FOURIER–MUKAI FUNCTORS: A SURVEY

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Abstract. This paper surveys some recent results about Fourier–Mukai functors. In particular, given an exact functor between the bounded derived categories of coherent sheaves on two smooth projective varieties, we deal with the question whether this functor is of Fourier–Mukai type. Several related questions are answered and many open problems are stated.

1. Introduction

Fourier–Mukai functors are ubiquitous in geometric contexts and the general belief is that they actually are *the* geometric functors. Essentially, all known exact functors are of Fourier–Mukai type in the setting of proper schemes. This paper may be seen as an attempt to survey some recent works addressing this expectation according to several points of view.

Let us first recall the definition of this kind of functors. Assume that X_1 and X_2 are smooth projective varieties over a field $\Bbbk$ and denote by $\mathrm{D}^b(X_i) := \mathrm{D}^b(\mathbf{Coh}(X_i))$ the bounded derived category of coherent sheaves on X_i. Given $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ we define the exact functor $\Phi_{\mathcal{E}} \colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ as

$$\Phi_{\mathcal{E}}(-) := \mathbf{R}(p_2)_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)), \tag{1.1}$$

where $p_i \colon X_1 \times X_2 \to X_i$ is the natural projection. An exact functor $\mathsf{F} \colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is a *Fourier–Mukai functor* (or of *Fourier–Mukai type*) if there exists $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$. The complex $\mathcal{E}$ is called a *kernel* of F. This definition will be extended to more general settings in the course of the paper allowing X_i to be singular or considering supported derived categories.

One of the first examples of these functors appeared in Mukai's seminal paper [40] dating 1981. Mukai studied what he originally called a *duality* between the bounded derived category $\mathrm{D}^b(A)$ of an abelian variety (or a complex torus) A and the one of its dual variety $\hat{A}$. Such a duality is nothing but an equivalence

$$\mathsf{F} \colon \mathrm{D}^b(A) \longrightarrow \mathrm{D}^b(\hat{A})$$

realized as a Fourier–Mukai functor whose kernel is precisely the universal Picard sheaf $\mathcal{P} \in \mathbf{Coh}(A \times \hat{A})$. In other words, the inverse of F sends a skyscraper sheaf $\mathcal{O}_{\mathrm{p}}$ (here p is a closed point of $\hat{A}$) on $\hat{A}$ to the degree 0 line bundle $L_{\mathrm{p}} \in \mathrm{Pic}^0(A)$ parametrized by p.

This discussion motivates the appearance of the word 'Mukai' in the name of these functors. On the other hand, Mukai himself clarified why they should be thought of as a sort of Fourier

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transforms. Indeed, the push forward along the projection is the analogue of the integration while the Fourier–Mukai kernel is the same as the kernel in a Fourier transform.

A more precise historical reconstruction of the origins of the notion of Fourier–Mukai functor should certainly point to the paper [50] where the notion of *Fourier–Sato transform* was introduced (see also Section 3.7 in [28]). This is probably one of the first attempts to 'categorify' the Fourier transform.

There are several possible directions along which to study these functors. In this paper, we are interested in the very specific but important question already mentioned at the beginning:

Are all exact functors between the bounded derived categories of smooth projective varieties of Fourier–Mukai type?

This is certainly one of the main open problems in the literature concerning the special geometric incarnation of the theory of derived categories. Our aim is to survey the more recent approaches to it and, at the same time, to analyze other related questions concerning, for example, the uniqueness of the Fourier–Mukai kernels. The relevance of the question above cannot be overestimated. Indeed, once we know that an exact functor is of Fourier–Mukai type and the base field is $\mathbb{C}$, then we can study its action on various cohomology groups and deform it along with the varieties. In Section 2 we survey some of these issues.

The main problems we want to consider are listed in Section 3.1. The breakthroughs in the theory are contained in [46] and, more recently, in [36], where new inputs from the theory of dg-categories are taken into account. Namely,

(A) **Orlov** [46]**:** If $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is a fully faithful functor and X_1, X_2 are smooth projective varieties, then there exists a unique (up to isomorphism) $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$ (see Theorem 3.1).

(B) **Lunts–Orlov** [36]**:** The same holds when X_1 and X_2 are projective schemes and we deal with the categories of perfect complexes on them (see Theorem 5.3).

These two results will provide the two leading references in this paper. They will be explained in Sections 3 and 5 and, at the same time, we will study to which extent we may expect that they can be extended and generalized. The examples that seem to be encouraging in this direction are roughly the following (more precise statements are given in the forthcoming sections):

(a) **Toën** [52]**:** Quasi-functors between dg-enhancements of the categories of perfect complexes on projective schemes (see Theorem 5.8).

(b) Exact functors between the abelian categories of coherent sheaves on smooth projective varieties (see Proposition 5.15 and [19]).

In both cases, one proves that these functors are of Fourier–Mukai type (in an appropriate sense) and that the kernel is unique (up to isomorphism). We also point to [6] (and [48]) for results extending those in [52].

The fact that an optimistic point of view about extending (A) and (B) in full generality may be too much is discussed in Section 4.

During the exposition we will explain and list several open problems appearing naturally in many geometric contexts. They will be presented all along the paper and, in particular, in Section 6.

Motivations are discussed in Section 2. Sections 3 and 5 deal with the main results and techniques now available in the literature. Of course, we do not pretend to be exhaustive and complete in our presentation. For example, other overviews on the subject but from completely different perspectives are in [1, 21] (and, of course, in [23]).

Notation. In the paper, $\Bbbk$ is a field. Unless otherwise stated, all schemes are assumed to be of finite type and separated over $\Bbbk$; similarly, all additive (in particular, triangulated) categories and all additive (in particular, exact) functors will be assumed to be $\Bbbk$-linear. An additive category will be called Hom-finite if the $\Bbbk$-vector space $\mathrm{Hom}(A, B)$ is finite dimensional for any two objects A and B. If $\mathbf{A}$ is an abelian (or more generally an exact) category, $\mathrm{D}(\mathbf{A})$ denotes the derived category of $\mathbf{A}$ and $\mathrm{D}^b(\mathbf{A})$ its full subcategory of bounded complexes. Unless stated otherwise, all functors are derived even if, for simplicity, we use the same symbol for a functor and its derived version.

2. Motivations

In this section we would like to motivate the relevance of Fourier–Mukai functors a bit more. We stress their appearance in moduli problems and we give indications concerning the way they induce actions on various cohomologies. The reader interested in an introduction about derived and triangulated categories in geometric contexts can have a look at [23].

2.1. **First properties and examples from moduli problems.** There are several instances where Fourier–Mukai functors appear. To make this clear, we discuss some examples.

Example 2.1. Let X_1 and X_2 be smooth projective varieties.

(i) Given an object $\mathcal{E} \in \mathrm{D}^b(X_1)$, the functor $\mathsf{F}(-) = \mathcal{E} \otimes (-)$ is of Fourier–Mukai type. Namely, its Fourier–Mukai kernel is the object $\Delta_*\mathcal{E}$, where $\Delta\colon X_1 \to X_1 \times X_1$ is the diagonal embedding.

A special example is provided by the Serre functor of X_i which is the exact equivalence $\mathsf{S}_{X_i}(-) = (-) \otimes \omega_{X_i}[\dim(X_i)]$, where ω_{X_i} is the dualizing sheaf of X_i. Hence S_{X_i} is of Fourier–Mukai type. For later use, set $S_{X_i} := \omega_{X_i}[\dim(X_i)]$.

(ii) For a given morphism $f\colon X_1 \to X_2$, denote by Γ_f its graph. Then f_* is a Fourier–Mukai functor with kernel $\mathcal{O}_{\Gamma_f}$. Analogously, one can show that f^* is a Fourier–Mukai functor whose kernel is always $\mathcal{O}_{\Gamma_f}$, providing now a functor $\mathrm{D}^b(X_2) \to \mathrm{D}^b(X_1)$.

We list here a number of useful properties.

Proposition 2.2. *Let X_1 and X_2 be smooth projective varieties over $\Bbbk$ and let $\Phi_{\mathcal{E}}$ be a Fourier–Mukai functor.*

(i) *The left and right adjoints of $\Phi_{\mathcal{E}}$ exist and are of Fourier–Mukai type with kernels $\mathcal{E}_L := \mathcal{E}^\vee \otimes p_2^* S_{X_2}$ and $\mathcal{E}_R := \mathcal{E}^\vee \otimes p_1^* S_{X_1}$ respectively, where $p_i\colon X_1 \times X_2 \to X_i$ is the projection.*

(ii) *The composition of two Fourier–Mukai functors is again of Fourier–Mukai type.*

We leave it to the reader to explicitly determine a kernel in (ii) above.

Let us now see some more complicated but interesting examples. Indeed, soon after [40], it was clear that Fourier–Mukai functors appear in many moduli problems. This is the case of K3 surfaces (i.e. smooth, compact, complex simply connected surfaces with trivial canonical bundle) and moduli spaces of stable sheaves on them. Following [41], let X be a projective K3 surface and M a fine moduli space of stable sheaves on X with topological invariants fixed in such a way that M is again a projective K3 surface. The universal family $\mathcal{E} \in \mathbf{Coh}(M \times X)$ associated to this moduli problem provides an equivalence of Fourier–Mukai type

$$\Phi_{\mathcal{E}} \colon \mathrm{D}^b(M) \longrightarrow \mathrm{D}^b(X)$$

sending a skyscraper sheaf to a stable sheaf on X. Most remarkably, it was observed in [46] that all K3 surfaces Y such that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ are actually isomorphic to moduli spaces of stable sheaves on X.

In higher dimensions the interplay between Fourier–Mukai functors, geometric problems and moduli interpretations of them have been extensively studied. There are many occurrences in the context of birational geometry and in the more modern theory of stability conditions due to Bridgeland. We refrain from discussing them in this paper.

2.2. **Action on (singular) cohomology.** Having a description of an exact functor as a Fourier–Mukai functor allows one to define an action on cohomologies and homologies of various types. This may be very useful to describe the groups of autoequivalences of the derived categories of smooth projective varieties, which are rather complicated algebraic objects as soon as the variety has trivial canonical bundle.

The first highly non-trivial example we have in mind is the group of autoequivalences of the derived category of a projective K3 surface X. This group has a very complicated structure coming from the presence of the so called *spherical objects* in $\mathrm{D}^b(X)$ (i.e. objects whose endomorphism graded algebra is isomorphic to the cohomology of a 2-sphere). The idea proposed in [46] is to approach the analysis of $\mathrm{Aut}(\mathrm{D}^b(X))$ by studying its action on singular cohomology.

To spell this out clearly, we start with some general remarks. Assume that X_1 and X_2 are smooth complex projective varieties and let $\Phi_{\mathcal{E}} \colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ be a Fourier–Mukai functor with kernel $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$. Then the induced morphism at the level of Grothendieck groups is given by the morphism $\Phi^K_{[\mathcal{E}]} \colon K(X_1) \to K(X_2)$ defined by

$$\Phi^K_{[\mathcal{E}]}(e) := (p_2)_*([\mathcal{E}] \cdot p_1^*(e)),$$

where $p_i \colon X_1 \times X_2 \to X_i$ is the natural projection.

Going further, for $\mathcal{G} \in \mathrm{D}^b(X_i)$, one can consider the *Mukai vector*

$$v([\mathcal{G}]) := \mathrm{ch}(\mathcal{G}) \cdot \sqrt{\mathrm{td}(X_i)}$$

of $\mathcal{G}$. When the context is clear, we write $v(\mathcal{G})$ instead of $v([\mathcal{G}])$. Now the morphism $\Phi^K_{[\mathcal{E}]} \colon K(X_1) \to K(X_2)$ gives rise to a map $\Phi^H_{v([\mathcal{E}])} \colon H^*(X_1, \mathbb{Q}) \to H^*(X_2, \mathbb{Q})$ such that

$$\Phi^H_{v([\mathcal{E}])} \colon b \longmapsto (p_2)_*(v([\mathcal{E}]) \cdot p_1^*(b)).$$

The Grothendieck–Riemann–Roch Theorem shows that the following diagram commutes:

$$\begin{tikzcd} K(X_1) \arrow[r, "\Phi^K_{[\mathcal{E}]}"] \arrow[d, "v(-)"'] & K(X_2) \arrow[d, "v(-)"] \\ H^*(X_1,\mathbb{Q}) \arrow[r, "\Phi^H_{v([\mathcal{E}])}"] & H^*(X_2,\mathbb{Q}). \end{tikzcd} \tag{2.1}$$

From now on, given a Fourier–Mukai functor $\Phi_{\mathcal{E}}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$, we denote $\Phi^H_{v([\mathcal{E}])}$ by $\Phi^H_{\mathcal{E}}$. The following is a fairly easy remark from [46].

Proposition 2.3. *With the above assumptions, the morphism* $\Phi^H_{\mathcal{E}}\colon H^*(X_1,\mathbb{Q}) \to H^*(X_2,\mathbb{Q})$ *is an isomorphism of* $\mathbb{Q}$*-vector spaces if* $\Phi_{\mathcal{E}}$ *is an equivalence.*

For a positive integer n, one may take the Hodge decomposition $H^n(X_i,\mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X_i)$. A Fourier–Mukai equivalence does not preserve such a decomposition as, in general, it does not preserve the grading of the cohomology rings. Nevertheless, one has the following.

Proposition 2.4. *If* $\Phi_{\mathcal{E}}$ *is an equivalence, the morphism* $\Phi^H_{\mathcal{E}}$ *induces isomorphisms*

$$\bigoplus_{p-q=i} H^{p,q}(X_1) \cong \bigoplus_{p-q=i} H^{p,q}(X_2)$$

for all integers i.

