

$$\alpha \overset{D}{\sim} \alpha' \Longleftrightarrow \mathrm{D^b}(X, \alpha) \cong \mathrm{D^b}(X, \alpha')$$

where the isomorphism on the right hand side means Fourier–Mukai equivalence.

Hodge equivalence *is defined by*

$$\alpha \overset{H}{\sim} \alpha' \iff \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X, B', \mathbb{Z}),$$

where the isomorphism on the right hand side means Hodge isometry.

We say that two Brauer classes α, α' *are* T-equivalent, $\alpha \overset{T}{\sim} \alpha'$ *if and only if there exists a Hodge isometry* $T(X, \alpha) \cong T(X, \alpha')$.

The Hodge structures $\widetilde{H}(X, B, \mathbb{Z})$ and $\widetilde{H}(X, B', \mathbb{Z})$ are defined in terms of $B, B' \in H^2(X, \mathbb{Q})$, which are chosen such that $\alpha = \alpha_B$ and $\alpha' = \alpha_{B'}$. Note that Hodge equivalence is well-defined, as different choices of the B-field lift of a Brauer class give rise to isomorphic Hodge structures. Derived equivalences will only be considered for algebraic K3 surfaces and we will assume that the equivalence is a Fourier–Mukai equivalence. Hodge equivalence and T-equivalence of Brauer classes make perfect sense also in the non-algebraic situation.

Remark 3.3. i) Clearly, two Hodge equivalent Brauer classes are also T-equivalent. Building upon the construction of the last section, we shall show that derived equivalence implies Hodge equivalence (see Corollary 4.4). So eventually, we will have

$$\alpha \overset{D}{\sim} \alpha' \Longrightarrow \alpha \overset{H}{\sim} \alpha' \Longrightarrow \alpha \overset{T}{\sim} \alpha'.$$

The first implication is expected to be almost an equivalence (see Conjecture 4.9). More precisely, if $\alpha \overset{H}{\sim} \alpha'$ such that the Hodge isometry in Definition 3.2 can be chosen orientation preserving then $\alpha \overset{D}{\sim} \alpha'$.

ii) Note that by Remark 3.1 T-equivalent Brauer classes are of the same order. This immediately shows that there always exists an infinite number of T-equivalence classes. The same holds true for Hodge equivalence (see i)) and for derived equivalence (see Corollary 4.7).

The automorphism group $\mathrm{Aut}(X)$ acts naturally on $\mathrm{Br}(X)$ via $\alpha \mapsto f^*\alpha$. Note that $f^*\alpha \overset{*}{\sim} \alpha$ for $* = D, H, T$. Indeed, the induced action of f on cohomology yields a Hodge isometry between the two Hodge structures, which proves the equivalence for $* = H, T$, and $E \mapsto f^*E$ defines an equivalence $\mathrm{D^b}(X, \alpha) \cong \mathrm{D^b}(X, f^*\alpha)$. In other words, $\mathrm{Aut}(X)$ preserves the equivalence classes of all three equivalence relations.

Since the automorphism group of a K3 surface might very well be infinite, the set

$$\{\alpha \in \mathrm{Br}(X) \mid \alpha \overset{*}{\sim} \alpha_0\}$$

with $\alpha_0 \in \mathrm{Br}(X)$ fixed will in general be infinite. Note however that for algebraic K3 surfaces the action of $\mathrm{Aut}(X)$ on $H^2(X, \mathcal{O}_X)$ and hence on $H^2(X, \mathcal{O}_X^*)$ factorizes over a finite group (see the proof of the following proposition).

Proposition 3.4. *Let X be a K3 surface (not necessarily algebraic) and $\alpha_0 \in \mathrm{Br}(X)$. Then*

$$\{\alpha \mid \alpha \overset{*}{\sim} \alpha_0\}_{/\mathrm{Aut}(X)}$$

is finite. Moreover, if X is algebraic then $\{\alpha \mid \alpha \overset{*}{\sim} \alpha_0\}$ is finite. In both cases, $* = H$ or $* = T$.

Proof. Since Hodge equivalence implies T-equivalence, it suffices to prove the assertions for $* = T$. The second follows easily from the first, as the action of the possibly infinite group $\mathrm{Aut}(X)$ on $H^2(X, \mathcal{O}_X)$ factorizes over a finite group if X is algebraic (see the comment at the end of the proof).

Let us suppose α_0 is of order k. Then the same holds true for any $\alpha \overset{T}{\sim} \alpha_0$ (see Remark 3.3, ii)). For any such α we fix a Hodge isometry $T(X,\alpha) \cong T(X,\alpha_0) =: T_0$.

As was explained earlier, for any Brauer class α of order k there exists a natural short exact sequence

$$0 \longrightarrow T(X,\alpha) \longrightarrow T(X) \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow 0.$$

Moreover, the class α itself is determined by the map $T(X) \to \mathbb{Z}/k\mathbb{Z}$.

Therefore, any class $\alpha \overset{T}{\sim} \alpha_0$ is determined by a Hodge embedding $T_0 \hookrightarrow T(X)$ compatible with the intersection form and an isomorphism $T(X)/T_0 \cong \mathbb{Z}/k\mathbb{Z}$.

Clearly, for a given embedding $T_0 \hookrightarrow T(X)$ there is only a finite number of isomorphisms $T(X)/T_0 \cong \mathbb{Z}/k\mathbb{Z}$. Thus, it suffices to show that the number of embeddings $T_0 \hookrightarrow T(X)$ that are compatible with Hodge structure and intersection form is finite up to the action of $\mathrm{Aut}(X)$.

Up to the action of the group of isometries $\mathrm{O}(T(X))$ the embedding $T_0 \hookrightarrow T(X)$ is determined by a subgroup of the finite group ${T_0}^\vee/T_0$. Hence, up to the action of $\mathrm{O}(T(X))$, there is only a finite number of possibilities for $T_0 \hookrightarrow T(X)$. On the other hand, any isometry $g \in \mathrm{O}(T(X))$ fixing T_0 is in fact a Hodge isometry of $T(X)$.

To conclude, one uses the fact that the image of the natural map $\mathrm{Aut}(X) \to \mathrm{Aut}(T(X))$ is a finite index subgroup. Here, $\mathrm{Aut}(T(X))$ is the group of Hodge isometries of the transcendental lattice $T(X)$.

For the reader's convenience we include a proof of this fact. Up to the action of $\mathrm{O}(\Lambda)$ there exists only finitely many primitive embeddings $T(X) \hookrightarrow \Lambda$. In particular, up to isometries of Λ there exist only finitely many Hodge isometries $T(X) \cong T(X)$. Now use that any $g \in \mathrm{O}(\Lambda)$ that is compatible with a Hodge isometry of $T(X)$, is in fact a Hodge isometry $H^2(X,\mathbb{Z})$. By the global Torelli theorem one knows that up to sign any Hodge isometry of $H^2(X,\mathbb{Z})$ modulo the action of the Weyl group, which acts trivially on the transcendental lattice $T(X)$, is induced by an automorphism of X. This proves the assertion.

Note that in the case of an algebraic K3 surface, i.e. when the signature of $T(X)$ is $(2,s)$, the group $\mathrm{Aut}(T(X))$ is isomorphic to a discrete subgroup of the compact group $\mathrm{O}(2)\times\mathrm{O}(s)$. Hence, the action of $\mathrm{Aut}(X)$ on $H^2(X,\mathcal{O}_X)$ factorizes over a finite group. $\square$

In Section 4 we shall see that the proposition implies the analogous statement for derived equivalence.

4. Twisted Fourier–Mukai equivalence on cohomology

We indicate how to modify the well-known arguments in order to make them work in the twisted situation. We start with an arbitrary smooth projective variety and shall restrict to algebraic K3 surfaces later on.

Definition 4.1. *The* Mukai vector *of* $E\in K(X,\alpha)$ *is*

$$v^B(E):=\mathrm{ch}^B(E)\cdot\sqrt{\mathrm{td}(X)}.$$

As before, $B\in H^2(X,\mathbb{Q})$ is a rational B-field such that $\exp(B^{0,2})=\alpha$.

Clearly,

$$v^B:K(X,\alpha)\longrightarrow H^*(X,\mathbb{Q})$$

takes again values in $\exp(B)\left(\bigoplus H^{p,p}(X)\right)$ (see Remark 1.3, ii)).

Remark 4.2. With this definition the Riemann–Roch formula still holds, i.e. for $E,F\in\mathbf{Coh}(X,\alpha)$ one has

$$\chi(E,F)=\langle v^B(E)^\vee,v^B(F)\rangle,$$

where $\langle\ ,\ \rangle$ is the Mukai pairing (or rather its generalization introduced by Căldăraru [10]). This follows easily from the observations $\chi(E,F)=\chi(X,E^\vee\otimes F)$, $\mathrm{ch}^B(E)^\vee=\mathrm{ch}^{-B}(E^\vee)$, and $\mathrm{ch}^{-B}(E^\vee).\mathrm{ch}^B(F)=\mathrm{ch}(E^\vee\otimes F)$.

Let X and X' be two smooth projective varieties equipped with topologically trivial Brauer classes α respectively α' and B-field lifts B respectively B'.

We then consider the natural B-field $(-B)\boxplus B':=q^*(-B)+p^*B'\in H^2(X\times X',\mathbb{Q})$ and the induced Brauer class $\alpha^{-1}\boxtimes\alpha'\in H^2(X\times X',\mathcal{O}^*)$.

