NON-UNIQUENESS OF FOURIER–MUKAI KERNELS

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Abstract. We prove that the kernels of Fourier–Mukai functors are not unique in general. On the other hand we show that the cohomology sheaves of those kernels are unique. We also discuss several properties of the functor sending an object in the derived category of the product of two smooth projective schemes to the corresponding Fourier–Mukai functor.

1. Introduction

All functors that appeared so far in the geometric applications of the theory of derived categories have a very special nature: they are *Fourier–Mukai functors*. Recall that if X_1 and X_2 are projective schemes, an exact functor $\mathsf{F} : \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$ is of Fourier–Mukai type if there exists $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ and an isomorphism of exact functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$, where, denoting by $p_i : X_1 \times X_2 \to X_i$ the natural projections, $\Phi_{\mathcal{E}} : \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$ is the exact functor defined by

$$\Phi_{\mathcal{E}} := \mathbf{R}(p_2)_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)).$$

Such a complex $\mathcal{E}$ is called a *kernel* of F. Recall that the category $\mathbf{Perf}(X_i)$ of perfect complexes is the full triangulated subcategory of the bounded derived category of coherent sheaves $\mathrm{D}^b(X_i) := \mathrm{D}^b(\mathbf{Coh}(X_i))$ consisting of complexes which are quasi-isomorphic to bounded complexes of locally free sheaves of finite type over X_i. Notice that $\mathbf{Perf}(X_i)$ coincides with $\mathrm{D}^b(X_i)$ if and only if X_i is regular.

There are many advantages of having a functor which is described in terms of an object in the derived category of the product. Among them is the study of the action of those functors on cohomology leading, for example, to a description of the group of autoequivalences of special projective varieties (see [12]). As Fourier–Mukai equivalences act also on Hochschild homology and cohomology one may also study deformations of smooth projective varieties together with deformations of equivalences between the corresponding bounded derived categories of coherent sheaves.

Despite the relevance of these functors, two important and basic questions remain open:

(Q1) *Are all exact functors between the bounded derived categories of coherent sheaves on smooth projective varieties of Fourier–Mukai type?*

(Q2) *Is the kernel of a Fourier–Mukai functor unique (up to isomorphism)?*

Obviously, the same questions may be reformulated more generally in terms of perfect complexes on projective schemes.

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The best evidence that the answer to both questions could be positive is due to some beautiful results of Toën concerning dg-categories. Indeed, in [14] it is shown that all dg (quasi-)functors between the dg-categories of perfect complexes on smooth proper schemes are of Fourier–Mukai type. This result, combined with the conjecture by Bondal, Larsen and Lunts in [3] saying that all exact functors between the bounded derived categories of coherent sheaves on smooth projective varieties should be liftable to dg (quasi-)functors between the corresponding dg-enhancements, would answer positively (Q1).

Contrary to the exhaustive picture for dg-categories, the results concerning derived categories are more fragmentary and essentially provide responses to (Q1) and (Q2) under some assumptions on the functor. In the seminal paper [12] (together with [4]) Orlov solved completely the case of fully faithful functors between the bounded derived categories of coherent sheaves on smooth projective varieties. Indeed, he proves that these functors are all of Fourier–Mukai type with unique (up to isomorphism) kernel. Various generalizations to quotient stacks and twisted categories were given by Kawamata in [9] and by the authors in [6] respectively. In particular, in [6] a condition much weaker than fully faithfulness is required for a functor to be of Fourier–Mukai type. More recently, a new approach involving dg-categories has been proposed by Lunts and Orlov in [11], where they deal with the case of fully faithful functors between the derived categories of perfect complexes on projective schemes. This approach allows them to avoid some of the assumptions made by Ballard in [2]. In [7], we extend further the results in [11] and study exact functors between supported derived categories.

Back to the questions above, the main result in this paper shows that the answer to (Q2) cannot be positive in general (see Section 3 for the proof).

Theorem 1.1. *For every elliptic curve X over an algebraically closed field there exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{D}^b(X \times X)$ such that $\mathcal{E}_1 \not\cong \mathcal{E}_2$ but $\Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2}$.*

More precisely, we get the following picture. Given two smooth projective varieties X_1 and X_2, denote by $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2))$ the category of exact functors between $\mathrm{D}^b(X_1)$ and $\mathrm{D}^b(X_2)$. Putting all together, we will see in Sections 2 and 3 that the natural functor

$$\Phi_{-}^{X_1 \to X_2} \colon \mathrm{D}^b(X_1 \times X_2) \longrightarrow \mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2)) \tag{1.1}$$

sending $\mathcal{E}$ to the functor $\Phi_{\mathcal{E}} = \Phi_{\mathcal{E}}^{X_1 \to X_2}$ is, in general, neither essentially injective (Theorem 1.1) nor faithful (see [5, Example 6.5]) nor full (Proposition 2.3). Moreover we cannot even expect that $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2))$ has a triangulated structure making the above functor exact (Corollary 2.7). Such a negative picture puts the optimistic hope to answer question (Q1) positively a bit in the shade.

On the positive side, in Section 4 we prove the following result, which provides our best substitute for the uniqueness of Fourier–Mukai kernels.

Theorem 1.2. *Let X_1 and X_2 be projective schemes and let $\mathsf{F} \colon \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$ be an exact functor. If $\mathsf{F} \cong \Phi_{\mathcal{E}}$ for some $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$, then the cohomology sheaves of $\mathcal{E}$ are uniquely determined (up to isomorphism) by F.*

Notice that, as a consequence, the class in the Grothendieck group $K(X_1 \times X_2)$ of a Fourier–Mukai kernel is uniquely determined by the functor.

After the final version of this paper was completed, we were informed that the example used in the proof of Theorem 1.1 had already been circulating among some people. As we could not find any mention of this result in the literature, we still believe that it is important to have it written down.

Notation. In the paper, $\Bbbk$ is a field and all schemes are assumed to be over $\Bbbk$. Notice that in Sections 2 and 3, the field $\Bbbk$ is assumed to be algebraically closed. All additive (in particular, triangulated) categories and all additive (in particular, exact) functors will be assumed to be $\Bbbk$-linear. An additive category will be called Hom-finite if the $\Bbbk$-vector space $\mathrm{Hom}(A, B)$ is finite dimensional for every objects A and B. If $f\colon A \to B$ is a morphism in a triangulated category, the *cone* of f, denoted by $\mathrm{C}(f)$, is an object (well defined up to isomorphism) fitting into a distinguished triangle $A \xrightarrow{f} B \to \mathrm{C}(f) \to A[1]$.