The vector space $H^*(X_i,\mathbb{C})$ can be endowed with some additional structure. Namely, for $v = \sum v_j \in \bigoplus_j H^j(X_i,\mathbb{C})$, set $v^\vee := \sum \sqrt{-1}^j v_j$. Then, for all $v,w \in H^*(X_i,\mathbb{C})$, we can define the *Mukai pairing*

$$\langle v,w\rangle_{X_i} := \int_{X_i} \exp(\mathrm{c}_1(X_i)/2).(v^\vee.w).$$

Proposition 2.5. *If* $\Phi_{\mathcal{E}}$ *is an equivalence, then the morphism* $\Phi^H_{\mathcal{E}}$ *preserves the Mukai pairing.*

Before going back to specific examples, let us mention a property that will be discussed later on in a different context. Here we assume that $\Phi_{\mathcal{E}},\Phi_{\mathcal{F}}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ are Fourier–Mukai functors and not necessarily equivalences.

Lemma 2.6. *If* $\Phi^H_{\mathcal{E}} = \Phi^H_{\mathcal{F}}$, *then* $v([\mathcal{E}]) = v([\mathcal{F}])$.

Proof. The morphisms $\Phi^H_{\mathcal{E}}$ and $\Phi^H_{\mathcal{F}}$ are induced by objects in $H^*(X_1 \times X_2,\mathbb{Q})$. Now apply the Künneth decomposition for the cohomology of the product to get $v(\mathcal{E}) = v(\mathcal{F})$. □

In particular, this means that the 'cohomological Fourier–Mukai kernel' of cohomological Fourier–Mukai functors is *always* uniquely determined. Due to what we will show in Section 4, one can speak about *the* action of a Fourier–Mukai functor, being independent of the choice of the Fourier–Mukai kernel.

Assume now that X_1 and X_2 are projective K3 surfaces and take a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$. A remark by Mukai shows that $\Phi^H_{\mathcal{E}}$ induces an isomorphism of $\mathbb{Z}$-modules $H^*(X_1,\mathbb{Z}) \cong H^*(X_2,\mathbb{Z})$ in this case. The total cohomology $H^*(X_i,\mathbb{Z})$ endowed with the Mukai pairing and the Hodge structure mentioned in Proposition 2.4, is called the *Mukai lattice* and denoted by $\widetilde{H}(X_i,\mathbb{Z})$. Using the action of equivalences on cohomology and a bit of lattice theory, one can prove the following.

Proposition 2.7. **([12], Proposition 5.3.)** *Given a projective K3 surface X, the number of isomorphism classes of K3 surfaces Y such that $\mathrm{D}^b(X)\cong\mathrm{D}^b(Y)$ is finite.*

Nevertheless such a number can be arbitrarily large.

Proposition 2.8. **([43] and [51])** *For any positive integer N, there exist non-isomorphic K3 surfaces $X_1,\ldots,X_N$ such that $\mathrm{D}^b(X_i)\cong\mathrm{D}^b(X_j)$ for $i,j=1,\ldots,N$.*

Two smooth projective varieties X_1 and X_2 such that $\mathrm{D}^b(X_1)\cong\mathrm{D}^b(X_2)$ are usually called *Fourier–Mukai partners.* Notice that Proposition 2.7 is a special instance of the following conjecture which is nothing but [29, Conj. 1.5].

Conjecture 2.9. (Kawamata) *The number of Fourier–Mukai partners up to isomorphism of a smooth projective variety is finite.*

Abelian varieties satisfy this prediction as well (see [45]). In [2], the authors provide further evidence for it.

To give one more important application of the discussion in this section, we can go back to the problem mentioned at the beginning of this section and use the structure of Fourier–Mukai functors to get a (partial) description of the group of autoequivalences of a K3 surface X. The following is the result of the papers [22, 24, 46].

Theorem 2.10. *For a K3 surface X, there exists a surjective morphism*

$$\mathrm{Aut}(\mathrm{D}^b(X))\longrightarrow \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$$

sending a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}$ to $\Phi^H_{\mathcal{E}}$.

Here $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$ is the group of Hodge isometries of the Mukai lattice preserving the orientation of some 4-dimensional (real) vector subspace of $H^*(X,\mathbb{R})$.

2.3. **Hochschild homology, cohomology and deformations.** For many geometric purposes, the cohomology theory one may want to consider is Hochschild cohomology (and homology). More precisely, assume that a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}\colon \mathrm{D}^b(X_1)\to\mathrm{D}^b(X_2)$ between the bounded derived categories of the smooth complex projective varieties X_1 and X_2 is given. Then one may want to study (first order) deformations of X_i compatible with deformations of the Fourier–Mukai kernel $\mathcal{E}\in\mathrm{D}^b(X_1\times X_2)$. To this end, we indeed have to study Hochschild cohomology and homology and the corresponding actions of $\Phi_{\mathcal{E}}$.

If X is a smooth projective variety and ω_X is its dualizing sheaf, we define S_X as in Example 2.1, $S_X^{-1}:=\omega_X^\vee[-\dim(X)]$ and $S_\Delta^{\pm1}:=(\Delta)_*S_X^{\pm1}$, where $\Delta\colon X\hookrightarrow X\times X$ is the diagonal embedding. The *i-th Hochschild homology and cohomology groups*, $i\in\mathbb{Z}$, are respectively (see, for example, [15])

$$\begin{aligned}\mathrm{HH}_i(X)&:=\mathrm{Hom}_{\mathrm{D}^b(X\times X)}(S_\Delta^{-1}[i],\mathcal{O}_\Delta)\cong\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X[i],\Delta^*\mathcal{O}_\Delta)\\ \mathrm{HH}^i(X)&:=\mathrm{Hom}_{\mathrm{D}^b(X\times X)}(\mathcal{O}_\Delta,\mathcal{O}_\Delta[i])\cong\mathrm{Hom}_{\mathrm{D}^b(X)}(\Delta^*\mathcal{O}_\Delta,\mathcal{O}_X[i]).\end{aligned}$$

Set $\mathrm{HH}_*(X) := \bigoplus_i \mathrm{HH}_i(X)$ and $\mathrm{HH}^*(X) := \bigoplus_i \mathrm{HH}^i(X)$. The *Hochschild–Kostant–Rosenberg isomorphisms* are graded isomorphisms

$$I^X_{\mathrm{HKR}} \colon \mathrm{HH}_*(X) \to \mathrm{H}\Omega_*(X) := \bigoplus_i \mathrm{H}\Omega_i(X)$$
$$I_X^{\mathrm{HKR}} \colon \mathrm{HH}^*(X) \to \mathrm{HT}^*(X) := \bigoplus_i \mathrm{HT}^i(X),$$

where $\mathrm{H}\Omega_i(X) := \bigoplus_{q-p=i} H^p(X, \Omega^q_X)$ and $\mathrm{HT}^i(X) := \bigoplus_{p+q=i} H^p(X, \wedge^q \mathcal{T}_X)$. One then defines the graded isomorphisms

$$I^X_K = (\mathrm{td}(X)^{1/2} \wedge (-)) \circ I^X_{\mathrm{HKR}}$$
$$I^K_X = (\mathrm{td}(X)^{-1/2} \lrcorner (-)) \circ I_X^{\mathrm{HKR}}.$$

From [14, 15], we get a functorial graded morphism $(\Phi_{\mathcal{E}})_{\mathrm{HH}} \colon \mathrm{HH}_*(X_1) \to \mathrm{HH}_*(X_2)$. The following shows the compatibility between this action and the one described in Section 2.2. It is based on [39].

Theorem 2.11. **([38], Theorem 1.2.)** *Let X_1 and X_2 be smooth complex projective varieties and let $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$. Then the following diagram*

$$\begin{tikzcd}
\mathrm{HH}_*(X_1) \arrow[r, "(\Phi_{\mathcal{E}})_{\mathrm{HH}}"] \arrow[d, "I^{X_1}_K"'] & \mathrm{HH}_*(X_2) \arrow[d, "I^{X_2}_K"] \\
H^*(X_1, \mathbb{C}) \arrow[r, "\Phi^H_{\mathcal{E}}"] & H^*(X_2, \mathbb{C})
\end{tikzcd}$$

commutes.

If $\Phi_{\mathcal{E}}$ is an equivalence, then there exists also an action $(\Phi_{\mathcal{E}})^{\mathrm{HH}}$ on Hochschild cohomology induced by the functor $\Phi_{\mathcal{E} \boxtimes \mathcal{P}} \colon \mathrm{D}^b(X_1 \times X_1) \to \mathrm{D}^b(X_2 \times X_2)$, where $\mathcal{P} \cong \mathcal{E}_L \cong \mathcal{E}_R$ is the kernel of the inverse of $\Phi_{\mathcal{E}}$, which sends $\mathcal{O}_{\Delta_{X_1}}$ to $\mathcal{O}_{\Delta_{X_2}}$ (see, for example, [23, Remark 6.3]).

Now the second Hochschild cohomolgy group controls first order deformations of a smooth projective variety. Hence, given a Fourier–Mukai equivalence $\Phi_{\mathcal{E}} \colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ and combining the actions $(\Phi_{\mathcal{E}})_{\mathrm{HH}}$, $(\Phi_{\mathcal{E}})^{\mathrm{HH}}$ and Theorem 2.11, one can control first order deformations of X_1 and X_2 compatible with deformations of the Fourier–Mukai functor $\Phi_{\mathcal{E}}$. This was done, for example, in [24].

Interesting recent developments are contained in [3], where the authors deal with fully faithful Fourier–Mukai functors whose kernel is a (shift of a) sheaf.

3. The main problems and the first improvements

In this section we list the main problems that we want to address. The answers to them which are available in the literature will be presented in Section 4. For the moment we content ourselves with a discussion of a celebrated result of Orlov about Fourier–Mukai functors. Various generalizations or attempts to weaken the hypotheses in this result are discussed in this section as well.

3.1. **The questions.** Assume for the moment that all the varieties are smooth and projective. The most important problems concerning Fourier–Mukai functors may be summarized by the following two questions:

(1) *Are all exact functors between the bounded derived categories of coherent sheaves on smooth projective varieties of Fourier–Mukai type?*

(2) *Is the kernel of a Fourier–Mukai functor unique (up to isomorphism)?*

A positive answer to the first one was conjectured in [8] as a consequence of a conjecture about the possibility to lift all exact funtors to the corresponding dg-enhancements. In these terms, a positive or negative answer to the second one implies the uniqueness or non-uniqueness of such dg-lifts.

We can now put these questions in a more general setting. Indeed, consider the category $\mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))$ of exact functors between $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$ (with morphisms the natural transformations compatible with shifts) and define the functor

$$\Phi_{-}^{X_1\to X_2}\colon \mathrm{D}^b(X_1\times X_2)\longrightarrow \mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2)) \tag{3.1}$$

by sending $\mathcal{E}\in\mathrm{D}^b(X_1\times X_2)$ to the Fourier–Mukai functor $\Phi_{\mathcal{E}}$. Thus we can formulate the following problems:

(Q1) *Is* $\Phi_{-}^{X_1\to X_2}$ *essentially surjective?*

(Q2) *Is* $\Phi_{-}^{X_1\to X_2}$ *essentially injective?*

(Q3) *Is* $\Phi_{-}^{X_1\to X_2}$ *faithful?*

(Q4) *Is* $\Phi_{-}^{X_1\to X_2}$ *full?*

(Q5) *Does* $\mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))$ *have a triangulated structure making* $\Phi_{-}^{X_1\to X_2}$ *exact?*

Clearly, (Q1) and (Q2) are precisely (1) and (2), respectively. Căldăraru provided a negative answer to (Q3) in [13, Example 6.5] (see also [47]), while a negative answer to (Q5) was expected already in [8, 52]. Nevertheless, in the seminal paper [46] a positive answer to (1) and (2) has been provided under some additional assumption on the exact functor. In the original formulation, it can be stated as follows:

Theorem 3.1. (Orlov) *Let X_1 and X_2 be smooth projective varieties and let $\mathsf{F}\colon \mathrm{D}^b(X_1)\to\mathrm{D}^b(X_2)$ be an exact fully faithful functor admitting a left adjoint. Then there exists a unique (up to isomorphim) $\mathcal{E}\in\mathrm{D}^b(X_1\times X_2)$ such that $\mathsf{F}\cong\Phi_{\mathcal{E}}$.*

A generalization to smooth stacks (actually obtained as global quotients) is contained in [30]. In the rest of this section and as a preparation for a complete discussion of (Q1)–(Q5) that will be carried out in Sections 4 and 5, we start discussing how one may try to weaken the hypotheses of the above result.

3.2. **Existence of adjoints.** Of course, in purely categorical terms, the existence of adjoints to a given functor is not automatic. In this section we will see a first approach, due to Bondal and Van den Bergh, to make this straightforward in the geometric setting we are dealing with.

Let us start from the more general setting where $\mathbf{T}$ is an Ext-finite triangulated category. This means that $\sum_n \dim_{\Bbbk} \mathrm{Hom}(A, B[n]) < \infty$, for all $A, B \in \mathbf{T}$. Denote by $\mathbf{Vect}$-$\Bbbk$ the category of $\Bbbk$-vector spaces. A contravariant functor $\mathsf{H}\colon \mathbf{T} \to \mathbf{Vect}$-$\Bbbk$ is *cohomological* if, given a distinguished triangle

$$A \longrightarrow B \longrightarrow C$$

in $\mathbf{T}$, the sequence

$$\mathsf{H}(C) \longrightarrow \mathsf{H}(B) \longrightarrow \mathsf{H}(A)$$

is exact in $\mathbf{Vect}$-$\Bbbk$. A cohomological contravariant functor H is *of finite type* if $\dim_{\Bbbk} \bigoplus_i \mathsf{H}(A[i]) < \infty$, for all $A \in \mathbf{T}$.

Definition 3.2. The triangulated category $\mathbf{T}$ is *(right) saturated* if every cohomological contravariant functor H of finite type is representable, i.e. there exists $A \in \mathbf{T}$ and an isomorphism of functors

$$\mathsf{H} \cong \mathrm{Hom}(-, A).$$

Remark 3.3. (i) By the Yoneda Lemma, if a cohomological functor H is representable, then the object representing it is unique (up to isomorphism).