Any $e\in K(X\times X',\alpha^{-1}\boxtimes\alpha')$ defines a Fourier–Mukai transformation

$$\Phi_e^K:K(X,\alpha)\longrightarrow K(X',\alpha').$$

As in the untwisted case, we obtain a commutative diagram

$$\begin{tikzcd} K(X,\alpha) \arrow[r, "\Phi_e^K"] \arrow[d, "v^B"'] & K(X',\alpha') \arrow[d, "v^{B'}"] \\ H^*(X,\mathbb{Q}) \arrow[r, "\Phi^H_{v^{(-B)\boxplus B'}(e)}"'] & H^*(X',\mathbb{Q}) \end{tikzcd}$$

The idea here is to write $\mathrm{ch}^B(E)$ as $\mathrm{ch}(E_B)$ as in the proof of Proposition 1.2. The original argument uses the Grothendieck–Riemann–Roch formula at this point. Fortunately, due to work of Atiyah and Hirzebruch, see [2, Thm. 1, Thm. 2], the same formula holds true also in the differentiable setting.

Suppose a Fourier–Mukai transform

$$\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \longrightarrow \mathrm{D}^{\mathrm{b}}(X',\alpha')$$

with kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X', \alpha^{-1} \boxtimes \alpha')$ is given (see [7] for the discussion of all the necessary derived functors). We will always assume that a rational B-field has been chosen inducing a given Brauer class $\alpha = \alpha_B$.

Many of the following arguments are taken directly from the original sources [7, 25]. We only indicate the necessary modifications.

1) *The right and left adjoints of a Fourier–Mukai functor* $\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \to \mathrm{D}^{\mathrm{b}}(X',\alpha')$ *exist. They are of Fourier–Mukai type.*

2) *The composition of two twisted Fourier–Mukai functors is again of Fourier–Mukai type.*

3) *If a Fourier–Mukai equivalence* $\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \to \mathrm{D}^{\mathrm{b}}(X,\alpha)$ *with kernel* $\mathcal{E}$ *is isomorphic to the identity, then* $\mathcal{E} \cong \mathcal{O}_\Delta$.

Observe that $\mathcal{O}_\Delta$ is indeed an object in $\mathrm{D}^{\mathrm{b}}(X \times X, \alpha^{-1} \boxtimes \alpha)$.

4) *To any Fourier–Mukai functor* $\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \to \mathrm{D}^{\mathrm{b}}(X',\alpha')$ *with kernel* $\mathcal{E}$ *one associates a homomorphism* $\Phi_*^{B,B'} : H^*(X,\mathbb{Q}) \to H^*(X',\mathbb{Q})$ *depending on the B-field lifts of* α *and* α'. *If* Φ *is a Fourier–Mukai equivalence, then* $\Phi_*^{B,B'}$ *is bijective.*

Here, the cohomological Fourier–Mukai transform is with respect to the Mukai vector $v^{(-B)\boxplus B'}(\mathcal{E})$. Check that $v^{(-B)\boxplus B}(\mathcal{O}_\Delta) = [\Delta]$. In fact, a little more can be said. As in the untwisted case, a Fourier–Mukai equivalence induces a cohomological Fourier–Mukai transform which yields isomorphisms $\bigoplus_{p-q=a} H^{p,q}(X,B,\mathbb{Q}) \cong \bigoplus_{p-q=a} H^{p,q}(X',B',\mathbb{Q})$ for all a.

5) *Any Fourier–Mukai equivalence* $\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \to \mathrm{D}^{\mathrm{b}}(X,\alpha)$ *induces a rational isometry* $\Phi_*^{B,B'}$ *with respect to Căldăraru's generalization of the Mukai pairing.*

So far, X was just any smooth projective variety. We now restrict to the case of K3 surfaces. We shall use the notation of the last section.

6) *Let* X *and* X' *be two K3 surfaces and* $e \in K(X \times X', \alpha^{-1} \boxtimes \alpha')$. *Then* $v^{(-B)\boxplus B'}(e) \in H^*(X \times X', \mathbb{Z})$.

Once more, one has to replace the Grothendieck–Riemann–Roch formula in Mukai's original proof by the differentiable version of it (cf. [2]).

Here comes the main result of this section. It is an immediate consequence of the above results.

Proposition 4.3. *Let* X *and* X' *be two algebraic K3 surfaces equipped with rational B-fields* B *respectively* B'. *The induced elements in the Brauer* *group are denoted* α *respectively* α'. *If*

$$\Phi : \mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$$

is a Fourier–Mukai equivalence, then the induced cohomological Fourier–Mukai map

$$\Phi_*^{B,B'} : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z})$$

is an isometry of integral weight-two Hodge structures. □

Corollary 4.4. *Let* $\alpha,\alpha' \in \mathrm{Br}(X)$. *Then* $\alpha \overset{D}{\sim} \alpha'$ *implies* $\alpha \overset{H}{\sim} \alpha'$. □

Together with Proposition 3.4 this also yields:

Corollary 4.5. *Let* X *be an algebraic K3 surface and* $\alpha_0 \in \mathrm{Br}(X)$. *Then the equivalence class* $\{\alpha \mid \alpha \overset{D}{\sim} \alpha_0\}$ *is a finite set.* □

The corollary is in fact a special case of the more general statement that up to automorphism the number of Fourier–Mukai partners is finite. Here a twisted K3 surface (X',α') is a *(twisted) Fourier–Mukai partner* of a given twisted K3 surface (X,α) if there exists a Fourier–Mukai equivalence $\mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$. In [5] it has been shown that for any given untwisted K3 surface there are only finitely many untwisted Fourier–Mukai partners. As a consequence of Proposition 4.3 and its Corollary 4.5 one deduces in the same way the finiteness in the twisted case:

Corollary 4.6. *Any twisted algebraic K3 surface* (X,α) *admits only finitely many Fourier–Mukai partners up to isomorphisms.*

Proof. Any Fourier–Mukai equivalence $\mathrm{D^b}(X,\alpha_B) \cong \mathrm{D^b}(X',\alpha_{B'})$ induces a Hodge isometry $T(X,B) \cong T(X',B')$. As was explained earlier, the transcendental lattice $T(X,B)$ can be embedded into $T(X)$ via $\exp(-B)$ and similarly for X'. Hence $T(X')$ sits between $T(X,B)$ and $T(X,B)^\vee$, i.e.

$$T(X,B) \cong T(X',B') \subset T(X') \subset T(X',B')^\vee \cong T(X,B)^\vee.$$

This shows that there are only finitely many possibilities for the isomorphism type of $T(X')$. Arguing as in [5] shows that there exist only finitely many isomorphism classes of K3 surfaces X' for which a Brauer class α' can be chosen such that $\mathrm{D^b}(X,\alpha_B)$ and $\mathrm{D^b}(X',\alpha')$ are Fourier–Mukai equivalent.

Eventually, Corollary 4.5 says that on any of the finitely many K3 surfaces X' one only has a finite number (up to isomorphisms) of Brauer classes realizing a derived category that is Fourier–Mukai equivalent to the given one $\mathrm{D^b}(X,\alpha)$. □

As another consequence of the proposition and Remark 3.3, ii) one obtains

Corollary 4.7. *Let* X *be an algebraic K3 surface. Then there exists an infinite number of pairwise inequivalent twisted derived categories* $\mathrm{D^b}(X,\alpha)$. □

We conclude this section with a detailed discussion of Căldăraru's conjecture. It turns out that the original formulation has to be modified. At the same time, we shall propose a version that relates Hodge isometries and derived equivalences in a more precise way.

Let us begin with the untwisted version. We shall state it as a conjecture, although most, but not all, of it has been proved already.

Conjecture 4.8. *Let X and X' be two algebraic K3 surfaces.*

a) *If $\Phi : \mathrm{D^b}(X) \cong \mathrm{D^b}(X')$ is a Fourier–Mukai equivalence, then $\Phi_* : \widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(X',\mathbb{Z})$ satisfies:*

- i) *Φ_* is a Hodge isometry and*
- ii) *Φ_* preserves the natural orientation of the four positive directions.*

b) *If $g : \widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(X',\mathbb{Z})$ satisfies* i) *and* ii) *then there exists a Fourier–Mukai equivalence Φ with $g = \Phi_*$.*

The known results are essentially due to Mukai and Orlov with complementary observations provided by [17, 31]. That the orientation should be important at this point was first observed in [33]. Part b) of the conjecture is essentially known. The only missing detail that we could not find in the literature is explained in the next section. In a) one only knows, for the time being, that Φ_* satisfies i). The problem of showing ii) can be reduced to the question whether the Hodge isometry $j = (-\mathrm{id}_{H^0}) \oplus \mathrm{id}_{H^2} \oplus (-\mathrm{id}_{H^4})$ can be realized by an autoequivalence.

Let us now state the twisted version.

Conjecture 4.9. *Let X and X' be two algebraic K3 surfaces with B-fields B and B'.*

a) *If $\Phi : \mathrm{D^b}(X,\alpha_B) \cong \mathrm{D^b}(X',\alpha_{B'})$ is a Fourier–Mukai equivalence, then $\Phi_*^{B,B'} : \widetilde{H}(X,B,\mathbb{Z}) \to \widetilde{H}(X',B',\mathbb{Z})$ satisfies:*

- i) *$\Phi_*^{B,B'}$ is a Hodge isometry and*
- ii) *$\Phi_*^{B,B'}$ preserves the natural orientation of the four positive directions.*

b) *If $g : \widetilde{H}(X,B,\mathbb{Z}) \to \widetilde{H}(X',B',\mathbb{Z})$ satisfies* i) *and* ii) *then there exists a Fourier–Mukai equivalence Φ with $g = \Phi_*^{B,B'}$.*

Proposition 4.3 shows that i) part a) holds. Later we will explain how to deduce part b) for large Picard number.

