STABILITY CONDITIONS FOR GENERIC K3 CATEGORIES

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Abstract. A K3 category is by definition a Calabi–Yau category of dimension two. Geometrically K3 categories occur as bounded derived categories of (twisted) coherent sheaves on K3 or abelian surfaces. A K3 category is generic if there are no spherical objects (or just one up to shift). We study stability conditions on K3 categories as introduced by Bridgeland and prove his conjecture about the topology of the stability manifold and the autoequivalences group for generic twisted projective K3, abelian surfaces, and K3 surfaces with trivial Picard group.

1. Introduction

A K3 category is by definition a Calabi–Yau category $\mathbf{T}$ of dimension two, i.e. a k-linear triangulated category with functorial isomorphisms $\mathrm{Hom}(E,F) \simeq \mathrm{Hom}(F,E[2])^*$ for all objects $E,F \in \mathbf{T}$.

Examples of K3 categories are provided by the bounded derived category $\mathrm{D^b}(X)$ of the abelian category $\mathbf{Coh}(X)$ of all coherent sheaves on a K3 or abelian surface X. Twisted analogues can be obtained by considering the bounded derived category $\mathrm{D^b}(X,\alpha)$ of the abelian category $\mathbf{Coh}(X,\alpha)$ of all α-twisted coherent sheaves, where α is a fixed Brauer class on the surface X.

The complexity of a K3 category $\mathbf{T}$ is reflected by its group of k-linear, exact autoequivalences $\mathrm{Aut}(\mathbf{T})$. It is in fact a highly non-trivial matter to produce non-trivial autoequivalences and only very few general techniques are known. Most importantly, there are the so-called spherical twists which have been studied in detail by Seidel and Thomas in [24].

Another way to gain more insight into the structure of a triangulated category has more recently introduced by Bridgeland. In [5] he defines the notion of a stability condition on a k-linear triangulated category $\mathbf{T}$ which allows one to decompose any object in terms of distinguished triangles in a unique way into semistable objects. Very roughly, a stability condition is a bounded t-structure together with a numerical function on its heart.

The truly wonderful aspect of Bridgeland's theory, which has been strongly influenced by work of Douglas on Π-stability in the context of mirror symmetry, is that the space $\mathrm{Stab}(\mathbf{T})$ of all stability conditions on a triangulated category $\mathbf{T}$ admits a natural topology. In general, however, not much is known about the structure of $\mathrm{Stab}(\mathbf{T})$. In particular, basic questions like whether the space is non-empty, connected or simply-connected are very hard to come by.

In the geometric situation, i.e. for the bounded derived category $\mathrm{D^b}(X)$ of all coherent sheaves on a complex variety X, the space $\mathrm{Stab}(X) := \mathrm{Stab}(\mathrm{D^b}(X))$ of all stability conditions

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is the algebraic version of the usual Kähler or ample cone. Very few stability conditions, i.e. points in $\mathrm{Stab}(X)$, have explicitly been described so far. A complete description of $\mathrm{Stab}(X)$ for a complex projective (or compact) manifold X has only been achieved when X is a curve [5, 16, 20]. For partial results for higher-dimensional Fano manifolds see [2, 16].

For compact Calabi–Yau manifolds of dimension at least three not much is known, not even whether stability conditions always exist. (Non-compact Calabi–Yau manifolds have been dealt with e.g. in [7].) In dimension two, i.e. for K3 and abelian surfaces, Bridgeland constructs in [6] explicit examples of stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ and investigates in detail one connected component $\mathrm{Stab}^{\dagger}(X)$ of $\mathrm{Stab}(X)$. Maybe the most fascinating aspect of the intriguing paper [6] is a conjecture that describes $\mathrm{Stab}(X)$ (or rather $\mathrm{Stab}^{\dagger}(X)$) as a universal cover of an explicit period domain such that the subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ of cohomologically trivial autoequivalences acts naturally as the group of deck transformations. This paper proves a stronger version of this conjecture for generic twisted K3 surfaces (and abelian surfaces) and generic non-projective K3 surfaces. This provides further evidence for Bridgeland's conjecture.

The group of autoequivalences of a triangulated category $\mathbf{T}$ naturally acts on the space of stability conditions $\mathrm{Stab}(\mathbf{T})$ and usually the complexity of the group is reflected by the topology of the quotient. For the special case of K3 categories it seems that solely spherical twists are responsible for the rich structure of the group of autoequivalences, e.g. for braid group actions as described in [24]. Studied from the point of view of stability conditions, spherical twists lead to a complicated topological structure of the quotient of the space of stability conditions by the action of the group of autoequivalences. This paper confirms this belief by showing that in the absence of spherical objects, the group of autoequivalences and the space of stability conditions become manageable. As generic twisted projective K3 surfaces give rise to K3 categories not containing any spherical objects, we call these categories generic K3 categories.

The next more complicated case would be a K3 category with only one (up to shift) spherical object. In this case one expects that the group of autoequivalences is generated by the associated spherical twist, the usual shift functor and a few other obvious autoequivalences. Geometrically this situation is realized by the bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a generic non-projective K3 surface. Indeed, any line bundle on X defines a spherical object in $\mathrm{D}^{\mathrm{b}}(X)$ and if X is generic non-projective, then the trivial line bundle is the only one there is.

Here are the main results proved in the present paper.

Theorem 1. *Let X be a complex projective K3 or abelian surface with a Brauer class $\alpha \in H^2(X, \mathcal{O}_X^*)$ such that the abelian category of α-twisted coherent sheaves $\mathbf{Coh}(X,\alpha)$ does not contain any spherical objects. Then the space $\mathrm{Stab}(X,\alpha)$ of all locally finite, numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X,\alpha) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X,\alpha))$ admits only one connected component of maximal dimension. Moreover, this component is simply-connected. (Theorem 3.15)*

It can be shown that in the period domain of all twisted K3 surfaces (X,α) those satisfying the assumption of Theorem 1 form a dense subset (Corollary 3.23). We expect $\mathrm{Stab}(X,\alpha)$ to be connected, but a technical problem prevents us, for the time being, from stating it in this generality. See Remark 3.16 for comments.

Theorem 2. *Suppose (X,α) satisfies the assumption of Theorem 1. Then a Hodge isometry $\widetilde{H}(X,\alpha,\mathbb{Z})\simeq\widetilde{H}(Y,\beta,\mathbb{Z})$ can be lifted to an equivalence $\mathrm{D^b}(X,\alpha)\simeq\mathrm{D^b}(Y,\beta)$ if and only if it preserves the orientation of the four positive directions.*

Moreover, there exists a natural surjection $\mathrm{Aut}(\mathrm{D^b}(X,\alpha))\twoheadrightarrow\mathrm{Aut}^+(\widetilde{H}(X,\alpha,\mathbb{Z}))$ onto the group of orientation preserving Hodge isometries. The kernel is spanned by the double shift $E\longmapsto E[2]$. (Theorem 3.17, Corollary 3.21)

The group of autoequivalences is certainly more complicated for an arbitrary (twisted) K3 surface, mainly due to the existence of various spherical objects. The first assertion, however, is expected to hold always, but even for untwisted K3 surface it is still open whether the Hodge isometry induced by an equivalence preserves the natural orientation of the positive directions.

Theorem 3. *Suppose X is a complex K3 surface with $\mathrm{Pic}(X)=0$. Then the space $\mathrm{Stab}(X)$ of all locally finite, numerical stability conditions on $\mathrm{D^b}(X):=\mathrm{D^b}(\mathbf{Coh}(X))$ is connected and simply-connected. (Theorem 4.8)*

Clearly, a K3 surface satisfying the hypothesis of the theorem cannot be projective. The abelian category $\mathbf{Coh}(X)$ and its bounded derived category $\mathrm{D^b}(X)$ are, in an imprecise sense, smaller than in the projective situation. However, although there are no non-trivial line bundles, one still has the point sheaves $k(x)$ and many vector bundles (necessarily with trivial determinant). In fact, stable bundles with trivial determinant on projective K3 surfaces survive the deformation to the generic K3 surface. So, Theorem 3 really describes $\mathrm{Stab}(X)$ in a geometric highly non-trivial case.

The techniques used to prove Theorem 1 and 3 can be combined to cover also the case of non-projective twisted K3 surfaces which are generic in an appropriate sense.

Theorem 4. *For a K3 surface as in Theorem 3 one has $\mathrm{Aut}(\mathrm{D^b}(X))=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathrm{Aut}(X)$. (Proposition 4.10)*

Here, $\mathrm{Aut}(\mathrm{D^b}(X))$ denotes the group of all autoequivalences of Fourier–Mukai type, which in general, and this is a difference to the projective case, is smaller than the full group of all autoequivalences. The two free factors are spanned by the spherical twist induced by the trivial line bundle respectively the shift functor.

We believe that these results for generic twisted and generic non-projective untwisted K3 surfaces will eventually lead to a better understanding of stability conditions and autoequivalences for $\mathrm{D^b}(X)$ when X is an arbitrary (projective) K3 surface.

The plan of the paper is as follows. In Section 2 we discuss abstract K3 categories and the role of rigid, semi-rigid, and spherical objects. In particular, it is shown that in the absence of spherical objects rigid objects do not exist (Proposition 2.9) and semi-rigid ones are automatically stable with respect to any stability condition (Corollary 2.10). If there is only one spherical object for which the spherical twist can be defined, then any semi-rigid object is stable up to the action of the corresponding spherical twist (Proposition 2.18). The discussion is later used in the geometric case, but should be applicable in other situations as well.

Section 3 is devoted to twisted projective K3 and abelian surfaces. We first outline how the arguments of [6] can be adapted to the twisted case (Section 3.1). Although dealing at the same time with B-field lifts of Brauer classes and B-fields complexifying the Kähler cone makes the situation more technical, most of Bridgeland's arguments go through unchanged and we will therefore be brief. In Section 3.2 we apply the general results of Section 2 to the case of a projective K3 or abelian surface endowed with a Brauer class α such that $\mathrm{D^b}(X,\alpha)$ does not contain any spherical object. The main result is Theorem 1 above. Section 3.3 presents two approaches to a complete description of $\mathrm{Aut}(\mathrm{D^b}(X,\alpha))$ in the generic case (Theorem 2). The first one follows Bridgeland's, i.e. studying $\mathrm{Aut}(\mathrm{D^b}(X,\alpha))$ via its action on $\mathrm{Stab}(X,\alpha)$, whereas the second one works more directly by showing that up to shift any Fourier–Mukai kernel is isomorphic to twisted sheaves.

Section 4 deals with K3 surfaces X which do not admit any non-trivial line bundles. Although these surfaces are certainly not projective, most aspects of the general theory go through. As we prove in Section 4.1, the only spherical object in the derived category $\mathrm{D^b}(X)$ is up to shift the trivial line bundle and thus the discussion of Section 2 applies. By an ad hoc argument we prove in Section 4.3 that $\mathrm{Stab}(X)$ is connected and simply-connected (Theorem 3). A more detailed description of $\mathrm{Stab}(X)$ as the universal covering of an appropriate period domain, whose definition differs from the one in the projective case, has been given in [17]. In order to determine the group of autoequivalences one could, as for generic twisted K3 surfaces, imitate Bridgeland's approach and study the action of $\mathrm{Aut}(\mathrm{D^b}(X))$ on $\mathrm{Stab}(X)$ or describe all Fourier–Mukai kernels explicitly. In Section 4.4 we do the latter and prove Theorem 4.

We would like to point out that similar questions are dealt with in recent articles by Okada [21] and Ishii, Ueda, and Uehara [15].

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2. K3 Categories: Spherical and (semi-)rigid objects

This section contains all the abstract arguments of the paper that are valid for all triangulated categories for which a Serre functor is provided by the double shift. Although we call these categories K3 categories, K3 surfaces actually never occur. We recall the definition of a stability condition and study the space of all stability conditions on a K3 category without (or just a few) spherical objects.

2.1. **Stability conditions, phases.**

Definition 2.1. A *K3 category* is a triangulated category $\mathbf{T}$ such that the double shift $E \longmapsto E[2]$ is a Serre functor.

All our categories are tacitly assumed to be linear over some base field k of characteristic $\neq 2$ (usually $k = \mathbb{C}$) and of finite type, i.e. for any two objects $E, F \in \mathbf{T}$ the k-vector space $\mathrm{Hom}^*(E,F) = \bigoplus \mathrm{Hom}(E, F[i])$ is finite-dimensional.

We shall use the following short hands:

$$(E,F)^i := \dim \operatorname{Hom}(E,F[i]) \text{ and } (E,F)^{\leq i} := \sum_{j\leq i}(-1)^j(E,F)^j.$$

Similarly, $(E,F)^{<i} := (E,F)^{\leq i-1}$. Thus, if $\mathbf{T}$ is a K3 category, then $(E,F)^i = (F,E)^{2-i}$. Moreover, $(E,E)^1$ is always even, as Serre duality defines a symplectic pairing on it (see [26]).

Recall that for two objects $E,F\in\mathbf{T}$ one defines

$$\chi(E,F) := \sum(-1)^i(E,F)^i.$$

Clearly, $\chi(E[i],F)=\chi(E,F[-i])=(-1)^i\chi(E,F)$ and if $\mathbf{T}$ is a K3 category, then $\chi(E,F)$ is symmetric.

Two objects $E_1,E_2\in\mathbf{T}$ are called *numerically equivalent*, $E_1\sim E_2$, if $\chi(E_1,F)=\chi(E_2,F)$ for all $F\in\mathbf{T}$. In particular, E is numerically trivial if $E\sim 0$. Note that for all $E\in\mathbf{T}$ one has $E\sim E[2]$. The *numerical Grothendieck group* of $\mathbf{T}$ is defined as $\mathcal{N}(\mathbf{T}) := K(\mathbf{T})/\sim$.

Definition 2.2. (**Bridgeland**) A *stability condition* $\sigma=(Z,\mathcal{P})$ on a k-linear triangulated category $\mathbf{T}$ consists of a linear map

$$Z:\mathcal{N}(\mathbf{T})\longrightarrow\mathbb{C}$$

(the *central charge*) and full additive subcategories

$$\mathcal{P}(\phi)\subset\mathbf{T}$$

for each $\phi\in\mathbb{R}$ satisfying the following conditions:
a) If $0\neq E\in\mathcal{P}(\phi)$, then $Z(E)=m(E)\exp(i\pi\phi)$ for some $m(E)\in\mathbb{R}_{>0}$.
b) $\mathcal{P}(\phi+1)=\mathcal{P}(\phi)[1]$ for all ϕ.
c) $\operatorname{Hom}(E_1,E_2)=0$ for all $E_i\in\mathcal{P}(\phi_i)$ with $\phi_1>\phi_2$.
d) Any $0\neq E\in\mathbf{T}$ admits a *Harder–Narasimhan filtration* given by a collection of distinguished triangles $E_{i-1}\longrightarrow E_i\longrightarrow A_i$ with $E_0=0$ and $E_n=E$ such that $A_i\in\mathcal{P}(\phi_i)$ with $\phi_1>\ldots>\phi_n$.