(ii) In [10], the authors provide examples of 'geometric' categories which are not saturated. Namely, if X is a smooth compact complex surface containing no compact curves, then $\mathrm{D}^b(X)$ is not saturated. Examples in higher dimensions are given in [42].

In the smooth proper case one has the following result.

Theorem 3.4. ([10]**, Theorem 1.1.)** *Assume that X is a smooth proper scheme over $\Bbbk$. Then $\mathrm{D}^b(X)$ is saturated.*

Now assume that X_1 and X_2 are smooth proper schemes. As an application of the above theorem, we get the following well-known result.

Proposition 3.5. *Any exact functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ has left and right adjoints.*

Proof. For any $\mathcal{F} \in \mathrm{D}^b(X_2)$ the functor $\mathrm{Hom}(\mathsf{F}(-), \mathcal{F})$ is representable by a unique $\mathcal{E} \in \mathrm{D}^b(X_1)$ due to Theorem 3.4. Setting $\mathsf{G}(\mathcal{F}) := \mathcal{E}$, by the Yoneda Lemma we get a functor $\mathsf{G}\colon \mathrm{D}^b(X_2) \to \mathrm{D}^b(X_1)$ which is right adjoint to F. Since $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$ have Serre functors, it is a very easy exercise to prove that F has also a left adjoint. □

Observe that, due to [7, Prop. 1.4], the right and left adjoints in the above statement are automatically exact.

3.3. **The algebricity assumption.** In this section we show in which sense it is important to work with algebraic varieties. In particular, we give examples of exact functors between the bounded derived categories of coherent sheaves on smooth compact complex manifolds which are not of Fourier–Mukai type.

For this, let X be a generic non-projective K3 surface. With this we mean a K3 surface X such that $\mathrm{Pic}(X) = 0$. The following surprising result shows that the abelian categories of coherent sheaves on those surfaces are not fine invariants (see, for example, [37] for a brief account about coherent sheaves and Chern characters in this setting).

Theorem 3.6. ([53]) *Let X_1 and X_2 be generic non-projective K3 surfaces. Then there exists an equivalence of abelian categories $\mathbf{Coh}(X_1) \cong \mathbf{Coh}(X_2)$.*

Remark 3.7. (i) In the case of smooth projective varieties X_1 and X_2 a result of Gabriel (see [23, Cor. 5.24] for an easy proof using Fourier–Mukai functors) asserts that exactly the converse holds. Namely $X_1 \cong X_2$ if and only if $\mathbf{Coh}(X_1) \cong \mathbf{Coh}(X_2)$.

(ii) The above result was proved in [54] for the case of generic non-projective complex tori as well.

Now take two non-isomorphic generic non-projective K3 surfaces X_1 and X_2. Theorem 3.6 implies that there exists an exact equivalence

$$\mathsf{F}\colon \mathrm{D}^b(X_1) \longrightarrow \mathrm{D}^b(X_2).$$

One may then wonder whether all such equivalences are of Fourier–Mukai type.

Proposition 3.8. *Let X_1 and X_2 be non-isomorphic generic non-projective K3 surfaces and let $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ be the exact equivalence induced by an exact equivalence $\mathbf{Coh}(X_1) \cong \mathbf{Coh}(X_2)$. Then F is not of Fourier–Mukai type.*

Proof. By assumption, F sends the minimal objects in $\mathbf{Coh}(X_1)$ to minimal objects in $\mathbf{Coh}(X_2)$ (recall that an object in an abelian category is minimal if it does not admit proper subobjects). In particular, following the same argument as in the proof of [23, Cor. 5.24], we get that F sends skyscraper sheaves to skyscraper sheaves. Hence if $\mathsf{F} \cong \Phi_{\mathcal{E}}$, for some $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$, then there should be an isomorphism $f\colon X_1 \to X_2$ and a line bundle $L \in \mathrm{Pic}(X_2)$ such that $\mathsf{F} \cong (L \otimes (-)) \circ f_*$ (see, for example, [23, Cor. 5.23]). But this contradicts the assumption $X_1 \ncong X_2$. □

3.4. **Non fully faithful functors.** Now we discuss how the fully faithfulness assumption can be removed. We first discuss a generalization of Theorem 3.1 while later we observe that the faithfulness assumption is redundant anyway. Indeed full functors turn out to be automatically faithful.

3.4.1. *Negative Hom's and sheaves.* We now see a way to reduce the assumptions on the functor F, that, to our knowledge, is the best one available in the literature in the context of smooth projective varieties. We will see later on how this has to be modified for perfect complexes on singular (projective) varieties. Some details about the key ingredients in the proof will be discussed in Section 4.

Theorem 3.9. ([19], Theorem 1.1.) *Let X_1 and X_2 be smooth projective varieties and let $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X_1)$,*

$$\mathrm{Hom}_{\mathrm{D}^b(X_2)}(\mathsf{F}(\mathcal{F}), \mathsf{F}(\mathcal{G})[j]) = 0 \quad \text{if } j < 0. \tag{3.2}$$

Then there exist $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

A class of exact functors satisfying (3.2) is clearly provided by full functors. Unfortunately this is not a really interesting case, as in Section 3.4.2 we will show that, in the present context, all full functors are actually automatically faithful.

Example 3.10. For a rather trivial example of a non-full exact functor satisfying (3.2), we can consider $\mathrm{id} \oplus \mathrm{id}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$, where X is a smooth projective variety. More generally, given a line bundle $L \in \mathrm{Pic}(X)$, we can take $\Phi_{\Delta_* L} \oplus \Phi_{\Delta_* L}$ (see Example 2.1).

Example 3.11. Notice that all exact functors $\mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ obtained by deriving an exact functor $\mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$ are examples of functors satisfying (3.2).

Remark 3.12. The original version of Theorem 3.9, stated in [19], deals with the more general notion of *twisted variety* where condition (3.2) can be stated as well.

3.4.2. *Full implies faithful.* In this section we assume that $\Bbbk$ is algebraically closed of characteristic 0. Let X_1 and X_2 be smooth projective varieties and assume that an exact functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is full and such that $\mathsf{F} \ncong 0$. By Theorem 3.9, F is a Fourier–Mukai functor. So $\mathsf{F} \cong \Phi_{\mathcal{E}}$, for some $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$.

There exists a very useful criterion to establish when a Fourier–Mukai functor $\Phi_{\mathcal{E}}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is fully faithful.

Theorem 3.13. ([9] and [11]) *Under the assumptions above, $\Phi_{\mathcal{E}}$ is fully faithful if and only if*

$$\mathrm{Hom}_{\mathrm{D}^b(X_2)}(\Phi_{\mathcal{E}}(\mathcal{O}_{x_1}), \Phi_{\mathcal{E}}(\mathcal{O}_{x_2})[i]) \cong \begin{cases} \Bbbk & \text{if } x_1 = x_2 \text{ and } i = 0 \\ 0 & \text{if } x_1 \neq x_2 \text{ or } i \notin [0, \dim(X_1)] \end{cases}$$

for all closed points $x_1, x_2 \in X_1$.

Thus, because of this result and the fact that F is full, to show that the functor is also faithful it is enough to prove that there are no closed points $x \in X_1$ such that $\mathrm{Hom}(\mathsf{F}(\mathcal{O}_x), \mathsf{F}(\mathcal{O}_x)) = 0$ or, in other words, such that $\mathsf{F}(\mathcal{O}_x) \cong 0$.

To see this, take the left adjoint $\mathsf{G}\colon \mathrm{D}^b(X_2) \to \mathrm{D}^b(X_1)$ of F and consider the composition $\mathsf{G} \circ \mathsf{F}$ which is again a Fourier–Mukai functor (see Proposition 2.2), hence isomorphic to $\Phi_{\mathcal{F}}$ for some $\mathcal{F} \in \mathrm{D}^b(X_1 \times X_1)$. Assume that there are $x_1, x_2 \in X_1$ such that $\mathsf{F}(\mathcal{O}_{x_1}) \ncong 0$ while $\mathsf{F}(\mathcal{O}_{x_2}) \cong 0$. By [9] (see, in particular, Proposition 1.5 there) the Mukai vector $v(\Phi_{\mathcal{F}}(\mathcal{O}_{x_1}))$ is not zero.

On the other hand, by Propositions 2.3, 2.4 and 2.5, the functor $\Phi_{\mathcal{F}}$ induces a morphism $\Phi_{\mathcal{F}}^H\colon H^*(X_1, \mathbb{Q}) \to H^*(X_1, \mathbb{Q})$ such that

$$0 \neq v(\Phi_{\mathcal{F}}(\mathcal{O}_{x_1})) = \Phi_{\mathcal{F}}^H(v(\mathcal{O}_{x_1})) = \Phi_{\mathcal{F}}^H(v(\mathcal{O}_{x_2})) = v(\Phi_{\mathcal{F}}(\mathcal{O}_{x_2})) = 0.$$

This contradiction proves that, if F were not faithful, then $\mathsf{F}(\mathcal{O}_x) \cong 0$ for every closed point $x \in X$.

We claim that if this is true, then $\mathsf{F} \cong 0$. Indeed let G and H be the left and right adjoints of F. Of course, $\mathsf{G} \circ \mathsf{F}(\mathcal{O}_x) \cong 0$, for all closed points x in X_1. In particular, for all $n \in \mathbb{Z}$ and any $\mathcal{B} \in \mathrm{D}^b(X_1)$, we have

$$0 = \mathrm{Hom}(\mathsf{G} \circ \mathsf{F}(\mathcal{O}_x), \mathcal{B}[n]) \cong \mathrm{Hom}(\mathcal{O}_x, \mathsf{H} \circ \mathsf{F}(\mathcal{B})[n]).$$

Therefore $\mathsf{H} \circ \mathsf{F}(\mathcal{B}) \cong 0$, for all $\mathcal{B} \in \mathrm{D}^b(X_1)$. But now

$$0 = \mathrm{Hom}(\mathcal{B}, \mathsf{H} \circ \mathsf{F}(\mathcal{B})) \cong \mathrm{Hom}(\mathsf{F}(\mathcal{B}), \mathsf{F}(\mathcal{B})).$$

Thus we would get $\mathsf{F}(\mathcal{B}) \cong 0$, for all $\mathcal{B} \in \mathrm{D}^b(X_1)$ and so we proved the following result.

Theorem 3.14. *Let X_1 and X_2 be smooth projective varieties over an algebraically closed field of characteristic* 0 *and assume that an exact functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is full. If $\mathsf{F} \ncong 0$, then F is faithful as well.*

Remark 3.15. (i) Notice that in [16] a more general result is proved. In particular, the target category can be any triangulated category while the source category can be the category of perfect (supported) complexes on a noetherian scheme.

(ii) One may easily extend the proof above to the case of twisted varieties. For this we just need to use the twisted version of the Chern character defined in [25] and again apply [9, Prop. 1.5]. We leave this to the reader.

4. The (partial) answers to (Q2)–(Q5)

We postpone for the moment the discussion about (Q1) which will be examined in Section 5. The remaining problems can be studied in a unitary way explained here below.

4.1. **Perfect complexes and good news.** We start our discussion with a case where all the above five questions have a positive answer. In particular, this implies that (in the smooth case) interesting examples answering these questions negatively have to be searched for in dimension greater than zero.

We begin by extending the setting explained in the previous section. In particular, let X be a projective (not necessarily smooth) scheme over $\Bbbk$. Denote by $\mathbf{Perf}(X)$ the category of *perfect complexes* on X consisting of the objects in $\mathrm{D}(\mathbf{Qcoh}(X))$ which are quasi-isomorphic to bounded complexes of locally free sheaves of finite type over X. Obviously, $\mathbf{Perf}(X) \subseteq \mathrm{D}^b(X)$ and the equality holds if and only if X is regular.

The category $\mathbf{Perf}(X)$ coincides with the full subcategory of compact objects in $\mathrm{D}(\mathbf{Qcoh}(X))$. Recall that an object A in a triangulated category $\mathbf{T}$ is *compact* if, for each family of objects $\{X_i\}_{i \in I} \subset \mathbf{T}$ such that $\bigoplus_i X_i$ exists in $\mathbf{T}$, the canonical map

$$\bigoplus_i \mathrm{Hom}(A, X_i) \longrightarrow \mathrm{Hom}\left(A, \oplus_i X_i\right)$$

is an isomorphism.

In the singular setting we redefine the notion of Fourier–Mukai functors once more since in general we cannot expect the Fourier–Mukai kernels of exact functors $\mathbf{Perf}(X_1) \to \mathbf{Perf}(X_2)$ to be objects in $\mathbf{Perf}(X_1 \times X_2)$, but rather in $\mathrm{D}^b(X_1 \times X_2)$. More precisely, one can show the following (see, for example, [18, Lemma 4.3] for the proof).

Lemma 4.1. *Let X_1 and X_2 be projective schemes and let $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$ be an object such that $\Phi_{\mathcal{E}} \colon \mathrm{D}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ (defined as in (1.1)) sends $\mathbf{Perf}(X_1)$ to $\mathrm{D}^b(X_2)$. Then $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$. Conversely, any $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ yields a Fourier–Mukai functor $\Phi_{\mathcal{E}} \colon \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$.*

Hence given two projective schemes X_1 and X_2 one can consider the functor

$$\Phi_{-}^{X_1 \to X_2} \colon \mathrm{D}^b(X_1 \times X_2) \longrightarrow \mathbf{ExFun}(\mathbf{Perf}(X_1), \mathrm{D}^b(X_2))$$

(which coincides with (3.1) in the smooth case) and for it one can again ask questions (Q1)–(Q5).