Remark 4.10. Căldăraru stated his conjecture originally as: $\mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$ if and only if there exists a Hodge isometry $T(X,\alpha) \cong T(X',\alpha')$.

It seems two problems may occur. First of all, unlike the untwisted case the existence of a Hodge isometry of the twisted transcendental lattices $T(X,B) \cong T(X,\alpha_B) \cong T(X',\alpha_{B'}) \cong T(X,B)$ does not necessarily yield the existence of a Hodge isometry $\widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z})$. Indeed, the orthogonal complement of the twisted transcendental lattice $T(X,B)$ does not, in general, contain a hyperbolic plane, so that Nikulin's results do not apply anymore. See Example 4.11 for a counterexample.

Secondly, even when a Hodge isometry $\widetilde{H}(X,B,\mathbb{Z})\cong\widetilde{H}(X',B',\mathbb{Z})$ can be found it might reverse the orientation of the positive directions. In the untwisted case, this is no problem, as one might compose with $-j$ in order to get a Hodge isometry that preserves the orientation of the positive directions. This trick does not work any longer in the twisted case, as $-j$ is a Hodge isometry between $\widetilde{H}(X,B,\mathbb{Z})$ and $\widetilde{H}(X,-B,\mathbb{Z})$.

On the level of twisted derived categories this is related to the problem that $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ and $\mathrm{D}^{\mathrm{b}}(X,\alpha^{-1})$ might a priori not be Fourier–Mukai equivalent if α is of order >2 (compare with Corollary 7.5).

Example 4.11. We shall construct a twisted K3 surface $(X,\alpha=\alpha_B)$ with α of order five such that $T(X,\alpha)\cong T(X,\alpha^2)$, but the two twisted Hodge structures $\widetilde{H}(X,B,\mathbb{Z})$ and $\widetilde{H}(X,2B,\mathbb{Z})$ are not Hodge isometric. The latter will be ensured by showing that $\mathrm{Pic}(X,B)$ and $\mathrm{Pic}(X,2B)$ are not isometric.

To be very explicit we write Λ as $\Lambda\cong U_1\oplus\Lambda'$, with U_1 isomorphic to the hyperbolic plane. The standard basis is denoted e_1,e_2. Let X be a K3 surface with maximal transcendental lattice $T(X)=\Lambda$ and choose $B=(1/5)(e_1+e_2)$. It is not difficult to check that α_B under these assumptions is indeed of order five. The twisted Picard groups $\mathrm{Pic}(X,B)$ and $\mathrm{Pic}(X,2B)$ can now be described as follows $\mathrm{Pic}(X,B)=u_1\mathbb{Z}\oplus(5u_2+(e_1+e_2))\mathbb{Z}$ respectively $\mathrm{Pic}(X,2B)=u_1\mathbb{Z}\oplus(5u_2+2(e_1+e_2))\mathbb{Z}$ (see Section 2). As abstract lattices they are given by the matrices

$$\begin{pmatrix}0&-5\\-5&2\end{pmatrix}\text{ respectively }\begin{pmatrix}0&-5\\-5&8\end{pmatrix}.$$

These two matrices define inequivalent lattices, for 2 is realized as $((0,1),(0,1))_1$ in the first, but there is no integral vector (a,b) such that $((a,b),(a,b))_2=2$.

Note that the K3 surface X is not algebraic, but it is possible to start with this example and construct an algebraic counterexample by adding an algebraic Picard group of the form $40\mathbb{Z}$. Indeed, the two lattices given by the symmetric matrices

$$\begin{pmatrix}0&-5&0\\-5&2&0\\0&0&40\end{pmatrix}\text{ respectively }\begin{pmatrix}0&-5&0\\-5&8&0\\0&0&40\end{pmatrix}$$

are not equivalent.

In order to see this, it suffices to verify that $-10xy+8y^2+40z^2=2$ or, equivalently, $-5xy+4y^2+20z^2=1$ has no integral solution.

Suppose (x,y,z) is an integral solution. Then x and y are both odd and hence $p:=(x-y)/2$, $q:=(5x-3y)/2$ are integral. Replacing $x=q-3p$ and $y=q-5p$ in the above equation yields $25p^2-1=q^2-20z^2$. Viewing this equation modulo four reveals that p is odd and q is even. We write $p=2a+1$ and $q=2b$ with $a,b\in\mathbb{Z}$. Thus, $(5a+2)(5a+3)=b^2-5z^2$.

Let $d := (b, z)$ and write $b = d\beta$, $z = d\zeta$. Then $d^2|(5a + 2)$ or $d^2|(5a + 3)$ and, therefore, $mn = \beta^2 - 5\zeta^2$ with either $5a + 2 = md^2$, $5a + 3 = n$ or, respectively, $5a + 2 = m$, $5a + 3 = nd^2$. As $(\zeta, m) = 1 = (\zeta, n)$, this shows $\left(\frac{5}{m}\right) = 1 = \left(\frac{5}{n}\right)$. On the other hand, one has $\left(\frac{m}{5}\right) = -1 = \left(\frac{n}{5}\right)$. Indeed, $\left(\frac{2}{5}\right) = -1 = \left(\frac{3}{5}\right)$ and $d^2 \equiv \pm 1(5)$. Eventually, a contradiction is obtained by applying the reciprocity law $\left(\frac{5}{m}\right) = \left(\frac{m}{5}\right)$, if $5a + 2$ and hence m is odd, or $\left(\frac{5}{n}\right) = \left(\frac{n}{5}\right)$, if $5a + 3$ and hence n is odd.

5. Moduli spaces yield orientation preserving equivalences

Let us start out with a fairly general discussion of Hodge isometries $\varphi : \widetilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X', \mathbb{Z})$ and the question when they are orientation preserving.

Firstly, since φ is an isomorphism of weight two Hodge structures, we may choose generators $\sigma \in H^{2,0}(X)$ and $\sigma' \in H^{2,0}(X')$ such that $\varphi(\sigma) = \sigma'$. The oriented plane $\langle 1 - \omega^2/2, \omega \rangle$, where ω is a Kähler class, is completely encoded by the complex line spanned by $\exp(i\omega)$. Moreover, as φ is an isometry, the image $\varphi(\exp(i\omega))$ is orthogonal to $\sigma' = \varphi(\sigma)$ and, therefore, of the form

$$\varphi(\exp(i\omega)) = \lambda \cdot \exp(b + ia)$$

with $\lambda \in \mathbb{C}^*$ and $a, b \in H^{1,1}(X', \mathbb{R})$.

Let us first compute the scalar λ. It will be expressed as a linear combination of the degree zero parts of certain natural Mukai vectors. We shall use the following short hand $r := \varphi(0, 0, 1)_0$, $\chi := \varphi(1, 0, 1)_0$, and $\chi_H := \varphi(0, \omega, -\omega^2/2)_0$. Here, we are anticipating the moduli space situation. Indeed, if $x \in X$ is a closed point and $H \subset X$ an ample divisor with fundamental class ω, then $v(k(x)) = (0, 0, 1)$, $v(\mathcal{O}_X) = (1, 0, 1)$, and $v(\mathcal{O}_H) = (0, \omega, -\omega^2/2)$.

Lemma 5.1. $\lambda = \chi - r\left(\frac{\omega^2}{2} + 1\right) + i\left(\chi_H + r\frac{\omega^2}{2}\right)$.

Proof. Write

$$\exp(i\omega) = v(\mathcal{O}_X) - \left(\frac{\omega^2}{2} + 1\right) \cdot v(k(x)) + i\left(v(\mathcal{O}_H) + \frac{\omega^2}{2} \cdot v(k(x))\right)$$

and use $\lambda = \varphi(\exp(i\omega))_0$. □

Let us introduce the basic classes

$$u_0 := -r \cdot [\mathcal{O}_X] + \chi \cdot [k(x)] \text{ and } u_1 := -r \cdot [\mathcal{O}_H] + \chi_H \cdot [k(x)]$$

as elements in the Grothendieck group $K(X)$ (cf. [19, Ex. 8.1.8]). Their Mukai vectors are given by

$$v(u_0) = (-r, 0, -r + \chi) \text{ resp. } v(u_1) = (0, -r\omega, r\frac{\omega^2}{2} + \chi_H).$$

Lemma 5.2. *Suppose $r \neq 0$. Then*

$$|\lambda|^2 \cdot a = \varphi\left(\left(\frac{\chi_H}{r} + \frac{\omega^2}{2}\right) \cdot v(u_0) + \left(\left(\frac{\omega^2}{2} + 1\right) - \frac{\chi}{r}\right) \cdot v(u_1)\right)_2.$$

Proof. The assertion follows from

$$\begin{aligned}|\lambda|^2\cdot a &= \operatorname{Im}(\bar\lambda\cdot\varphi(\exp(i\omega)))_2\\ &= \left(\chi - r\left(\frac{\omega^2}{2}+1\right)\right)\varphi(\omega)_2 - \left(\chi_H + r\frac{\omega^2}{2}\right)\varphi\left(1,0,-\frac{\omega^2}{2}\right)_2.\end{aligned}$$

□

If $r=0$ and $\chi_H\neq 0$, then a cannot be written as the image of a linear combination of $v(u_0)$ and $v(u_1)$.