This is what Bridgeland calls a numerical stability condition, but we will only work with stability conditions of this type.

The category $\mathcal{P}(\phi)$ is the category of *semistable* objects of phase ϕ. The objects A_i in condition d) are called the *semistable factors* or *Harder–Narasimhan factors* of E. They are unique up to isomorphism. The minimal objects of $\mathcal{P}(\phi)$ are called *stable* of phase ϕ. Note that a non-trivial homomorphism between stable objects of the same phase is an isomorphism. In particular, for any stable E the endomorphism algebra $\operatorname{End}(E)$ is a division algebra. Also, if E is stable, then $(E,E)^i=0$ for $i<0$ and $i>2$.

Bridgeland shows in [5, Prop. 5.3] that a stability condition can equivalently be described by a bounded t-structure and a centered stability function $Z:\mathcal{A}\longrightarrow\mathbb{C}$ on its heart $\mathcal{A}$ which has the Harder–Narasimhan property. More precisely, one has $\mathcal{A}=\mathcal{P}((0,1])$, i.e. $\mathcal{A}$ contains all objects $E\in\mathbf{T}$ whose semistable factors A_i satisfy $\phi(A_i)\in(0,1]$. We call $\mathcal{A}$ also the heart of the stability condition σ.

Following Bridgeland, all stability conditions we will consider are locally finite [6]. In particular, any semistable object $E \in \mathcal{P}(\phi)$ admits a finite Jordan–Hölder filtration, i.e. a finite filtration $E_0 \subset \ldots \subset E_n = E$ with stable quotients $E_{i+1}/E_i \in \mathcal{P}(\phi)$.

The space of all locally finite stability conditions is denoted $\mathrm{Stab}(\mathbf{T})$. For later use we state the following observation, which will be needed to ensure equality of phases of a distinguished family of stable objects.

Lemma 2.3. *Suppose* $\mathbf{T}$ *is a K3 category endowed with a stability condition* σ *and suppose* $E_1 \sim E_2$ *are numerically equivalent* σ*-stable objects. If there exists a* σ*-stable object* $E_0 \in \mathbf{T}$ *with*

$$(E_0, E_1)^0 \neq 0 \neq (E_0, E_2)^0,$$

then

$$\text{either } \phi(E_1) = \phi(E_2) \text{ or } E_1 \simeq E_2[\pm 2].$$

Proof. Let $\phi_i := \phi(E_i)$. By assumption and Serre duality there exist non-trivial homomorphisms $E_0 \longrightarrow E_i \longrightarrow E_0[2]$, which due to stability implies $\phi_0 \leq \phi_i \leq \phi_0 + 2$.

As $E_1 \sim E_2$, the values of the stability function satisfy $Z(E_1) = Z(E_2)$ and hence $\phi_1 = \phi_2 + 2\ell$ for some integer ℓ. Thus, either $\phi_1 = \phi_2$ or one has one of the following possibilities i) $\phi_1 = \phi_0 = \phi_2 - 2$ or ii) $\phi_2 = \phi_0 = \phi_1 - 2$. Due to the stability of all three objects, the latter two cases lead to i) $E_1 \simeq E_0 \simeq E_2[-2]$ respectively ii) $E_2 \simeq E_0 \simeq E_1[-2]$. $\square$

Remark 2.4. In the geometric context, numerical equivalence of two objects (e.g. sheaves) E_1 and E_2 is often caused by the existence of a 'flat family of objects (sheaves)' connecting E_1 and E_2. In this context, for E_1 and E_2 stable objects, $\phi(E_1) = \phi(E_2)$ should follow from the 'openness of stability' as proved, under natural assumptions, in [1, Prop. 3.3.2].

2.2. **Spherical, rigid, and semi-rigid objects.**

Following the standard terminology in [18] and [24], we give the following definition:

Definition 2.5. i) An object $E \in \mathbf{T}$ is *rigid* if $(E, E)^1 = 0$;

ii) An object $E \in \mathbf{T}$ is *semi-rigid* if $(E, E)^1 = 2$;

iii) An object $E \in \mathbf{T}$ is *spherical* if $(E, E)^i = 1$ for $i = 0, 2$ and otherwise zero.

iv) An object $E \in \mathbf{T}$ is *quasi-spherical* if $(E, E)^i = 0$ for $i \neq 0, 2$ and $\mathrm{End}(E)$ is a division algebra.

Remark 2.6. Since any finite-dimensional division algebra over an algebraically closed field k is isomorphic to k itself, the notions of quasi-spherical and spherical coincide in the case of a k-linear category over an algebraically closed field k.

In fact, for the results mentioned in the introduction it would be enough to work over an algebraically closed field, but for the sequel to this paper it is important to consider the more general situation as well. Indeed, in [14] we use the techniques in the present paper to prove that any autoequivalence of the derived category of a smooth projective K3 surface induces an orientation preserving Hodge isometry on cohomology. Our argument involves the passage to a triangulated category which is linear over a field which is not algebraically closed.

In order to control how these notions behave in distinguished triangles, the following lemma will be crucial. It is the analogue of Lemma 5.2 in [6], but the main idea goes back to Mukai's paper [18].

Lemma 2.7. *Consider in* $\mathbf{T}$ *the distinguished triangle*

$$A \xrightarrow{i} E \xrightarrow{j} B \xrightarrow{\delta} A[1]$$

such that

$$(A,B)^r = (B,B)^s = 0 \text{ for } r \leq 0 \text{ and } s < 0.$$

Then

$$(A,A)^1 + (B,B)^1 \leq (E,E)^1.$$

Proof. First of all we note that $\chi(E,E) = \chi(A,A)+\chi(B,B)+2\chi(A,B)$. Due to our hypotheses and using Serre duality this can be rewritten as

$$(A,A)^1 + (B,B)^1 = (E,E)^1 + 2\underbrace{\left((A,A)^{\leq 0} + (B,B)^0 - (E,E)^{\leq 0} + (B,A)^{\leq 1}\right)}_{=N}. \tag{2.1}$$

The result is proved once we show that $N \leq 0$.

Since $\mathrm{Hom}(A,B) = \mathrm{Hom}(A,B[-1]) = 0$, for any $f \in \mathrm{Hom}(E,E)$, there exist unique $g \in \mathrm{Hom}(A,A)$ and $h \in \mathrm{Hom}(B,B)$ making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"] & E \arrow[r, "j"] \arrow[d, "f"] & B \arrow[r, "\delta"] \arrow[d, "h"] & A[1] \arrow[d, "g[1]"] \\ A \arrow[r, "i"] & E \arrow[r, "j"] & B \arrow[r, "\delta"] & A[1]. \end{tikzcd} \tag{2.2}$$

In particular, we get a map $\mathrm{Hom}(E,E) \to \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(B,B)$.

If $g = h = 0$, then f factorizes through a morphism $f_1 : E \to A$. Moreover, since $i \circ f_1 \circ i = i \circ g = 0$, the morphism $f_1 \circ i$ can be factorized further through some $f_2 : A \to B[-1]$ which must be trivial due to our assumption. Thus $f_1 \circ i = 0$ and hence f_1 admits a factorization through $f_3 : B \to A$. This discussion leads to an exact sequence

$$\mathrm{Hom}(B,A) \longrightarrow \mathrm{Hom}(E,E) \xrightarrow{\tau} \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(B,B).$$

Consider now the morphism $\eta : \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(B,B) \to \mathrm{Hom}(B,A[1])$ which we define as $\eta((g,h)) = \delta \circ h - g[1] \circ \delta$. Suppose that $(g,h) = \tau(f)$, for some $f \in \mathrm{Hom}(E,E)$. Then $\eta((g,h)) = 0$ and it is very easy to check that the previous exact sequence can be completed to the following:

$$\mathrm{Hom}(B,A) \xrightarrow{\zeta} \mathrm{Hom}(E,E) \xrightarrow{\tau} \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(B,B) \xrightarrow{\eta} \mathrm{Hom}(B,A[1]).$$

Next, take $g_1 : B \to A$ such that $\zeta(g_1) = 0$. By definition this means that $i \circ g_1 \circ j = 0$ and then $g_1 \circ j$ factorizes through some $g_2 : E \to B[-1]$, which must be trivial. Indeed, otherwise g_2 would give rise to a non-trivial $B \to B[-1]$ or a non-trivial $A \to B[-1]$, the existence of which is in both cases excluded by assumption. Hence, $g_1 \circ j = 0$ and, therefore, g_1 factorizes through some $g_3 : A[1] \longrightarrow A$. Thus we obtain the exact sequence

$$\mathrm{Hom}(A[1],A) \xrightarrow{\theta} \mathrm{Hom}(B,A) \xrightarrow{\zeta} \mathrm{Hom}(E,E) \xrightarrow{\tau} \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(B,B) \xrightarrow{\eta} \mathrm{Hom}(B,A[1]).$$

Reasoning as before, consider $g' \in \mathrm{Hom}(A[1],A)$ such that $\theta(g') = 0$. Then $g' \circ \delta = 0$ and there exists $g'_1 : E[1] \longrightarrow A$ such that $g' = g'_1 \circ i[1]$. Using the exact sequence

$$\mathrm{Hom}(E[1],B[-1]) \longrightarrow \mathrm{Hom}(E[1],A) \longrightarrow \mathrm{Hom}(E[1],E) \longrightarrow \mathrm{Hom}(E[1],B)$$

and the assumption $(E[1],B)^m = 0$ for $m \leq 0$ we see that g'_1 induces a unique $g'_2 : E[1] \longrightarrow E$. Thus the sequence

$$\mathrm{Hom}(E[1],E) \longrightarrow \mathrm{Hom}(A[1],A) \xrightarrow{\theta} \mathrm{Hom}(B,A)$$

is exact.

Iterating the previous argument we obtain the exact sequence

$$\begin{array}{c} \cdots \longrightarrow \mathrm{Hom}^{-m}(E,E) \longrightarrow \mathrm{Hom}^{-m}(A,A) \longrightarrow \mathrm{Hom}^{-m+1}(B,A) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Hom}^0(E,E) \longrightarrow \mathrm{Hom}^0(A,A) \oplus \mathrm{Hom}^0(B,B) \xrightarrow{\eta} \mathrm{Hom}^1(B,A). \end{array} \tag{2.3}$$

Therefore, $-(E,E)^{\leq 0} + (A,A)^{\leq 0} + (B,B)^0 + (B,A)^{\leq 0} \leq (B,A)^1$, i.e. $N \leq 0$. □

Remark 2.8. If the base field is not algebraically closed, we have to introduce the following technical condition which ensures that the theory goes through smoothly. Situations where this condition does not hold seem rather pathological. In any case, in all applications dealt with in this paper and its sequel, it can be easily verified. See e.g. Proposition 2.14.

($*$) *The category* $\mathbf{T}$ *does not contain any object* $B \in \mathbf{T}$ *such that* $\mathrm{End}(B)$ *is a division algebra,* $(B,B)^0 = (B,B)^1$ *and* $(B,B)^i = 0$ *for any integer* $i < 0$.

Due to the existence of the symplectic structure on $\mathrm{Hom}(B,B[1])$, we know that it is an even-dimensional k-vector space. So, if k is algebraically closed, this condition is automatic, but in general $\mathrm{Hom}(B,B[1])$ need not be even-dimensional over $\mathrm{End}(B)$.

Proposition 2.9. *Suppose* $\mathbf{T}$ *is a K3 category satisfying condition* ($*$) *and let* $\sigma \in \mathrm{Stab}(\mathbf{T})$ *be a stability condition.*

i) *If* $E \in \mathbf{T}$ *is rigid, then all* σ*-stable factors of* E *are quasi-spherical (and thus spherical if* k *is algebraically closed).*

ii) *If* $E \in \mathbf{T}$ *is semi-rigid, then all* σ*-stable factors of* E *are either quasi-spherical or semi-rigid. In fact, there exists at most one semi-rigid stable factor, which moreover occurs with multiplicity one.*

Proof. We start out with the following technical observation, which is irrelevant if k is algebraically closed. Suppose that the endomorphism algebra $\mathrm{End}(B)$ of an object $B \in \mathbf{T}$ is a division algebra. Then $\mathrm{Hom}(B,B[1])$ is a vector space over $\mathrm{End}(B)$. Note however that multiplication from the right and from the left do not necessarily commute. In particular, as an abstract $\mathrm{End}(B)$-vector space one has $\mathrm{Hom}(B,B[1]) \simeq \mathrm{End}(B)^{\oplus r}$ for some r and, therefore, $(B,B)^1 = r(B,B)^0$.

Let $A_1, \ldots, A_n$ be the semistable factors of an object $E \in \mathbf{T}$. Then Lemma 2.7 implies that $\sum (A_i, A_i)^1 \leq (E, E)^1$.

i) Suppose now that E is rigid. Then the A_i are as well rigid. Thus, since an object that is stable and rigid is quasi-spherical, it suffices to show that the stable factors of a semistable rigid $A \in \mathbf{T}$ are also rigid.

Let us first assume that the stable factors of A are all isomorphic to the same object B. Then $\chi(A, A) = \ell^2 \chi(B, B)$ for some positive integer ℓ. By rigidity of A one has $\chi(A, A) = 2(A, A)^0 > 0$ and for B one has $\chi(B, B) = 2(B, B)^0 - (B, B)^1$. Since B is stable, $\mathrm{End}(B)$ is a division algebra and the above observation applies. Hence, $\chi(B, B) = (2 - r)(B, B)^0$ for some $r \geq 0$ and $(2 - r)\ell^2 = 2(A, A)^0/(B, B)^0 > 0$ then shows that $r = 0$ or $r = 1$, but the latter is excluded by condition $(*)$.