2. Properties of the functor $\Phi_{-}^{X_1 \to X_2}$

In this section we deal with some preliminary results concerning the functor defined in (1.1) from the derived category of the product of two smooth projective varieties to the category of exact functors between the corresponding derived categories of coherent sheaves. The base field $\Bbbk$ is assumed to be algebraically closed.

2.1. **Counterexamples to faithfulness and fullness.** Following the notation in the introduction, if $\mathbf{T}_1$ and $\mathbf{T}_2$ are two triangulated categories, we denote by $\mathbf{ExFun}(\mathbf{T}_1, \mathbf{T}_2)$ the category whose objects are the exact functors from $\mathbf{T}_1$ to $\mathbf{T}_2$ and whose morphisms are the natural transformations compatible with shifts. Clearly $\mathbf{ExFun}(\mathbf{T}_1, \mathbf{T}_2)$ is additive and has a natural shift functor, but, due to the non-functoriality of the cone, it is not known if in general it can be endowed with any triangulated structure. In particular, it is not expected to possess a natural one.

Now assume that X_i for $i = 1, 2$ are two smooth projective varieties of dimension d_i. It is easy to see that the map $\mathcal{E} \mapsto \Phi_{\mathcal{E}} = \Phi_{\mathcal{E}}^{X_1 \to X_2}$ extends to the functor (1.1), which is obviously additive and compatible with shifts. It is natural to study properties of this functor, in particular one can ask if it is faithful, full, essentially injective (i.e. if a kernel of a Fourier–Mukai functor is unique up to isomorphism), essentially surjective (i.e. if every exact functor is of Fourier–Mukai type) or if $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2))$ admits a triangulated structure such that $\Phi_{-}^{X_1 \to X_2}$ is exact. We are going to see that, at least for some choices of X_1 and X_2, the answers to most of these questions are negative. Unfortunately we were unable to prove anything new about essential surjectivity, which is certainly a very intriguing problem.

Remark 2.1. The functor $\Phi_{-}^{X_2 \to X_1}$ satisfies one of the properties we are interested in if and only if $\Phi_{-}^{X_1 \to X_2}$ does: this follows from the fact that $\Phi_{-}^{X_2 \to X_1}$ can be identified with the opposite functor of $\Phi_{-}^{X_1 \to X_2}$ under the equivalences $\mathrm{D}^b(X_1 \times X_2) \to \mathrm{D}^b(X_1 \times X_2)^{\circ}$ (defined on the objects by $\mathcal{E} \mapsto \mathcal{E}^{\vee} \otimes p_1^* \omega_{X_1}[d_1]$) and $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2)) \to \mathbf{ExFun}(\mathrm{D}^b(X_2), \mathrm{D}^b(X_1))^{\circ}$ (defined on the objects by $\mathsf{F} \mapsto \mathsf{F}_*$, the right adjoint of F). Notice that we are using the fact that, in this context, any exact functor has right and left adjoint by [4] (see also [6, Rmk. 2.1]).

Remark 2.2. The functor $\Phi_{-}^{X_1 \to X_2}$ is an equivalence (hence it has all the good properties we are investigating) if d_1 or d_2 is 0. Indeed, by Remark 2.1 we can assume $d_1 = 0$ (so that $X_1 = \operatorname{Spec} \Bbbk$ is a point, being $\Bbbk$ algebraically closed), and then it is easy to see that a quasi-inverse is the functor defined on objects by $\mathsf{F} \mapsto \mathsf{F}(\Bbbk)$.

So the interesting case to study is when $d_1, d_2 > 0$, but we can prove something only when d_1 or d_2 is 1. The reason for this is that if X is a smooth projective curve, then the abelian category $\mathbf{Coh}(X)$ is hereditary (i.e. $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for every $i > 1$ and for every $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X)$), which implies that every object of $\mathrm{D}^b(X)$ is isomorphic to the direct sum of its cohomology sheaves. Being X proper, by [1, Thm. 2], the Krull–Schmidt theorem holds for the abelian category $\mathbf{Coh}(X)$. Namely, each object in $\mathbf{Coh}(X)$ can be written in a unique way (up to reordering and isomorphism) as a finite direct sum of indecomposable objects. Moreover, being X a smooth curve, every indecomposable object in $\mathbf{Coh}(X)$ is either a vector bundle or a torsion sheaf of the form $\mathcal{O}_{n\mathrm{p}}$ with n a positive integer and p a closed point of X. Since a natural transformation between additive functors is always additive in the obvious sense, we see in particular that a natural transformation of exact functors from $\mathrm{D}^b(X)$ is determined by its values on the indecomposable objects of $\mathbf{Coh}(X)$. This property is essential in the proof of the following result, whose statement about non-faithfulness is a generalization of [5, Example 6.5] (where only the particular case in which $X_1 = X_2$ is an elliptic curve is considered).

Proposition 2.3. *If* $\min\{d_1, d_2\} = 1$*, then* $\Phi_{-}^{X_1 \to X_2}$ *is neither faithful nor full.*

Proof. By Remark 2.1 we can assume that $1 = d_1 \leq d_2$. Choose a finite morphism $f\colon X_1 \to \mathbb{P}^{d_2}$ and a finite and surjective (hence flat) morphism $g\colon X_2 \to \mathbb{P}^{d_2}$. Then $\mathsf{F} := g^* \circ f_*\colon \mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$ is an exact functor, which trivially extends to an exact functor again denoted by $\mathsf{F}\colon \mathrm{D}^b(X_1) \to \mathrm{D}^b(X_2)$. Clearly there exists $0 \not\cong \mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ such that $\mathsf{F} \cong \Phi_{\mathcal{E}}$.

In order to prove that $\Phi_{-}^{X_1 \to X_2}$ is not faithful, notice that, by Serre duality,

$$\operatorname{Hom}_{\mathrm{D}^b(X_1 \times X_2)}(\mathcal{E}, \mathcal{E}) \cong \operatorname{Hom}_{\mathrm{D}^b(X_1 \times X_2)}(\mathcal{E}, \mathcal{E} \otimes \omega_{X_1 \times X_2}[1 + d_2])^\vee,$$

so there exists $0 \neq \alpha \in \operatorname{Hom}_{\mathrm{D}^b(X_1 \times X_2)}(\mathcal{E}, \mathcal{E} \otimes \omega_{X_1 \times X_2}[1 + d_2])$. Since $\omega_{X_1 \times X_2} \cong p_1^*\omega_{X_1} \otimes p_2^*\omega_{X_2}$, this induces for any $\mathcal{F} \in \mathbf{Coh}(X_1)$ a morphism

$$\Phi_\alpha(\mathcal{F})\colon \Phi_{\mathcal{E}}(\mathcal{F}) \cong \mathsf{F}(\mathcal{F}) \to \Phi_{\mathcal{E} \otimes \omega_{X_1 \times X_2}[1+d_2]}(\mathcal{F}) \cong \mathsf{F}(\mathcal{F} \otimes \omega_{X_1}) \otimes \omega_{X_2}[1 + d_2].$$

As $\mathsf{F}(\mathcal{F})$ and $\mathsf{F}(\mathcal{F} \otimes \omega_{X_1})$ are objects of $\mathbf{Coh}(X_2)$, it follows that $\Phi_\alpha(\mathcal{F}) = 0$, whence $\Phi_\alpha = 0$.