Proposition 5.3. *Let $\varphi:\widetilde H(X,\mathbb{Z})\xrightarrow{\sim}\widetilde H(X',\mathbb{Z})$ be a Hodge isometry with $\varphi(0,0,1)_0\neq 0$. Suppose $H\subset X$ is an ample divisor. Then φ is orientation preserving if and only if*

$$\left(\frac{\chi_H}{r}+\frac{\omega^2}{2}\right)\cdot\varphi(v(u_0))_2 + \left(\left(\frac{\omega^2}{2}+1\right)-\frac{\chi}{r}\right)\cdot\varphi(v(u_1))_2$$

is contained in the positive cone $\mathcal{C}_{X'}\subset H^{1,1}(X',\mathbb{R})$.

Recall that the positive cone $\mathcal{C}_{X'}$ is the connected component of the cone of all classes $\alpha\in H^{1,1}(X,\mathbb{R})$ with $\alpha^2>0$ that contains the ample (respectively Kähler) classes.

Proof. Using the notation introduced before, this linear combination of $\varphi(v(u_0))_2$ and $\varphi(v(u_1))_2$ is (up to the positive real scalar $|\lambda|$) the class a.

Now use the following easy facts:

i) Real and imaginary part of $\exp(ia)$ induce the same orientation of the two positive directions of $\widetilde H^{1,1}(X',\mathbb{R})$ as the natural one given by $\exp(i\omega')$ with ω' an ample class if and only if $a\in\mathcal{C}_{X'}$.

ii) Real and imaginary part of $\exp(ia)$ induce the same orientation of the two positive directions in $\widetilde H^{1,1}(X',\mathbb{R})$ as $\lambda\cdot\exp(b+ia)$ for any $\lambda\in\mathbb{C}^*$ and any $b\in H^{1,1}(X',\mathbb{R})$. □

Remark 5.4. Either by applying the above proposition or by any other means, it is easy to check that the following standard equivalences are orientation preserving: i) $F\mapsto f_*(F)$, where f is an isomorphism, ii) Line bundle twists $F\mapsto L\otimes F$ with $L\in\operatorname{Pic}(X)$, iii) Shift functor $F\mapsto F[i]$, iv) Twist functor $F\mapsto T_E(F)$, where $E\in\mathrm{D}^{\mathrm{b}}(X)$ is an arbitrary spherical object (e.g. $\mathcal{O}$).

All known (auto)equivalences can be written as compositions of the above ones and the ones induced by universal families of stable sheaves. Thus, in order to decide whether at least all known (auto)equivalences are orientation preserving, it suffices to consider the case of a fine moduli space and the equivalence it induces.

Let us now consider a fine moduli space $X'=M(v)$ of stable sheaves E with $v(E)=v$, where $v\in\widetilde H^{1,1}(X,\mathbb{Z})$ with $\langle v,v\rangle=0$. Here stability is meant with respect to a chosen ample divisor $H\subset X$.

Mukai has shown that in such a situation the moduli space $M(v)$ is, if not empty, again a K3 surface and that the Fourier–Mukai transform $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(M(v))$ with kernel the universal sheaf $\mathcal{E} = \mathcal{E}(v)$ on $X \times M(v)$ is an equivalence (see [25]).

Proposition 5.5. *The induced Hodge isometry*

$$\Phi_* : \widetilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(M(v), \mathbb{Z})$$

is orientation preserving.

Proof. The idea of the proof is of course to apply Proposition 5.3 and to show that the class a introduced in the general context is contained in the positive cone. We will however show a more precise result for the case of a fine moduli space of stable sheaves of positive rank, namely that in this case a is in fact ample.

Suppose $m \in M(v)$ corresponds to a stable sheaf E on X. Then $v = v(E) = (\mathrm{rk}(E), \mathrm{c}_1(E), \chi(E) - \mathrm{rk}(E))$, which we will write as (r, ℓ, s).

The invariants r, χ, and χ_H introduced above, can in this special situation geometrically be interpreted as:

$$\begin{aligned} r = \Phi_*(v(k(x))_0 &= \mathrm{rk}(\mathcal{E}|_{\{x\}\times M(v)}) = \mathrm{rk}(E) \\ \chi = \Phi_*(v(\mathcal{O}_X))_0 &= \mathrm{rk}(Rp_*\mathcal{E}) = \chi(E), \text{ and} \\ \chi_H = \Phi_*(v(\mathcal{O}_H))_0 &= \mathrm{rk}(Rp_*(\mathcal{E}|_{\{H\}\times M(v)})) = \chi(E|_H). \end{aligned}$$

Now observe that that twisting $E \mapsto E(nH)$ defines an isomorphism

$$t(n) : M(v) \xrightarrow{\sim} M(\exp(n[H]) \cdot v)$$

under which the universal families can be compared by

$$(\mathrm{id}_X \times t(n))^* \mathcal{E}(\exp(n[H]) \cdot v) \cong q^* \mathcal{O}(nH) \otimes \mathcal{E}(v).$$

Clearly, Φ induces an orientation preserving Hodge isometry if and only if $t(n)_* \circ \Phi \circ (\mathcal{O}(nH) \otimes (\ ))$ does (cf. Remark 5.4), but the latter is nothing but the Fourier–Mukai transform with respect to the universal family $\mathcal{E}(\exp(n[H]) \cdot v)$ on $X \times M(\exp(n[H]) \cdot v)$. (Similarly, the ampleness of the class a for $\mathcal{E}$ is equivalent to the ampleness of the analogous class with respect to $\mathcal{E}(\exp(n[H]) \cdot v)$.)

Thus, in order to prove the assertion we may first twist by $\mathcal{O}(nH)$ for $n \gg 0$. Under the additional assumptions that $r > 0$ we may thus arrange things such that $\left(\frac{\chi_H}{r} + \frac{[H]^2}{2}\right) > 0$ and $\left(\left(\frac{[H]^2}{2} + 1\right) - \frac{\chi}{r}\right) > 0$. Moreover, we may assume from the very beginning that all sheaves $E \in M(v)$ are so positive that the standard GIT construction of the moduli space applies directly. In particular, we may assume that the hypothesis of Theorem 8.1.11 in [19] are fulfilled. Again, one has to assume that $r > 0$.

Thus, the line bundle

$$\mathcal{L}_0 := \det(\Phi(u_0)) \in \mathrm{Pic}(M(v))$$

is ample. Its Chern class is $\mathrm{c}_1(\mathcal{L}_0) = \Phi_*(v(u_0))_2$. (The line bundle $\mathcal{L}_0$ is the descent of the standard polarization of the Quot-scheme.) Combined with [19, Prop. 8.1.10] we deduce from this that also the line bundle

$$\mathcal{L}_1 := \det(\Phi(u_1))$$

is nef (see also the comments in [19, Sect. 8.2]). Its Chern class is $\mathrm{c}_1(\mathcal{L}_1) = \Phi_*(v(u_1))_2$. Hence, $\left(\frac{\chi_H}{r} + \frac{\omega^2}{2}\right) \cdot \Phi_*(v(u_0))_2 + \left(\left(\frac{\omega^2}{2} + 1\right) - \frac{\chi}{r}\right) \cdot \Phi_*(v(u_1))_2$ is an ample class and thus contained in the positive cone $\mathcal{C}_{M(v)}$. Proposition 5.3 yields the assertion.

It remains to prove the assertion for $r = 0$. Here, it will only be shown that a is contained in the positive cone (and not that it is ample). A geometric proof can be given by viewing the moduli space $M(v)$ as a relative moduli space over the linear system of curves that occur as support of a stable sheaf (the case of sheaves concentrated in points being trivial), but a more elegant argument was suggested to us by Yoshioka. Instead of working with the line bundles $\mathcal{L}_i$, $i = 0, 1$, considered above he suggested to work directly with the ample line bundle on the moduli space that naturally occurs in the construction of Simpson. Here are the details of the argument.

We know that $a \in \pm\mathcal{C}_M$. In order to show that $a \in \mathcal{C}_M$ it suffices to show that $\langle a, \beta\rangle > 0$ for one ample class β on M. By the very construction of the moduli space à la Simpson, the line bundle $\mathcal{L} := \det(p_*(\mathcal{E} \otimes q^*u))$ is ample for $m \gg 0$, where $u := \chi \cdot [\mathcal{O}(mH)] - \chi(E(m)) \cdot [\mathcal{O}]$. Again, we suppose from the very beginning that the sheaves have been twisted such that the moduli space can directly be described as a GIT-quotient of a certain Quot-scheme parametrizing all stable sheaves $[E] \in M$.

Revisiting the proof of Lemma 5.2 we find that the class a is a positive multiple of $\left(\chi - r\left(\frac{\omega^2}{2} + 1\right)\right)\varphi(\omega)_2 - \left(\chi_H + r\frac{\omega^2}{2}\right)\varphi\left(1, 0, -\frac{\omega^2}{2}\right)_2$. So if we introduce $\gamma := \left(\chi - r\left(\frac{\omega^2}{2} + 1\right)\right)\omega - \left(\chi_H + r\frac{\omega^2}{2}\right)\left(1, 0, -\frac{\omega^2}{2}\right)$, then $a = \varphi(\gamma)_2$ up to a positive scalar. One checks that $\varphi(\gamma)_0 = 0$ by using $\chi_H = -\langle(0, \omega, -\omega^2/2), v\rangle$. Similarly, $\varphi(u)_0 = 0$. Hence, $\langle\varphi(\gamma)_2, \varphi(u)_2\rangle = \langle\varphi(\gamma), \varphi(u)\rangle = \langle\gamma, u\rangle$. A straightforward calculation shows that $\langle\gamma, u\rangle > 0$ which concludes the proof.