Suppose now that A has at least two non-isomorphic stable factors. Then there exists a distinguished triangle

$$C \longrightarrow A \longrightarrow D \longrightarrow C[1] \tag{2.4}$$

with C and D semistable of the same phase and such that $(C, D)^i = 0$, for any $i \leq 0$, and such that all stable factors of D are isomorphic to each other. Again by Lemma 2.7, C and D are rigid. Then by the above argument the stable factors of D are quasi-spherical and one continues with C.

ii) If E is semi-rigid, then all its semistable factors will be either rigid or semi-rigid. Due to i), the stable factors of a rigid semistable A will all be quasi-spherical. If A is semi-rigid and semistable with just one type of stable factor B, then as before, $\chi(A, A) = \ell^2 \chi(B, B)$. This time one has only the weak inequality $\chi(A, A) \geq 0$, which yields $(2 - r)(B, B)^0 \geq 0$, i.e. $r = 0$, $r = 1$, or $r = 2$. If $r = 2$, then $\chi(B, B) = 0$ and hence $\chi(A, A) = 0$. As A is semi-rigid, the latter implies that A is simple and hence $A = B$. Again the case $r = 1$ is excluded by condition $(*)$. If $r = 0$, then B is rigid and thus A would be direct sum of copies of B which is not the case.

If A has at least two non-isomorphic stable factors, then use the distinguished triangle (2.4) as above. At most one of the two objects C or D can be semi-rigid. More precisely, A has only one semi-rigid stable factor which can only occur in either C or D. Combined with the discussion before one sees that all stable factors of E are spherical up to at most one which is semi-rigid and occurs with multiplicity one. □

For $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathbf{T})$ and a σ-semistable $0 \neq E \in \mathcal{P}(1)$, an analogous result is proved in [6, Lemma 12.2]. In the applications the following corollary will be combined with Lemma 2.3.

Corollary 2.10. *Let* $\mathbf{T}$ *be a K3 category satisfying* $(*)$ *and not containing any quasi-spherical objects. If* $E \in \mathbf{T}$ *is semi-rigid, then* E *is stable with respect to any stability condition.* □

The above discussion can be generalized to yield a relation between $(E, E)^1$ and the number of (semi)stable factors of E. For this we denote by $\ell_{\mathrm{HN}}(E)$ the number of semistable (Harder–Narasimhan) factors of an object E and, similarly, by $\ell_{\mathrm{JH}}(E)$ the number of stable (Jordan–Hölder) factors counted without(!) multiplicity. Then one has

Corollary 2.11. *Let* $\mathbf{T}$ *be a K3 category satisfying* $(*)$ *and not containing any quasi-spherical objects. Then* $\ell_{\mathrm{HN}}(E) \le \ell_{\mathrm{JH}}(E) \le (1/2)(E,E)^1$. □

2.3. **Only one spherical object.**

So far we have dealt with K3 categories not containing any spherical objects. Let us now pass to the slightly more complicated situation when there exists a spherical object $E \in \mathbf{T}$ which is unique up to shift.

Let us first explain how to construct inductively indecomposable rigid objects starting with an arbitrary spherical object E. This goes as follows: The unique morphism

$$E \longrightarrow E[2]$$

can be completed to a distinguished triangle

$$E_2 \longrightarrow E \longrightarrow E[2] \longrightarrow E_2[1].$$

One checks easily that $(E,E_2)^2 = 1$ and that therefore there exists a natural distinguished triangle

$$E_3 \longrightarrow E \longrightarrow E_2[2] \longrightarrow E_3[1].$$

Due to the following lemma this procedure can be iterated and produces distinguished triangles

$$E_{n+1} \longrightarrow E \longrightarrow E_n[2] \longrightarrow E_{n+1}[1]. \tag{2.5}$$

Thus we obtain a sequence of rigid objects $E_1 := E, E_2, E_3, \ldots$.

Lemma 2.12. *If* $m \le n$ *are two positive integers, then*

$$(E_m,E_n)^i = \begin{cases} 1 & \textit{if } i = -n+1,\ldots,-n+m \\ 1 & \textit{if } i = 2,\ldots,m+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Applying $\mathrm{Hom}(E,-)$ to (2.5) we obtain the long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}^i(E,E_n[1]) \longrightarrow \mathrm{Hom}^i(E,E_{n+1}) \longrightarrow \mathrm{Hom}^i(E,E) \longrightarrow \ldots$$

and so by induction

$$(E,E_n)^i = \begin{cases} 1 & \text{if } i = -n+1 \\ 1 & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Applying the exact sequence (2.3) in the proof of Lemma 2.7 to the distinguished triangle $E_{n-1}[1] \to E_n \to E$, we obtain the exact sequence

$$\ldots \to \mathrm{Hom}^{-i}(E,E_{n-1}[1]) \to \mathrm{Hom}^{-i}(E_n,E_n) \to \mathrm{Hom}^{-i}(E_{n-1}[1],E_{n-1}[1]) \to \ldots,$$

if $i > 0$ and

$$\dots \longrightarrow \mathrm{Hom}^0(E, E_{n-1}[1]) \longrightarrow \mathrm{Hom}^0(E_n, E_n) \longrightarrow \mathrm{Hom}^0(E_{n-1}[1], E_{n-1}[1]) \oplus \mathrm{Hom}^0(E, E) \xrightarrow{\eta} \mathrm{Hom}^1(E, E_{n-1}[1]).$$

Since η is surjective, we get the following

$$(E_n, E_n)^i = \begin{cases} 1 & \text{if } i = -n+1, \dots, 0 \\ 1 & \text{if } i = 2, \dots, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the lemma follows by applying $\mathrm{Hom}(E_m, -)$ to (2.5). □

An immediate consequence of the above is

Corollary 2.13. *If $E \in \mathbf{T}$ is a spherical object, then the objects E_n are rigid.*

Proposition 2.14. *Suppose our K3 category $\mathbf{T}$ is the bounded derived category $\mathrm{D^b}(\mathcal{A})$ of some abelian category $\mathcal{A}$ and that $\mathcal{A}$ contains a spherical object $E \in \mathcal{A}$ which is the only indecomposable rigid object in $\mathcal{A}$. Then:*
i) *Any indecomposable rigid object $F \in \mathbf{T}$ is isomorphic to a shift of some E_n,*
ii) *E is up to shift the only quasi-spherical object also in $\mathbf{T}$, and*
iii) *$\mathbf{T}$ satisfies condition $(*)$ in Remark 2.8.*

Proof. The assertion is proved by induction on the length of the complex F. Since F is rigid, all its cohomology objects $H^i(F)$ are rigid (see e.g. [4, Lemma 2.9] or Lemma 2.7). Suppose k is maximal with $H^k(F) \neq 0$ and for simplicity we assume $k = 0$. Then $H^0(F) = E^{\oplus r}$ for some r and one has the natural distinguished triangle

$$F_1 \longrightarrow F \longrightarrow E^{\oplus r} \longrightarrow F_1[1]$$

with $H^i(F_1) = 0$ for $i \geq 0$. In particular, the assumptions of Lemma 2.7 are satisfied and, therefore, F_1 is again rigid. By induction $F_1 = \bigoplus E_i[n_i]^{\oplus r_i}$.

Therefore, $\mathrm{Hom}(E, E_i[n_i + 1]) \neq 0$, as otherwise $E_i[n_i]$ would be a direct summand of F, which contradicts the assumption that F is indecomposable. By Lemma 2.12 the only non-trivial morphisms $E \longrightarrow E_i[n_i + 1]$ exist for $n_i + 1 = 2$ and $n_i + 1 = -i + 1$. In the first case, E_{i+1} would be a direct summand of F and hence $F = E_{i+1}$, for F is indecomposable. In the latter case $E_i[-i]$ would be a summand of F_1, which contradicts $H^i(F_1) = 0$ for $i \geq 0$. (Note $H^i(E_i[-i]) = H^0(E_i) \neq 0$.)

Let us now verify condition $(*)$. Suppose $B \in \mathrm{D^b}(\mathcal{A})$ is an object such that $\mathrm{End}(B)$ is a division algebra, $(B, B)^0 = (B, B)^1$ and $(B, B)^i = 0$, for any integer $i < 0$. We shall derive a contradiction as follows. Consider a non-trivial extension $B \longrightarrow A \longrightarrow B \longrightarrow B[1]$. (In fact, the isomorphism type of A is unique, but we will not need this.) The boundary morphisms $\mathrm{End}(B) \longrightarrow \mathrm{Hom}(B, B[1])$ and $\mathrm{Hom}(B, B[1]) \longrightarrow \mathrm{Hom}(B, B[2])$ are bijective, because of Serre duality and linearity under the action of $\mathrm{End}(B)$. This shows $\mathrm{Hom}(B, B) \simeq \mathrm{Hom}(B, A)$ and

$\mathrm{Hom}(A, A[i]) = 0$ for $i \neq 0, 2$. We leave it to the reader to show that A is also indecomposable. Hence, $A \simeq E$ (up to shift), which yields the contradiction $2 = \chi(E, E) = \chi(A, A) = 4\chi(B, B) = 4(B, B)^0 > 2$. □

Proposition 2.15. *Suppose* $\mathbf{T}$ *satisfies* $(*)$ *and* $E \in \mathbf{T}$ *is a spherical object which is up to shift the only quasi-spherical object in* $\mathbf{T}$*. Then* E *is stable with respect to any stability condition on* $\mathbf{T}$ *and the stable factors of* E_n *are* $E[n-1], \ldots, E[1], E$.

Proof. Consider the stable factors $A_1, \ldots, A_\ell$ of E. By Proposition 2.9 they are all quasi-spherical and hence isomorphic to some $E[i]$. In particular, E is stable. The description of the stable factors of the E_n follows immediately. □

Remark 2.16. Indecomposable rigid objects in a K3 category with only one spherical object can be classified in general. The precise result is the following: Let $\mathbf{T}$ be a K3 category satisfying $(*)$ and containing a spherical object E which is up to shift the only quasi-spherical object in $\mathbf{T}$. Suppose there exists a stability condition on $\mathbf{T}$. Then any indecomposable rigid object in $\mathbf{T}$ is isomorphic to a shift of some E_n which are constructed as above.

In all relevant examples of K3 categories a spherical object $E \in \mathbf{T}$ gives rise to a *spherical twist*, a particular exact autoequivalence of $\mathbf{T}$ (see [24] or [13, Ch. 8]). In the following, we shall thus assume that the spherical twist can be constructed, i.e. that there exists an autoequivalence

$$T_E : \mathbf{T} \longrightarrow \mathbf{T},$$

such that for any object $F \in \mathbf{T}$ one has a distinguished triangle

$$T_E(F)[-1] \longrightarrow \mathrm{Hom}(E, F[*]) \otimes E \longrightarrow F \longrightarrow T_E(F)$$

which is functorial in F. Recall that the inverse can be described by a distinguished triangle

$$T_E^{-1}(F) \longrightarrow F \longrightarrow E[2] \otimes \mathrm{Hom}(E, F[*]) \longrightarrow T_E^{-1}(F)[1].$$

Also recall that $T_E^k(E) \simeq E[-k]$.

Remark 2.17. i) We shall be interested in the spherical twist $T := T_E$ applied to objects $F \in \mathbf{T}$ with $\sum(E, F)^i = 1$. After shifting F (or E) appropriately, this reduces to the case $(E, F)^0 = 1$ and $(E, F)^i = 0$ for $i \neq 0$.

Then for all k one has $(E, T^k(F))^k = 1$ and $(E, T^k(F))^i = 0$ for $i \neq k$. Indeed,

$$(E, T^k(F))^i = (T^k(E)[k], T^k(F))^i = (E[k], F)^i = (E, F)^{i-k}.$$

Using the distinguished triangles $E \longrightarrow T^k(F)[k] \longrightarrow T^{k+1}(F)[k] \longrightarrow E[1]$, one also proves by induction $(F, T^k(F))^0 = 1$ for $k \geq 0$.

ii) Suppose the spherical object E, $T^n(F)$, and $T^k(F)$ are all σ-stable for some stability condition σ on $\mathbf{T}$. If as before $\sum(F, E)^i = 1$, then $|n - k| < 2$.

Clearly, E is also stable with respect $T^n(\sigma)$, so that we may assume that $n = 0$, $k > 0$, and $(E, F)^0 = 1$.

Then using the non-triviality of $(F, T^k(F))^0$, $(T^k(F)[k], E)^2$, and $(E, F)^0$ one obtains the following inequality of phases

$$\phi(F) \leq \phi(T^k(F)) \leq \phi(E) - k + 2 \leq \phi(F) - k + 2.$$

Hence $k \leq 2$. Moreover, if $k = 2$ then $\phi(E) = \phi(F)$ and hence $E \simeq F$, which is absurd.

iii) For later use we rephrase the last observation as follows: Let E be a spherical object in a K3 category $\mathbf{T}$ satisfying $(*)$, which induces a spherical twist $T := T_E : \mathbf{T} \simeq \mathbf{T}$. Suppose E is up to shift the only quasi-spherical object. If $F \in \mathbf{T}$ with $\sum (E, F)^i = 1$ and $W_F := \{\sigma \in \mathrm{Stab}(\mathbf{T}) \mid F \text{ is } \sigma\text{-stable}\}$, then

$$T^n W_F \cap T^k W_F = \emptyset$$

if $|n - k| \geq 2$.

Proposition 2.18. *Let $\mathbf{T}$ be a K3 category satisfying $(*)$ with a spherical object E inducing a spherical twist $T := T_E : \mathbf{T} \simeq \mathbf{T}$. Suppose there is no other quasi-spherical object up to shift. Then for any semi-rigid object F with $(F, E)^* := \sum (F, E)^i = 1$ and any stability condition σ, one finds an integer n such that $T^n(F)$ is stable.*

Proof. By assumption, we may assume that $(E, F)^0 = (F, E)^2 = 1$ and $(E, F)^i = (F, E)^{2-i} = 0$ for all $i \neq 0$.

Consider the Harder–Narasimhan factors $A_1, \ldots, A_n$ of F with respect to a given stability condition. Then by Proposition 2.9 at most one A_i is semi-rigid and all others are rigid.

Suppose first that A_1 is rigid. Then, again by Proposition 2.9, all stable factors of A_1 are quasi-spherical and hence isomorphic to some shift of E. Since A_1 itself is semistable, all the shifts are the same, say $E[k]$. In fact $k = 0$, for there exists a non-trivial morphism $A_1 \longrightarrow F$. On the other hand, the indecomposable factors of A_1 are as well rigid and hence isomorphic to some $E_j[n_j]$, where the E_j are as above. However, due to Proposition 2.15 the E_j are semistable only for $j = 1$. Combining both observations yields $A_1 \simeq E^{\oplus r}$.

Next apply $\mathrm{Hom}(E, -)$ to the distinguished triangle

$$A_1 \longrightarrow F \longrightarrow F' \longrightarrow A_1[1].$$

Since $(A_1, F')^{-1} = (A_1, F')^0 = 0$ and $(E, F)^0 = 1$, this shows $r = 1$. Hence, the first factor in the Harder–Narasimhan filtration $A_1 \longrightarrow F$ can be interpreted as the natural map $\mathrm{Hom}(E, F) \otimes E \longrightarrow F$. Thus, the cone F' is isomorphic to $T(F)$.