Now we are going to show that $\Phi_{-}^{X_1 \to X_2}$ is not full. We start by observing that for every closed point $\mathrm{p} \in X_1$ we can define a natural transformation $\zeta_{\mathrm{p}}\colon \mathrm{id} \to [1]$ of exact functors on $\mathrm{D}^b(X_1)$ by setting $\zeta_{\mathrm{p}}(\mathcal{F}) := 0$ for every indecomposable object of $\mathbf{Coh}(X_1)$ not isomorphic to $\mathcal{O}_{\mathrm{p}}$ and taking $\zeta_{\mathrm{p}}(\mathcal{O}_{\mathrm{p}}) \neq 0$ (note that the latter is an element of $\operatorname{Hom}(\mathcal{O}_{\mathrm{p}}, \mathcal{O}_{\mathrm{p}}[1]) \cong \operatorname{Hom}(\mathcal{O}_{\mathrm{p}}, \mathcal{O}_{\mathrm{p}})^\vee \cong \Bbbk$ by Serre duality), and then extending additively and by shifts in the obvious way. It is easy to see that in this way ζ_{p} is really a natural transformation, namely that $\phi[1] \circ \zeta_{\mathrm{p}}(\mathcal{F}) = \zeta_{\mathrm{p}}(\mathcal{G}) \circ \phi$ for every morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{D}^b(X)$: indeed, it is enough to assume that $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X)$ are indecomposable, in which case the required equality follows from Lemma 3.3 below if $\mathcal{F}$ and $\mathcal{G}$ are supported at p, and is otherwise trivial. (Indeed, we use Lemma 3.3 identifying F_n with $\mathcal{O}_{n\mathrm{p}}$, as it is explained in the paragraph before the lemma.)

Composing with F clearly defines a natural transformation from $\mathsf{F} \cong \Phi_{\mathcal{E}}$ to $\mathsf{F}[1] \cong \Phi_{\mathcal{E}[1]}$, hence an element $\zeta'_{\mathrm{p}} \in \mathrm{Hom}_{\mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))}(\Phi_{\mathcal{E}}, \Phi_{\mathcal{E}[1]})$. It is not difficult to see that $\zeta'_{\mathrm{p}}(\mathcal{O}_{\mathrm{p}}) \neq 0$, which implies that

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathbf{ExFun}(\mathrm{D}^b(X_1),\mathrm{D}^b(X_2))}(\Phi_{\mathcal{E}}, \Phi_{\mathcal{E}[1]}) = \infty,$$

thereby proving that $\Phi_{-}^{X_1 \to X_2}$ is not full. □

2.2. **Projective line.** We start by proving the uniqueness (up to isomorphism) of Fourier–Mukai kernels for the projective line. This has to be compared with the more interesting case of elliptic curves (Section 3).

Proposition 2.4. *If X_1 or X_2 is $\mathbb{P}^1$, then $\Phi_{-}^{X_1 \to X_2}$ is essentially injective.*

Proof. As usual, by Remark 2.1 we can assume that $X_1 = \mathbb{P}^1$. Since on $\mathbb{P}^1 \times \mathbb{P}^1$ there is a resolution of the diagonal of the form

$$0 \to \mathcal{O}(-1,-1) \xrightarrow{x_0 \boxtimes x_1 - x_1 \boxtimes x_0} \mathcal{O} \to \mathcal{O}_{\Delta} \to 0,$$

the argument in [6, Sect. 4.3] shows that, for every exact functor $\mathsf{F} \colon \mathrm{D}^b(\mathbb{P}^1) \to \mathrm{D}^b(X_2)$, any object $\mathcal{E}$ in $\mathrm{D}^b(\mathbb{P}^1 \times X_2)$ such that $\mathsf{F} \cong \Phi_{\mathcal{E}}$ is necessarily a convolution of the complex

$$\mathcal{O}(-1) \boxtimes \mathsf{F}(\mathcal{O}(-1)) \xrightarrow{\varphi := x_0 \boxtimes \mathsf{F}(x_1) - x_1 \boxtimes \mathsf{F}(x_0)} \mathcal{O} \boxtimes \mathsf{F}(\mathcal{O}),$$

hence it is uniquely determined up to isomorphism as the cone of φ. □

We conclude this section showing that, in some cases, the category $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2))$ cannot have a suitable triangulated structure. For this we need some preliminary results.

Lemma 2.5. *Let $\mathbf{T}$ be a* Hom*-finite triangulated category and let $f \colon A \to B$ be a morphism of $\mathbf{T}$. Then $\mathrm{C}(f) \cong A[1] \oplus B$ if and only if $f = 0$.*

Proof. The other implication being well-known, we assume that $\mathrm{C}(f) \cong A[1] \oplus B$. Applying the cohomological functor $\mathrm{Hom}(-, B)$ to the distinguished triangle $A \xrightarrow{f} B \to A[1] \oplus B \to A[1]$, one gets an exact sequence of finite dimensional $\Bbbk$-vector spaces

$$\mathrm{Hom}(A[1], B) \to \mathrm{Hom}(A[1] \oplus B, B) \to \mathrm{Hom}(B, B) \xrightarrow{(-)\circ f} \mathrm{Hom}(A, B).$$

For dimension reasons, the last map must be 0, hence $f = 0$. □

Lemma 2.6. *Let $\mathsf{F} \colon \mathbf{T} \to \mathbf{T}'$ be an exact functor between triangulated categories, and assume that $\mathbf{T}$ is* Hom*-finite. If F is essentially injective, then F is faithful, too.*

Proof. Let $f \colon A \to B$ be a morphism of $\mathbf{T}$ such that $\mathsf{F}(f) = 0$. Then

$$\mathsf{F}(\mathrm{C}(f)) \cong \mathrm{C}(\mathsf{F}(f)) \cong \mathsf{F}(A)[1] \oplus \mathsf{F}(B) \cong \mathsf{F}(A[1] \oplus B)$$

in $\mathbf{T}'$, whence $\mathrm{C}(f) \cong A[1] \oplus B$ in $\mathbf{T}$ because F is essentially injective. It follows from Lemma 2.5 that $f = 0$. □