$\square$

Remark 5.6. The case of a fine moduli space of stable sheaf of positive rank is special in the sense that the class a given by $\Phi_{\mathcal{E}}^H(\exp(i[H])) = \lambda\cdot\exp(b+ia)$ is not only contained in the positive cone, but is in fact ample. This does not hold true for arbitrary Fourier–Mukai equivalences as is shown e.g. by the twist with respect to the spherical object $\mathcal{O}_C(k)$, where $C \subset X$ is a smooth rational curve.

Remark 5.7. We will use the main result of this section not only in the case of fine moduli spaces, but as well for coarse moduli spaces. More precisely, the $(1, \alpha_B)$-twisted universal family $\mathcal{E}$ on $X \times M(v)$ induces an orientation preserving Hodge isometry $\Phi_*^{0,B} : \widetilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(M(v), B, \mathbb{Z})$.

The above proof carries over to coarse moduli spaces by the following rather standard arguments. The main idea is to use the fact that the line bundles $\mathcal{L}_i$ and $\mathcal{L}$ are also defined for coarse moduli spaces (even semi-stable sheaves are allowed). Attention has to be paid to problems with the twist on the level of cohomology.

Here are some of the details. The moduli space $M(v)$ is constructed as a quotient $\pi : R^s \to M(v)$ of a certain open subset R^s of a certain Quot-scheme. In particular, there always exists a universal sheaf $\tilde{\mathcal{E}}$ on $X \times R^s$. In fact, even in the case of a fine moduli space one first constructs $\mathcal{L}_0$ on R^c. Then one shows that it is the restriction of an ample line bundle on the ambient Quot-scheme and that its restriction to R^s descends to an ample line bundle on $M(v)$.

If a twisted universal sheaf $\mathcal{E}$ is given by $\{X \times U_i, \varphi_{ij} \in \mathcal{O}^*(X \times U_{ij})\}$ then $\tilde{\mathcal{E}}$ can be thought of as given by $\{X \times \pi^{-1}(U_i), \tilde{\varphi}_{ij} := (1 \times \pi)^* \varphi_{ij} \cdot \lambda_{ij}\}$. Here, $\lambda_{ij} \in \mathcal{O}^*(X \times \pi^{-1}(U_{ij}))$ satisfy $\lambda_{ij} \cdot \lambda_{jk} \cdot \lambda_{ki} = (1 \times \pi)^* \alpha_{ijk}^{-1}$.

Recall that also the sheaf $\mathcal{E}_B$ needed to define $\mathrm{ch}^B(\mathcal{E})$ was obtained by untwisting with invertible functions $\exp(a_{ij})$ satisfying a similar cocycle condition as the λ_{ij} but already on $X \times M(v)$. The functions a_{ij} are defined on $X \times U_{ij}$, but they are not holomorphic. Thus, one finds that $\tilde{\mathcal{E}}$ and $(1 \times \pi)^* \mathcal{E}_B$ differ by a (non-holomorphic) line bundle (the one given by the transition functions $\lambda_{ij} \cdot \exp(-a_{ij})$). For the same reason that shows that for fine moduli spaces the line bundles $\mathcal{L}_i$ do not depend on the chosen universal family (which is only unique up to twist by line bundles on the moduli space) one concludes that $\pi^* \Phi_*^{0,B}(u_i)_2 = \pi^* \mathrm{c}_1(\mathcal{L}_i)$ as well as $\pi^* \Phi_*^{0,B}(u)_2 = \pi^* \mathrm{c}_1(\mathcal{L})$ (using the notation of the proof of Proposition 5.5). This is enough to conclude also in the case of coarse moduli spaces.

6. (Twisted) derived isometries

Let us begin with a few comments on the group $\mathrm{O}(\widetilde{\Lambda})$ of isometries of the lattice $\widetilde{\Lambda} = \Lambda \oplus U$. Classical results due to C. T. C. Wall show that the three natural subgroups $\mathrm{O}(\Lambda), \mathrm{O}(U), \exp(\Lambda) \subset \mathrm{O}(\widetilde{\Lambda})$ generate $\mathrm{O}(\widetilde{\Lambda})$ (see [34]). Here, $\exp(B)$ with $B \in \Lambda$ acts by multiplication with $1 + B + B^2/2$ in $\widetilde{\Lambda} = H^*(M, \mathbb{Z})$. Clearly, the subgroups $\mathrm{O}(U)$ and $\mathrm{O}(\Lambda)$ commute. For $g \in \mathrm{O}(\Lambda)$ and $\exp(B) \in \exp(\Lambda)$ one has

$$g \circ \exp(B) = \exp(g(B)) \circ g.$$

Hence, any element $g \in \mathrm{O}(\widetilde{\Lambda})$ can be written as

$$g = g_1 \circ g_2 \text{ with } g_1 \in \langle \mathrm{O}(U), \exp(\Lambda) \rangle \text{ and } g_2 \in \mathrm{O}(\Lambda).$$

Also note that $j = -\mathrm{id}_U$ commutes with $\mathrm{O}(U), \mathrm{O}(\Lambda)$ and satisfies $\exp(B) \circ j = j \circ \exp(-B)$. The group of orientation preserving isometries $\mathrm{O}_+(\widetilde{\Lambda})$ can similarly be generated by $\exp(\Lambda)$, $\mathrm{O}_+(\Lambda)$, and $\mathrm{O}_+(U)$.

In the following, we will be interested in the following subsets

$$\begin{aligned} H &:= \{g \in \mathrm{O}_+(\widetilde{\Lambda}) \mid g(Q) \cap Q \neq \emptyset\} \\ H_{\mathrm{alg}} &:= \{g \in \mathrm{O}_+(\widetilde{\Lambda}) \mid g(Q_{\mathrm{alg}}) \cap Q_{\mathrm{alg}} \neq \emptyset\}, \end{aligned}$$

where $Q \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ is the K3 surface period domain $\{x \mid x^2 = 0, (x, \bar{x}) > 0\}$ and $Q_{\mathrm{alg}} \subset Q$ is the dense subset of periods of algebraic K3 surfaces. In other words, Q_{alg} is the set of those $x \in Q$ for which there exists a class $B \in x^{\perp} \cap \Lambda$ with $B^2 > 0$.

Proposition 6.1. *Both sets $H_{\mathrm{alg}} \subset H \subset \mathrm{O}_+(\widetilde{\Lambda})$ contain the generating subgroups $\mathrm{O}_+(\Lambda), \mathrm{O}_+(U), \exp(\Lambda) \subset \mathrm{O}_+(\widetilde{\Lambda})$ and thus generate $\mathrm{O}_+(\widetilde{\Lambda})$. However, $H \neq \mathrm{O}_+(\widetilde{\Lambda})$ or, equivalently, the subsets H_{alg} and H do not form subgroups of $\mathrm{O}_+(\widetilde{\Lambda})$.*

Proof. In fact, $\mathrm{O}_+(\Lambda)$ and $\mathrm{O}_+(U)$ both respect the period domain Q. If $x \in Q_{\mathrm{alg}} \cap B^{\perp}$ for a given $B \in \Lambda$, then $\exp(B)x = x$ and hence $\exp(B) \in H_{\mathrm{alg}}$. The existence of x follows from the fact that $B^{\perp}$ contains a positive plane.

In order to show the second assertion, it suffices to construct one $g \in \mathrm{O}_+(\widetilde{\Lambda})$ with $g(Q) \cap Q = \emptyset$.

Choose $B_0, B_1 \in \Lambda$ such that $\langle B_0, B_1 \rangle \subset \Lambda_{\mathbb{R}}$ is a positive plane. Then consider the isometry $g := \exp(B_1) \circ i \circ \exp(B_0)$ where $i \in \mathrm{O}_+(U)$ is the isometry that maps the generators u_2 and u_1 of $H^0(M, \mathbb{Z})$ respectively $H^4(M, \mathbb{Z})$ to $-u_1$ respectively $-u_2$.

Suppose $g(x) \in Q$ for some $x \in Q$. Writing $g(x) = -(B_0, x)u_2 + (x - (B_0, x)B_1) + (B_1 - \frac{B_1^2}{2}B_0, x)u_1$ then shows that $x \in B_0^{\perp} \cap B_1^{\perp}$. This is impossible, as real and imaginary part of x together with B_0 and B_1 would span a positive four-space in $\Lambda_{\mathbb{R}}$ that does not exist. □

Let us now introduce the following notation:

$$\begin{aligned} H' &:= \{g \in \mathrm{O}_+(\widetilde{\Lambda}) \mid g(\exp(\Lambda_{\mathbb{Q}}) \cdot Q) \cap (\exp(\Lambda_{\mathbb{Q}}) \cdot Q) \neq \emptyset\} \\ H'_{\mathrm{alg}} &:= \{g \in \mathrm{O}_+(\widetilde{\Lambda}) \mid g(\exp(\Lambda_{\mathbb{Q}}) \cdot Q_{\mathrm{alg}}) \cap (\exp(\Lambda_{\mathbb{Q}}) \cdot Q_{\mathrm{alg}}) \neq \emptyset\}. \end{aligned}$$

Proposition 6.2. $H'_{\mathrm{alg}} = H' = \mathrm{O}_+(\widetilde{\Lambda})$. *In particular, both sets are groups.*

Proof. We will actually show slightly more, namely that for any $g \in \mathrm{O}(\widetilde{\Lambda})$ one has $g(Q_{\mathrm{alg}}) \cap (\exp(\Lambda_{\mathbb{Q}}) \cdot Q_{\mathrm{alg}}) \neq \emptyset$.