Now one continues with F', whose Harder–Narasimhan filtration has smaller length than the one of F. Indeed, F' is again semi-rigid and $(F', E)^* = (T(F), E)^* = (F, T^{-1}(E))^* = (F, E)^* = 1$, i.e. F' satisfies all the assumptions on F.

If the first Harder–Narasimhan factor A_1 is only semi-rigid, then we rather work with the last one A_n which is necessarily rigid. The same reasoning as above shows that $A_n \simeq E[2]$ and this time the last quotient $F \longrightarrow E[2]$ of the Harder–Narasimhan filtration can be interpreted as the natural morphism $F \longrightarrow \mathrm{Hom}^2(F, E) \otimes E[2]$. Hence, the kernel is isomorphic to $T^{-1}(F)$. As above, one proceeds then with $T^{-1}(F)$, which still meets all the requirements.

Thus, there exists an integer n such that $T^n(F)$ is semistable. (Of course, working through the Harder–Narasimhan filtration only either T or T^{-1} will occur.)

We leave it to the reader to repeat the same arguments once more for appropriate Jordan–Hölder filtrations. Compare the arguments in the proof of Proposition 2.9. $\square$

In the application we shall need stability of $T^n(F)$ for several F at a time, but with the same n. The assertion that will be needed is the following:

Corollary 2.19. *If $F_1 \sim F_2$ are as above with $(E,F_1)^i = (E,F_2)^i$ and $(F_1,F_2)^i = 0$ for all i, then there exists an integer n such that $T^n(F_1)$ and $T^n(F_2)$ are both stable.*

Proof. As in the proof of the proposition, we shall assume that $(E,F_1)^0 = (E,F_2)^0 = 1$. After interchanging the role of F_1 and F_2 if necessary, we may assume that F_1 and $T^n(F_2)$ are stable for some $n > 0$ (see Remark 2.17, ii)). We shall show that in fact F_2 is already semistable. The proof of the stability of F_2 is similar and uses appropriate Jordan–Hölder filtrations (instead of Harder–Narasimhan).

Following the arguments in the above proof, we shall assume that the unique $E \longrightarrow F_2$ is the first factor in the Harder–Narasimhan filtration.

Let us first make the following observations, which can be proved easily by induction: For any $k > 0$ one has:
i) $(E, T^k(F_2))^i \neq 0$ only for $i = k$ (see Remark 2.17, i)).
ii) $(F_1, T^k(F_2))^k \neq 0$ and $(F_1, T^k(F_2))^i = 0$ for $i > k$.
iii) $(F_1, T^k(F_2))^1 \neq 0$.
(Use the distinguished triangle $T^k(F_2)[k] \longrightarrow E[2] \longrightarrow T^{k-1}(F_2)[k+1]$ and the non-trivial morphism $F_1 \longrightarrow E[2]$ for ii) and $E[1-k] \longrightarrow T^k(F_2)[1] \longrightarrow T^{k+1}(F_2)[1]$ and $(F_1, E[1-k])^1 = 0$ for iii).)

Hence, by ii) $(F_1, T^n(F_2))^n \neq 0$, which by stability of F_1 and $T^n(F_2)$ shows

$$\phi(F_1) \leq \phi(T^n(F_2)) + n \leq \phi(F_1) + 2. \tag{2.6}$$

On the other hand, iii) implies

$$\phi(F_1) \leq \phi(T^n(F_2)) + 1 \leq \phi(F_1) + 2. \tag{2.7}$$

This leaves us with the possibilities $n = 1, 2, 3$.

Next, $(E, F_1)^0 \neq 0$ and $(E, T^n(F_2))^n \neq 0$ combined with Serre duality yields

$$\phi(E) < \phi(F_1) < \phi(E) + 2 \text{ and } \phi(E) < \phi(T^n(F_2)) + n < \phi(E) + 2. \tag{2.8}$$

(This time equality can be excluded, as the spherical E cannot be isomorphic to a rigid $F_1[i]$ or $T^n(F_2)[i]$.)

If $n = 3$, then $\phi(F_1) = \phi(T^n(F_2)) + 1$, which yields a contradiction in (2.8).

If $n = 2$, then (2.6) and (2.7) show that $\phi(T^n(F_2)) = \phi(F_1)$. (Use that F_1, F_2, and $T^2(F_2)$ are numerically equivalent and that therefore $\phi(T^2(F_2)) - \phi(F_1)$ is an even integer.) Both cases contradict (2.8).

If $n = 1$, we shall derive a contradiction as follows. For notational simplicity assume $\phi(F_1) = 0$, i.e. $Z(F_1) = Z(F_2) \in \mathbb{R}_{>0}$. Now (2.6) reads $-1 \leq \phi(T(F_2)) \leq 1$ and (2.8) shows $\phi(E) \in (-2, 0)$. Since $E \longrightarrow F_2 \longrightarrow T(F_2)$ is the Harder–Narasimhan filtration of F_2, one has $0 > \phi(E) \geq \phi(T(F_2)) \geq -1$. The latter contradicts $Z(E) + Z(T(F_2)) = Z(F_2) \in \mathbb{R}_{>0}$. □

3. Generic twisted K3 and twisted abelian surfaces

It is extremely difficult to obtain any information about the space of stability conditions on a general triangulated category. Even the most basic questions, e.g. whether it is non-empty or connected, are usually very hard. In this section we aim at a complete description of the space of stability conditions on the bounded derived category of coherent sheaves on a surface, which is a generic twisted K3 surface or an arbitrary twisted abelian surface. Due to a technical problem, we are, for the time being, only able to deal with the part of maximal dimension. In particular, we will see that this space is connected and simply-connected (see Theorem 3.15). Moreover, following ideas of Bridgeland, this can be used to describe the group of autoequivalences in these cases (see Theorem 3.17 and Corollary 3.21).

3.1. **Stability conditions on twisted surfaces.**

For the bounded derived category of coherent sheaves on a projective K3 surface Bridgeland describes in [6] one connected component $\mathrm{Stab}^\dagger(X)$ of the space $\mathrm{Stab}(X)$ of stability conditions on $\mathrm{D}^{\mathrm{b}}(X) = \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ as a covering of a certain period domain. In this section we will describe an analogous connected component $\mathrm{Stab}^\dagger(X,\alpha)$ of the space of stability conditions on the bounded derived category $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ of α-twisted coherent sheaves on a K3 or abelian surface X endowed with an additional Brauer class α. The construction follows closely the original one in [6] and we will therefore be brief and only explain the necessary modifications.

Recall that the (cohomological) *Brauer group* $\mathrm{Br}(X)$ of a smooth complex projective variety X is the torsion part of the cohomology group $H^2(X,\mathcal{O}_X^*)$ in the analytic topology (or, equivalently, $H^2_{\text{ét}}(X,\mathcal{O}_X^*)$ in the tale topology). A *twisted K3* (or *abelian*) *surface* is a pair (X,α) consisting of a K3 (abelian) surface X and a Brauer class $\alpha \in \mathrm{Br}(X)$.

Any $\alpha \in \mathrm{Br}(X)$ can be represented by a Čech cocycle on an open analytic cover $\{U_i\}_{i\in I}$ of X using the sections $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)$. An *$\alpha$-twisted coherent sheaf* $\mathcal{F}$ consists of a collection $(\{\mathcal{F}_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$, where $\mathcal{F}_i$ is a coherent sheaf on U_i and $\varphi_{ij} : \mathcal{F}_j|_{U_i\cap U_j} \longrightarrow \mathcal{F}_i|_{U_i\cap U_j}$ is an isomorphism satisfying the following conditions:

$$(1)\ \varphi_{ii} = \mathrm{id}; \qquad (2)\ \varphi_{ji} = \varphi_{ij}^{-1}; \qquad (3)\ \varphi_{ij}\circ\varphi_{jk}\circ\varphi_{ki} = \alpha_{ijk}\cdot \mathrm{id}.$$

By $\mathbf{Coh}(X,\alpha)$ we shall denote the abelian category of α-twisted coherent sheaves on X. Note that for different Čech cocycles representing α the abelian categories will be equivalent, although not canonical. For a discussion of this see [10, 11].

Definition 3.1. The bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X,\alpha))$ and the space of numerical, locally finite stability conditions on it shall be denoted

$$\mathrm{D}^{\mathrm{b}}(X,\alpha) \ \text{ resp. } \ \mathrm{Stab}(X,\alpha).$$

In order to study the twisted categories $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ and equivalences between them, we have introduced in [10] the twisted Hodge structure $\widetilde{H}(X,\alpha,\mathbb{Z})$ and the twisted Chern character $\mathrm{ch}^\alpha : \mathrm{D}^{\mathrm{b}}(X,\alpha) \longrightarrow \widetilde{H}(X,\alpha,\mathbb{Z})$. To 'materialize' both structures one needs to fix a B-field lift B of the Brauer class α and a cocycle representing B. So, the above twisted Chern character ch^α stands for $\mathrm{ch}^B : \mathrm{D}^{\mathrm{b}}(X,\alpha_B) \longrightarrow \widetilde{H}(X,B,\mathbb{Z})$. We will largely ignore this issue here, but the details, sometimes a little confusing, can be found in [11] (see also Remark 3.3).

The twisted *Néron–Severi group* is by definition

$$\mathrm{NS}(X,\alpha) := \widetilde{H}^{1,1}(X,\alpha,\mathbb{Z}).$$

Note that for the trivial twist $\alpha = 1$ the Néron–Severi group $\mathrm{NS}(X, \alpha = 1)$ differs from the classical Néron–Severi group by the additional sum $(H^0 \oplus H^4)(X, \mathbb{Z})$. The Chern character ch^α and the Mukai vector $v^\alpha := \mathrm{ch}^\alpha \cdot (1, 0, 1)$ take values only in $\mathrm{NS}(X, \alpha)$.

It is not difficult to show that ch^α identifies the numerical Grothendieck group with the Néron–Severi group, i.e. $\mathcal{N}(\mathrm{D}^{\mathrm{b}}(X, \alpha)) \simeq \mathrm{NS}(X, \alpha)$. In particular, the stability function Z of a numerical stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ is of the form $Z(E) = \langle v^\alpha(E), \varphi\rangle$ for some $\varphi \in \mathrm{NS}(X, \alpha) \otimes \mathbb{C}$. This gives rise to the *period map*

$$\pi : \mathrm{Stab}(X, \alpha) \longrightarrow \mathrm{NS}(X, \alpha) \otimes \mathbb{C}, \quad \sigma = (Z, \mathcal{P}) \longmapsto \varphi.$$

Definition 3.2. Denote by

$$P(X, \alpha) \subset \mathrm{NS}(X, \alpha) \otimes \mathbb{C}$$

the open subset of vectors φ such that real part and imaginary part of φ generate a positive plane in $\mathrm{NS}(X, \alpha) \otimes \mathbb{R}$.

Remark 3.3. i) Suppose $B_0 \in H^2(X, \mathbb{Q})$ is a B-field lift of α, i.e. α is the image of B_0 under the exponential map $H^2(X, \mathbb{Q}) \to H^2(X, \mathcal{O}_X^*)$.

To any real ample class $\omega \in H^{1,1}(X, \mathbb{Z}) \otimes \mathbb{R}$ one associates

$$\varphi := \exp(B_0 + i\omega) = 1 + (B_0 + i\omega) + (B_0^2 - \omega^2)/2 + i(B_0.\omega) \in \mathrm{NS}(X, \alpha) \otimes \mathbb{C}.$$

Here we use that for a chosen B-field lift B_0 of α the twisted cohomology $\widetilde{H}^{1,1}(X, \alpha, \mathbb{Z})$ can be identified with the integral part of $\exp(B_0) \cdot \widetilde{H}^{1,1}(X, \mathbb{Q})$.

The real part $1 + B_0 + (B_0^2 - \omega^2)/2$ and the imaginary part $\omega + (\omega.B_0)$ of φ span a plane which is positive due to $\omega^2 > 0$.

As in the untwisted case, $P(X, \alpha)$ has two connected components and we shall denote the one that contains $\varphi = \exp(B_0 + i\omega)$ as above by $P^+(X, \alpha)$. Thus, we have

$$P^+(X, \alpha) \subset P(X, \alpha) \subset \mathrm{NS}(X, \alpha) \otimes \mathbb{C}$$

and one proves that the fundamental group $\pi_1(P^+(X, \alpha)) \simeq \mathbb{Z}$ is generated by the loop induced by the natural $\mathbb{C}^*$-action $(\lambda, \varphi) \mapsto \lambda \cdot \varphi$ on $P(X, \alpha)$.

ii) Note that if B_0 and ω are as in i), $B_1 \in \mathrm{NS}(X) \otimes \mathbb{R}$, and $B := B_1 + B_0$, then

$$\varphi := \exp(B + i\omega) = \exp(B_1) \cdot \exp(B_0 + i\omega) \in \mathrm{NS}(X, \alpha) \otimes \mathbb{C}$$

is also contained in $P^+(X, \alpha)$.

iii) It is worth pointing out that in the twisted case there are two sorts of B-fields. First, there are B-field lifts B of the Brauer class α. It is the $(0, 2)$-part of B that matters in this case. Second, as in the untwisted case, one needs B-fields in order to 'complexify' the polarization (or Kähler class) on X. As a B-field lift of a given Brauer class α can always be changed by a rational $(1, 1)$-class without changing α, the difference between these two classes is not always clear cut.

Following Bridgeland, we shall associate to any $\varphi \in P^+(X, \alpha)$ a torsion theory and thus a t-structure on $\mathrm{D}^{\mathrm{b}}(X, \alpha)$. Under additional conditions this will lead to a stability condition, whose stability function is

$$Z_\varphi(E) := \langle v^\alpha(E), \varphi\rangle \,. \tag{3.1}$$

Fix $\varphi \in P^+(X,\alpha)$ and let $\mathcal{T},\mathcal{F} \subset \mathbf{Coh}(X,\alpha)$ be the following two full additive subcategories: The non-trivial objects in $\mathcal{T}$ are the twisted sheaves E such that every non-trivial torsion free quotient $E \twoheadrightarrow E'$ satisfies $\mathrm{Im}(Z_\varphi(E')) > 0$. A non-trivial twisted sheaf E is an object in $\mathcal{F}$ if E is torsion free and every non-zero subsheaf E' satisfies $\mathrm{Im}(Z_\varphi(E')) \le 0$.

It is easy to see that $(\mathcal{T},\mathcal{F})$ does define a torsion theory. In particular, $\mathrm{Hom}(E,F) = 0$ for $E \in \mathcal{T}$ and $F \in \mathcal{F}$ and any sheaf $G \in \mathbf{Coh}(X,\alpha)$ can be written in a unique way as an extension

$$0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0$$

with $E \in \mathcal{T}$ and $F \in \mathcal{F}$.