Corollary 2.7. *If $d_1, d_2 > 0$ and X_1 or X_2 is $\mathbb{P}^1$, then $\mathbf{ExFun}(\mathrm{D}^b(X_1), \mathrm{D}^b(X_2))$ does not admit a triangulated structure such that $\Phi_{-}^{X_1 \to X_2}$ is exact.*

Proof. This follows from Lemma 2.6, since we know that in this case $\Phi_{-}^{X_1 \to X_2}$ is essentially injective by Proposition 2.4, but not faithful by Proposition 2.3. □

3. Elliptic curves and non-uniqueness

In this section we provide the proof of Theorem 1.1, so we assume that $\Bbbk$ is algebraically closed and that X is an elliptic curve. Up to replacing $\mathrm{D}^b(X)$ with an equivalent category, we can assume that there is exactly one object in every isomorphism class, and more precisely, as explained in Section 2.1, that every object is a finite direct sum of shifts of coherent sheaves and that every object of $\mathbf{Coh}(X)$ is uniquely (up to reordering) a finite direct sum of indecomposable sheaves. Recall that the indecomposable objects of $\mathbf{Coh}(X)$ are either vector bundles or torsion sheaves of the form $\mathcal{O}_{n\mathrm{p}}$ with $n > 0$ and p a closed point of X. The following result summarizes some properties of indecomposable vector bundles over an elliptic curve.

Proposition 3.1. ([15]) *For every $r > 0$ and $d \in \mathbb{Z}$ there is an indecomposable vector bundle $E_{r,d}$ of rank r and degree d on X such that:*

- (i) *All indecomposable vector bundles of rank r and degree d are those of the form $E_{r,d} \otimes L$ with $L \in \mathrm{Pic}^0(X)$, and they are all distinct;*
- (ii) *If $k > 0$,then $F_k := E_{k,0}$ is the only indecomposable vector bundle of rank k and degree 0 having global sections (in particular, $F_1 = \mathcal{O}_X$), and if $k > 1$ there is an exact sequence*

$$0 \to F_1 \to F_k \to F_{k-1} \to 0; \tag{3.1}$$

- (iii) *If $n = \gcd(r, d)$, $E_{r,d} = E_{r/n,d/n} \otimes F_n$;*
- (iv) *$E_{r,d}$ (hence also $E_{r,d} \otimes L$ for every $L \in \mathrm{Pic}^0(X)$) is semistable, and it is stable if and only if $\gcd(r, d) = 1$.*

Corollary 3.2. *Let E_i (for $i = 1, 2$) be indecomposable vector bundles of rank r_i and degree d_i on X with the property that $\mathrm{Hom}(E_1, E_2) \neq 0 \neq \mathrm{Hom}(E_2, E_1)$. Then, setting $n_i := \gcd(r_i, d_i)$, there exists a stable vector bundle E of rank $r_1/n_1 = r_2/n_2$ and degree $d_1/n_1 = d_2/n_2$ such that $E_i = E \otimes F_{n_i}$ for $i = 1, 2$.*

Proof. The hypothesis, together with the fact that E_1 and E_2 are semistable, implies that $d_1/r_1 = d_2/r_2$, from which it is immediate to deduce that $r_1/n_1 = r_2/n_2$ and $d_1/n_1 = d_2/n_2$. Set $r := r_1/n_1$ and $d := d_1/n_1$. As $E_i = E_{r_i,d_i} \otimes L_i$ for some $L_i \in \mathrm{Pic}^0(X)$, we get $E_i = E_{r,d} \otimes L_i \otimes F_{n_i}$ for $i = 1, 2$ (see parts (i) and (iii) of Proposition 3.1). It remains to prove that $L_1 = L_2$, because then we can conclude setting $E := E_{r,d} \otimes L_i$ (which, due to part (iv) of Proposition 3.1, is stable since $\gcd(r, d) = 1$). Assuming instead that $L_1 \neq L_2$, we will reach a contradiction by showing that $\mathrm{Hom}(E_1, E_2) = 0$. We proceed by induction on $n_1 + n_2$: the case $n_1 = n_2 = 1$ follows from the fact $E_i = E_{r,d} \otimes L_i$ for $i = 1, 2$ are distinct stable vector bundles (see parts (i) and (iv) of Proposition 3.1). As for the inductive step, we suppose $n_2 > 1$ (the case $n_1 > 1$ is similar) and apply the functor $\mathrm{Hom}(E_1, E_{r,d} \otimes L_2 \otimes -)$ to (3.1) with $k = n_2$. This yields an exact sequence

$$\mathrm{Hom}(E_1, E_{r,d} \otimes L_2 \otimes F_1) \to \mathrm{Hom}(E_1, E_{r,d} \otimes L_2 \otimes F_{n_2}) \to \mathrm{Hom}(E_1, E_{r,d} \otimes L_2 \otimes F_{n_2-1}).$$

By induction, the first and the third terms in the sequence are 0, whence the second one is 0 as well. But $E_{r,d} \otimes L_2 \otimes F_{n_2} = E_2$ and this provides the desired contradiction. □

We will denote by $\mathbf{T}_E$ for E a stable vector bundle on X (respectively $\mathbf{T}_{\mathrm{p}}$ for p a closed point of X) the full triangulated subcategory of $\mathrm{D}^b(X)$ classically generated by E (respectively $\mathcal{O}_{\mathrm{p}}$), namely the smallest strictly full triangulated subcategory of $\mathrm{D}^b(X)$ containing E (respectively $\mathcal{O}_{\mathrm{p}}$) and closed under direct summands. Since $\mathbf{R}\mathrm{Hom}(E,E)\cong\mathbf{R}\mathrm{Hom}(\mathcal{O}_{\mathrm{p}},\mathcal{O}_{\mathrm{p}})\cong\Bbbk\oplus\Bbbk[1]$ (so that E and $\mathcal{O}_{\mathrm{p}}$ are 1-spherical objects), it follows from [10, Thm. 2.1] that these categories are all equivalent; it is also clear that the indecomposable sheaves of $\mathbf{T}_E$ (respectively $\mathbf{T}_{\mathrm{p}}$) are $E\otimes F_n$ (respectively $\mathcal{O}_{n\mathrm{p}}$) for $n>0$. In the following we will also denote by $\mathbf{T}$ any of the equivalent categories $\mathbf{T}_E$ or $\mathbf{T}_{\mathrm{p}}$, but for simplicity of notation we will identify it with $\mathbf{T}_{\mathcal{O}_X}$. As $\mathbf{T}$ is equivalent to a (derived) category of $\Bbbk[x]$–modules, F_n corresponding to $\Bbbk[x]/(x^n)$ (this is perhaps easier to see regarding $\mathbf{T}$ as $\mathbf{T}_{\mathrm{p}}$), it is clear that $\dim_\Bbbk\mathrm{Hom}(F_m,F_n)=\min\{m,n\}$, for $m,n>0$, and there are (non-split) distinguished triangles in $\mathbf{T}$ (the second one is induced by (3.1) with $k=n+1$)

$$F_n\xrightarrow{\pi'_{n+1,1}}F_{n+1}\xrightarrow{\pi_{n+1,1}}F_1\xrightarrow{\pi''_{n+1,1}}F_n[1]\tag{3.2}$$

$$F_1\xrightarrow{\pi'_{n+1,n}}F_{n+1}\xrightarrow{\pi_{n+1,n}}F_n\xrightarrow{\pi''_{n+1,n}}F_1[1]\tag{3.3}$$

where $\pi_{m,n}\colon F_m\to F_n$ for $m>n$ denotes the natural projection.