Now, if X is a projective scheme over $\Bbbk$, it is an easy exercise to show that every exact functor $\mathsf{F} \colon \mathbf{Perf}(\operatorname{Spec} \Bbbk) = \mathrm{D}^b(\operatorname{Spec} \Bbbk) \to \mathrm{D}^b(X)$ is of Fourier–Mukai type. More precisely, there exists an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$, where

$$\mathcal{E} := \mathsf{F}(\mathcal{O}_{\operatorname{Spec} \Bbbk}) \in \mathrm{D}^b(X) = \mathrm{D}^b(\operatorname{Spec} \Bbbk \times X).$$

It is also straightforward to see that the functor $\Phi_{-}^{\operatorname{Spec} \Bbbk \to X}$ is an equivalence of categories, so that all the above questions have a positive answer in this case.

If we exchange the role of X and $\operatorname{Spec} \Bbbk$ above, the situation becomes slightly more complicated but nevertheless it is not difficult to see that $\Phi_{-}^{X \to \operatorname{Spec} \Bbbk}$ is an equivalence as well. Indeed, as an easy consequence of [49, Cor. 7.50] (see also [4, Thm. 3.3]), there is an equivalence

$$\mathrm{D}^b(X) \longrightarrow \mathbf{ExFun}(\mathbf{Perf}(X)^{\circ}, \mathrm{D}^b(\operatorname{Spec} \Bbbk))$$

and one can check that $\Phi_{-}^{X \to \operatorname{Spec} \Bbbk}$ is induced from this by the exact anti-equivalence $\mathbf{Perf}(X) \cong \mathbf{Perf}(X)^{\circ}$ sending $\mathcal{F}$ to $\mathcal{F}^{\vee}$.

4.2. **Non-uniqueness of Fourier–Mukai kernels.** The aim of this section is to prove that, even in the smooth case, (Q2) has a negative answer in general. First observe that the functor $\Phi_{-}^{X_2 \to X_1}$ satisfies any of (Q1)–(Q5) if and only if $\Phi_{-}^{X_1 \to X_2}$ does. To see this, one identifies $\Phi_{-}^{X_2 \to X_1}$ with the opposite functor of $\Phi_{-}^{X_1 \to X_2}$ under the equivalences $\mathrm{D}^b(X_1 \times X_2) \to \mathrm{D}^b(X_1 \times X_2)^{\circ}$ (defined on the objects by $\mathcal{E} \mapsto \mathcal{E}^{\vee} \otimes p_1^* \omega_{X_1}[d_1]$) and $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2)) \to \mathbf{ExFun}(\mathrm{D}^b(X_2), \mathrm{D}^b(X_1))^{\circ}$ (defined on the objects by $\mathsf{F} \mapsto \mathsf{F}_*$, the right adjoint of F). A key ingredient for this is Proposition 3.5. Here we set $d_i := \dim(X_i)$.

For later use, we start studying the case of the projective line which provides a positive result related to (Q2).

Lemma 4.2. *If X_1 or X_2 is $\mathbb{P}^1$, then $\Phi_{-}^{X_1 \to X_2}$ is essentially injective.*

Proof. As observed above, we can assume that $X_1 = \mathbb{P}^1$. Since on $\mathbb{P}^1 \times \mathbb{P}^1$ there is a resolution of the diagonal of the form

$$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1) \xrightarrow{x_0 \boxtimes x_1 - x_1 \boxtimes x_0} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathcal{O}_{\Delta} \to 0,$$

the argument in [19, Sect. 4.3] shows that, for every exact functor $\mathsf{F} \colon \mathrm{D}^b(\mathbb{P}^1) \to \mathrm{D}^b(X_2)$, any object $\mathcal{E}$ in $\mathrm{D}^b(\mathbb{P}^1 \times X_2)$ such that $\mathsf{F} \cong \Phi_{\mathcal{E}}$ is necessarily a convolution of the complex

$$\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathsf{F}(\mathcal{O}_{\mathbb{P}^1}(-1)) \xrightarrow{\varphi := x_0 \boxtimes \mathsf{F}(x_1) - x_1 \boxtimes \mathsf{F}(x_0)} \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathsf{F}(\mathcal{O}_{\mathbb{P}^1}),$$

hence it is uniquely determined up to isomorphism as the cone of φ. $\square$

As soon as the genus of the curve grows, the situation becomes more complicated and, in a sense, more interesting. Indeed, we have the following result that is [18, Thm. 1.1].

Theorem 4.3. *For every elliptic curve X over an algebraically closed field there exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{D}^b(X \times X)$ such that $\mathcal{E}_1 \not\cong \mathcal{E}_2$ but $\Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2}$.*

There is no space to explain the proof of this result in detail. Let us just mention how the two kernels are defined. By Serre duality,

$$0 \neq \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)^\vee \cong \mathrm{Hom}(\mathcal{O}_\Delta[-1], \mathcal{O}_\Delta[1]),$$

where $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_X \in \mathrm{D}^b(X \times X)$. For $0 \neq \alpha \in \mathrm{Hom}(\mathcal{O}_\Delta[-1], \mathcal{O}_\Delta[1])$, we set

$$\mathcal{E}_1 := \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[1] \qquad \mathcal{E}_2 := \mathsf{Cone}(\alpha).$$

It makes then perfect sense to pose the following.

Problem 4.4. *Extend the non-uniqueness result in Theorem 4.3 to any curve of genus ≥ 1.*

In [18] we provided our best approximation to the uniqueness of the Fourier–Mukai kernels.

Theorem 4.5. ([18], **Theorem 1.2.**) *Let X_1 and X_2 be projective schemes and let $\mathsf{F}\colon \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$ be an exact functor. If $\mathsf{F} \cong \Phi_{\mathcal{E}}$ for some $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$, then the cohomology sheaves of $\mathcal{E}$ are uniquely determined (up to isomorphism) by F.*

Using the discussion in Section 2.2 we can derive the following straightforward consequence from the above result. We will always assume that X_1 and X_2 are smooth projective varieties.

Corollary 4.6. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{D}^b(X_1 \times X_2)$ be such that $\Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$. Then $[\mathcal{E}_1] = [\mathcal{E}_2]$ in $K(X_1 \times X_2)$ and so $\Phi^K_{\mathcal{E}_1} = \Phi^K_{\mathcal{E}_2}$ and $\Phi^H_{\mathcal{E}_1} = \Phi^H_{\mathcal{E}_2}$.*

4.3. **The remaining questions (Q3)–(Q5).** Let us first consider the case of smooth projective curves.

Proposition 4.7. ([18], **Proposition 2.3.**) *Set $d_i := \dim(X_i)$. If $\min\{d_1, d_2\} = 1$, then $\Phi_{-}^{X_1 \to X_2}$ is neither faithful nor full.*

Proof. We give a full proof only of the non-faithfulness, as it plays a role in the study of (Q5) below. As above, we can assume that $1 = d_1 \leq d_2$. Hence take a finite morphism $f\colon X_1 \to \mathbb{P}^{d_2}$ and a finite and surjective (hence flat) morphism $g\colon X_2 \to \mathbb{P}^{d_2}$. Then $\mathsf{F} := g^* \circ f_*\colon \mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$ is an exact functor, which trivially extends to an exact functor again denoted by $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$. Clearly there exists $0 \not\cong \mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ such that $\mathsf{F} \cong \Phi_{\mathcal{E}}$ (see Example 2.1 and Proposition 2.2).

Now observe that, by Serre duality,

$$\mathrm{Hom}_{\mathrm{D}^b(X_1 \times X_2)}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Hom}_{\mathrm{D}^b(X_1 \times X_2)}(\mathcal{E}, \mathcal{E} \otimes \omega_{X_1 \times X_2}[1 + d_2])^\vee,$$

so there exists $0 \neq \alpha \in \mathrm{Hom}_{\mathrm{D}^b(X_1 \times X_2)}(\mathcal{E}, \mathcal{E} \otimes \omega_{X_1 \times X_2}[1 + d_2])$. Since $\omega_{X_1 \times X_2} \cong p_1^* \omega_{X_1} \otimes p_2^* \omega_{X_2}$, this induces, for any $\mathcal{F} \in \mathbf{Coh}(X_1)$, a morphism

$$\Phi_\alpha(\mathcal{F})\colon \Phi_{\mathcal{E}}(\mathcal{F}) \cong \mathsf{F}(\mathcal{F}) \to \Phi_{\mathcal{E} \otimes \omega_{X_1 \times X_2}[1+d_2]}(\mathcal{F}) \cong \mathsf{F}(\mathcal{F} \otimes \omega_{X_1}) \otimes \omega_{X_2}[1 + d_2].$$

As $\mathsf{F}(\mathcal{F})$ and $\mathsf{F}(\mathcal{F}\otimes\omega_{X_1})$ are objects of $\mathbf{Coh}(X_2)$, it follows that $\Phi_\alpha(\mathcal{F})=0$, whence $\Phi_\alpha=0$ because every object of $\mathrm{D}^b(X_1)$ is isomorphic to the direct sum of its (shifted) cohomology sheaves (since the abelian category $\mathbf{Coh}(X_1)$ is hereditary).

As for non-fullness, we prove it only when $X_1=X_2=X$ is an elliptic curve and $\Bbbk$ is algebraically closed. By Theorem 4.3 there exist $\mathcal{E}_1,\mathcal{E}_2\in\mathrm{D}^b(X\times X)$ with $\mathcal{E}_1\not\cong\mathcal{E}_2$ and an isomorphism $\psi\colon\Phi_{\mathcal{E}_1}\xrightarrow{\sim}\Phi_{\mathcal{E}_2}$. Then we claim that there is no morphism $f\colon\mathcal{E}_1\to\mathcal{E}_2$ such that $\psi=\Phi_f^{X\to X}$. Indeed, assume that such an f exists. Then it can be completed to a distinguished triangle

$$\mathcal{E}_1\xrightarrow{f}\mathcal{E}_2\longrightarrow\mathcal{G},$$

for some $\mathcal{G}\in\mathrm{D}^b(X_1\times X_2)$. By assumption $\Phi_{\mathcal{G}}(\mathcal{A})=0$, for all $\mathcal{A}\in\mathrm{D}^b(X_1)$. Therefore $\Phi_{\mathcal{G}}\cong 0$, whence $\mathcal{G}\cong 0$ by Theorem 3.9. But then f would be an isomorphism, contradicting the assumption $\mathcal{E}_1\not\cong\mathcal{E}_2$. □

We finally recall how (Q5) is studied in [18]. For this we need a couple of easy lemmas.

Lemma 4.8. *Let $\mathbf{T}$ be a* Hom*-finite triangulated category and let $f\colon A\to B$ be a morphism of $\mathbf{T}$. Then $\mathsf{Cone}(f)\cong A[1]\oplus B$ if and only if $f=0$.*

Proof. The other implication being well-known, we assume that $\mathsf{Cone}(f)\cong A[1]\oplus B$. Applying the cohomological functor $\mathrm{Hom}(-,B)$ to the distinguished triangle $A\xrightarrow{f}B\to A[1]\oplus B\to A[1]$, one gets an exact sequence of finite dimensional $\Bbbk$-vector spaces

$$\mathrm{Hom}(A[1],B)\to\mathrm{Hom}(A[1]\oplus B,B)\to\mathrm{Hom}(B,B)\xrightarrow{(-)\circ f}\mathrm{Hom}(A,B).$$

For dimension reasons, the last map must be 0, hence $f=0$. □

Lemma 4.9. *Let $\mathsf{F}\colon\mathbf{T}\to\mathbf{T}'$ be an exact functor between triangulated categories and assume that $\mathbf{T}$ is* Hom*-finite. If F is essentially injective, then F is faithful, too.*

Proof. Let $f\colon A\to B$ be a morphism of $\mathbf{T}$ such that $\mathsf{F}(f)=0$. Then

$$\mathsf{F}(\mathsf{Cone}(f))\cong\mathsf{Cone}(\mathsf{F}(f))\cong\mathsf{F}(A)[1]\oplus\mathsf{F}(B)\cong\mathsf{F}(A[1]\oplus B)$$

in $\mathbf{T}'$, whence $\mathsf{Cone}(f)\cong A[1]\oplus B$ in $\mathbf{T}$ because F is essentially injective. It follows from Lemma 4.8 that $f=0$. □

Recollecting the above results, we get the following.

Proposition 4.10. **([18], Corollary 2.7.)** *If $d_1,d_2>0$ and X_1 or X_2 is $\mathbb{P}^1$, then there is no triangulated structure on $\mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))$ such that $\Phi_-^{X_1\to X_2}$ is exact.*

Proof. This follows from Lemma 4.9, since we know that in this case $\Phi_-^{X_1\to X_2}$ is essentially injective by Lemma 4.2, but not faithful by Proposition 4.7. □

Notice that, as observed in [52], there is no natural triangulated structure on the category $\mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))$. One can then pose the following question.

Problem 4.11. *Understand whether there may be smooth projective varieties X_1 and X_2 of positive dimension such that (Q5) has a positive answer.*

5. Existence of Fourier–Mukai kernels and (Q1)

We are now ready to discuss the partial answers to (Q1) actually present in the literature. As we have already observed, we need to impose rather strong conditions on the exact functors in order to get nice results.

5.1. **The non-smooth case.** The idea of studying Fourier–Mukai functors between triangulated categories associated to singular varieties explained in the baby examples in Section 4.1 has been extensively analyzed in [36] using new ideas coming from dg-categories. Let us start from the following result.

Proposition 5.1. ([36], Corollary 9.12.) *Let X_1 and X_2 be quasi-compact separated schemes over $\Bbbk$. Assume that X_1 has enough locally free sheaves and let $\mathsf{F}\colon \mathbf{Perf}(X_1) \to \mathrm{D}(\mathbf{Qcoh}(X_2))$ be a fully faithful exact functor that commutes with direct sums. Then there is an $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$ such that the functor $\Phi_{\mathcal{E}}$ is fully faithful and*

$$\Phi_{\mathcal{E}}(\mathcal{A}) \cong \mathsf{F}(\mathcal{A}) \tag{5.1}$$

for any $\mathcal{A} \in \mathbf{Perf}(X_1)$.