The heart of the induced t-structure is the abelian category

$$\mathcal{A}(\varphi) := \left\{ E \in \mathrm{D}^{\mathrm{b}}(X,\alpha) : \begin{array}{l} \bullet\ \mathcal{H}^i(E) = 0 \text{ for } i \notin \{-1,0\}, \\ \bullet\ \mathcal{H}^{-1}(E) \in \mathcal{F}, \\ \bullet\ \mathcal{H}^0(E) \in \mathcal{T} \end{array} \right\}.$$

Recall that a function $Z : \mathcal{A} \longrightarrow \mathbb{C}$ on an abelian category is called a *stability function* if for all $0 \ne E \in \mathcal{A}$ either $\mathrm{Im}(Z(E)) > 0$ or $Z(E) \in \mathbb{R}_{<0}$. The following is the analogue of [6, Lemma 6.2].

Lemma 3.4. *For $\varphi = \exp(B + i\omega)$ as in Remark 3.3, ii), the induced homomorphism*

$$Z_\varphi : \mathcal{A}(\varphi) \longrightarrow \mathbb{C}$$

is a stability function on $\mathcal{A}(\varphi)$ if and only if for any spherical twisted sheaf $E \in \mathbf{Coh}(X,\alpha)$ one has $Z_\varphi(E) \notin \mathbb{R}_{\le 0}$.

Proof. Let $E \in \mathcal{A}(\varphi)$ with cohomology sheaves $\mathcal{H}^i := \mathcal{H}^i(E)$, $i = 0, -1$. Then

$$Z_\varphi(E) = Z_\varphi(\mathcal{H}^0) - Z_\varphi(\mathcal{H}^{-1}) = Z_\varphi(\mathcal{H}^0_{\mathrm{tor}}) + Z_\varphi(\mathcal{H}^0/\mathcal{H}^0_{\mathrm{tor}}) - Z_\varphi(\mathcal{H}^{-1}),$$

where $\mathcal{H}^0_{\mathrm{tor}}$ denotes the torsion part of $\mathcal{H}^0$.

The first thing to notice, is that for ω ample any torsion sheaf $\mathcal{S} \ne 0$ satisfies $\mathrm{Im}(Z_\varphi(\mathcal{S})) > 0$ if $\dim(\mathrm{supp}(\mathcal{S})) = 1$ and $Z_\varphi(\mathcal{S}) \in \mathbb{R}_{<0}$ if $\dim(\mathrm{supp}(\mathcal{S})) = 0$. Secondly, by construction, $\mathrm{Im}(Z_\varphi(\mathcal{H}^0/\mathcal{H}^0_{\mathrm{tor}})) > 0$ if $\mathcal{H}^0$ is not torsion. Thus, Z_φ is a stability function on $\mathcal{A}(\varphi)$ if and only if for any $E \in \mathcal{F}$ with $\mathrm{Im}(Z_\varphi(E)) = 0$ one has $Z_\varphi(E) \in \mathbb{R}_{>0}$.

If one writes $\mathrm{ch}^\alpha(E)(1,0,1) = (r,\ell,s)$ with $r > 0$, then $\mathrm{Im}(Z_\varphi(E)) = (\omega.\ell) - r(\omega.B)$ and

$$\mathrm{Re}(Z_\varphi(E)) = (1/2r)\left((\ell^2 - 2rs) + r^2\omega^2 - (\ell - rB)^2\right).$$

Thus, $\mathrm{Im}(Z_\varphi(E)) = 0$ if and only if $(\ell - rB)$ is orthogonal to ω. The Hodge index theorem, applied to the class $(\ell - rB)$, which really is of type $(1,1)$ on the untwisted(!) surface, yields $(\ell - rB)^2 \le 0$. On the other hand, $\ell^2 - 2rs = -\chi(E,E)$. Using the existence of the usual Jordan–Hölder filtration, one can restrict to the case that E is simple and hence $\ell^2 - 2rs \ge 0$ or $= -2$ if E is spherical. This proves the assertion. $\square$

Remark 3.5. i) In fact, the proof shows that the condition is satisfied if e.g. $\omega^2 > 2$.

ii) Later we shall be interested in the case when there are no spherical objects. Then the lemma simply says that any $\varphi = \exp(B + i\omega)$ defines a stability function. Here ω varies in the Kähler cone which coincides with the positive cone, since there would not be any (-2)-curves.

The following two results are the twisted analogues of Bridgeland's result in [6]. For the first proposition we assume that $\varphi = \exp(B + i\omega)$ is constructed as in Remark 3.3, ii) and satisfies the condition of Lemma 3.4, i.e. $Z_\varphi(E) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves.

Proposition 3.6. *The stability function Z_φ on the abelian category $\mathcal{A}(\varphi)$ has the Harder–Narasimhan property and therefore defines a stability condition σ_φ on $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ which, moreover, is locally finite.* $\square$

Note that the proof of this result is rather round about already in the untwisted case. For a direct proof when B and ω are rational see [6, Prop. 7.1]. The argument works as well in the twisted case.

Definition 3.7. One denotes by $\mathrm{Stab}^\dagger(X, \alpha)$ the connected component of $\mathrm{Stab}(X, \alpha)$ that contains the stability conditions described by Proposition 3.6 (which form a connected set). Furthermore, $U(X, \alpha) \subset \mathrm{Stab}^\dagger(X, \alpha)$ shall denote the space of all stability conditions in $\mathrm{Stab}^\dagger(X, \alpha)$ for which all point sheaves $k(x)$ are stable all of the same phase.

We say that a connected component of $\mathrm{Stab}(X, \alpha)$ is of maximal dimension when the restriction of the period map $\pi : \mathrm{Stab}(X, \alpha) \longrightarrow \mathrm{NS}(X, \alpha) \otimes \mathbb{C}$ to it is locally homeomorphic.

Clearly, $\mathrm{Stab}^\dagger(X, \alpha)$ is a connected component of maximal dimension. Also note that by definition $U(X, \alpha)$ is connected.

Proposition 3.8. *Suppose $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X, \alpha)$ is contained in a connected component of maximal dimension, e.g. in $\mathrm{Stab}^\dagger(X, \alpha)$, and that for any closed point $x \in X$ the skyscraper sheaf $k(x)$ is σ-stable of phase one with $Z(k(x)) = -1$. Then there exists $\varphi = \exp(B + i\omega) \in P^+(X, \alpha)$ as in Remark 3.3, ii) such that the heart of σ coincides with $\mathcal{A}(\varphi)$, i.e. $\mathcal{P}((0, 1]) = \mathcal{A}(\varphi)$.* $\square$

The result presumably continues to hold for any $\sigma \in \mathrm{Stab}(X, \alpha)$. Note that the proposition does not assert equality $\sigma = \sigma_\varphi$ of stability conditions, but only of their hearts. The assumption that σ is contained in a connected component of maximal dimension is needed in order to ensure the existence of generic deformations of σ and to eventually ensure that ω is really ample (see Step 1 in the proof of [6, Prop. 10.3]).

Remark 3.9. In fact, Bridgeland shows moreover that for 'good' stability conditions σ one has $\sigma = \sigma_\varphi$ up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$. For our purpose we need the following stronger form of Proposition 3.8: Suppose σ satisfies the conditions of the proposition and its stability function Z is of the form $Z_{\varphi'}$ with $\varphi' \in \mathrm{NS}(X, \alpha) \otimes \mathbb{Q}(i)$. Then φ in the assertion can be chosen such that up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ one has $\sigma_\varphi = \sigma$.

We sketch the argument in the untwisted case. Clearly, as $Z(k(x)) = -1$, one has $\varphi' = 1 + B' + i\omega' + (a + ib)$ with $B', \omega' \in \mathrm{NS}(X)_\mathbb{R}$ and $a, b \in H^4(X, \mathbb{R})$. Modulo the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ we can assume $b = (B'.\omega')$. Indeed, if $\widetilde{g}$ acts on the stability function Z by $g^{-1} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ with $\gamma = ((B'.\omega') - b)/(\omega.\omega)$, then $g^{-1}Z$ has this property. Once this is achieved, Bridgeland shows that ω' is ample, so he sets $\omega = \omega'$, and that one can choose $B = B'$. The difference between the two stability functions $Z_{\varphi'}$ and Z_{φ}, where $\varphi = \exp(B + i\omega)$, is the real degree four part.

Now one uses the additional assumption B', ω', a, and b are all rational. The special g used to modify φ' does not affect this. One first checks that then $a < (B.B)/2$. In order to see this, pick $r > 0$ such that rB is integral and such that $r^2(B.B)$ is divisible by $2r$. Then there exists a μ-stable vector bundle E with $\mathrm{c}_1(E) = rB$, such that $\langle v(E), v(E)\rangle = 0$. Then $E[1] \in \mathcal{A}(\varphi)$ and $\mathrm{Im}(Z_{\varphi'}(E)) = 0$. On the other hand, $\mathrm{Re}(Z_{\varphi'}(E)) > 0$ if and only if $a < (B.B)/2$.

Eventually apply an element of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ which acts by $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (B.B) - 2a \end{pmatrix}$ on $Z_{\varphi'}$. The resulting stability conditions is of the form Z_φ with $\varphi = \exp(B + i((B.B) - 2a)\omega)$.

Proposition 3.10. *The period map defines a covering map* $\pi : \mathrm{Stab}^\dagger(X, \alpha) \twoheadrightarrow P_0^+(X, \alpha)$ *onto some open subset* $P_0^+(X, \alpha) \subset P^+(X, \alpha)$. □

Moreover, the subgroup of autoequivalences that preserve the component $\mathrm{Stab}^\dagger(X, \alpha)$ acts freely on $\mathrm{Stab}^\dagger(X, \alpha)$.

Of course, as in the untwisted case $P_0^+(X, \alpha)$ is cut out by all hyperplanes orthogonal to (-2)-classes, but we will not need this.

Let $\sigma \in U(X, \alpha)$. Then σ can be changed by an element in $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ such that all point sheaves $k(x)$ are stable of phase one with $Z(k(x)) = -1$.

Remark 3.11. In complete analogy to the discussion in Sections 11, 12, and 13 in [6] one also proves the following result: If (X, α) is a twisted K3 or abelian surface such that $\mathbf{Coh}(X, \alpha)$ does not contain any spherical objects, then $U(X, \alpha) = \mathrm{Stab}^\dagger(X, \alpha)$.

However, a stronger result with a more direct proof relying only on the discussion of Section 2 will be established further below (see Theorem 3.15).

For untwisted abelian surfaces, which never admit any spherical sheaves, this was remarked in [6, Sect. 15].

3.2. **Twisted surfaces without spherical sheaves.**

Typical examples of spherical sheaves on a K3 surface X are the trivial line bundle $\mathcal{O}_X$ and the structure sheaves $\mathcal{O}_C$ of (-2)-curves $\mathbb{P}^1 \simeq C \subset X$. The latter do not exist on a generic projective K3 surface, but e.g. $\mathcal{O}_X$, of course, persists. The situation is different in the twisted case.

Definition 3.12. A twisted K3 or abelian surface (X, α) will be called *generic* if $\mathbf{Coh}(X, \alpha)$ (or, equivalently, $\mathrm{D^b}(X, \alpha)$) does not contain any spherical objects.

Remark 3.13. Here we collect a few observations, that will be useful throughout this section and in particular explain why in the above definition the two formulations are really equivalent. All the assertions can be proved by adapting the arguments of Section 2.2. One replaces stability conditions on triangulated categories by ordinary (Gieseker) stability of twisted coherent sheaves (see [28]). So, in the following, semistable and stable factors of a sheaf are meant with respect to the usual Harder–Narasimhan respectively Jordan–Hölder filtrations in the context of Gieseker stability.

i) Let us first work in the abelian category $\mathbf{Coh}(X,\alpha)$ (cf. Proposition 2.9):
• The stable factors of a rigid sheaf $E \in \mathbf{Coh}(X,\alpha)$ are spherical.
• At most one stable factor of a semi-rigid sheaf $E \in \mathbf{Coh}(X,\alpha)$ is semi-rigid and all others are spherical.

In particular, if there are no spherical sheaves in $\mathbf{Coh}(X,\alpha)$, then there are no rigid sheaves and semi-rigid ones are stable (cf. Corollary 2.10).

ii) In the derived category one has the following:
• If $E \in \mathrm{D}^{\mathrm{b}}(X,\alpha)$ is rigid, then all cohomology sheaves $\mathcal{H}^i(E)$ are rigid. Thus, if there are no spherical sheaves in $\mathbf{Coh}(X,\alpha)$, then $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ does not contain any rigid or spherical objects. To prove this one applies [4, Lemma 2.9], valid also in the twisted context, to show $\sum(\mathcal{H}^i(E),\mathcal{H}^i(E))^1 \leq (E,E)^1 = 0$.
• Similarly, if $E \in \mathrm{D}^{\mathrm{b}}(X,\alpha)$ is semi-rigid, then at most one cohomology sheaf $\mathcal{H}^i(E)$ is semi-rigid and all others are rigid. Thus, if there are no spherical objects in $\mathbf{Coh}(X,\alpha)$, a semi-rigid object in $\mathrm{D}^{\mathrm{b}}(X,\alpha)$ is up to shift isomorphic to a semi-rigid, stable sheaf.

Twisted abelian surfaces are always generic in this sense, as they never admit any spherical sheaves. (Note that the twist α in a twisted surface (X,α) might very well be trivial, i.e. (X,α) might simply be the surface X. Thus, ordinary abelian surfaces are covered by our discussion.)

As we shall see in Section 3.4 generic twisted K3 surfaces are dense among all twisted K3 surfaces. Thus, the following consequence of Proposition 2.9 (see also Corollary 2.10) applies to a dense subset of twisted K3 surfaces and all twisted abelian surfaces.

Corollary 3.14. *Suppose (X,α) is a generic twisted K3 surface or an arbitrary twisted abelian surface. Then the skyscraper sheaves $k(x)$ for closed points $x \in X$ are σ-stable for all $\sigma \in \mathrm{Stab}(X,\alpha)$.*

Proof. Indeed, for a closed point the skyscraper sheaf $k(x)$ is semi-rigid. □

This then yields

Theorem 3.15. *Let (X,α) be a generic twisted K3 surface or an arbitrary twisted abelian surface. Then $\mathrm{Stab}(X,\alpha)$ admits only one connected component of maximal dimension.*

Moreover, this maximal component is simply-connected and the restriction of the period map $\pi : \mathrm{Stab}(X,\alpha) \longrightarrow \mathrm{NS}(X,\alpha)\otimes\mathbb{C}$ to it can be viewed as the universal cover of $P^+(X,\alpha)$.