Lemma 3.3. *If $0<m\leq n$, every morphism $F_m\to F_{n+1}$ factors through $\pi'_{n+1,1}$ and every morphism $F_{n+1}\to F_m$ factors through $\pi_{n+1,n}$. Moreover, every morphism $F_{n+1}\to F_{n+1}$ is uniquely the sum of $\lambda\mathrm{id}$ for some $\lambda\in\Bbbk$ and of a morphism which factors through $\pi'_{n+1,1}$ and $\pi_{n+1,n}$.*

Proof. By (3.2) the map $\mathrm{Hom}(F_m,F_n)\xrightarrow{\pi'_{n+1,1}\circ(-)}\mathrm{Hom}(F_m,F_{n+1})$ is injective for every $m>0$. In particular, if $m\leq n$ the map is also surjective because both spaces have dimension m, whereas if $m=n+1$ the first space has dimension n, the second $n+1$ and clearly id is not in the image of the map. This proves both statements involving $\pi'_{n+1,1}$, and those involving $\pi_{n+1,n}$ can be proved in a similar way using (3.3) instead of (3.2). □

Since $\mathrm{Hom}(\mathcal{O}_\Delta[-1],\mathcal{O}_\Delta[1])\cong\mathrm{Hom}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)^\vee\cong\Bbbk$ by Serre duality, an object $\mathcal{E}$ of $\mathrm{D}^b(X\times X)$ obtained as the cone of a nonzero morphism $\mathcal{O}_\Delta[-1]\to\mathcal{O}_\Delta[1]$ is well defined up to isomorphism. Notice that such a morphism is the same as the one considered in [5, Example 6.5]. Setting also $\mathcal{E}_0:=\mathcal{O}_\Delta\oplus\mathcal{O}_\Delta[1]$, we have $\mathcal{E}\not\cong\mathcal{E}_0$ by Lemma 2.5. We are going to prove that

$$\Phi_{\mathcal{E}}\cong\Phi_{\mathcal{E}_0}\colon\mathrm{D}^b(X)\to\mathrm{D}^b(X).$$

To this purpose, we start by observing that $\Phi_{\mathcal{E}_0}$ is just $\mathrm{id}\oplus[1]$ and that $\Phi_{\mathcal{E}}$ coincides with $\Phi_{\mathcal{E}_0}$ on objects. As explained in the following example, in general, this is not enough to conclude that the two functors are isomorphic.

Example 3.4. An easy calculation shows that, on $\mathbb{P}^1\times\mathbb{P}^1$, there is an isomorphism of $\Bbbk$-vector spaces $\mathrm{Hom}(\Delta_*\mathcal{O}_{\mathbb{P}^1}[-1],\Delta_*(\omega_{\mathbb{P}^1}^{\otimes 2})[1])\cong\Bbbk$. Take $0\neq f\in\mathrm{Hom}(\Delta_*\mathcal{O}_{\mathbb{P}^1}[-1],\Delta_*(\omega_{\mathbb{P}^1}^{\otimes 2})[1])$ and consider the objects $\mathcal{F}_0:=\Delta_*\mathcal{O}_{\mathbb{P}^1}\oplus\Delta_*(\omega_{\mathbb{P}^1}^{\otimes 2})[1]$ and $\mathcal{F}:=\mathrm{C}(f)$ in $\mathrm{D}^b(\mathbb{P}^1\times\mathbb{P}^1)$. Obviously $\Phi_{\mathcal{F}_0}$ and $\Phi_{\mathcal{F}}$ coincide on objects because $\mathcal{G}\oplus(\mathcal{G}\otimes\omega_{\mathbb{P}^1}^{\otimes 2})[1]\cong\Phi_{\mathcal{F}_0}(\mathcal{G})\cong\Phi_{\mathcal{F}}(\mathcal{G})$, for every $\mathcal{G}\in\mathrm{D}^b(\mathbb{P}^1)$. On the other hand $\mathcal{F}_0\not\cong\mathcal{F}$ (use again Lemma 2.5) and so, by Proposition 2.4, the functors $\Phi_{\mathcal{F}_0}$ and $\Phi_{\mathcal{F}}$ are not isomorphic.

Back to the genus 1 case, to prove that $\Phi_{\mathcal{E}} \cong \Phi_{\mathcal{E}_0}$ we have to take care of morphisms as well. To this end observe that $\Phi_{\mathcal{E}}$ is defined on every morphism $f\colon \mathcal{A} \to \mathcal{B}$ of $\mathrm{D}^b(X)$ by

$$\Phi_{\mathcal{E}}(f) = \begin{pmatrix} f & 0 \\ \epsilon(f) & f[1] \end{pmatrix} : \mathcal{A} \oplus \mathcal{A}[1] \to \mathcal{B} \oplus \mathcal{B}[1]$$

for some $\epsilon(f)\colon \mathcal{A} \to \mathcal{B}[1]$. Notice that ϵ is $\Bbbk$–linear in the obvious sense (because $\Phi_{\mathcal{E}}$ is $\Bbbk$–linear), $\epsilon(\mathrm{id}_{\mathcal{A}}) = 0$ for every object $\mathcal{A}$ of $\mathrm{D}^b(X)$ (because $\Phi_{\mathcal{E}}(\mathrm{id}_{\mathcal{A}}) = \mathrm{id}_{\Phi_{\mathcal{E}}(\mathcal{A})}$) and $\epsilon(g \circ f) = \epsilon(g) \circ f + g[1] \circ \epsilon(f)$ for every pair of composable morphisms f and g of $\mathrm{D}^b(X)$ (because $\Phi_{\mathcal{E}}(g \circ f) = \Phi_{\mathcal{E}}(g) \circ \Phi_{\mathcal{E}}(f)$). It is also evident that if $\mathcal{A}$ and $\mathcal{B}$ are sheaves and $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B}[1])$, then $\epsilon(f) = 0$.