Needless to say, the existence of the isomorphism (5.1) is a rather weak condition because, already in the smooth case, it may not extend to an isomorphism of functors. To show that this is possible, consider the case of $\mathbb{P}^1 \times \mathbb{P}^1$. Exactly as in Section 4.2, observe that, by Serre duality,

$$0 \neq \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)^\vee \cong \mathrm{Hom}(\mathcal{O}_\Delta[-1], \mathcal{O}_\Delta \otimes \omega_{\mathbb{P}^1\times\mathbb{P}^1}[1]).$$

Hence take a non-trivial $\alpha\colon \mathcal{O}_\Delta[-1] \to \mathcal{O}_\Delta \otimes \omega_{\mathbb{P}^1\times\mathbb{P}^1}[1] \cong \Delta_*\omega_{\mathbb{P}^1}^{\otimes 2}[1]$ and consider the objects

$$\mathcal{E}_1 := \mathcal{O}_\Delta \oplus \Delta_*\omega_{\mathbb{P}^1}^{\otimes 2}[1] \qquad \mathcal{E}_2 := \mathsf{Cone}(\alpha).$$

Then one has the following easy result.

Lemma 5.2. *For every $\mathcal{A} \in \mathrm{D}^b(\mathbb{P}^1)$ we have $\Phi_{\mathcal{E}_1}(\mathcal{A}) \cong \Phi_{\mathcal{E}_2}(\mathcal{A})$ but $\Phi_{\mathcal{E}_1} \not\cong \Phi_{\mathcal{E}_2}$.*

Proof. The existence of an isomorphism $\Phi_{\mathcal{E}_1}(\mathcal{A}) \cong \Phi_{\mathcal{E}_2}(\mathcal{A})$ for any $\mathcal{A} \in \mathrm{D}^b(\mathbb{P}^1)$ is obvious. The fact that $\Phi_{\mathcal{E}_1} \not\cong \Phi_{\mathcal{E}_2}$ follows from the uniqueness of Fourier–Mukai kernels for $\mathbb{P}^1$ (see Lemma 4.2) and the fact that $\mathcal{E}_1 \not\cong \mathcal{E}_2$. □

On the other hand, putting some more hypotheses on the schemes, we get a global isomorphism, as stated in the following theorem which is [36, Cor. 9.13]. For a scheme X, denote by $T_0(\mathcal{O}_X)$ the maximal 0-dimensional torsion subsheaf of $\mathcal{O}_X$.

Theorem 5.3. (Lunts–Orlov) *Let X_1 be a projective scheme with $T_0(\mathcal{O}_{X_1}) = 0$ and assume that X_2 is a noetherian separated scheme over $\Bbbk$. Given an exact fully faithful functor $\mathsf{F}\colon \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$, there are an $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of exact functors $\Phi_{\mathcal{E}} \cong \mathsf{F}$.*

Remark 5.4. The kernel turns out to be unique in perfect analogy with Theorem 3.1. This is observed in [17], following a suggestion by Orlov.

There is another approach to the Fourier–Mukai functors in the non-smooth case due to Ballard.

Theorem 5.5. ([4], **Theorem 1.2.**) *Let X_1 and X_2 be projective schemes such that $T_0(\mathcal{O}_{X_1}) = 0$. If $\mathsf{F}\colon \mathbf{Perf}(X_1) \to \mathbf{Perf}(X_2)$ is a fully faithful exact functor with left and right adjoints, then there are an $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of exact functors $\Phi_{\mathcal{E}} \cong \mathsf{F}$.*

As remarked in [4], contrary to the smooth case, the existence of the adjoints is not automatic at all. On the other hand, the proof of Theorem 5.5 differs from the one of Theorem 5.3 as it does not make use of dg-categories and is closer to the spirit of the one of Theorem 3.1.

5.2. **Some ingredients in the proof of Theorem 5.3.** A complete account of the details of the proof of Theorem 5.3 is far beyond the scope of this paper. Nevertheless, there are at least three main steps in it which we want to highlight as they provide sources of interesting (and difficult) open problems.

5.2.1. *Dg-categories.* First one wants to find an object $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ to compare the functors F and $\Phi_{\mathcal{E}}$. This is done by passing to dg-ehnacements and using a celebrated result of Toën.

Recall that a *dg-category* is an additive category $\mathbf{A}$ such that, for all $A, B \in \mathrm{Ob}(\mathbf{A})$, the morphism spaces $\mathrm{Hom}(A, B)$ are $\mathbb{Z}$-graded $\Bbbk$-modules with a differential $d\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$ of degree 1 compatible with the composition.

Given a dg-category $\mathbf{A}$ we denote by $H^0(\mathbf{A})$ its *homotopy* category. The objects of $H^0(\mathbf{A})$ are the same as those of $\mathbf{A}$ while the morphisms are obtained by taking the 0-th cohomology $H^0(\mathrm{Hom}_{\mathbf{A}}(A, B))$ of the complex $\mathrm{Hom}_{\mathbf{A}}(A, B)$. If $\mathbf{A}$ is pre-triangulated (see [31] for the definition), then $H^0(\mathbf{A})$ has a natural structure of triangulated category.

A *dg-functor* $\mathsf{F}\colon \mathbf{A} \to \mathbf{B}$ is the datum of a map $\mathrm{Ob}(\mathbf{A}) \to \mathrm{Ob}(\mathbf{B})$ and of morphisms of dg $\Bbbk$-modules $\mathrm{Hom}_{\mathbf{A}}(A, B) \to \mathrm{Hom}_{\mathbf{B}}(\mathsf{F}(A), \mathsf{F}(B))$, for $A, B \in \mathrm{Ob}(\mathbf{A})$, which are compatible with the composition and the units.

For a small dg-category $\mathbf{A}$, one can consider the pre-triangulated dg-category $\mathcal{M}od\text{-}\mathbf{A}$ of *right dg* $\mathbf{A}$*-modules.* A right dg $\mathbf{A}$-module is a dg-functor $\mathsf{M}\colon \mathbf{A}^\circ \to \mathcal{M}od\text{-}\Bbbk$, where $\mathcal{M}od\text{-}\Bbbk$ is the dg-category of dg $\Bbbk$-modules. The full dg-subcategory of acyclic right dg-modules is denoted by $\mathcal{A}c(\mathbf{A})$, and $H^0(\mathcal{A}c(\mathbf{A}))$ is a full triangulated subcategory of the homotopy category $H^0(\mathcal{M}od\text{-}\mathbf{A})$. Hence the *derived category* of the dg-category $\mathbf{A}$ is the Verdier quotient

$$\mathrm{D}^{\mathrm{dg}}(\mathbf{A}) := H^0(\mathcal{M}od\text{-}\mathbf{A})/H^0(\mathcal{A}c(\mathbf{A})).$$

According to [31, 52], given two dg-categories $\mathbf{A}$ and $\mathbf{B}$, we denote by $\mathrm{rep}(\mathbf{A}, \mathbf{B})$ the full subcategory of the derived category $\mathrm{D}^{\mathrm{dg}}(\mathbf{A}^\circ \otimes \mathbf{B})$ of $\mathbf{A}$-$\mathbf{B}$-bimodules C such that the functor $(-) \otimes_{\mathbf{A}} \mathsf{C}\colon \mathrm{D}^{\mathrm{dg}}(\mathbf{A}) \to \mathrm{D}^{\mathrm{dg}}(\mathbf{B})$ sends the representable $\mathbf{A}$-modules to objects which are isomorphic to representable $\mathbf{B}$-modules. A *quasi-functor* is an object in $\mathrm{rep}(\mathbf{A}, \mathbf{B})$ which is represented by a dg-functor $\mathbf{A} \to \mathcal{M}od\text{-}\mathbf{B}$ whose essential image consists of dg $\mathbf{B}$-modules quasi-isomorphic to representable $\mathbf{B}$-modules. Notice that a quasi-functor $\mathsf{M} \in \mathrm{rep}(\mathbf{A}, \mathbf{B})$ defines a functor $H^0(\mathsf{M})\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$.

Given two pre-triangulated dg-categories $\mathbf{A}$ and $\mathbf{B}$, a *dg-lift* of an exact functor $\mathsf{F}\colon H^0(\mathbf{A}) \to H^0(\mathbf{B})$ is a quasi-functor $\mathsf{G} \in \mathrm{rep}(\mathbf{A}, \mathbf{B})$ such that $H^0(\mathsf{G}) \cong \mathsf{F}$.

An *enhancement* of a triangulated category $\mathbf{T}$ is a pair $(\mathbf{A}, \alpha)$, where $\mathbf{A}$ is a pre-triangulated dg-category and $\alpha\colon H^0(\mathbf{A}) \to \mathbf{T}$ is an exact equivalence. The enhancement $(\mathbf{A}, \alpha)$ of $\mathbf{T}$ is *unique* if for any enhancement $(\mathbf{B},\beta)$ of $\mathbf{T}$ there exists a quasi-functor $\gamma\colon \mathbf{A}\to\mathbf{B}$ such that $H^0(\gamma)\colon H^0(\mathbf{A})\to H^0(\mathbf{B})$ is an exact equivalence.

Example 5.6. For X a quasi-compact quasi-separated scheme, let $\mathrm{C}^{\mathrm{dg}}(X)$ be the dg-category of unbounded complexes of objects in $\mathbf{Qcoh}(X)$. Denote by $\mathrm{Ac}^{\mathrm{dg}}(X)$ the full dg-subcategory of $\mathrm{C}^{\mathrm{dg}}(X)$ consisting of acyclic complexes. Following [20], we take the quotient $\mathrm{D}^{\mathrm{dg}}(X) := \mathrm{C}^{\mathrm{dg}}(X)/\mathrm{Ac}^{\mathrm{dg}}(X)$ which is again a dg-category. This dg-category $\mathrm{D}^{\mathrm{dg}}(X)$ is pre-triangulated and $H^0(\mathrm{D}^{\mathrm{dg}}(X))\cong\mathrm{D}(\mathbf{Qcoh}(X))$ (see [31, 52]). Therefore it is an enhancement of $\mathrm{D}(\mathbf{Qcoh}(X))$.

Consider then the full dg-subcategory $\mathrm{Perf}^{\mathrm{dg}}(X)$ whose objects are all the perfect complexes in $\mathrm{D}(\mathbf{Qcoh}(X))$. It turns out (see, for example, [36, Sect. 1]) that $\mathrm{Perf}^{\mathrm{dg}}(X)$ is an enhancement of $\mathbf{Perf}(X)$.

The following result answers positively a conjecture in [8]. The reader can have a look at [36, Sect. 9] for stronger statements.

Theorem 5.7. **([36], Theorem 7.9.)** *The triangulated category $\mathbf{Perf}(X)$ on a quasi-projective scheme X has a unique enhancement.*

Given a functor $\mathsf{F}\colon\mathbf{Perf}(X_1)\to\mathrm{D}^b_{\mathbf{Coh}}(\mathbf{Qcoh}(X_2))$ as in the statement of Theorem 5.3, Lunts and Orlov construct in a highly non-trivial way a quasi-functor $\mathsf{F}^{\mathrm{dg}}\colon\mathrm{Perf}^{\mathrm{dg}}(X_1)\to\mathrm{D}^{\mathrm{dg}}(X_2)$. Now one can use the following.

Theorem 5.8. **([52], Theorem 8.9.)** *Let X_1 and X_2 be quasi-compact and separated schemes over $\Bbbk$. Then we have a canonical quasi-equivalence*

$$\mathrm{D}^{\mathrm{dg}}(X_1\times X_2)\xrightarrow{\sim}\mathbf{R}\mathcal{H}om_c(\mathrm{D}^{\mathrm{dg}}(X_1),\mathrm{D}^{\mathrm{dg}}(X_2)),$$

where $\mathbf{R}\mathcal{H}om_c$ denotes the dg-category formed by the direct sums preserving quasi-functors (i.e. their homotopy functors do).

Hence there are an $\mathcal{E}\in\mathrm{D}^{\mathrm{dg}}(X_1\times X_2)$ and an isomorphism $\mathsf{F}^{\mathrm{dg}}\cong\Phi^{\mathrm{dg}}_{\mathcal{E}}$ and it remains to show that $\mathsf{F}\cong H^0(\mathsf{F}^{\mathrm{dg}})\cong\Phi_{\mathcal{E}}$.

5.2.2. *Ample sequences.* The projectivity assumption in the statement has a rather important role. Indeed one needs to work with ample sequences according to the following.

Definition 5.9. Given a Hom-finite abelian category $\mathbf{A}$, a subset $\{P_i\}_{i\in\mathbb{Z}}\subset\mathrm{Ob}(\mathbf{A})$ is an *ample sequence* if, for any $B\in\mathrm{Ob}(\mathbf{A})$, there exists an integer $i(B)$ such that, for any $i\le i(B)$,

(1) the natural morphism $\mathrm{Hom}_{\mathbf{A}}(P_i,B)\otimes P_i\to B$ is surjective;
(2) if $j\neq 0$ then $\mathrm{Hom}_{\mathrm{D}^b(\mathbf{A})}(P_i,B[j])=0$;
(3) $\mathrm{Hom}_{\mathbf{A}}(B,P_i)=0$.

If X is a projective scheme and H is an ample line bundle on X, then one may consider the set $\mathbf{C}$ (often identified with the corresponding full subcategory of $\mathbf{Coh}(X)$) consisting of objects of the form $\mathcal{O}_X(iH)$, where i is any integer.