Proof. Let $\sigma \in \mathrm{Stab}(X,\alpha)$ be a stability condition contained in a connected component of maximal dimension. In particular, after a small deformation the period of σ will be rational, i.e. contained in $\mathrm{NS}(X,\alpha)\otimes\mathbb{Q}(i)$.

Due to Corollary 3.14 all point sheaves $k(x)$ are stable. Now choose a torsion free semi-rigid twisted sheaf E on (X,α). The existence of such an E can be deduced from standard existence results, see e.g. [28] for the twisted case. Note that Corollary 2.10 also applies to E and shows that E is stable with respect to σ.

Since E is torsion free, there exists a non-trivial $E \twoheadrightarrow k(x)$ for any closed point $x \in X$. Hence, the assumptions of Lemma 2.3 are satisfied for any two points $x_1 \neq x_2 \in X$. Thus, either $\phi(k(x_1)) = \phi(k(x_2))$ or $k(x_1) \simeq k(x_2)[\pm 2]$, but the latter is absurd. (For the equality of the phases of the point sheaves $k(x)$ see also Remark 2.4, which explains how to avoid to work with E.) Together with Remark 3.9, this shows that $\sigma \in U(X,\alpha)$. In particular, $U(X,\alpha) = \mathrm{Stab}^\dagger(X,\alpha)$ and $\mathrm{Stab}^\dagger(X,\alpha)$ is the only connected component of maximal dimension.

Next, consider the period map $\pi : U(X,\alpha) \twoheadrightarrow P^+(X,\alpha)$. As mentioned before, the fundamental group of $P^+(X,\alpha)$ is the free cyclic group generated by the loop produced by the $\mathbb{C}^*$-action which lifts to the double shift $\sigma \mapsto \sigma[2]$ on $\mathrm{Stab}(X,\alpha)$. Thus, in order to prove the remaining assertions, it suffices to show that $\pi : U(X,\alpha) \twoheadrightarrow P^+(X,\alpha)$ is surjective.

As there are no spherical objects in $\mathbf{Coh}(X,\alpha)$, the surface X does not, in particular, contain any (-2)-curve. Thus, Kähler cone and positive cone coincide. Therefore, any element in $P^+(X,\alpha)$ is up to the action $\mathrm{Gl}_2^+(\mathbb{R})$ of the form φ with $\varphi = \exp(B + i\omega)$ as in Remark 3.3, ii). Hence, due to Lemma 3.4 and Proposition 3.6 all the periods are in the image of $\pi : U(X,\alpha) \twoheadrightarrow P^+(X,\alpha)$. □

Remark 3.16. The result can be strengthened as follows. All stability conditions which can be deformed to a stability condition with rational period form one connected component. This component, moreover, is simply-connected and the universal cover of $P^+(X,\alpha)$.

Moreover, following [6], one shows that if $\sigma \in \mathrm{Stab}(X,\alpha)$ and $\pi(\sigma) \in \mathcal{P}^+(X,\alpha)$ then σ is in the unique component $\mathrm{Stab}^\dagger(X,\alpha)$ of maximal dimension.

3.3. **(Auto)equivalences.**

As autoequivalences of the derived category $\mathrm{D^b}(X,\alpha)$ act naturally on the space of stability conditions $\mathrm{Stab}(X,\alpha)$, the results of the previous section can be used to fully determine the group of autoequivalences $\mathrm{Aut}(\mathrm{D^b}(X,\alpha))$ for a generic twisted K3 surface (X,α). Everything is known for arbitrary (twisted) abelian surfaces, so we will concentrate on K3 surfaces here. In particular, we shall show that autoequivalences of $\mathrm{D^b}(X,\alpha)$ induce orientation preserving Hodge isometries of $\widetilde{H}(X,\alpha,\mathbb{Z})$, i.e. that the orientation of the four-space of positive directions is not changed. This had first been conjectured by Szendrői, but even for untwisted K3 surfaces it is still not known in general (see e.g. [13]).

In the following we shall tacitly use that any equivalence is of Fourier–Mukai type, which was proved by Orlov in [22] in the untwisted case and in [8] in general.

Theorem 3.17. *Suppose (X,α) is a generic twisted K3 surface. Then there exists a short exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X,\alpha)) \longrightarrow \mathrm{Aut}^+(\widetilde{H}(X,\alpha,\mathbb{Z})) \longrightarrow 0,$$

where $\mathrm{Aut}^+(\widetilde{H}(X,\alpha,\mathbb{Z}))$ denotes the group of orientation preserving Hodge isometries.

Proof. Let $\mathrm{id} \neq \Phi \in \mathrm{Aut}(\mathrm{D^b}(X,\alpha))$. Then the cohomological Fourier–Mukai transform defined in terms of the twisted Mukai vector of the Fourier–Mukai kernel of Φ yields a Hodge isometry $\Phi_* \in \mathrm{Aut}(\widetilde{H}(X,\alpha,\mathbb{Z}))$ (see [10]). Clearly, Φ_* preserves the orientation of the positive plane $(\widetilde{H}^{2,0} \oplus \widetilde{H}^{0,2})(X,\mathbb{R})$. Thus, in order to show that it preserves the orientation of all four positive directions, it suffices to prove that the orientation of the two remaining positive directions in $\mathrm{NS}(X,\alpha)$ is preserved.

If $\sigma \in \mathrm{Stab}(X,\alpha)$ is the stability condition associated to $\varphi = \exp(B+i\omega)$ as in Proposition 3.6, then the stability function of $\Phi(\sigma)$ is $Z_{\Phi_*\varphi}$. Since there is only one connected component of maximal dimension in $\mathrm{Stab}(X,\alpha)$ and π is continuous, $\pi(\sigma)$ and $\pi(\Phi(\sigma))$ are contained in the same connected component of $P(X,\alpha)$, i.e. in $P^+(X,\alpha)$. In other words, real and imaginary part of φ and of $\Phi_*(\varphi)$ yield the same orientation of the positive directions. Hence $\Phi_* \in \mathrm{Aut}^+(\widetilde{H}(X,\alpha,\mathbb{Z}))$.

If $\Phi_* = \mathrm{id}$, then for any stability condition σ the induced stability condition $\Phi(\sigma)$ has the same stability function. Thus, Φ acts as a non-trivial deck transformation of the covering map $\pi : U(X,\alpha) \twoheadrightarrow P^+(X,\alpha)$. As a generator of $\pi_1(P^+(X,\alpha)) \simeq \mathbb{Z}$ lifts to the shift $E \mapsto E[2]$ acting as a deck transformation of the simply-connected space $U(X,\alpha)$, one finds that Φ is an even shift $E \mapsto E[2n]$. $\square$

The fact that connectivity of the space of stability conditions can be used to prove that autoequivalences preserve the orientation of the positive directions in $\widetilde{H}(X,\mathbb{Z})\otimes\mathbb{R}$ was observed by Bridgeland [6, Conj. 1.2]. In the following we shall give another proof of this fact which is more direct and does not rely on the concept of a stability condition and, in particular, not on Theorem 3.15.

Proposition 3.18. *Let (X,α) and (Y,β) be generic twisted K3 surfaces. Suppose*

$$\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X,\alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y,\beta)$$

is a Fourier–Mukai equivalence with $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X\times Y, \alpha^{-1}\boxtimes\beta)$. Then there exists a twisted sheaf(!) $\mathcal{F} \in \mathbf{Coh}(X\times Y,\alpha^{-1}\boxtimes\beta)$ such that $\mathcal{E}\simeq\mathcal{F}[n]$, for some $n\in\mathbb{Z}$.

Proof. For $x \in X$ a closed point, let $\mathcal{E}_x := \mathcal{E}|_{\{x\}\times Y} = \Phi_{\mathcal{E}}(k(x))$. Since $\Phi_{\mathcal{E}}$ is an equivalence, $\mathcal{E}_x$ is a semi-rigid object in $\mathrm{D}^{\mathrm{b}}(Y,\beta)$.

By Remark 3.13, ii) there exists at most one $i\in\mathbb{Z}$ such that $\mathcal{H}^i(\mathcal{E}_x)\neq 0$. It is not difficult to see that i does not depend on the point $x\in X$. This proves that there exists a sheaf $\mathcal{F}\in\mathbf{Coh}(X\times Y,\alpha^{-1}\boxtimes\beta)$ and an integer n such that $\mathcal{E}=\mathcal{F}[n]$. $\square$

Corollary 3.19. *Suppose (X,α) and (Y,β) are generic twisted K3 surfaces.*

Then any Fourier–Mukai equivalence $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X,\alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y,\beta)$ induces an orientation preserving Hodge isometry $\Phi_{\mathcal{E}*} : \widetilde{H}(X,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\beta,\mathbb{Z})$.

Proof. Note again that in order to define the cohomological twisted Fourier–Mukai functor $\Phi_{\mathcal{E}*}$ one needs to fix B-field lifts of α and β, but we will suppress all technical details here.

By Proposition 3.18 the kernel $\mathcal{E}$ is, up to shift, a flat family of semi-rigid and hence, by Remark 3.13, ii), stable sheaves on Y parametrized by X. Now one argues as in [10, Prop. 5.5, Rem. 5.7] in order to deduce that the Hodge isometry $\Phi_{\mathcal{E}*} : \widetilde{H}(X,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y,\beta,\mathbb{Z})$ is orientation preserving. $\square$

Remark 3.20. Due to the previous proposition, there are twisted K3 surfaces (X, α) such that the Hodge isometry

$$j := \mathrm{id}_{H^0 \oplus H^4} \oplus -\mathrm{id}_{H^2} : \widetilde{H}(X, \alpha, \mathbb{Z}) \longrightarrow \widetilde{H}(X, \alpha^{-1}, \mathbb{Z})$$

is not induced by any (Fourier–Mukai) equivalence.

In [11, Thm. 0.1] we proved that any orientation preserving Hodge isometry $\widetilde{H}(X, \alpha, \mathbb{Z}) \simeq \widetilde{H}(Y, \beta, \mathbb{Z})$ between arbitrary twisted K3 surfaces can be lifted to a derived equivalence.

Thus, as a consequence of Corollary 3.19 and of [11, Thm. 0.1], we obtain the following more precise result, which is expected to hold without any genericity assumption.

Corollary 3.21. *Let (X, α) and (Y, β) be two generic twisted K3 surfaces. Then a Hodge isometry*

$$g : \widetilde{H}(X, \alpha, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(Y, \beta, \mathbb{Z})$$

can be lifted to a Fourier–Mukai equivalence

$$\mathrm{D^b}(X, \alpha) \xrightarrow{\sim} \mathrm{D^b}(Y, \beta)$$

if and only if g is orientation preserving. □

3.4. **Density of generic twisted K3 surfaces.**

Here we shall explain that generic twisted K3 surfaces in the sense of Definition 3.12 are dense in the moduli space. Unfortunately, for technical reasons this information seems difficult to use in order to pass by deformation from the generic case to the case of an arbitrary K3 surface.

Lemma 3.22. *For any K3 surface X with Picard number one, there exist infinitely many $\alpha \in \mathrm{Br}(X)$ such that (X, α) is generic, i.e.* **Coh**(X, α) *does not contain any spherical objects.*

Proof. We shall work with specific B-field lifts B of the Brauer class $\alpha = \alpha_B$ and with the Mukai vector $v^B = \mathrm{ch}^B \cdot (1, 0, 1)$. For the notation compare [10, 11].

Let $\mathrm{Pic}(X) \simeq \mathbb{Z}H$ and suppose $H^2 = 2d > 2$. Consider B-fields $B \in H^2(X, \mathbb{Q})$ such that:

(1) $(B.H) = (B.B) = 0$ and
(2) There exists a prime number p dividing d and the order of α_B.

Under these hypotheses, the quadratic form associated to $\mathrm{NS}(X, \alpha_B)$ is represented by the matrix

$$M = \begin{pmatrix} H^2 & 0 & 0 \\ 0 & 0 & mp \\ 0 & mp & 0 \end{pmatrix},$$

for some integer m. Of course, M does not represent -2, which excludes the existence of spherical objects $E \in \mathrm{D^b}(X, \alpha_B)$, for a spherical E satisfies $\langle v^B(E), v^B(E) \rangle = -2$.

If $H^2 = 2$, then consider B-fields B satisfying (1) and such that -1 is not a square modulo the order of α_B. The intersection form on $\mathrm{NS}(X, \alpha_B)$ has matrix form M and does not represent -2. □

Let $\Lambda := U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ and $\widetilde{\Lambda} := U^{\oplus 4} \oplus (-E_8)^{\oplus 2}$. The space $Q := \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) : x^2 = 0, (x, \bar{x}) > 0\}$ is the period domain of all K3 surfaces while $Q_{\mathrm{alg}} \subset Q$ is the dense subset of periods of algebraic K3 surfaces. In other words, Q_{alg} is the set of those $x \in Q$ for which there exists a class $h \in x^{\perp} \cap \Lambda$ with $h^2 > 0$. According to [9], the space $\widetilde{Q} := \exp(\Lambda \otimes \mathbb{R})(Q_{\mathrm{alg}}) \subset \mathbb{P}(\widetilde{\Lambda} \otimes \mathbb{C})$ is the period domain of the moduli space of generalized Calabi–Yau structures on algebraic K3 surfaces obtained by (real) B-field shifts. Using Lemma 3.22 it is very easy to derive the following result.

Corollary 3.23. *The periods of generic twisted K3 surfaces (X, α) are dense in $\widetilde{Q}$.* □

More precisely, the corollary says that the periods of K3 surfaces X shifted by B-fields B inducing a generic twisted K3 surface (X, α_B) form a dense subset of $\widetilde{Q}$.

4. Generic non-algebraic K3 surfaces

In this section we shall deal with generic non-algebraic K3 surfaces. The abelian category $\mathbf{Coh}(X)$ and hence its derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a non-algebraic K3 surface is smaller than in the projective case and contrary to Gabriel's classical result for algebraic varieties $\mathbf{Coh}(X)$ usually does not determine X (see [27]).

However, although the generic non-algebraic K3 surface X has trivial Picard group, there always is the trivial line bundle $\mathcal{O}_X$ which gives rise to a spherical object in $\mathbf{Coh}(X)$. In this sense, the generic non-algebraic case is 'less generic' than the generic twisted projective case discussed in the previous section. This makes a study of the generic non-algebraic situation interesting; it leads to K3 categories with exactly one spherical object (up to shift) and the method of Section 2.3 will be applicable.