Proposition 3.5. *With the above notation, there is an isomorphism $\Phi_{\mathcal{E}} \cong \Phi_{\mathcal{E}_0}$.*

Proof. We will show that there is an isomorphism $\eta\colon \Phi_{\mathcal{E}_0} \xrightarrow{\sim} \Phi_{\mathcal{E}}$ such that

$$\eta(\mathcal{A}) = \begin{pmatrix} \mathrm{id} & 0 \\ \beta(\mathcal{A}) & \mathrm{id} \end{pmatrix} : \mathcal{A} \oplus \mathcal{A}[1] \to \mathcal{A} \oplus \mathcal{A}[1]$$

for every object $\mathcal{A}$ of $\mathrm{D}^b(X)$. It is clearly enough to define $\eta(\mathcal{A})$ for every indecomposable sheaf $\mathcal{A}$ (and then extend additively and by shifts in the obvious way) so that $\Phi_{\mathcal{E}}(f) \circ \eta(\mathcal{A}) = \eta(\mathcal{B}) \circ \Phi_{\mathcal{E}_0}(f)$ for every morphism of indecomposable sheaves $f\colon \mathcal{A} \to \mathcal{B}$. Now, this equality is equivalent to $\epsilon(f) = \beta(\mathcal{B}) \circ f - f[1] \circ \beta(\mathcal{A})$, so it is certainly satisfied if either $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = 0$ or $\mathrm{Hom}(\mathcal{A}, \mathcal{B}[1]) = 0$. On the other hand, if the indecomposable sheaves $\mathcal{A}$ and $\mathcal{B}$ are such that $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \neq 0 \neq \mathrm{Hom}(\mathcal{A}, \mathcal{B}[1])$, then $\mathcal{A}$ and $\mathcal{B}$ belong to the same subcategory $\mathbf{T}_E$ (E a vector bundle) or $\mathbf{T}_{\mathrm{p}}$ (p a closed point). This follows from Corollary 3.2 if both $\mathcal{A}$ and $\mathcal{B}$ are vector bundles (taking into account that $\mathrm{Hom}(\mathcal{A}, \mathcal{B}[1]) \cong \mathrm{Hom}(\mathcal{B}, \mathcal{A})^\vee$ by Serre duality), whereas it is trivial in the other cases.

So, setting $\mathsf{F} := \Phi_{\mathcal{E}}|_{\mathbf{T}}$ and $\mathsf{F}_0 = \Phi_{\mathcal{E}_0}|_{\mathbf{T}}$, it is enough to prove that there is an isomorphism $\eta\colon \mathsf{F}_0 \to \mathsf{F}$ of the above form. In order to do that, we are going to define inductively for every $n > 0$ (exact) functors $\mathsf{F}_n\colon \mathbf{T} \to \mathbf{T}$ and morphisms of $\mathbf{T}$

$$\alpha_n = \begin{pmatrix} \mathrm{id} & 0 \\ \beta_n & \mathrm{id} \end{pmatrix} : F_n \oplus F_n[1] \to F_n \oplus F_n[1]$$

with the following properties:

(a) $\mathsf{F}_1 = \mathsf{F}$ and $\alpha_1 = \mathrm{id}$;

(b) for every $n > 0$ the functor F_n coincides with F_0 on objects, $\mathsf{F}_n(f) = \begin{pmatrix} f & 0 \\ \epsilon_n(f) & f[1] \end{pmatrix}$ for every morphism f of $\mathbf{T}$ and $\mathsf{F}_n|_{\mathbf{T}_n} = \mathsf{F}_0|_{\mathbf{T}_n}$, where $\mathbf{T}_n$ denotes the full additive and closed under shifts (but not triangulated) subcategory of $\mathbf{T}$ generated by F_i for $0 < i \leq n$;

(c) for every $n > 1$ the morphisms $\eta_n(F_m)\colon F_m \oplus F_m[1] \to F_m \oplus F_m[1]$ (for $m > 0$) defined by $\eta_n(F_n) = \alpha_n$ and $\eta_n(F_m) = \mathrm{id}$ if $m \neq n$, extend to an isomorphism $\eta_n\colon \mathsf{F}_n \xrightarrow{\sim} \mathsf{F}_{n-1}$.

Once this is done, it is then straightforward to check that the morphisms $\eta(F_n) := \alpha_n$ (for $n > 0$) extend to an isomorphism $\eta\colon \mathsf{F}_0 \xrightarrow{\sim} \mathsf{F}$ as wanted.

In order to perform the inductive step from n to $n+1$, notice that for an arbitrary choice of β_{n+1} (hence of α_{n+1} and of η_{n+1}) and setting $\mathsf{F}_{n+1}(f) := \eta_{n+1}(B)^{-1} \circ \mathsf{F}_n(f) \circ \eta_{n+1}(A)$ for every morphism $f\colon A \to B$ of $\mathbf{T}$, all the required properties are satisfied, except possibly $\mathsf{F}_{n+1}|_{\mathbf{T}_{n+1}} = \mathsf{F}_0|_{\mathbf{T}_{n+1}}$.

Since $\mathsf{F}_{n+1}|_{\mathbf{T}_n} = \mathsf{F}_n|_{\mathbf{T}_n}$ by construction and $\mathsf{F}_n|_{\mathbf{T}_n} = \mathsf{F}_0|_{\mathbf{T}_n}$ by the inductive hypothesis, in view of Lemma 3.3 this last condition holds if and only if the diagram

$$\begin{array}{ccccc} \mathsf{F}(F_n) & \xrightarrow{\mathsf{F}_0(\pi'_{n+1,1})} & \mathsf{F}(F_{n+1}) & \xrightarrow{\mathsf{F}_0(\pi_{n+1,n})} & \mathsf{F}(F_n) \\ & \searrow^{\mathsf{F}_n(\pi'_{n+1,1})} & \downarrow{\scriptstyle \alpha_{n+1}} & \nearrow_{\mathsf{F}_n(\pi_{n+1,n})} & \\ & & \mathsf{F}(F_{n+1}) & & \end{array}$$

commutes. Clearly this is true if and only if

$$\epsilon_n(\pi'_{n+1,1}) = \beta_{n+1} \circ \pi'_{n+1,1}, \tag{3.4}$$

$$\epsilon_n(\pi_{n+1,n}) = -\pi_{n+1,n}[1] \circ \beta_{n+1}. \tag{3.5}$$

As $\epsilon_n(\pi'_{n+1,1}) \circ \pi''_{n+1,1}[-1] \in \mathrm{Hom}(F_1[-1], F_{n+1}[1]) = 0$, from the distinguished triangle (3.2) we deduce that there exists $\beta_{n+1} \colon F_{n+1} \to F_{n+1}[1]$ such that (3.4) is satisfied, and we claim that then (3.5) is automatically true, namely that $\gamma := \epsilon_n(\pi_{n+1,n}) + \pi_{n+1,n}[1] \circ \beta_{n+1} = 0$. Indeed, using (3.4) and the fact that $\epsilon_n|_{\mathbf{T}_n} = 0$,

$$\begin{aligned} \gamma \circ \pi'_{n+1,1} &= \epsilon_n(\pi_{n+1,n}) \circ \pi'_{n+1,1} + \pi_{n+1,n}[1] \circ \beta_{n+1} \circ \pi'_{n+1,1} \\ &= \epsilon_n(\pi_{n+1,n}) \circ \pi'_{n+1,1} + \pi_{n+1,n}[1] \circ \epsilon_n(\pi'_{n+1,1}) = \epsilon_n(\pi_{n+1,n} \circ \pi'_{n+1,1}) = 0. \end{aligned}$$

