STABILITY CONDITIONS IN FAMILIES

AREND BAYER, MARTÍ LAHOZ, EMANUELE MACRÌ, HOWARD NUER,
ALEXANDER PERRY, PAOLO STELLARI

ABSTRACT. We develop a theory of Bridgeland stability conditions and moduli spaces of semistable objects for a family of varieties. Our approach is based on and generalizes previous work by Abramovich–Polishchuk, Kuznetsov, Lieblich, and Piyaratne–Toda. Our notion includes openness of stability, semistable reduction, a support property uniformly across the family, and boundedness of semistable objects. We show that such a structure exists whenever stability conditions are known to exist on the fibers.

Our main application is the generalization of Mukai's theory for moduli spaces of semistable sheaves on K3 surfaces to moduli spaces of Bridgeland semistable objects in the Kuznetsov component associated to a cubic fourfold. This leads to the extension of theorems by Addington–Thomas and Huybrechts on the derived category of special cubic fourfolds, to a new proof of the integral Hodge conjecture, and to the construction of an infinite series of unirational locally complete families of polarized hyperkähler manifolds of K3 type.

Other applications include the deformation-invariance of Donaldson–Thomas invariants counting Bridgeland stable objects on Calabi–Yau threefolds, and a method for constructing stability conditions on threefolds via degeneration.

CONTENTS

1. Introduction 2
2. Setup, notation, and some terminology 8

Part I. Semiorthogonal decompositions and t-structures in families 11
3. Semiorthogonal decompositions and base change 11
4. Local t-structures 21
5. Base change of local t-structures 26

2010 *Mathematics Subject Classification.* 14A22, 14D20, 14F05, 14J28, 14M20, 14N35, 18E30.

Key words and phrases. Bridgeland stability conditions, base change for semiorthogonal decompositions, base change for t-structures, relative moduli spaces, non-commutative varieties, K3 surfaces, cubic fourfolds.

A. B. was supported by the ERC Starting Grant ERC-2013-StG-337039-WallXBirGeom, by the ERC Consolidator Grant ERC-2018-CoG-819864-WallCrossAG, and by the NSF Grant DMS-1440140 while the author was in residence at the MSRI in Berkeley, during the Spring 2019. M. L. was supported by a Ramón y Cajal fellowship and partially by the Spanish MINECO research project PID2019-104047GB-I00. E. M. was partially supported by the NSF grant DMS-1700751, by a Poincaré Chair from the Institut Henri Poincaré and the Clay Mathematics Institute, by the Institut des Hautes Études Scientifiques (IHÉS), by a Poste Rouge CNRS at Université Paris-Sud, by the ERC Synergy Grant ERC-2020-SyG-854361-HyperK, and by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA-INdAM). H. N. was partially supported by the NSF postdoctoral fellowship DMS-1606283, by the NSF RTG grant DMS-1246844, and by the NSF FRG grant DMS-1664215. A. P. was partially supported by the NSF postdoctoral fellowship DMS-1606460 and the NSF grant DMS-2002709. P. S. was partially supported by the ERC Consolidator Grant ERC-2017-CoG-771507-StabCondEn, by the research project PRIN 2017 "Moduli and Lie Theory", and by research project FARE 2018 HighCaSt (grant number R18YA3ESPJ).

6. Flat, torsion, and torsion free objects 34
7. Inducing local t-structures on semiorthogonal components 39

Part II. Moduli spaces 43
8. Moduli of complexes 43
9. Moduli of objects in a subcategory 45
10. Fiberwise t-structures 48
11. Quot spaces 51

Part III. Harder–Narasimhan structures over a curve 59
12. Stability conditions and base change 59
13. Harder–Narasimhan structures over a curve 67
14. Weak stability conditions, tilting, and base change 71
15. Weak Harder–Narasimhan structures over a curve 75
16. Semistable reduction 80
17. Torsion theories and Harder–Narasimhan structures 84
18. Harder–Narasimhan structures via stability conditions on fibers 87
19. Tilting weak Harder–Narasimhan structures over a curve 90

Part IV. Stability conditions over a higher-dimensional base 93
20. Flat families of fiberwise (weak) stability conditions 93
21. Stability conditions over S 99
22. Deforming stability conditions over S 106
23. Inducing stability conditions over S 108

Part V. Construction 111
24. Main construction statements 111
25. Tilting slope-stability in families 112
26. Tilting tilt-stability in families of threefolds 115
27. Stability conditions from degeneration 117
28. Donaldson–Thomas invariants 117

Part VI. Moduli spaces for Kuznetsov components of cubic fourfolds 118
29. Main applications to cubic fourfolds 118
30. Stability conditions on families of Kuznetsov components 121
31. Generalized Mukai's theorem 126
32. Specializing Kuznetsov components to twisted K3 surfaces 129
33. Proofs of the main results 132
References 137
Index of Notations 143

1. Introduction

Stability conditions on triangulated categories were introduced by Bridgeland in [Bri07] and further studied by Kontsevich and Soibelman in [KS08]; originally based on work by

Douglas [Dou02] in string theory, they have found many applications to algebraic geometry via wall-crossing. In this paper we develop the corresponding theory for derived categories in a family of varieties over a base. Our definition is guided by the requirement that it should come with a notion of *relative moduli spaces*, but also be flexible enough to allow deformations.