Fix a basis $x_1, \ldots, x_{22} \in \Lambda$. Then any $g \in \mathrm{O}(\widetilde{\Lambda})$ can be written as

$$g(x) = \sum_{i=1}^{22} \lambda_i x_i + \mu_1 u_1 + \mu_2 u_2$$

with $\lambda_i, \mu_j \in \mathbb{Z}$ depending linearly on $x \in \Lambda$. (Here, u_2 and u_1 span $H^0(M, \mathbb{Z})$ respectively $H^4(M, \mathbb{Z})$.) In other words, there exist elements $A_i, B_1, B_2 \in \Lambda$ with

$$g(x) = \sum_{i=1}^{22} (A_i, x)x_i + (B_1, x)u_1 + (B_2, x)u_2$$

for any $x \in \Lambda$. Of course, by linear extension the same formula holds for any $x \in \Lambda_{\mathbb{C}}$.

Now choose a positive plane inside $B_2^{\perp} \subset \Lambda_{\mathbb{Q}}$ and an orthogonal basis $x_1, x_2 \in \Lambda_{\mathbb{Q}}$ of it with $x_1^2 = x_2^2$. Then set $x := x_1 + ix_2$. With this definition $x \in Q$ corresponds to a K3 surface of Picard number 20 which is necessarily projective, i.e. $x \in Q_{\mathrm{alg}}$.

As $(B_1, x) = (B_1, x_1) + i(B_1, x_2) \in \mathbb{Q}(i)$, one finds $B \in \Lambda_{\mathbb{Q}}$ such that $(B_1, x) = (B, y)$, where $y \in Q$ is given by $g(x) = y + (B_1, x)u_1$. As $x \in Q_{\mathrm{alg}}$ immediately yields $y \in Q_{\mathrm{alg}}$, this proves $g(x) \in \exp(\Lambda_{\mathbb{Q}}) \cdot Q_{\mathrm{alg}}$.

This then shows that $g(Q_{\mathrm{alg}}) \cap (\exp(\Lambda_{\mathbb{Q}}) \cdot Q_{\mathrm{alg}}) \neq \emptyset$. $\square$

Remark 6.3. i) Note that the K3 surface corresponding to the period x considered in the above proof, which is defined over $\mathbb{Q}$, has maximal Picard number $\rho = 20$.

ii) The above result can be improved to the following statement: For any $g \in \mathrm{O}_+(\widetilde{\Lambda})$ one has $g(Q_{\mathrm{alg}}) \cap (\exp(\Lambda) \cdot Q_{\mathrm{alg}}) \neq \emptyset$. Indeed, we may choose integral classes $x_1, x_2 \in B_2^{\perp}$ with $x_1^2 = x_2^2 = 2$ and $(x_1, x_2) = 0$. (Note that, after applying isometries of $\widetilde{\Lambda}$, any primitive vector can be assumed to be contained in one copy of U in the decomposition $\widetilde{\Lambda} = U^{\oplus 4} \oplus (-E_8)^{\oplus 2}$. Thus $B_2^{\perp}$ contains in particular two other copies of the hyperbolic plane, which ensures the existence of x_1, x_2.) Then, real and imaginary parts y_1, y_2 of y, defined as in the proof, have the same properties. It is now possible to find an integral element $B \in \widetilde{\Lambda}$ with $(B, y_i) = (B_1, x_i)$. Indeed, the primitive sublattice generated by y_1, y_2 is isometric to the sublattice generated by $e_1 + e_2$ and $e_1' + e_2'$ inside the direct sum of two copies of the hyperbolic plane $U \oplus U' \subset \widetilde{\Lambda}$, which form a direct summand of $\widetilde{\Lambda}$. Standard results of Nikulin show that this isometry can be extended to an isometry of $\widetilde{\Lambda}$. The inverse image under this isometry of $(B_1, x_1)e_1 + (B_1, x_2)e_1'$ can be taken for B. Note that $x = x_1 + ix_2$ is again algebraic.

We continue to present a K3 surface by a complex structure I on the fixed manifold M. In particular, a marking $H^2(X, \mathbb{Z}) \cong H^2(M, \mathbb{Z}) \cong \Lambda$ is automatically given.

Definition 6.4. *An element $g \in \mathrm{O}_+(\widetilde{\Lambda})$ is called a* derived isometry *if there exist two algebraic K3 surfaces X and X' and a Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(X) \cong \mathrm{D^b}(X')$ with $\Phi_* = g$.*

An element $g \in \mathrm{O}_+(\widetilde{\Lambda})$ is called a twisted derived isometry *if there exist two algebraic K3 surfaces X, X', B-fields $B \in H^2(X, \mathbb{Z})$, $B' \in H^2(X', \mathbb{Z})$ and a twisted Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(X', \alpha_{B'})$ such that $\Phi_*^{B,B'} = g$.*

In the twisted as well as in the untwisted situation, the difficult question seems to be whether any (twisted) derived equivalence induces a (twisted) derived isometry in this sense, i.e. whether it is orientation preserving. For all known examples this is the case, as was shown in the previous section.

In order to emphasize the difference between these two notions we sometimes speak of untwisted derived isometry in the first case.

Corollary 6.5. *The set* $G \subset \mathrm{O}_+(\widetilde{\Lambda})$ *of all derived isometries generates* $\mathrm{O}_+(\widetilde{\Lambda})$. *However, G is not a group, i.e.* $G \neq \mathrm{O}_+(\widetilde{\Lambda})$.

Proof. By Borcea's result any element in $\mathrm{O}_+(\Lambda)$ is realized as an isomorphism of K3 surfaces. Hence $\mathrm{O}_+(\Lambda) \subset G$.

The subgroup $\mathrm{O}_+(U) = \mathrm{O}_+((H^0 \oplus H^4)(M, \mathbb{Z}))$ consists of the two elements id and i, where i is as in the proof of Proposition 6.1. If X is an arbitrary K3 surface, then the reflection $T_{\mathcal{O}}$ associated to the spherical object $\mathcal{O}$ acts as i on cohomology. Hence, G contains $\mathrm{O}_+(U)$.

Eventually, any $\exp(B)$ with $B \in \Lambda$ is contained in G. One way to see this is to choose an algebraic K3 surface X for which B is of type $(1,1)$, i.e. $B = \mathrm{c}_1(L)$ for some $L \in \mathrm{Pic}(X)$. Then the Fourier–Mukai equivalence given by $\otimes L$ acts as multiplication with $\exp(B) = \mathrm{ch}(L)$ on cohomology.

Thus, G contains all three subgroups $\mathrm{O}_+(\Lambda)$, $\mathrm{O}_+(U)$, and $\{\exp(B)\}_{B\in\Lambda}$ and hence generates $\mathrm{O}_+(\widetilde{\Lambda})$.

The last assertion follows from Proposition 6.1 and the fact that $G \subset H_{\mathrm{alg}}$. □

The positive result in the twisted case we are going to present confirms once more the difference between the twisted and the untwisted situation. It is not a direct consequence of Proposition 6.2, as it uses Căldăraru's conjecture for large Picard number which will be established only in the next section.

Proposition 6.6. *Let* $G' \subset \mathrm{O}_+(\widetilde{\Lambda})$ *be the set of all twisted isometries. Then* $G' = \mathrm{O}_+(\widetilde{\Lambda})$.

Proof. Let $g \in \mathrm{O}(\widetilde{\Lambda})$. By Proposition 6.2 there exist twisted algebraic K3 surfaces (X, α_B) and $(X', \alpha_{B'})$ such that g defines an Hodge isometry $\widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$. If one could use Căldăraru's conjecture as stated in 4.9, then one immediately would deduce the existence of a twisted derived Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{B'})$ with $\Phi_*^{B,B'} = g$.

Fortunately, in the proof of Proposition 6.2 we actually constructed X and X' of Picard number $\rho \geq 12$ (cf. Remark 6.3) and for those a variant of Căldăraru's conjecture, that suffices for this argument, will be proved in the next section.

One could also argue without evoking Căldăraru's conjecture at all by using ii), Remark 6.3. Indeed, this remark shows that for any $g \in \mathrm{O}(\widetilde{\Lambda})$ there exist two algebraic K3 surfaces X and X', an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X')$, and an integral B-field $B' \in H^2(X', \mathbb{Z})$ such that $g = \exp(B') \cdot \Phi_*$. □

7. Căldăraru's conjecture for large Picard number

Let us start out by recalling the following

Proposition 7.1. (Mukai) *Suppose X_1 and X_2 are two K3 surfaces with Picard number $\rho(X_i) \geq 12$. Then up to sign any Hodge isometry $T(X_1) \cong T(X_2)$ is induced by an isomorphism $X_1 \cong X_2$.*

Remark 7.2. i) As $(H^0 \oplus H^4)(M, \mathbb{Z})$ forms a hyperbolic plane contained in the orthogonal complement of any transcendental lattice $T(X) \subset \widetilde{H}(M, \mathbb{Z})$, any Hodge isometry $T(X_1) \cong T(X_2)$ can be extended to a Hodge isometry $\widetilde{H}(X_1, \mathbb{Z}) \cong \widetilde{H}(X_2, \mathbb{Z})$. (This follows from [27, Thm.1.14.4], see Remark 7.11, and no additional assumption on the Picard number is needed for this.)