It follows again from (3.2) that $\gamma = \gamma' \circ \pi_{n+1,1}$ for some $\gamma' \colon F_1 \to F_n[1]$. Now, by Serre duality, $\mathrm{Hom}(F_1, F_n[1]) \cong \mathrm{Hom}(F_n, F_1)^\vee \cong \Bbbk$, so there exists $\lambda \in \Bbbk$ such that $\gamma' = \lambda \pi''_{n+1,1}$, whence $\gamma = \lambda \pi''_{n+1,1} \circ \pi_{n+1,1} = 0$. □

Corollary 3.6. *For every elliptic curve X the functor $\Phi_{-}^{X \to X}$ is not essentially injective.*

4. The uniqueness of the cohomology sheaves

In this section we prove Theorem 1.2, hence we assume that X_1 and X_2 are projective schemes with ample divisors H_1 and H_2 on X_1 and X_2 respectively. For $l \in \mathbb{Z}$, denote by $\mathbf{C}_l$ the full subcategory with objects $\{\mathcal{O}_{X_1}(mH_1) : m > l\} \subset \mathbf{Coh}(X_1)$. Consider Fourier–Mukai functors

$$\Phi_{\mathcal{E}_1}, \Phi_{\mathcal{E}_2} : \mathbf{Perf}(X_1) \longrightarrow \mathrm{D}^b(X_2)$$

where $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$, and such that there exists an isomorphism

$$\beta \colon \Phi_{\mathcal{E}_1}|_{\mathbf{C}_l} \xrightarrow{\sim} \Phi_{\mathcal{E}_2}|_{\mathbf{C}_l}, \tag{4.1}$$

for some integer l.

The following easy lemma shows that we can be more precise about the Fourier–Mukai kernels above.

Lemma 4.1. *Under the above assumptions, $\mathcal{E}_i \in \mathrm{D}^b(X_1 \times X_2)$, for $i = 1, 2$. Conversely, any $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ yields a Fourier–Mukai functor $\Phi_{\mathcal{E}} \colon \mathbf{Perf}(X_1) \to \mathrm{D}^b(X_2)$.*

Proof. The second part of the statement is clear. For the first one, we can apply the argument in [11, Cor. 9.13 (4)] where the assumption that $\Phi_{\mathcal{E}_i}$ is fully faithful is not used.

For the convenience of the reader, we provide a different easy argument. Indeed, due to [13, Lemma 7.47], $\mathcal{E}_i \in \mathrm{D}^b(X_1 \times X_2)$ if and only if, for all $\mathcal{F} \in \mathbf{Perf}(X_1 \times X_2)$, we have

$$\dim \bigoplus_j \mathrm{Hom}(\mathcal{F}, \mathcal{E}_i[j]) < \infty.$$

Let G_i be a compact generator of $\mathrm{D}(\mathbf{Qcoh}(X_i))$, for $i = 1, 2$. By [4, Lemma 3.4.1], $G_1 \boxtimes G_2$ is a compact generator of $\mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$ (see [4, 13] for the definition of compact generator). As $\bigoplus_j \mathrm{Hom}(G_1 \boxtimes G_2, \mathcal{E}_i[j]) \cong \bigoplus_j \mathrm{Hom}(G_2, \Phi_{\mathcal{E}_i}(G_1^\vee)[j])$ is finite dimensional because $\Phi_{\mathcal{E}_i}(G_1^\vee) \in \mathrm{D}^b(X_2)$, we can conclude using the fact that $G_1 \boxtimes G_2$ classically generates $\mathbf{Perf}(X_1 \times X_2)$ (see, for example, [4, Thm. 2.1.2]). □

The first step in the proof of of Theorem 1.2 is the following.

Lemma 4.2. *If $\mathcal{E}_1, \mathcal{E}_2 \in \mathbf{Coh}(X_1 \times X_2)$, then $\mathcal{E}_1 \cong \mathcal{E}_2$.*

Proof. By [8, Thm. 3.4.4], for $i = 1, 2$, there is an isomorphism between $\mathcal{E}_i$ and the sheaf associated to $M_i := \bigoplus_{m \in \mathbb{Z}} (p_2)_*(\mathcal{E}_i \otimes p_1^* \mathcal{O}_{X_1}(mH_1))$, where $(p_2)_*$ is not derived. Since for $m \gg 0$ there are functorial isomorphisms

$$(p_2)_*(\mathcal{E}_i \otimes p_1^* \mathcal{O}_{X_1}(mH_1)) \cong \Phi_{\mathcal{E}_i}(\mathcal{O}_{X_1}(mH_1)),$$

by (4.1) the graded modules M_1 and M_2 are isomorphic in sufficiently high degrees. Hence, taking the associated sheaves, we get $\mathcal{E}_1 \cong \mathcal{E}_2$. □

If the Fourier–Mukai kernels are not sheaves, we have the following result concerning their cohomologies. Notice that due to the weaker assumptions on the functors $\Phi_{\mathcal{E}_1}$ and $\Phi_{\mathcal{E}_2}$ in (4.1), this may be seen as a stronger version of Theorem 1.2.