4.1. Spherical objects on generic K3 surfaces.

We will henceforth call a K3 surface X *generic* if its Picard group is trivial. Clearly, with this definition a generic K3 surface is never algebraic and $H^{1,1}(X, \mathbb{Z}) = 0$. The latter is equivalent to $\mathcal{N}(X) \simeq (H^0 \oplus H^4)(X, \mathbb{Z})$.

Notice that, in principle, when dealing with smooth compact analytic (non-projective) varieties X, we should work with the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X\text{-}\mathbf{Mod})$ of $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_X\text{-}\mathbf{Mod})$ consisting of complexes of $\mathcal{O}_X$-modules with coherent cohomology. Indeed due to the general results in [25] the usual derived functors (e.g. push-forwards, tensor products and pull-backs) are well-defined in this category and it makes perfect sense to introduce the notion of Fourier-Mukai functors. Moreover Serre duality holds true in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X\text{-}\mathbf{Mod})$ (see, for example, [3, Prop. 5.1.1]). If X is now a smooth compact analytic surface, then the natural functor $\mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathcal{O}_X\text{-}\mathbf{Mod})$ induces an equivalence $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X\text{-}\mathbf{Mod})$ (see for example [3, Prop. 5.2.1]) and any $E \in \mathbf{Coh}(X)$ admits a locally free resolution of finite length (see [23]). Since for the rest of this paper we will just consider generic K3 surfaces, we will continue to work with the triangulated category $\mathrm{D}^{\mathrm{b}}(X)$.

Lemma 4.1. *If X is a generic K3 surface, then the trivial line bundle $\mathcal{O}_X$ is up to shift the only spherical object in $\mathrm{D}^{\mathrm{b}}(X)$.*

Proof. Suppose $E \in \mathbf{Coh}(X)$ is a rigid sheaf and let $v(E) = (r, 0, s)$ be its Mukai vector. Then $2 \leq \chi(E, E) = 2rs$ and since $r \geq 0$, one finds $r, s > 0$. In particular, E cannot be torsion.

Next, we prove that a rigid E is torsion free. To this end, consider the short exact sequence $0 \to E_{\rm tor} \to E \to E' \to 0$ with E' torsion free and $E_{\rm tor}$ the torsion subsheaf of E. In particular, $\mathrm{Hom}(E_{\rm tor}, E') = 0$ and [18, Cor. 2.8] applies (the abelian analogue of Lemma 2.7). Thus, $E_{\rm tor}$ is rigid as well, which contradicts the above observation.

Again following Mukai, one proves that a rigid E is in fact locally free. Indeed, if not, then the surjection $E^{\vee\vee} \to E^{\vee\vee}/E$ from the locally free reflexive hull $E^{\vee\vee}$ to the torsion sheaf $E^{\vee\vee}/E \neq 0$ concentrated in dimension zero can be deformed providing non-trivial deformations of E which is excluded by rigidity.

We shall prove that any rigid sheaf is isomorphic to $\mathcal{O}_X^{\oplus r}$ for some r. This will be done by induction on the rank. The case $r = 1$ is obvious.

Observe that $\chi(\mathcal{O}_X, E) = r + s > 0$. Hence, $\mathrm{Hom}(\mathcal{O}_X, E) \neq 0$ or $\mathrm{Hom}(E, \mathcal{O}_X) \neq 0$, but it suffices to consider the first case. Indeed if $\mathrm{Hom}(E, \mathcal{O}_X) \neq 0$, then $\mathrm{Hom}(\mathcal{O}_X, E^\vee) \neq 0$ and $E^\vee$ is still rigid. Now suppose $\mathcal{O}_X \to E$ is non-trivial and hence injective, for E is torsion free. By induction we may assume that we have a short exact sequence $0 \to \mathcal{O}_X^{\oplus r} \to E \to F \to 0$ with $r > 0$. We shall show that $\mathrm{Hom}(\mathcal{O}_X, F) = 0$ if r is chosen maximal.

We first prove that F is torsion free as well. Consider the natural short exact sequence $0 \to \mathcal{O}_X^{\oplus r} \to G \to F_{\rm tor} \to 0$, where G is the kernel of the surjection $E \twoheadrightarrow F \twoheadrightarrow F/F_{\rm tor}$. Since $\mathrm{Ext}^1(F_{\rm tor}, \mathcal{O}_X) = 0$ and E is torsion free, this yields $F_{\rm tor} = 0$. Suppose now that $\mathrm{Hom}(\mathcal{O}_X, F) \neq 0$. Using $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = 0$, any $\mathcal{O}_X \to F$ can be lifted to a morphism $\mathcal{O}_X \to E$. Since F is torsion free, one thus obtains an injection $\mathcal{O}_X^{\oplus r+1} \hookrightarrow E$. Clearly, this process will terminate.

If $\mathrm{Hom}(\mathcal{O}_X, F) = 0$, then we are once more in the situation of [18, Cor. 2.8]. Hence F is rigid of smaller rank and thus $F \simeq \mathcal{O}_X^{\oplus r'}$ and therefore $E \simeq \mathcal{O}_X^{\oplus r+r'}$.

Proposition 2.14 then concludes the proof. $\square$

4.2. Construction of stability conditions on generic K3 surfaces.

Instead of adapting Bridgeland's discussion to the non-projective case, we shall give an ad hoc approach that works well for generic K3 surfaces. So, in the following, X will denote a K3 surface with trivial Picard group. Throughout we will use that torsion sheaves on such a K3 surface are concentrated in points and that the reflexive hull of an arbitrary torsion free sheaf is a μ-semistable vector bundle with trivial determinant.

The first step consists of an explicit construction of certain stability conditions. This immitates arguments in [6] but differs at a few places. In particular, the actual check of all the conditions is more direct and in some cases also the heart has a different shape.

Consider the open subset

$$R := \mathbb{C} \setminus \mathbb{R}_{\geq -1} = R_+ \cup R_- \cup R_0, \tag{4.1}$$

where

$$R_+ := R \cap \mathbb{H},\ R_- := R \cap (-\mathbb{H}),\ \text{and}\ R_0 := R \cap \mathbb{R}$$

with $\mathbb{H}$ denoting the upper half-plane.

For any $z = u + iv \in R$ we consider a torsion theory

$$\mathcal{F}(z), \mathcal{T}(z) \subset \mathbf{Coh}(X)$$

which is defined uniformly as follows: We let $\mathcal{F}(z)$ be the full subcategory of all torsion free sheaves of degree $\leq v$, whereas $\mathcal{T}(z)$ is the full subcategory that contains all torsion sheaves and all torsion free sheaves of degree $> v$. The degree is taken with respect to any Kähler structure, but this does not matter at all. In fact, if $z \in R_+ \cup R_0$ then $\mathcal{F}(z)$ and $\mathcal{T}(z)$ are simply the full subcategories of all torsion free sheaves respectively torsion sheaves and if $z \in R_-$ then $\mathcal{F}(z)$ is trivial and $\mathcal{T}(z) = \mathbf{Coh}(X)$.

Consequently, the tilt $\mathcal{A}(z)$ of $\mathbf{Coh}(X)$ with respect to the torsion theory $(\mathcal{F}(z), \mathcal{T}(z))$ yields the abelian category (the heart of the corresponding t-structure):

$$\mathcal{A} := \mathcal{A}(z) = \left\{ E \mid \mathcal{H}^{-1}(E) \text{ torsion free, } \mathcal{H}^0(E) \text{ torsion}, \mathcal{H}^i(E) = 0 \text{ for } i \neq 0, -1 \right\}$$

for $z \in R_+ \cup R_0$ and

$$\mathcal{A}(z) = \mathbf{Coh}(X)$$

for $z \in R_-$.

For every $z = u + iv \in R$ one defines the function

$$Z : \mathcal{A}(z) \longrightarrow \mathbb{C} \ , \quad E \longmapsto \langle v(E), (1, 0, z) \rangle = -u \cdot r - s - i(r \cdot v).$$

Here, as usual $v(E) = (r, 0, s = r - \mathrm{c}_2(E))$.

Lemma 4.2. *For any $z \in R$ the function Z defines a stability function on $\mathcal{A}(z)$.*

Proof. We have to show that for $0 \neq E \in \mathcal{A}(z)$ one has $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$.

For $z \in R_+$ one has $\mathrm{Im}(Z(E[1])) > 0$ for any non-trivial torsion free sheaf E. If E is a torsion sheaf, i.e. E is a finite length sheaf concentrated in dimension zero, then $Z(E) = \mathrm{c}_2(E) = -\chi(E) < 0$.

For $z \in R_0$ torsion sheaves can be dealt with in the same way. For a torsion free sheaf E, one has this time $\mathrm{Im}(Z(E)) = 0$. But the real part of $Z(E)$ is $-u \cdot r - s = -(u+1) \cdot r + \mathrm{c}_2(E)$. Using the additivity of the Mukai vector for short exact sequences one can easily reduce to the case that neither E nor its dual contains $\mathcal{O}_X$. At that point one applies $0 \geq \chi(E) = -\mathrm{c}_2(E) + 2\mathrm{rk}(E)$, which yields the even stronger inequality $\mathrm{c}_2(E) \geq 2\mathrm{rk}(E)$.

If $z \in R_-$, then $\mathrm{Im}(Z(E)) = -r \cdot v > 0$ for any non-torsion sheaf E and for torsion sheaves one concludes as before. $\square$

Remark 4.3. Let us next pass to the classification of minimal and stable objects in $\mathcal{A}(z)$.

i) For $z \in R_0$ every object in $\mathcal{A}(z) = \mathcal{A}$ is semistable. For $z \in R_-$ an object $E \in \mathcal{A}(z) = \mathbf{Coh}(X)$ is semistable if E is torsion or torsion free. Finally, for $z \in R_+$ torsion sheaves and shifted vector bundles $E[1]$ define semistable objects in $\mathcal{A}(z) = \mathcal{A}$. There are, however, other semistable objects. As an example one can consider for any closed point $x \in X$ the unique non-trivial extension $0 \to \mathcal{O}_X[1] \to F \to k(x) \to 0$.

ii) It is well-known that the minimal objects of $\mathbf{Coh}(X)$ are the point sheaves $k(x)$. This covers the case $z \in R_-$.

For $z = u + iv \in R_+ \cup R_0$, i.e. for $v \geq 0$, the minimal objects of the category are classified as follows (see e.g. the discussion in [12]): They consist of the point sheaves $k(x)$ and the shifted

μ-stable vector bundles $E[1]$. Note that a torsion free sheaf is μ-stable if and only if it has no proper subsheaf. (The degree of all sheaves is trivial!)

Stable objects and minimal objects coincide for $z \in R_0$, but this does not hold for $z \in R_\pm$.

iii) Later in Lemma 4.9 it will be important that for $z \in R_0$ the only stable semi-rigid objects are the point sheaves $k(x)$. Here one uses that on a generic K3 surface any stable vector bundle E of rank > 1 has $(E, E)^1 \geq 4$.

The same assertion does not hold for $z \in R_\pm$. Indeed, for $z \in R_+$ the objects given by a non-split exact sequence $0 \longrightarrow \mathcal{O}_X[1] \longrightarrow F \longrightarrow k(x) \longrightarrow 0$ are semi-rigid and stable in $\mathcal{A}$ and the ideal sheaves $I_x \in \mathbf{Coh}(X)$ of points $x \in X$ are semi-rigid and stable in $\mathbf{Coh}(X)$.

Proposition 4.4. *For $z \in R$ the stability function $Z : \mathcal{A}(z) \longrightarrow \mathbb{C}$ has the Harder–Narasimhan property. Moreover, the induced stability condition is locally finite and in particular all Jordan–Hölder filtrations are finite.*

Proof. For $z \in R_-$, this follows from the existence of the usual Harder–Narasimhan filtration in $\mathcal{A}(z) = \mathbf{Coh}(X)$. In fact, as $\mathrm{Pic}(X) = 0$ the Harder–Narasimhan filtration consists of the torsion part and the rest. Furthermore, due to the special form of the stability function one can easily verify that the stability condition is locally finite.

For $z \in R_+ \cup R_0$ one easily proves that (independently of any stability condition) the abelian category $\mathcal{A} = \mathcal{A}(z)$ is of finite length, i.e. any object admits a finite filtration whose quotients are minimal objects of the category. This follows from the description of the minimal objects given above. Consequently, any filtration of an object in $\mathcal{A}$ becomes stationary. This applies in particular to Harder–Narasimhan and Jordan–Hölder filtrations. □

So, we can conclude that for any $z \in R$ we have defined a locally finite stability condition

$$\sigma_z \in \mathrm{Stab}(\mathrm{D^b}(X))$$

given by the t-structure associated to the torsion theory $(\mathcal{F}(z), \mathcal{T}(z))$ and the stability function Z on its heart.

This allows us to consider R in a natural way as a subset of the space of stability conditions:

$$R \subset \mathrm{Stab}(X).$$

For any $z \in R$ the point sheaves $k(x)$ are σ_z-stable of phase one with $Z(k(x)) = -1$ and if, moreover, $z \in R_0$, then they are the only semi-rigid stable objects (up to shift).

Remark 4.5. Note that due to [6], $\mathbf{Coh}(X)$ never occurs as the heart of a stability condition on a projective K3 surface. Indeed, if $\mathbf{Coh}(X) = \mathcal{P}((0,1])$, then the point sheaves $k(x)$, which are minimal objects in $\mathbf{Coh}(X)$, would all be stable of the same maximal phase $\phi \in (0,1]$. Then one easily shows $\phi(k(x)) = 1$ and [6, Prop. 10.3] yields $\mathcal{P}((0,1]) = \mathcal{A}(\varphi)$ for some $\varphi = \exp(B + i\omega)$. Clearly, if X is projective, then $\mathcal{A}(\varphi) \neq \mathbf{Coh}(X)$.

4.3. **The space of all stability conditions on a generic K3 surface.**

In the next step, again following Bridgeland, one tries to classify all stability conditions $\sigma = (Z, \mathcal{P})$ for which all point sheaves $k(x)$ are stable of phase one. Up to scaling we may assume that $Z(k(x)) = -1$. Hence the stability function Z is of the form considered above: $Z(E) = -u \cdot r - s - i(v \cdot r)$.

By Proposition 2.15 the trivial line bundle $\mathcal{O}_X$, which is up to shift the only spherical object in $\mathrm{D^b}(X)$, is stable with respect to σ. Since $(\mathcal{O}_X, k(x))^0 \neq 0 \neq (k(x), \mathcal{O}_X)^2$, one has $\phi(\mathcal{O}_X) < 1 < \phi(\mathcal{O}_X) + 2$. Hence, either $\mathcal{O}_X \in \mathcal{P}((0,1])$ or $\mathcal{O}_X[1] \in \mathcal{P}((0,1])$ depending on whether $v < 0$ or $v \geq 0$. If $v = 0$, then necessarily $u < -1$. Note that so far we have only used that there is one point sheaf $k(x)$ which is σ-stable of phase one.