It is this fact that allows one to phrase Orlov's result in terms of the transcendental lattices as: $\mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$ if and only if $T(X_1) \cong T(X_2)$. Thus, as isomorphic K3 surfaces have equivalent derived categories, this result of Mukai's proves in particular Orlov's result under the additional assumption $\rho \geq 12$.

ii) Using the Global Torelli theorem, the proof of the above proposition reduces to the purely lattice theoretical problem to extend a given Hodge isometry $T(X_1) \cong T(X_2)$ to a Hodge isometry $H^2(X_1, \mathbb{Z}) \cong H^2(X_2, \mathbb{Z})$. This is possible by results of Nikulin [27], as the orthogonal complement of $T(X_i)$ in $H^2(X_i, \mathbb{Z})$, i.e. the Picard group, is big enough by assumption.

Note that the occurrence of a possible sign is missing in Prop. 6.3 in [25].

The principal result of this section establishes Căldăraru's conjecture under a similar assumption on the Picard number. The slight difference to Mukai's original result is that in the twisted case only derived equivalence holds and not isomorphism of the twisted K3 surfaces. For examples see Section 8.

Before coming to this, let us show how to reduce the twisted problem for large Picard number to the untwisted situation.

Proposition 7.3. *Let (X, α_B) be a twisted K3 surface with $\rho(X) \geq 12$. Then there exists a K3 surface Z and a Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(Z) \cong \mathrm{D^b}(X, \alpha_B)$. Moreover, Φ can be chosen such that $\Phi_*^{0,B}$ is orientation preserving.*

Proof. The proof follows almost directly from results of Mukai and Căldăraru, so we will be brief.

As $\mathrm{rk}(T(X, B)) = \mathrm{rk}(T(X)) \leq 10$, [27, Thm.1.14.4] applies (see Remark 7.11) and shows that there exists a primitive embedding $T(X, B) \hookrightarrow \Lambda$ (see the discussion in Remark 7.11). By the surjectivity of the period map, one finds a K3 surface Z and a Hodge isometry $T(Z) \cong T(X, B)$.

The composition of this Hodge isometry with $\exp(-B)$ yields an embedding $i : T(Z) \hookrightarrow T(X)$ which is the kernel of the natural map $\alpha : T(X) \to$

$\mathbb{Z}/n\mathbb{Z}$ defined by $\alpha = \alpha_B$, where n is the order of $\alpha \in \mathrm{Br}(X)$. Choose an element $\ell \in T(X)$ with $\alpha(\ell) = 1 \in \mathbb{Z}/n\mathbb{Z}$ and denote by $t \in T(Z)$ the element with $i(t) = n \cdot \ell$.

Next, we use arguments of Mukai (see Section 6 in [25]). He shows that there exists a compact, smooth, two-dimensional moduli space M of stable sheaves on Z such that the inclusion $\varphi : T(Z) \to T(M)$, which is defined in terms of a quasi-universal family, maps t to an element in $T(M)$ which is divisible by n and such that $(1/n)\varphi(t)$ generates the quotient $\mathrm{Coker}(\varphi) \cong \mathbb{Z}/n\mathbb{Z}$.

Căldăraru continued Mukai's discussion and showed that the Brauer class $\beta \in \mathrm{Br}(M)$ defined by $\mathrm{Coker}(\varphi) \cong \mathbb{Z}/n\mathbb{Z}$, $(1/n)\varphi(t) \mapsto 1$ is the obstruction class for the existence of a universal sheaf. Moreover, he showed that a $(1, \beta^{-1})$-twisted universal sheaf $\mathcal{E}$ exists and that the induced Fourier–Mukai functor defines an equivalence $\Phi : \mathrm{D^b}(Z) \cong \mathrm{D^b}(M, \beta^{-1})$ which is orientation preserving (see Section 5). Căldăraru checked this equivalence by applying the standard criterion due to Bondal and Orlov [3, 6] testing a Fourier–Mukai functor on points.

In the penultimate step, one remarks that the two inclusions $i : T(Z) \hookrightarrow T(X)$ and $\varphi : T(Z) \hookrightarrow T(M)$ can be identified via an isomorphism $\psi : T(X) \cong T(M)$ which sends $\ell = (1/n)i(t)$ to $(1/n)\varphi(t)$. This yields a commutative diagram

$$\begin{array}{ccccc} T(Z) & \overset{i}{\hookrightarrow} & T(X) & \twoheadrightarrow & \mathbb{Z}/n\mathbb{Z} \\ \| & & \downarrow{\scriptstyle\psi} & & \| \\ T(Z) & \overset{\varphi}{\hookrightarrow} & T(M) & \twoheadrightarrow & \mathbb{Z}/n\mathbb{Z}. \end{array}$$

Eventually, we use Mukai's result Proposition 7.1 to find an isomorphism $f : M \to X$ with $f^*|_{T(X)} = \pm\psi$. Let us first suppose that $f^*|_{T(X)} = -\psi$. In view of the commutativity of the above diagram this morphism satisfies $f^*\alpha = \beta^{-1}$. Which shows that there exists a Fourier–Mukai equivalence

$$\mathrm{D^b}(Z) \xrightarrow[\Phi]{\sim} \mathrm{D^b}(M, \beta^{-1}) \xrightarrow[f_*]{\sim} \mathrm{D^b}(X, \alpha)$$

that preserves the orientation. If $f^*|_{T(X)} = \psi$, then use the composition

$$\mathrm{D^b}(Z) \xrightarrow[\Psi]{\sim} \mathrm{D^b}(M, \beta^{-1}) \xrightarrow[f_*]{\sim} \mathrm{D^b}(X, \alpha).$$

Here, Ψ is the Fourier–Mukai transform with kernel $\mathcal{E}^*$ that is obtained by dualizing the $(1, \beta^{-1})$-twisted universal family $\mathcal{E}$ on $Z \times M$. Thus, $\mathcal{E}^*$ is a $(1, \beta)$-object. Using the standard criterion it is easy to see that the Fourier–Mukai transform $\mathrm{D^b}(Z) \cong \mathrm{D^b}(M, \beta)$ induced by it is also an equivalence. This is essentially due to the fact that $\mathrm{Ext}^i(\mathcal{E}_x, \mathcal{E}_y) \cong \mathrm{Ext}^i(\mathcal{E}_y^*, \mathcal{E}_x^*)$ for all $x, y \in Z$. (The statement reminds of the fact that $\Phi_{\mathcal{E}^*} : \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ is an equivalence if $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ is one. A proof of this fact not relying on the point criterion can be found in [30].)

A quick look at the induced cohomological Fourier–Mukai transform reveals that Ψ_* is as well orientation preserving. Indeed, if $\Phi_*(\exp(i\omega)) = \lambda\exp(b+ia)$ and $\Psi_*(\exp(i\omega)) = \lambda^*\exp(b^*+ia^*)$ then $\lambda^* = \bar{\lambda}$ and $\lambda^*\exp(b^*+ia^*) = -\overline{\lambda\exp(b+ia)}$ (we use the notation of Section 5). Hence, the imaginary part of the degree two part of the image of $\exp(i\omega)$ does not change. $\square$

Corollary 7.4. *Let (X,α) and (X',α') be twisted K3 surfaces. Assume $\rho(X) = \rho(X') \geq 12$. Then any equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha) \cong \mathrm{D}^{\mathrm{b}}(X',\alpha')$ is of Fourier–Mukai type.*

Proof. The proposition provides Fourier–Mukai equivalences $\Phi : \mathrm{D}^{\mathrm{b}}(Z) \cong \mathrm{D}^{\mathrm{b}}(X,\alpha)$ and $\Phi' : \mathrm{D}^{\mathrm{b}}(Z') \cong \mathrm{D}^{\mathrm{b}}(X',\alpha')$. If $\Psi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \cong \mathrm{D}^{\mathrm{b}}(X',\alpha')$ is any equivalence, then $\Phi'^{-1}\circ\Psi\circ\Phi : \mathrm{D}^{\mathrm{b}}(Z) \cong \mathrm{D}^{\mathrm{b}}(Z')$ is an equivalence between untwisted derived categories and hence, due to the result of Orlov, of Fourier–Mukai type. This is enough to conclude that also Ψ is of Fourier–Mukai type. $\square$

There is a natural Hodge isometry $\widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X,-B,\mathbb{Z})$ provided by $j = (-\mathrm{id}_{H^0}) \oplus \mathrm{id}_{H^2} \oplus (-\mathrm{id}_{H^4})$. However, it is not preserving the orientation of the positive directions. So, according to the modified Căldăraru conjecture 4.9 we do not expect this Hodge isometry to lift to a Fourier-Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X,\alpha_{-B})$. In fact, in general the two twisted derived categories $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ and $\mathrm{D}^{\mathrm{b}}(X,\alpha^{-1})$ are probably not equivalent (except if α is of order two). However, if the Picard group is big, this is true and follows from the proof of the proposition:

Corollary 7.5. *Let (X,α) be a twisted K3 surface with $\rho(X) \geq 12$. Then there exists an orientation preserving Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha) \cong \mathrm{D}^{\mathrm{b}}(X,\alpha^{-1})$.* $\square$

Remark 7.6. The arguments given in the proof also show that any Fourier–Mukai equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X,\alpha) \cong \mathrm{D}^{\mathrm{b}}(X',\alpha')$ induces a Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha^{-1}) \cong \mathrm{D}^{\mathrm{b}}(X',\alpha'^{-1})$.