Proposition 5.10. **([36], Proposition 9.2.)** *If X is a projective scheme such that $T_0(\mathcal{O}_X)=0$, then $\mathbf{C}$ forms an ample sequence in the abelian category $\mathbf{Coh}(X)$.*

Notice that this is the place where the assumption about the maximal torsion subsheaf plays a distinguished role. Thus there is space for further work:

Problem 5.11. *Remove the assumption $T_0(\mathcal{O}_X) = 0$ and, in particular, find a way to extend Theorem 5.3 when X_1 is a 0-dimensional projective scheme.*

At this point Lunts and Orlov show that the Fourier–Mukai functor $\Phi_{\mathcal{E}}$, with kernel found in Section 5.2.1, and the given functor F are such that there is an isomorphism

$$\theta_1\colon \mathsf{F}|_{\mathbf{C}} \xrightarrow{\sim} \Phi_{\mathcal{E}}|_{\mathbf{C}}. \tag{5.2}$$

Before discussing how this isomorphism can be extended, let us formulate the following rather general problem.

Problem 5.12. *Avoid the use of ample sequences and relax the projectivity assumptions.*

Both Problem 5.11 and 5.12 are widely open but we believe that any improvement in these directions may give new important impulses to the theory.

5.2.3. *Convolutions.* The extension of (5.2) is achieved in two steps. First the extension takes place on the level of sheaves. And for this one writes every perfect sheaf (i.e. a coherent sheaf which is a perfect object as well) as a convolution of objects in the ample sequence $\mathbf{C}$ on X_1 described in the previous section.

Following [30, 46], recall that a bounded complex in a triangulated category $\mathbf{T}$ is a sequence of objects and morphisms in $\mathbf{T}$

$$A_m \xrightarrow{d_m} A_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} A_0 \tag{5.3}$$

such that $d_j \circ d_{j+1} = 0$ for $0 < j < m$. A *right convolution* of (5.3) is an object A together with a morphism $d_0\colon A_0 \to A$ such that there exists a diagram in $\mathbf{T}$

$$\begin{tikzcd}
A_m \arrow[rr, "d_m"] \arrow[dr, "\mathrm{id}"'] & & A_{m-1} \arrow[rr, "d_{m-1}"] \arrow[dr] & & \cdots \arrow[rr, "d_2"] & & A_1 \arrow[rr, "d_1"] \arrow[dr] & & A_0 \arrow[dr, "d_0"] & \\
& A_m \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & & C_{m-1} \arrow[ll, "{[1]}"] \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & & \cdots \arrow[ll, "{[1]}"] & & C_1 \arrow[ll, "{[1]}"] \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & & A, \arrow[ll, "{[1]}"]
\end{tikzcd}$$

where the triangles with a $\circlearrowleft$ are commutative and the others are distinguished.

Roughly speaking, in this part of the argument, we have $A \in \mathbf{Coh}(X_1) \cap \mathbf{Perf}(X_1)$ while A_i is a finite direct sum of objects in $\mathbf{C}$, for all i. Unfortunately, to use convolutions one needs to make assumptions on the functor F. The hypothesis in Theorem 5.3 that F is fully faithful goes exactly in this direction. Thus, if we want to substantially improve Theorem 5.3, one has to address the following:

Problem 5.13. *Avoid the use of convolutions.*

All in all, we get an isomorphism

$$\theta_2\colon \mathsf{F}|_{\mathbf{Coh}(X_1)\cap\mathbf{Perf}(X_1)} \xrightarrow{\sim} \Phi_{\mathcal{E}}|_{\mathbf{Coh}(X_1)\cap\mathbf{Perf}(X_1)}.$$

To produce the desired isomorphism

$$\theta_3 \colon \mathsf{F} \xrightarrow{\sim} \Phi_{\mathcal{E}}$$

one argues by induction on the length of the interval to which the non-trivial cohomologies of an object $\mathcal{F} \in \mathbf{Perf}(X_1)$ belong.

Remark 5.14. The techniques used to get the extension θ_3 were improved in [17] (see, in particular, Sections 3.2 and 3.3 of that paper). Indeed, we consider a wider class of triangulated categories and we deal with extensions of natural transformations rather than isomorphisms of functors. These ingredients play a role in the results of Sections 5.3 and 5.4.

5.3. **Exact functors between the abelian categories of coherent sheaves.** As pointed out in Example 3.11, if X_1 and X_2 are smooth projective varieties, then the functors induced by exact functors from $\mathbf{Coh}(X_1)$ to $\mathbf{Coh}(X_2)$ satisfy (3.2), hence Theorem 3.9 holds for them. This suggests that questions analogous to (Q1)–(Q5) should be easier to answer for exact functors between the abelian categories of coherent sheaves. Indeed, for them one can prove the following result, improving [19, Prop. 5.1].

As a matter of notation, if X_1 and X_2 are smooth projective varieties we denote by $\mathbf{K}(X_1, X_2)$ the full subcategory of $\mathbf{Coh}(X_1 \times X_2)$ having as objects the sheaves $\mathcal{E}$ which are flat over X_1 and such that $p_2|_{\mathrm{Supp}(\mathcal{E})} \colon \mathrm{Supp}(\mathcal{E}) \to X_2$ is a finite morphism.

Proposition 5.15. *Let X_1 and X_2 be smooth projective varieties. If $\mathcal{E}$ is in $\mathbf{Coh}(X_1 \times X_2)$, then the additive functor*

$$\Psi_{\mathcal{E}} := (p_2)_*(\mathcal{E} \otimes p_1^*(-)) \colon \mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$$

(where $(p_2)_*$ and $\otimes$ are not derived) is exact if and only if $\mathcal{E} \in \mathbf{K}(X_1, X_2)$.

Moreover, if we denote by $\mathbf{ExFun}(\mathbf{Coh}(X_1), \mathbf{Coh}(X_2))$ the category of exact functors from $\mathbf{Coh}(X_1)$ to $\mathbf{Coh}(X_2)$, the functor

$$\Psi_{-}^{X_1 \to X_2} \colon \mathbf{K}(X_1, X_2) \longrightarrow \mathbf{ExFun}(\mathbf{Coh}(X_1), \mathbf{Coh}(X_2))$$

sending $\mathcal{E} \in \mathbf{K}(X_1, X_2)$ to $\Psi_{\mathcal{E}}$ is an equivalence of categories.

Proof. We just stick to the second part of the statement and we invite the reader interested in a proof of the first part to have a look at [19].

We sketch the proof that $\Psi_{-}^{X_1 \to X_2}$ is essentially surjective (again, for more details see [19]). Hence assume that $\mathsf{F} \colon \mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$ is an exact functor. By Theorem 3.9 there exists (unique up to isomorphism) $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ such that the extension of F to the level of derived categories is isomorphic to $\Phi_{\mathcal{E}}$, and $\mathcal{E} \in \mathbf{Coh}(X_1 \times X_2)$ (to see that $\mathcal{E}$ is a sheaf, one can use, for example, [19, Lemma 2.5]). From the fact that $\Phi_{\mathcal{E}}(\mathbf{Coh}(X_1)) \subseteq \mathbf{Coh}(X_2)$ it is easy to deduce that $\mathsf{F} \cong \Phi_{\mathcal{E}}|_{\mathbf{Coh}(X_1)} \cong \Psi_{\mathcal{E}}$.

In order to prove that $\Psi_{-}^{X_1 \to X_2}$ is fully faithful, denoting by $\mathcal{S}$ the sheaf $\bigoplus_{m \geq 0} (p_2)_*(p_1^* \mathcal{O}_{X_1}(mH))$ of graded algebras on X_2 (H being an ample line bundle on X_1), we will use the relative version of the Serre correspondence between graded $\mathcal{S}$-modules and sheaves on $\mathrm{Proj}\, \mathcal{S} \cong X_1 \times X_2$. More precisely, denoting by $\mathbf{gmod}\text{-}\mathcal{S}$ the category of graded $\mathcal{S}$-modules of finite type (meaning finitely generated in sufficiently high degrees), one considers the associated sheaf functor $\mathsf{H} \colon \mathbf{gmod}\text{-}\mathcal{S} \to \mathbf{Coh}(X_1 \times X_2)$ and the functor $\mathsf{G} \colon \mathbf{Coh}(X_1 \times X_2) \to \mathbf{gmod}\text{-}\mathcal{S}$ defined on objects by $\mathsf{G}(\mathcal{E}) :=$

$\bigoplus_{m\in\mathbb{Z}} \Psi_{\mathcal{E}}(\mathcal{O}_{X_1}(mH))$. They satisfy $\mathsf{H}\circ\mathsf{G} \cong \mathrm{id}$ and, moreover, an object or a morphism of **gmod-**$\mathcal{S}$ is sent to 0 by H if and only if it is 0 in sufficiently high degrees.

Now, given $\mathcal{E}_1, \mathcal{E}_2 \in \mathbf{K}(X_1, X_2)$, morphisms in **gmod-**$\mathcal{S}$ from $\mathsf{G}(\mathcal{E}_1)$ to $\mathsf{G}(\mathcal{E}_2)$ can be identified with natural transformations from $\Psi_{\mathcal{E}_1}|_{\mathbf{C}}$ to $\Psi_{\mathcal{E}_2}|_{\mathbf{C}}$, where $\mathbf{C}$ is the full subcategory of $\mathbf{Coh}(X_1)$ with objects $\{\mathcal{O}_{X_1}(iH)\}_{i\in\mathbb{Z}}$. By [17, Prop. 3.6] (applied to the functors $\Phi_{\mathcal{E}_1}$ and $\Phi_{\mathcal{E}_2}$) such natural transformations correspond bijectively to natural transformations from $\Psi_{\mathcal{E}_1}$ to $\Psi_{\mathcal{E}_2}$. Therefore, in view of the properties of G and H mentioned above, the fully faithfulness of $\Psi_{-}^{X_1\to X_2}$ amounts to the following: if $\alpha\colon \Psi_{\mathcal{E}_1} \to \Psi_{\mathcal{E}_2}$ is a natural transformation such that $\alpha_m := \alpha(\mathcal{O}_{X_1}(mH)) = 0$ for $m \gg 0$, then $\alpha_m = 0$ for every $m \in \mathbb{Z}$. Clearly to this purpose it is enough to show that $\alpha_m = 0$ implies $\alpha_{m-1} = 0$. To see this, take a monomorphism $f\colon \mathcal{O}_{X_1}((m-1)H) \hookrightarrow \mathcal{O}_{X_1}(mH)$ and just observe that in the commutative diagram

$$\begin{tikzcd}
\Psi_{\mathcal{E}_1}(\mathcal{O}_{X_1}((m-1)H)) \arrow[r, "\Psi_{\mathcal{E}_1}(f)"] \arrow[d, "\alpha_{m-1}"] & \Psi_{\mathcal{E}_1}(\mathcal{O}_{X_1}(mH)) \arrow[d, "\alpha_m=0"] \\
\Psi_{\mathcal{E}_2}(\mathcal{O}_{X_1}((m-1)H)) \arrow[r, "\Psi_{\mathcal{E}_2}(f)"] & \Psi_{\mathcal{E}_2}(\mathcal{O}_{X_1}(mH))
\end{tikzcd}$$

$\Psi_{\mathcal{E}_2}(f)$ is a monomorphism, because $\Psi_{\mathcal{E}_2}$ is exact. $\square$

In particular, this shows that for the functor $\Psi_{-}^{X_1\to X_2}$ questions (Q1)–(Q4) can be answered positively. As for (Q5), notice that in general $\mathbf{K}(X_1, X_2)$ is an additive but not an abelian subcategory of $\mathbf{Coh}(X_1 \times X_2)$.

5.4. **The supported case.** In this section we want to show how Theorem 5.3 can be extended both considering a more general categorical setting and weakening the assumptions on the exact functor.

Indeed, let X be a separated scheme of finite type over $\Bbbk$ and let Z be a subscheme of X which is proper over $\Bbbk$. We denote by $\mathrm{D}_Z(\mathbf{Qcoh}(X))$ the derived category of unbounded complexes of quasi-coherent sheaves on X with cohomologies supported on Z. Using this, we can define the triangulated categories

$$\mathrm{D}^b_Z(\mathbf{Qcoh}(X)) := \mathrm{D}_Z(\mathbf{Qcoh}(X)) \cap \mathrm{D}^b(\mathbf{Qcoh}(X))$$
$$\mathrm{D}^b_Z(X) := \mathrm{D}_Z(\mathbf{Qcoh}(X)) \cap \mathrm{D}^b(X).$$

We also set

$$\mathbf{Perf}_Z(X) := \mathrm{D}_Z(\mathbf{Qcoh}(X)) \cap \mathbf{Perf}(X).$$

Example 5.16. These categories appear naturally studying the so called *open Calabi-Yau's*. Examples of them are local resolutions of A_n-singularities ([26, 27]) and the total space $\mathrm{tot}(\omega_{\mathbb{P}^2})$ of the canonical bundle of $\mathbb{P}^2$ ([5]). In the latter case, if Z denotes the zero section of the projection $\mathrm{tot}(\omega_{\mathbb{P}^2}) \to \mathbb{P}^2$, the derived category $\mathbf{Perf}_Z(\mathrm{tot}(\omega_{\mathbb{P}^2})) = \mathrm{D}^b_Z(\mathrm{tot}(\omega_{\mathbb{P}^2}))$ is a Calabi–Yau category of dimension 3 and may be seen as an interesting example to test predictions about Mirror Symmetry and the topology of the space of stability conditions according to Bridgeland's definition (see [5] for results in this direction). Moreover, as a consequence of [26, 27], all autoequivalences of the supported derived categories of A_n-singularities are of Fourier–Mukai type and the group of such autoequivalences can be explicitly described. See [17] for more details.