Proposition 4.3. *For any $j \in \mathbb{Z}$, we have isomorphisms $H^j(\mathcal{E}_1) \cong H^j(\mathcal{E}_2)$ in $\mathbf{Coh}(X_1 \times X_2)$.*

Proof. We first prove that, given $j \in \mathbb{Z}$, we have $H^j(\mathcal{E}_1) = 0$ if and only if $H^j(\mathcal{E}_2) = 0$. Indeed, observe that $H^j(\mathcal{E}_i) = 0$ if and only if $\mathrm{Hom}(\mathcal{O}_{X_1}(mH_1) \boxtimes \mathcal{O}_{X_2}(mH_2), \mathcal{E}_i[j]) = 0$ for $m \ll 0$. But

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_{X_1}(mH_1) \boxtimes \mathcal{O}_{X_2}(mH_2), \mathcal{E}_1[j]) &\cong \mathrm{Hom}(\mathcal{O}_{X_2}(mH_2), \Phi_{\mathcal{E}_1}(\mathcal{O}_{X_1}(-mH_1))[j]) \\ &\cong \mathrm{Hom}(\mathcal{O}_{X_2}(mH_2), \Phi_{\mathcal{E}_2}(\mathcal{O}_{X_1}(-mH_1))[j]) \cong \mathrm{Hom}(\mathcal{O}_{X_1}(mH_1) \boxtimes \mathcal{O}_{X_2}(mH_2), \mathcal{E}_2[j]).\end{aligned}$$

We are now ready to prove the statement by induction on the number of non-trivial cohomologies. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are the shift of a sheaf, we can just apply Lemma 4.2. Thus assume that $\mathcal{E}_i$ has at least two non-trivial cohomologies and that the last non-trivial one is in degree n while the first non-trivial one is in degree $n' < n$. In particular, we have distinguished triangles

$$\begin{aligned}\mathcal{E}_1' &\longrightarrow \mathcal{E}_1 \longrightarrow H^n(\mathcal{E}_1)[-n] \\ \mathcal{E}_2' &\longrightarrow \mathcal{E}_2 \longrightarrow H^n(\mathcal{E}_2)[-n],\end{aligned} \tag{4.2}$$

where $\mathcal{E}_1'$ and $\mathcal{E}_2'$ have cohomologies concentrated in the interval $[n', n-1]$ which is strictly smaller than the one of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Now observe that if $\mathcal{E} \in \mathrm{D}^b(X_1 \times X_2)$ is such that $H^j(\mathcal{E}) = 0$ if $j \notin [a, b]$, then we have $H^j(\Phi_{\mathcal{E}}(\mathcal{O}_{X_1}(mH_1))) = 0$ if $j \notin [a, b]$ and $m \gg 0$. Indeed,

$$0 = \mathrm{Hom}(\mathcal{O}_{X_1}(-mH_1) \boxtimes \mathcal{O}_{X_2}(-m'H_2), \mathcal{E}[j]) \cong \mathrm{Hom}(\mathcal{O}_{X_2}(-m'H_2), \Phi_{\mathcal{E}}(\mathcal{O}_{X_1}(mH_1))[j]),$$

if $m' \gg 0$ and under the above assumptions on m and j.

For $m \gg 0$ from (4.2) we get the diagram

$$\begin{array}{ccccc} \Phi_{\mathcal{E}_1'}(\mathcal{O}_{X_1}(mH_1)) & \longrightarrow & \Phi_{\mathcal{E}_1}(\mathcal{O}_{X_1}(mH_1)) & \longrightarrow & \Phi_{H^n(\mathcal{E}_1)}(\mathcal{O}_{X_1}(mH_1))[-n] \\ & & \downarrow{\scriptstyle \beta_m} & & \\ \Phi_{\mathcal{E}_2'}(\mathcal{O}_{X_1}(mH_1)) & \longrightarrow & \Phi_{\mathcal{E}_2}(\mathcal{O}_{X_1}(mH_1)) & \longrightarrow & \Phi_{H^n(\mathcal{E}_2)}(\mathcal{O}_{X_1}(mH_1))[-n], \end{array} \tag{4.3}$$

where the two rows are distinguished triangles and $\beta_m := \beta(\mathcal{O}_{X_1}(mH_1))$ is the isomorphism induced by (4.1).

Using the remark above, we get that $\Phi_{\mathcal{E}_1'}(\mathcal{O}_{X_1}(mH_1))$ has non-trivial cohomologies concentrated in degrees $[n', n-1]$ while $\Phi_{H^n(\mathcal{E}_2)}(\mathcal{O}_{X_1}(mH_1))[-n]$ is a sheaf in degree n. Hence

$$\mathrm{Hom}(\Phi_{\mathcal{E}_1'}(\mathcal{O}_{X_1}(mH_1))[k], \Phi_{H^n(\mathcal{E}_2)}(\mathcal{O}_{X_1}(mH_1))[-n]) = 0,$$

for $m \gg 0$ and $k = 0, 1$. It follows that (4.3) can be completed to a commutative diagram in a unique way. Thus, for some $l' > l$, we get natural transformations $\alpha : \Phi_{\mathcal{E}_1'}|_{\mathbf{C}_{l'}} \xrightarrow{\sim} \Phi_{\mathcal{E}_2'}|_{\mathbf{C}_{l'}}$ and $\gamma : \Phi_{H^n(\mathcal{E}_1)}|_{\mathbf{C}_{l'}} \xrightarrow{\sim} \Phi_{H^n(\mathcal{E}_2)}|_{\mathbf{C}_{l'}}$, which are easily seen to be isomorphisms (applying the same argument to β_m^{-1}). By Lemma 4.2, we have $H^n(\mathcal{E}_1) \cong H^n(\mathcal{E}_2)$ and, by induction, $H^j(\mathcal{E}_1') \cong H^j(\mathcal{E}_2')$, for all $j \in \mathbb{Z}$. This is enough, as $H^j(\mathcal{E}_i) \cong H^j(\mathcal{E}_i')$, for $j < n$. □

Denoting by $K(X_1 \times X_2)$ the Grothendieck group of the abelian category $\mathbf{Coh}(X_1 \times X_2)$, we clearly get the following result.

Corollary 4.4. *Let X_1 and X_2 be projective schemes. Consider two isomorphic Fourier–Mukai functors*

$$\Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2} : \mathbf{Perf}(X_1) \longrightarrow \mathrm{D}^b(X_2)$$

Then $H^j(\mathcal{E}_1) \cong H^j(\mathcal{E}_2)$ for all $j \in \mathbb{Z}$. In particular, $[\mathcal{E}_1] = [\mathcal{E}_2]$ in $K(X_1 \times X_2)$.

Notice that, if $\mathbb{k} = \mathbb{C}$, it is well-known and very easy to see, under the assumptions of Corollary 4.4, that $\mathrm{ch}(\mathcal{E}_1) = \mathrm{ch}(\mathcal{E}_2) \in H^*(X_1 \times X_2, \mathbb{Q})$. Indeed $\Phi_{\mathcal{E}_i}$ induces a correspondence between $H^*(X_1, \mathbb{Q})$ and $H^*(X_2, \mathbb{Q})$ given by the object $\mathrm{ch}(\mathcal{E}_i) \cdot \sqrt{\mathrm{td}(X_1 \times X_2)}$ (see [12]). The Künneth decomposition for the cohomology of the product yields then $\mathrm{ch}(\mathcal{E}_1) = \mathrm{ch}(\mathcal{E}_2)$.

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