We emphasize that the equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha) \cong \mathrm{D}^{\mathrm{b}}(X,\alpha^{-1})$ is by no means canonical, as it depends on the choice of the intermediate K3 surface Z and the kernel yielding the equivalence $\mathrm{D}^{\mathrm{b}}(Z) \cong \mathrm{D}^{\mathrm{b}}(X,\alpha)$. This becomes evident if one considers a case where α is trivial, $X = Z$, and $\mathrm{D}^{\mathrm{b}}(Z) \cong \mathrm{D}^{\mathrm{b}}(X)$ is given by a kernel $\mathcal{E}$ different from the diagonal. Then the autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ constructed by the above methods would be $\Phi_{\mathcal{E}}^2$, which is in no way natural.

Let us now come to the main result of this section.

Proposition 7.7. *Let X_1, X_2 be two algebraic K3 surfaces with rational B-fields B_1 respectively B_2. Assume $\rho(X_1) \geq 12$.*

If $g : \widetilde{H}(X_1,B_1,\mathbb{Z}) \cong \widetilde{H}(X_2,B_2,\mathbb{Z})$ is an orientation preserving Hodge isometry, then there exists a Fourier–Mukai equivalence

$$\Phi : \mathrm{D}^{\mathrm{b}}(X_1,\alpha_{B_1}) \cong \mathrm{D}^{\mathrm{b}}(X_2,\alpha_{B_2})$$

with $\Phi_*^{B_1,B_2} = g$.

Proof. The previous proposition provides K3 surfaces Z_1 and Z_2 together with Fourier–Mukai equivalences $\Phi_1 : \mathrm{D^b}(Z_1) \cong \mathrm{D^b}(X_1, \alpha_{B_1})$ and $\Phi_2 : \mathrm{D^b}(Z_2) \cong \mathrm{D^b}(X_2, \alpha_{B_2})$ both preserving the orientation of the positive directions in cohomology. On the level of cohomology this yields an orientation preserving Hodge isometry (cf. Proposition 4.3)

$$h : \widetilde{H}(Z_1,\mathbb{Z}) \xrightarrow{\Phi_{1*}^{0,B_1}} \widetilde{H}(X_1,B_1,\mathbb{Z}) \xrightarrow{g} \widetilde{H}(X_2,B_2,\mathbb{Z}) \xrightarrow{(\Phi_{2*}^{0,B_2})^{-1}} \widetilde{H}(Z_2,\mathbb{Z}).$$

Hence, by results of Mukai, Orlov et al, there exists an equivalence $\Psi : \mathrm{D^b}(Z_1) \cong \mathrm{D^b}(Z_2)$ with $\Psi_* = h$. $\square$

Corollary 7.8. *Let X_1 and X_2 be algebraic K3 surfaces with $\rho(X_i) \geq 12$ and $B_i \in H^2(X_i,\mathbb{Q})$, $i = 1, 2$. Then the following conditions are equivalent.*
i) *There exists a Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(X_1, \alpha_{B_1}) \cong \mathrm{D^b}(X_2, \alpha_{B_2})$.*
ii) *There exists a Hodge isometry $\widetilde{H}(X_1,B_1,\mathbb{Z}) \cong \widetilde{H}(X_2,B_2,\mathbb{Z})$.*
iii) *There exists a Hodge isometry $T(X_1,B_1) \cong T(X_2,B_2)$.*

Proof. By Proposition 4.3, it is clear that ii) and iii) follow from i) (without any assumption on the Picard number). That ii) implies i) follows from Proposition 7.7 and Corollary 7.5.

In order to see that iii) implies ii) one remarks that the natural embedding

$$T(X,B) \hookrightarrow \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{\Lambda}$$

is unique by the results of Nikulin [27, Thm.1.14.4] (see Remark 7.11). Hence, any Hodge isometry between the twisted transcendental lattices extends to a Hodge isometry of the full twisted Hodge structures. $\square$

The following result answers a question of Căldăraru (see 5.5.3 in [7]) affirmatively in the case of large Picard number, although the answer in the general case should be negative (see Example 4.11). At the same time, it generalizes Corollary 7.5.

Corollary 7.9. *Let X be a K3 surface with $\rho(X) \geq 12$ and $\alpha \in \mathrm{Br}(X)$. If k is prime to the order of α, then there exists a Fourier–Mukai equivalence $\mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X,\alpha^k)$.*

Proof. Just observe that the kernel of $\alpha : T(X) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ and $\alpha^k : T(X) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ are Hodge isometric and apply the previous corollary. $\square$

Remark 7.10. A priori the order of the two Brauer classes α_{B_1} and α_{B_2} in the proposition might be different (cf. Remark 8.2). Although we have seen earlier that Brauer classes on the same K3 surface defining equivalent twisted derived categories are of the same order (see Remark 3.1).

Remark 7.11. As it turns out, all results of this section hold true under a slightly technical but weaker lattice theoretical condition. Let us briefly explain this here. First we recall the original result of Nikulin [27, Thm.1.14.4] as given in [24]: Let T and Λ be an even respectively an even unimodular lattice such that $t_\pm < s_\pm$ and

$$\ell(T) \leq \mathrm{rk}(\Lambda) - \mathrm{rk}(T) - 2. \tag{1}$$

Then there exists a unique (up to isometries of Λ) primitive embedding of T into Λ. Here, (t_+, t_-) and (s_+, s_-) are the signatures of the lattices T respectiveley Λ and $\ell(T)$ is the minimal number of generators of the discriminant group $A_T = T^\vee/T$.

If $\mathrm{rk}(T) \leq 10$ then (1) holds true with Λ the K3 lattice. But clearly there are other cases when these assumptions are satisfied.

8. Counting twisted Fourier–Mukai partners

In the untwisted as well as in the twisted case it has been shown that the number of non-isomorphic Fourier–Mukai partners of a given algebraic (twisted) K3 surface is always finite. But the number itself can be arbitrarily large. More precisely, it has been shown in [28, 32] that for any N there exist pairwise non-isomorphic K3 surfaces $X_1, \ldots, X_N$ with

$$\mathrm{D^b}(X_1) \cong \ldots \cong \mathrm{D^b}(X_N).$$

However, in the untwisted case the Picard number of these K3 surfaces has to be small, i.e. $\rho(X_i) < 12$. Indeed, if $\mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$ and $\rho(X_i) \geq 12$, then due to Proposition 7.1 one automatically has $X_1 \cong X_2$. Moreover, following [18], this also holds true if $\rho \geq 3$ and the determinant of $\mathrm{Pic}(X)$ is square free. For the calculation of the number of Fourier–Mukai partners in the untwisted case see [18, 32].

Passing to the twisted situation allows one to construct arbitrarily many (say N) pairwise non-isomorphic twisted K3 surfaces $(X_1, \alpha_1), \ldots, (X_N, \alpha_N)$ of large Picard number.

More precisely, one has

Proposition 8.1. *For any N there exist N pairwise non-isomorphic algebraic K3 surfaces $X_1, \ldots, X_N$ of Picard number $\rho(X_i) = 20$, endowed with Brauer classes $\alpha_1, \ldots, \alpha_N$, respectively, such that the twisted derived categories $\mathrm{D^b}(X_i, \alpha_i)$, $i = 1, \ldots, n$, are all Fourier–Mukai equivalent.*

Proof. Let $p_1, \ldots, p_N$ be the first N primes and consider the following diagonal positive definite (2×2)-matrices: $C_N := \mathrm{diag}(2, 4\prod p_i^2)$ and $B_i := \mathrm{diag}(2, 4p_i^2)$. We denote the lattices defined by them as T respectively T_i and their natural generators by e_1, e_2 and $f_{i,1}, f_{i,2}$, respectively. Clearly, the lattice T can be embedded (non-primitively) into each of the T_i by $e_1 \mapsto f_{i,1}$ and $e_2 \mapsto (\prod_{k \neq i} p_k) f_{i,2}$.

Using primitive embeddings of T and T_i into Λ and the surjectivity of the period map, one finds K3 surfaces Z and X_i realizing T respectively T_i as their transcendental lattices. In particular, the Picard number of all of them is 20.

The methods of Proposition 7.3 apply and yield equivalences $\mathrm{D}^{\mathrm{b}}(Z) \cong \mathrm{D}^{\mathrm{b}}(X_i, \alpha_i)$. As $|\mathrm{disc}(T(X_i))| = 8p_i^2$, all K3 surfaces X_i are pairwise non-isomorphic. □

The above proof can be modified for other Picard numbers ($\rho \geq 8$) and with additional geometric conditions on the surfaces X_i, e.g. to be elliptic or Kummer. We leave this to the reader.

Remark 8.2. The examples constructed above are also interesting from another point of view. Namely, they provide examples of derived equivalent twisted K3 surfaces (X_1, α_1) and (X_2, α_2) with $\mathrm{ord}(\alpha_1) \neq \mathrm{ord}(\alpha_2)$. Compare the discussion in Remark 7.10.

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D.H.: Mathematisches Institut, Universität Bonn, Beringstr. 1, D- 53115 Bonn, Germany

E-mail address: `huybrech@math.uni-bonn.de`

P.S.: Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy

E-mail address: `paolo.stellari@unimi.it`

URL: `https://sites.unimi.it/stellari`