The category $\mathrm{D}_Z(\mathbf{Qcoh}(X))$ is a full subcategory of $\mathrm{D}(\mathbf{Qcoh}(X))$ and let

$$\iota\colon \mathrm{D}_Z(\mathbf{Qcoh}(X)) \longrightarrow \mathrm{D}(\mathbf{Qcoh}(X))$$

be the inclusion. This functor has a right adjoint

$$\iota^!\colon \mathrm{D}(\mathbf{Qcoh}(X)) \to \mathrm{D}_Z(\mathbf{Qcoh}(X)) \qquad \iota^!(\mathcal{E}) := \varinjlim_n \mathbf{R}\mathcal{H}om(\mathcal{O}_{nZ}, \mathcal{E}),$$

where nZ is the n-th infinitesimal neighborhood of Z in X (see [35, Prop. 3.2.2]). Due to [35, Cor. 3.1.4], the functor $\iota^!$ sends bounded complexes to bounded complexes and $\iota^! \circ \iota \cong \mathrm{id}$.

Now, let X_1 and X_2 be separated schemes of finite type over $\Bbbk$ containing, respectively, two subschemes Z_1 and Z_2 which are proper over $\Bbbk$. The following generalizes the standard definition of Fourier–Mukai functor.

Definition 5.17. An exact functor

$$\mathsf{F}\colon \mathrm{D}_{Z_1}(\mathbf{Qcoh}(X_1)) \to \mathrm{D}_{Z_2}(\mathbf{Qcoh}(X_2))$$

is a *Fourier–Mukai functor* if there exists $\mathcal{E} \in \mathrm{D}_{Z_1\times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$ and an isomorphism of exact functors

$$\mathsf{F} \cong \Phi^{\mathrm{s}}_{\mathcal{E}} := \iota^!(p_2)_*((\iota\times\iota)\mathcal{E} \otimes p_1^*(\iota(-))) \tag{5.4}$$

where $p_i\colon X_1 \times X_2 \to X_i$ is the projection.

An analogous definition can be given for functors defined between bounded derived categories of quasi-coherent, coherent or perfect complexes. As always, the object $\mathcal{E}$ is called *Fourier–Mukai kernel*. It should be noted that, contrary to the smooth non-supported case, the Fourier–Mukai kernel cannot be assumed to be a bounded coherent complex. This is clarified by the following example dealing with the identity functor.

Example 5.18. We want to show that a Fourier–Mukai kernel of the identity functor $\mathrm{id}\colon \mathrm{D}^b_Z(X) \to \mathrm{D}^b_Z(X)$ is

$$(\iota\times\iota)^!\mathcal{I} \in \mathrm{D}^b_{Z\times Z}(\mathbf{Qcoh}(X\times X)),$$

where, denoting by $\Delta\colon X \to X \times X$ the diagonal embedding,

$$\mathcal{I} := \Delta_* \circ \iota \circ \iota^!(\mathcal{O}_X).$$

Indeed, according to [17], we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(\mathcal{A}, \iota^!\Phi_{\mathcal{I}}(\iota\mathcal{B})) &\cong \mathrm{Hom}(\iota\mathcal{A}, (p_2)_*(\Delta_* \circ \iota \circ \iota^!(\mathcal{O}_X) \otimes p_1^*(\iota\mathcal{B})))\\
&\cong \mathrm{Hom}((\iota\mathcal{B})^\vee \boxtimes \iota\mathcal{A}, \Delta_* \circ \iota \circ \iota^!(\mathcal{O}_X))\\
&\cong \mathrm{Hom}((\iota\mathcal{B})^\vee \otimes \iota\mathcal{A}, \mathcal{O}_X)\\
&\cong \mathrm{Hom}(\mathcal{A}, \iota^!\iota\mathcal{B}) \cong \mathrm{Hom}(\mathcal{A}, \mathcal{B}),
\end{aligned}$$

for any $\mathcal{A}, \mathcal{B} \in \mathrm{D}^b_Z(X)$. Here $p_i\colon X \times X \to X$ is the natural projection. For the first and the fourth isomorphism we used the adjunction between ι and $\iota^!$. The same adjunction together with the one between Δ^* and Δ_* and the fact that ι is fully faithful and $(\iota\mathcal{B})^\vee \otimes \iota\mathcal{A}$ has support in Z explains the third isomorphism.

Obviously $(\iota\times\iota)^!\mathcal{I}$ does not belong to $\mathrm{D}^b_{Z\times Z}(X\times X)$. Suppose that there exists $\mathcal{E}\in\mathrm{D}^b_{Z\times Z}(X\times X)$ such that

$$\Phi^{\mathrm{s}}_{\mathcal{E}}\cong\mathrm{id}\colon\mathrm{D}^b_Z(X)\to\mathrm{D}^b_Z(X).$$

By [49, Lemma 7.41], there exist $n>0$ and $\mathcal{E}_n\in\mathrm{D}^b(nZ\times nZ)$ such that $(\iota\times\iota)\mathcal{E}\cong(i_n\times i_n)_*\mathcal{E}_n$, where $i_n\colon nZ\to X$ is the embedding. For any $\mathcal{F}_n\in\mathrm{D}^b(nZ)$, we have

$$(i_n)_*\mathcal{F}_n\cong\Phi_{\mathcal{E}}((i_n)_*\mathcal{F}_n)\cong(i_n)_*\Phi_{\mathcal{E}_n}((i_n)^*(i_n)_*\mathcal{F}_n). \tag{5.5}$$

Take now $X=\mathbb{P}^k$, $Z=\mathbb{P}^{k-1}$ and $\mathcal{F}_n:=\mathcal{O}_{nZ}(m)$, for $m\in\mathbb{Z}$. An easy calculation shows that $(i_n)^*(i_n)_*\mathcal{F}_n\cong\mathcal{O}_{nZ}(m)\oplus\mathcal{O}_{nZ}(m-n)[1]$. Hence to have (5.5) verified, we should have either $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m))=0$ or $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m-n))=0$. But the following isomorphisms should hold at the same time

$$\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m))\oplus\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m-n))[1]\cong\mathcal{O}_{nZ}(m),$$
$$\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m+n))\oplus\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m))[1]\cong\mathcal{O}_{nZ}(m+n).$$

If $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m-n))=0$, then from the second one we would have that $\mathcal{O}_{nZ}(m)[1]$ is a direct summand of $\mathcal{O}_{nZ}(m+n)$ which is absurd. Thus $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m))=0$. As this holds for all $m\in\mathbb{Z}$, we get a contradiction.

Now let X_1 be a quasi-projective scheme containing a projective subscheme Z_1 such that $\mathcal{O}_{iZ_1}\in\mathbf{Perf}(X_1)$, for all $i>0$, and let X_2 be a separated scheme of finite type over $\Bbbk$ with a subscheme Z_2 which is proper over $\Bbbk$.

Remark 5.19. Notice that under these assumptions, and having fixed an ample divisor H_1 on X_1, the objects $\mathcal{O}_{|i|Z_1}(jH_1)$ are in $\mathbf{Perf}_{Z_1}(X_1)$, for all $i,j\in\mathbb{Z}$. Special cases in which $\mathcal{O}_{iZ_1}\in\mathbf{Perf}(X_1)$ are when $X_1=Z_1$ or X_1 is smooth.

One can consider exact functors $\mathsf{F}\colon\mathbf{Perf}_{Z_1}(X_1)\to\mathbf{Perf}_{Z_2}(X_2)$ such that

($*$) (1) $\mathrm{Hom}(\mathsf{F}(\mathcal{A}),\mathsf{F}(\mathcal{B})[k])=0$, *for any* $\mathcal{A},\mathcal{B}\in\mathbf{Coh}_{Z_1}(X_1)\cap\mathbf{Perf}_{Z_1}(X_1)$ *and any integer* $k<0$*;*

(2) *For all* $\mathcal{A}\in\mathbf{Perf}_{Z_1}(X_1)$ *with trivial cohomologies in (strictly) positive degrees, there is* $N\in\mathbb{Z}$ *such that*

$$\mathrm{Hom}(\mathsf{F}(\mathcal{A}),\mathsf{F}(\mathcal{O}_{|i|Z_1}(jH_1)))=0,$$

for any $i<N$ *and any* $j\ll i$*, where* H_1 *is an ample divisor on* X_1*.*

Then we have the following.

Theorem 5.20. **([17], Theorem 1.1.)** *Let* X_1*,* X_2*,* Z_1 *and* Z_2 *be as above and let*

$$\mathsf{F}\colon\mathbf{Perf}_{Z_1}(X_1)\longrightarrow\mathbf{Perf}_{Z_2}(X_2)$$

be an exact functor.

If F *satisfies* ($*$)*, then there exist* $\mathcal{E}\in\mathrm{D}^b_{Z_1\times Z_2}(\mathbf{Qcoh}(X_1\times X_2))$ *and an isomorphism of exact functors* $\mathsf{F}\cong\Phi^{\mathrm{s}}_{\mathcal{E}}$*. Moreover, if* X_i *is smooth quasi-projective, for* $i=1,2$*, and* $\Bbbk$ *is perfect, then* $\mathcal{E}$ *is unique up to isomorphism.*

Back to Remark 5.19, the above theorem can be applied in at least two interesting geometric contexts. If $X_1 = Z_1$, then we get back (a generalization of) Theorem 5.3. On the other hand, if X_1 is smooth, then we can apply the above result to the autoequivalences of the categories described in Example 5.16 proving that they are all of Fourier–Mukai type. As noticed in [17], if $X_i = Z_i$, $\dim(X_1) > 0$ and they are smooth, then $(*)$ is equivalent to (3.2). Thus, Theorem 5.20 recovers Theorem 3.9 as well.

Remark 5.21. In the same vein as in [36], it is proved in [17, Thm. 1.2] that $\mathbf{Perf}_Z(X)$ has a (strongly) unique dg-enhancement if X and Z have the same properties as X_1 and Z_1 in Theorem 5.20 and $T_0(\mathcal{O}_Z) = 0$. See [36] for the definition of strongly unique dg-enhancement which is not needed here.

6. More open problems

The list of problems mentioned in the above sections can be extended further. The main sources are actually very concrete geometric settings where they appear naturally. We try to list some of them below, although a complete clarification of their geometric meaning goes far beyond the scope of this paper.

6.1. **Does full imply essentially surjective?** In Section 3.4.2 we have seen that a full functor between the bounded derived categories of coherent sheaves on smooth projective varieties is automatically faithful. Assume now that we are given an exact endofunctor $\mathsf{F}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$, where X is again a smooth projective variety. In this section we want to discuss the following.

Conjecture 6.1. *If* F *is full, then it is an autoequivalence.*

Notice that we only need to show that F is essentially surjective.

Remark 6.2. The conjecture is true if ω_X is trivial, because in that case every fully faithful exact endofunctor of $\mathrm{D}^b(X)$ is an equivalence (see, for example, [23, Cor. 7.8]).

The above conjecture is implied by another conjecture about admissible subcategories that we want to explain here.

Given a triangulated category $\mathbf{T}$ and a strictly full triangulated subcategory $\mathbf{S}$, we say that $\mathbf{S}$ is *left-* (resp. *right-*) *admissible* in $\mathbf{T}$ if the inclusion functor $\eta\colon \mathbf{S} \to \mathbf{T}$ has a left (resp. right) adjoint $\eta^*\colon \mathbf{T} \to \mathbf{S}$ (resp. $\eta^!\colon \mathbf{T} \to \mathbf{S}$). If a subcategory is left and right admissible, we say that it is *admissible.*

Remark 6.3. By [7, Prop. 1.6], an admissible subcategory $\mathbf{S} \subseteq \mathbf{T}$ is thick as well.

We can use the notion of admissible subcategory to 'decompose' triangulated categories. More generally, one can give the following.

Definition 6.4. A *semi-orthogonal decomposition* of a triangulated category $\mathbf{T}$ is given by a sequence of full triangulated subcategories $\mathbf{A}_1, \dots, \mathbf{A}_n \subseteq \mathbf{T}$ such that $\mathrm{Hom}_{\mathbf{T}}(\mathbf{A}_i, \mathbf{A}_j) = 0$, for $i > j$ and, for all $K \in \mathbf{T}$, there exists a chain of morphisms in $\mathbf{T}$

$$0 = K_n \to K_{n-1} \to \dots \to K_1 \to K_0 = K$$

with $\mathsf{Cone}(K_i \to K_{i-1}) \in \mathbf{A}_i$, for all $i = 1, \ldots, n$. We will denote such a decomposition by $\mathbf{T} = \langle \mathbf{A}_1, \ldots, \mathbf{A}_n \rangle$.

The easiest examples of semi-orthogonal decompositions are constructed via exceptional objects.

Definition 6.5. Assume that $\mathbf{T}$ is a $\Bbbk$-linear triangulated category. An object $E \in \mathbf{T}$ is called *exceptional* if $\mathrm{Hom}_{\mathbf{T}}(E, E) \cong \Bbbk$ and $\mathrm{Hom}_{\mathbf{T}}(E, E[p]) = 0$, for all $p \neq 0$. A sequence $(E_1, \ldots, E_m)$ of objects in $\mathbf{T}$ is called an *exceptional sequence* if E_i is an exceptional object, for all i, and $\mathrm{Hom}_{\mathbf{T}}(E_i, E_j[p]) = 0$, for all p and all $i > j$. An exceptional sequence is *full* if it generates $\mathbf{T}$.

Remark 6.6. If $(E_1, \ldots, E_m)$ is a full exceptional sequence in $\mathbf{T}$, then we get a semi-orthogonal decomposition $\mathbf{T} = \langle E_1, \ldots, E_n \rangle$, where for simplicity we write E_i for the triangulated subcategory generated by E_i, which is equivalent to $\mathrm{D}^b(\mathrm{Spec}\, \Bbbk)$ and is admissible in $\mathbf{T}$.

Example 6.7. A celebrated result of Beilinson shows that $\mathrm{D}^b(\mathbb{P}^n)$ has a full exceptional sequence $(\mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n})$ (see, for example, [23, Sect. 8.3]).