Now one copies the proof of Lemma 10.1 and Proposition 10.3 in [6]. In fact the arguments simplify, as $\mathrm{Pic}(X) = 0$. In particular, the proof of [6, Prop. 10.3] does not only show equality of the hearts, but right away equality of the stability conditions.

Proposition 4.6. *Suppose $\sigma = (Z, \mathcal{P})$ is a stability condition on $\mathrm{D^b}(X)$ for a K3 surface X with trivial Picard group. If all point sheaves $k(x)$ are stable of phase one with $Z(k(x)) = -1$, then $\sigma = \sigma_z$ as above for some $z \in R$.* $\square$

For later use we point out that the heart $\mathcal{P}((0,1])$ of σ is $\mathcal{A}$ if $\mathrm{Im}(Z(\mathcal{O}_X)) \leq 0$ and $\mathbf{Coh}(X)$ if $\mathrm{Im}(Z(\mathcal{O})) > 0$.

Combining Proposition 4.6 with Corollary 2.19 we get a complete classification of all stability conditions on $\mathrm{D^b}(X)$ for a generic K3 surface X. As before, T denotes the spherical shift $T_{\mathcal{O}_X}$.

Corollary 4.7. *Suppose σ is a stability condition on $\mathrm{D^b}(X)$ for a K3 surface X with trivial Picard group. Then up to the action of $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ the stability condition σ is of the form $T^n(\sigma_z)$ for some $z \in R$ and $n \in \mathbb{Z}$.* $\square$

Theorem 4.8. *If X is a K3 surface with trivial Picard group, then the space $\mathrm{Stab}(X)$ of stability conditions on $\mathrm{D^b}(X)$ is connected and simply-connected.*

Proof. Let us consider

$$W(X) := \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})(R) \subset \mathrm{Stab}(X),$$

which can also be written as the union $W(X) = W_+ \cup W_- \cup W_0$ according to (4.1). Then Corollary 4.7 says

$$\mathrm{Stab}(X) = \bigcup_n T^n W(X). \tag{4.2}$$

Recall that the group $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ can be thought of as the set of pairs $\tilde{g} = (g, f)$ of a linear map $g \in \mathrm{Gl}_2^+(\mathbb{R})$ and an increasing map $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(\phi + 1) = f(\phi) + 1$ inducing the same map on $\mathrm{S}^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2/\mathbb{R}_{>0}$. Using the natural action of $\mathrm{Gl}_2^+(\mathbb{R})$ on $\mathbb{C}$, one defines the action of an element $\tilde{g}$ on $\mathrm{Stab}(X)$ as follows: If $\sigma = (Z, \mathcal{P})$, then $\tilde{g}(\sigma) = (Z', \mathcal{P}')$ satisfies $Z'(E) = g^{-1}Z(E)$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$. The action is clearly continuous. See [5, Lemma 8.2].

i) Suppose that for some $\sigma_z \in R$ and $\tilde{g} \in \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ one has $\tilde{g}(\sigma_z) \in R$. Then $g^{-1} = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ with $b > 0$. Thus, if $\sigma_{z'} = \tilde{g}(\sigma_z)$, then $z' = u + av + ibv$. In particular, the sign of the imaginary part does not change. Hence, $W(X) = W_+ \cup W_- \cup W_0$ is disjoint.

The calculation also shows that W_+ and W_- are contained in two different orbits of the $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$-action, whereas two points in R_0 are contained in two different orbits. More precisely, if $z \in R_\pm$, then the continuous surjection $\{g \in \mathrm{Gl}_2^+(\mathbb{R}) \mid g^{-1} = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}\} \twoheadrightarrow R_\pm$, $g \longmapsto g^{-1}z$ also describes the inclusion $R_\pm \subset \mathrm{Stab}(X)$ by lifting g to $\tilde{g} = (g, f)$ such that $f((0,1]) = (0,1]$ and mapping σ_z to $\tilde{g}(\sigma_z) = \sigma_{g^{-1}z}$.

ii) We claim that $W(X) \subset \mathrm{Stab}(X)$ is an open connected subset.

Let us first prove that the inclusion $R \subset \mathrm{Stab}(X)$ is continuous. The arguments in i) show that the restrictions $R_\pm \longrightarrow \mathrm{Stab}(X)$ are continuous. Thus, it suffices to prove that the inclusion is continuous in points $z \in R_0$.

For this consider a σ_z-stable object $0 \neq F$ of phase ϕ. Hence $\phi = \ell$ for some $\ell \in \mathbb{Z}$ and $F \simeq k(x)[\ell - 1]$ with $x \in X$ or $F = E[\ell]$ with E stable and locally free (see Remark 4.3, ii)). Obviously both types of objects stay $\sigma_{z'}$-semistable for $z' \in R_\pm$ close to z with phase ϕ' close to $\phi = \ell$. The important point to notice here is that how close ϕ' is to ϕ only depends on z' and not on F. This is clear for $F \simeq k(x)[\ell - 1]$, whose phase stays ℓ. For $F \simeq E[\ell]$ this is a priori not so clear. In order to see this, one has to check that $vr/(ru + s)$ is uniformly small (independent of $r = \mathrm{rk}(E)$ and $s = \mathrm{rk}(E) - \mathrm{c}_2(E)$) for $|u| < \delta \ll 0$. This follows e.g. from $\mathrm{c}_2(E) \geq 2\mathrm{rk}(E)$ (see the proof of Lemma 4.2), as $vr/(ru + s) = v/(u + 1 - \mathrm{c}_2(E)/\mathrm{rk}(E))$.

Thus, R and hence $W(X) = \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})(R)$ are connected subsets of $\mathrm{Stab}(X)$. Moreover, $W(X) \longrightarrow \mathcal{N}(X)^* \otimes \mathbb{C} \simeq \mathbb{C}^2$ is a local homeomorphism. Since the same is true for the period map $\mathrm{Stab}(X) \longrightarrow \mathcal{N}(X)^* \otimes \mathbb{C}$ (see [5, Thm. 1.2]), this suffices to conclude the openness of $W(X)$ inside $\mathrm{Stab}(X)$ from the openness of its image in $\mathbb{C}^2$.

iii) Due to Remark 2.17, ii) one knows that $T^nW(X)$ and $T^kW(X)$ are disjoint for $|n-k| \geq 2$. We now claim that

$$T^nW(X) \cap T^{n+1}W(X) = T^nW_-. \tag{4.3}$$

Of course, it is enough to prove $W(X) \cap TW(X) = W_-$. More precisely, we shall show

$$TW_+ = W_- \text{ and } T(W_0 \cup W_-) \cap W(X) = \emptyset.$$

Let us consider $\sigma := \sigma_i = (Z, \mathcal{P}) \in R_+$ and $\sigma' := \sigma_{-i} = (Z', \mathcal{P}') \in R_-$. We shall show that $T\sigma = \tilde{i} \cdot \sigma'$, where $\tilde{i} \in \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ acts by $\tilde{i}(Z')(E) = -iZ'(E)$ and $(\tilde{i}\mathcal{P}')(\phi) = \mathcal{P}'(\phi + 1/2)$.

The verification of $T(Z)(E) = -iZ'(E)$ can be done easily by using that the spherical twist acts on cohomology by $(r, 0, s) \longmapsto -(s, 0, r)$. For the second equality it will be enough to show that $T\mathcal{P}((0,1]) = (\tilde{i}\mathcal{P}')((0,1]) = \mathcal{P}'((1/2, 3/2])$. Clearly, two stability conditions with identical hearts and stability functions coincide.

By construction $\mathcal{A} = \mathcal{P}((0,1])$ and $\mathbf{Coh}(X) = \mathcal{P}'((0,1])$. Moreover, for any closed point $x \in X$ one has $T(k(x)) = I_x[1] \in \mathcal{P}'(1/2)[1] = \mathcal{P}'(3/2)$. Thus, the heart $T^{-1}\mathcal{P}'((1/2, 3/2])$ of the stability condition $T^{-1}(\tilde{i}\sigma')$ contains all point sheaves $k(x)$ which, moreover, have central charge -1. By Proposition 4.6 this is enough to conclude $T^{-1}\mathcal{P}'((1/2,3/2]) = \mathcal{A}$. (A priori one could have $T^{-1}\mathcal{P}'((1/2,3/2]) = \mathbf{Coh}(X)$, but this is impossible as $T\mathcal{O} = \mathcal{O}[-1] \in \mathcal{P}'(-1/4) \not\subset \mathcal{P}'((1/2,3/2])$.)

Next, $TW_- = T^2W_+$, but due to Remark 2.17, ii) one has $T^2W_+ \cap W(X) = \emptyset$. Eventually, $TW_0 \cap W(X) = \emptyset$, because the stability function of $T\sigma_z$ with $z = u < -1$ is given by $(r, 0, s) \longmapsto u \cdot s + r$, which is not of the type realized by any stability condition in $W(X)$.

iv) As an immediate consequence of (4.3) in iii) and the connectedness of $W(X)$ proved in ii), one concludes that $\mathrm{Stab}(X) = \bigcup T^n W(X)$ is connected.

v) In the final step one applies the van Kampen theorem to the open cover (4.2). The intersections $T^nW(X) \cap T^kW(X) \subset \mathrm{Stab}(X)$ are either empty for $|n-k| \geq 2$ or homeomorphic to the connected W_-. Thus, it suffices to verify that the open sets $T^nW(X) \simeq W(X)$ are simply-connected.

For this purpose let us consider its image in $\mathbb{C}^2$. Clearly, $\mathrm{Gl}_2^+(\mathbb{R})(R_+ \cup R_-) \subset \mathbb{C}^2$ is the set of all $\mathbb{R}$-linearly independent pairs $(z_0, z_1) \in \mathbb{C}^2$ and $\mathrm{Gl}_2^+(\mathbb{R})(R_0)$ consists of pairs (z_0, uz_0) with $z \in \mathbb{C}^*$ and $u \in (-\infty, -1)$. Thus, $\mathrm{Gl}_2^+(\mathbb{R})(R)$ is the complement of $\{(z, uz) \mid z \in \mathbb{C}, u \in [-1, \infty)\} \bigcup \{(0, z) \mid z \in \mathbb{C}\}$, whose fundamental group is generated by the loop around the real codimension-one component $\{(0, z)\}$. This loop can be written as the rotation g_t by πt, $t \in [0, 2]$ which lifts to $\tilde{g}_t = (g_t, f_t) \in \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})(R)$ with $f_t(\phi) = \phi + t$. In particular, $\widetilde{g}_2$ is the double shift acting as a deck transformation on $W(X)$. Thus, $W(X)$ is simply-connected. $\square$

Some of the arguments in the above proof are inspired by a very detailed description of all stability conditions on a generic K3 surface due to S. Meinhardt [17]. In analogy to [6], he introduces a period domain and presents $\mathrm{Stab}(X)$ via a period map as the universal cover of it. The main difference compared to [6] is that the naive definition of the period domains $P(X)$ or $P^+(X)$ makes no sense, as there are no positive planes in $\mathcal{N}(X) \otimes \mathbb{R}$.

4.4. **Autoequivalences.**

A complete description of the group of autoequivalences of $\mathrm{D^b}(X)$ in the generic case can now either be obtained by following the general approach in [6] or by a more direct argument given below, which does not rely on the description of $\mathrm{Stab}(X)$ in Theorem 4.8.

Lemma 4.9. *If $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$ is a Fourier–Mukai equivalence between two K3 surfaces X and Y with $\mathrm{Pic}(X) = 0$, then up to shift*

$$\Phi_{\mathcal{E}} \simeq T^n \circ f_*$$

for some $n \in \mathbb{Z}$. Here, T is the spherical twist with respect to $\mathcal{O}_Y$ and $f : X \xrightarrow{\sim} Y$ is an isomorphism.

Proof. Suppose $\sigma = \sigma_{(u,v=0)}$ is one of the distinguished stability conditions constructed above with $(u, v) \in R$ and $v = 0$. Let $\tilde{\sigma}$ be its image under $\Phi_{\mathcal{E}}$.

Then by Proposition 2.18 and Corollary 2.19 there exists an integer n such that all point sheaves $k(x)$ are stable of the same phase with respect to $T^n(\tilde{\sigma})$. Thus the composition $\Psi := T^n \circ \Phi_{\mathcal{E}}$, which is again of Fourier–Mukai type, i.e. $\Psi = \Psi_{\mathcal{F}}$ for some $\mathcal{F} \in \mathrm{D^b}(X \times Y)$, sends the stability condition σ to a stability condition σ' for which all point sheaves are stable of the same phase. Shifting the kernel $\mathcal{F}$ allows one to assume that $\phi_{\sigma'}(k(y)) \in (0,1]$ for all $y \in Y$. Thus, the heart $\mathcal{P}((0,1])$ of σ', which under $\Psi_{\mathcal{F}}$ is identified with $\mathcal{A}(u)$, contains as stable objects the images $\Psi(k(x))$ of all points $x \in X$ and all point sheaves $k(y)$.

But as was remarked earlier (see Remark 4.3, iii), the only semi-rigid stable objects in $\mathcal{A}(u)$ are the point sheaves. Thus, $\Psi^{-1}(k(y))$ must be of the form $k(x)$. In other words, for all $y \in Y$ there exists a point $x \in X$ such that $\Psi(k(x)) \simeq k(y)$.

This suffices to conclude that the Fourier–Mukai equivalence $\Psi_{\mathcal{F}}$ is a composition of f_* for some isomorphism $f : X \xrightarrow{\sim} Y$ and a line bundle twist (cf. [13, Cor. 5.23]), but there are no non-trivial line bundles on Y. $\square$

This immediately leads to the following complete description of all Fourier–Mukai equivalences in the generic case.

Proposition 4.10. *If* $\mathrm{Aut}(\mathrm{D^b}(X))$ *denotes the group of autoequivalences of Fourier–Mukai type of a K3 surface* X *with* $\mathrm{Pic}(X) = 0$, *then*

$$\mathrm{Aut}(\mathrm{D^b}(X)) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Aut}(X).$$

The first two factors are generated respectively by the shift functor and the spherical twist $T_{\mathcal{O}_X}$. $\square$

Maybe it is worth pointing out that in the non-algebraic case not every derived equivalence is of Fourier–Mukai type (see [27]). Also, the automorphism group of a K3 surface with trivial Picard group can be explicitly described. Is either trivial or isomorphic to $\mathbb{Z}$. See [19, Cor. 1.6].

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