For a triangulated subcategory $\mathbf{S}$ of a triangulated category $\mathbf{T}$, we can define the strictly full triangulated subcategories (i.e. full and closed under isomorphism)

$$\mathbf{S}^{\perp} := \{A \in \mathbf{T} : \mathrm{Hom}(S, A) = 0,\ \text{for all } S \in \mathbf{S}\}$$

called *right orthogonal* to $\mathbf{S}$ and its *left orthogonal*

$$^{\perp}\mathbf{S} := \{A \in \mathbf{T} : \mathrm{Hom}(A, S) = 0,\ \text{for all } S \in \mathbf{S}\}.$$

One can formulate the following conjecture due to A. Kuznetsov and contained in [34].

Conjecture 6.8. (Noetherianity conjecture) *Let X be a smooth projective variety and assume that there exists a sequence*

$$\mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \ldots \subseteq \mathbf{A}_i \subseteq \ldots \subseteq \mathrm{D}^b(X)$$

of admissible subcategories. Then there is a positive integer N such that $\mathbf{A}_i = \mathbf{A}_N$, for all $i \geq N$.

Remark 6.9. Considering the strictly full triangulated subcategories $\mathbf{B}_i := \mathbf{A}_i^{\perp}$, the above conjecture can be equivalently reformulated in terms of stabilizing descending chains.

Proposition 6.10. *Conjecture 6.8 implies Conjecture 6.1.*

Proof. The functor F is automatically faithful. Thus

$$\mathbf{I} := \mathrm{im}\, \mathsf{F} := \{\mathcal{E} \in \mathrm{D}^b(X) : \mathcal{E} \cong \mathsf{F}(\mathcal{F}) \text{ for some } \mathcal{F} \in \mathrm{D}^b(X)\}$$

is a strictly full triangulated subcategory of $\mathrm{D}^b(X)$. By Proposition 3.5, the functor F has left and right adjoints and so $\mathbf{I}$ is admissible. Using the above notation, set $\mathbf{J} = \mathbf{I}^{\perp}$. Hence we have a semi-orthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathbf{J}, \mathbf{I} \rangle.$$

As $\mathbf{I} \cong \mathrm{D}^b(X)$, we can think of F as an exact endofunctor of $\mathbf{I}$. Hence, reasoning as above we get a semi-orthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathbf{J}, \mathbf{J}, \mathbf{I} \rangle$$

Hence, given a positive integer n, repeating this argument n times we get that

$$\mathbf{A}_n := \langle\underbrace{\mathbf{J},\ldots,\mathbf{J}}_{n\text{times}}\rangle$$

is a strictly full admissible triangulated subcategory of $\mathrm{D}^b(X)$.

Since $\mathbf{A}_p \subseteq \mathbf{A}_q \subseteq \mathrm{D}^b(X)$ whenever $p \leq q$, by Conjecture 6.8, this sequence must stabilize. Hence $\mathbf{J} = 0$ and so F is essentially surjective. $\square$

Due to the following easy result, a full endofunctor is automatically an equivalence when X has dimension at most 1.

Proposition 6.11. *Conjecture 6.8 holds true when X is a smooth projective variety of dimension smaller or equal to* 1.

Proof. Obviously the Conjecture is trivially true if $\mathrm{D}^b(X)$ does not admit a non-trivial semi-orthogonal decomposition and this is the case if $\dim(X) = 0$.

If X is a curve of genus 1, Serre duality and [11, Example 3.2] implies that $\mathrm{D}^b(X)$ cannot be decomposed. The same is true when X is a curve of genus $g \geq 2$ due to [44]. Thus the only case that has to be checked is $X \cong \mathbb{P}^1$.

For this assume that $\mathrm{D}^b(\mathbb{P}^1) = \langle \mathbf{A}_1, \mathbf{A}_2\rangle$, where $\mathbf{A}_i$ is not trivial, for $i = 1, 2$ (i.e. non-zero and not the whole category $\mathrm{D}^b(\mathbb{P}^1)$). It is clear that either $\mathbf{A}_1$ or $\mathbf{A}_2$ must contain a locally free sheaf E. As on $\mathbb{P}^1$ any locally free sheaf is the direct sum of line bundles and $\mathbf{A}_i$ is thick (see Remark 6.3(i)), there is $j \in \mathbb{Z}$ such that $\mathcal{O}_{\mathbb{P}^1}(j) \in \mathbf{A}_i$, for $i = 1$ or $i = 2$. We assume $i = 1$ as the argument in the other case is similar.

Now $\mathbf{A}_2 = {}^{\perp}\mathbf{A}_1 \subseteq {}^{\perp}\langle\mathcal{O}_{\mathbb{P}^1}(j)\rangle = \langle\mathcal{O}_{\mathbb{P}^1}(j+1)\rangle$. But $\langle\mathcal{O}_{\mathbb{P}^1}(j+1)\rangle \cong \mathrm{D}^b(\operatorname{Spec}\Bbbk)$ and so it does not contain proper thick subcategories. Thus $\mathbf{A}_2 = \langle\mathcal{O}_{\mathbb{P}^1}(j+1)\rangle$ and $\mathbf{A}_1 = \langle\mathcal{O}_{\mathbb{P}^1}(j)\rangle$. Therefore, there cannot be non-stabilizing ascending chains of admissible subcategories. $\square$

6.2. **Splitting functors.** Kuznetsov introduced in [33] the notion of splitting functor as a natural generalization of fully faithful functor. The expectation was that, in this context, one should get a representability result similar to Theorem 3.1. Let us clarify the situation a bit more.

More precisely, given two triangulated categories $\mathbf{T}_1$ and $\mathbf{T}_2$ and an exact functor $\mathsf{F}\colon \mathbf{T}_1 \to \mathbf{T}_2$, we can define the following full subcategories

$$\ker\mathsf{F} := \{A \in \mathbf{T}_1 : \mathsf{F}(A) \cong 0\} \qquad \operatorname{im}\mathsf{F} := \{A \cong \mathsf{F}(B) : B \in \mathbf{T}_1\}.$$

Remark 6.12. The subcategory $\ker\mathsf{F}$ is always triangulated while $\operatorname{im}\mathsf{F}$, in general, is not. It becomes triangulated if F is fully faithful.

Hence we can give the following.

Definition 6.13. An exact functor $\mathsf{F}\colon \mathbf{T}_1 \to \mathbf{T}_2$ is *right* (respectively *left*) *splitting* if $\ker\mathsf{F}$ is a right (respectively left) admissible subcategory in $\mathbf{T}_1$, the restriction of F to $(\ker\mathsf{F})^{\perp}$ (respectively ${}^{\perp}(\ker\mathsf{F})$) is fully faithful, and the category $\operatorname{im}\mathsf{F}$ is right (respectively left) admissible in $\mathbf{T}_2$.

An exact functor is *splitting* if it is both right and left splitting.

Remark 6.14. As observed in [33, Lemma 3.2], a right (respectively left) splitting functor F has a right (respectively left) adjoint functor $\mathsf{F}^!$ (respectively F^*).

We summarize the basic properties of these functors in the following.

Theorem 6.15. **([33], Theorem 3.3.)** *Let $\mathsf{F}\colon \mathbf{T}_1 \to \mathbf{T}_2$ be an exact functor. Then the following conditions are equivalent:*

(i) F *is right splitting;*

(ii) F *has a right adjoint functor* $\mathsf{F}^!$ *and the composition of the canonical morphism of functors* $\mathrm{id}_{\mathbf{T}_1} \to \mathsf{F}^! \circ \mathsf{F}$ *with* F *gives an isomorphism* $\mathsf{F} \cong \mathsf{F} \circ \mathsf{F}^! \circ \mathsf{F}$*;*

(iii) F *has a right adjoint functor* $\mathsf{F}^!$*, there are semi-orthogonal decompositions*

$$\mathbf{T}_1 = \langle \operatorname{im} \mathsf{F}^!, \ker \mathsf{F} \rangle, \qquad \mathbf{T}_2 = \langle \ker \mathsf{F}^!, \operatorname{im} \mathsf{F} \rangle,$$

and the functors F *and* $\mathsf{F}^!$ *give quasi-inverse equivalences* $\operatorname{im} \mathsf{F}^! \cong \operatorname{im} \mathsf{F}$*;*

(iv) *There exists a triangulated category* $\mathbf{S}$ *and fully faithful functors* $\mathsf{G}_1\colon \mathbf{S} \to \mathbf{T}_1$, $\mathsf{G}_2\colon \mathbf{S} \to \mathbf{T}_2$, *such that* G_1 *admits a left adjoint* G_1^*, G_2 *admits a right adjoint and* $\mathsf{F} \cong \mathsf{G}_2 \circ \mathsf{G}_1^*$.

Clearly, one can formulate analogous conditions for left splitting functors. The main conjecture is now the following:

Conjecture 6.16. **([33], Conjecture 3.7.)** *Let X_1 and X_2 be smooth projective varieties. Then any exact splitting functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is of Fourier–Mukai type.*

One may first wonder why the strategy outlined in Section 5.2 may not be applied in this case. The main problem is that convolutions do not work for this kind of functors. Alternatively, one would need to define an analogue of the ample sequence in Section 5.2.2 for the subcategory $\mathbf{S}$ in part (iv) of Theorem 6.15. Hence, the solution to Conjecture 6.16 is closely related to Problems 5.12 and 5.13.

Nevertheless, there are several instances in which the conjecture is verified. The easiest one is when the category $\mathbf{S}$ mentioned in Theorem 6.15(iv) is such that $\mathbf{S} \cong \mathrm{D}^b(Y)$, for some smooth projective variety Y. Indeed, in this case, one reduces the proof to Theorem 3.1 (using Proposition 2.2).

Moreover, it is not difficult to observe that, using the same type of arguments as in the proof of Proposition 6.11, one can show the following (the zero-dimensional case is trivial).

Proposition 6.17. *Let either X_1 or X_2 be a smooth projective curve. Then any splitting functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is of Fourier–Mukai type.*

For less trivial situations where Conjecture 6.16 can be verified, one has to refer to [32]. For this consider a full admissible subcategory $\eta\colon \mathbf{S} \hookrightarrow \mathrm{D}^b(X)$, for a smooth projective variety X. Thus we get the left and right adjoints $\eta^*\colon \mathrm{D}^b(X) \to \mathbf{S}$ and $\eta^!\colon \mathrm{D}^b(X) \to \mathbf{S}$.

Take now the functors $\mathsf{F}_1 := \eta \circ \eta^!\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ and $\mathsf{F}_2 := \eta \circ \eta^*\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$. It is not difficult to see (using, for example, Theorem 6.15 above) that F_1 and F_2 are splitting functors. A non-trivial argument allows one to prove the following:

Theorem 6.18. **([32], Theorem 7.1.)** *The functors F_1 and F_2 are of Fourier–Mukai type.*

6.3. **Relative Fourier–Mukai functors.** In [33], Kuznetsov drove the attention to a slightly more general version of the classical Fourier–Mukai functors. For sake of simplicity, take a pair of smooth projective varieties X_1 and X_2 over the same smooth projective variety S. To fix the notation, this means that, for $i = 1, 2$, there is a morphism $f_i \colon X_i \to S$. Clearly, one may want to relax the assumptions on X_i and S but this is not in order here.

Definition 6.19. (i) A functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ is *S-linear* if

$$\mathsf{F}(\mathcal{A} \otimes f_1^*(\mathcal{C})) \cong \mathsf{F}(\mathcal{A}) \otimes f_2^*(\mathcal{C}),$$

for all $\mathcal{A} \in \mathrm{D}^b(X_1)$ and for all $\mathcal{C} \in \mathrm{D}^b(S)$.

(ii) A strictly full subcategory $\mathbf{S} \subseteq \mathrm{D}^b(X_i)$ is *S-linear* if for all $\mathcal{C} \in \mathbf{S}$ and all $\mathcal{A} \in \mathrm{D}^b(S)$ we have $f_i^*(\mathcal{A}) \otimes \mathcal{C} \in \mathbf{S}$.

These functors have reasonable properties listed in the following proposition and proved in [33] (see, in particular, Section 2.7 there).

Proposition 6.20. (i) *If F is exact, S-linear and admits a right adjoint functor $\mathsf{F}^!$, then $\mathsf{F}^!$ is also S-linear.*

(ii) *If $\mathbf{S} \subseteq \mathrm{D}^b(X_i)$ is a strictly full admissible S-linear subcategory, then its right and left orthogonals are S-linear as well.*

As pointed out in, for example, [32, 33], the relative functors play important roles in various geometric situations. Thus it makes perfect sense to wonder whether the machinery developed for Fourier–Mukai functors in the non-relative setting can be applied.

It is clear that any full exact S-linear functor or rather any exact S-linear functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$ satisfying (3.2) is of Fourier–Mukai type in view of Theorem 3.9. In particular, there is a unique (up to isomorphism) $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism $\mathsf{F} \cong \Phi_{\mathcal{E}}$.

On the other hand, we may consider the fibre product $X_1 \times_S X_2$ and the closed embedding $i\colon X_1 \times_S X_2 \hookrightarrow X_1 \times X_2$.

Lemma 6.21. ([33], Lemma 2.32.) *If $\mathcal{E} \in \mathrm{D}^b(X_1 \times_S X_2)$, then the Fourier–Mukai functor $\Phi_{i_*\mathcal{E}}$ is S-linear.*

It is not difficult to observe that the Fourier–Mukai kernel of an S-linear Fourier–Mukai functor has to be set theoretically supported on the fibre product $X_1 \times_S X_2$. The scheme theoretical point of view is more complicated to be dealt with and thus, following [33], it makes sense to pose the following questions:

Question 6.22. (i) *Given a full exact S-linear functor $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$, do there exist an $\mathcal{E} \in \mathrm{D}^b(X_1 \times_S X_2)$ and an isomorphism of functors $\mathsf{F} \cong \Phi_{i_*\mathcal{E}}$?*

(ii) *Is the choice of the Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(X_1 \times_S X_2)$ in (i) unique (up to isomorphism)?*

To our knowledge, no general answer to these problems is present in the literature.

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