Stability conditions over a base. Let $X \to S$ be a flat family of projective varieties over some base scheme S. Consider a collection of stability conditions σ_s on the derived categories $\mathrm{D}^{\mathrm{b}}(X_s)$ of the fibers. When does this form a well-behaved family $\underline{\sigma}$ of stability conditions? Our proposed answer is the main content of this article. It can be paraphrased as follows; see Definitions 20.5 and 21.15 for the precise formulation.

Definition 1.1. A collection of numerical stability conditions

$$\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s \in S}$$

on the fibers is a *stability condition on* $\mathrm{D}^{\mathrm{b}}(X)$ *over* S if it satisfies the following conditions:

(1) The central charge is locally constant in families of objects.
(2) Geometric stability is open in families of objects.
(3) After base change $C \to S$ to any Dedekind scheme C, the stability conditions σ_c on the fibers over C are induced by a *HN structure* on $\mathrm{D}^{\mathrm{b}}(X_C)$.
(4) Each stability condition σ_s satisfies the support property, an inequality on classes of semistable objects, in a form that is uniform across all fibers.
(5) The set of semistable objects in the fibers of $X \to S$ satisfies a boundedness condition.

Our precise setup is quite general, see Main Setup; for instance S could be defined over a non-algebraically closed field, or it could be a scheme of mixed characteristic. In the definition, we consider *all* points of S, closed or non-closed. We work with numerical stability conditions, as they allow base change under field extensions, see Theorem 12.17. This gives a stability condition σ_t on X_t for any point t over S.

For a base change $T \to S$, the notion of a family of objects over T appearing in (1) and (2) is formalized by Lieblich's notion of a relatively perfect object $E \in \mathrm{D}^{\mathrm{b}}(X_T)$ [Lie06a]; the fibers E_t of such an object lie in $\mathrm{D}^{\mathrm{b}}(X_t)$.

Condition (2) is the glue tying together the stability conditions on different fibers; for example, we will see in Proposition 20.11 that in the case where X is a product $X_0 \times S$, it forces the collection of stability conditions on X_0 parametrized by $s \in S$ to be constant.

The concept of a Harder–Narasimhan (HN) structure in (3) will be introduced in Section 13 of this paper (see Definition 13.3 for a precise definition) and studied for the remainder of Part III. The condition (3) requires that the hearts $\mathcal{A}_c$ of the stability conditions σ_c "integrate" to a global heart $\mathcal{A}_C$; in the product case, such an $\mathcal{A}_C$ has been constructed by Abramovich–Polishchuk [AP06, Pol07]. Moreover, every object in $\mathrm{D}^{\mathrm{b}}(X_C)$ is required to have a global HN filtration; this combines the classical notion of generic HN filtrations with the existence of semistable reductions, and will thus imply the valuative criterion of properness for relative moduli spaces.

Let us elaborate on conditions (4) and (5). We fix a finite rank free abelian group Λ, and a morphism $v_s \colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X_s)) \to \Lambda$ for every $s \in S$, such that $v_t(E_t)$ is locally constant in families. Then $\underline{\sigma}$ is a stability condition with respect to Λ if each Z_s factors as $Z \circ v_s$ for some fixed $Z \in \mathrm{Hom}(\Lambda, \mathbb{C})$. In (4), we require that there is a quadratic form Q on $\Lambda_{\mathbb{R}}$ such that

(1) the kernel $(\ker Z) \subset \Lambda$ is negative definite with respect to Q, and
(2) for every $s \in S$ and for every σ_s-semistable object $E \in \mathcal{D}_s$, we have $Q(v_s(E)) \geqslant 0$.

Our boundedness condition (5) says that given $\mathbf{v} \in \Lambda$, there is a finite type family parametrizing the union for all $s \in S$ of objects $E \in \mathrm{D}^{\mathrm{b}}(X_s)$ that are σ_s-semistable with $v_s(E) = \mathbf{v}$.

Our first main result is a version of Bridgeland's Deformation Theorem.

Theorem 1.2 (Theorem 22.2)**.** *The space* $\mathrm{Stab}_\Lambda(\mathrm{D}^{\mathrm{b}}(X)/S)$ *of stability conditions on* $\mathrm{D}^{\mathrm{b}}(X)$ *over* S *with respect to* Λ *is a complex manifold, and the forgetful map*

$$\mathcal{Z}\colon \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)/S) \to \mathrm{Hom}(\Lambda, \mathbb{C}),$$

is a local isomorphism.

In the absolute case where S is a point, the support property (4) is enough to prove Bridgeland's Deformation Theorem for stability conditions [Bri07]. In our setting, the main step is to show that openness of geometric stability is preserved under small deformations of the central charge Z; this both requires boundedness of semistable objects (5) (to obtain boundedness of destabilizing quotients, and thereby constructibility of the unstable locus) and the existence of HN structures over DVRs (3) (to show that the unstable locus is closed under specialization).

Our second main result (which is the content of Part V) is that, whenever stability conditions are known to exist fiberwise, they also exist in families.

Theorem 1.3 (Theorem 24.1)**.** *Let* $g\colon X \to S$ *be a polarized flat family of smooth projective varieties.*

(1) *If the fibers of* g *are one- or two-dimensional, then the standard construction of stability conditions on curves or surfaces produces a stability condition* $\underline{\sigma}$ *on* $\mathrm{D}^{\mathrm{b}}(X)$ *over* S.
(2) *If the fibers of* g *are three-dimensional and satisfy the conjectural Bogomolov–Gieseker inequality of* [BMT14, BMS16]*, then the construction of stability conditions proposed in [ibid.] produces a stability condition* $\underline{\sigma}$ *on* $\mathrm{D}^{\mathrm{b}}(X)$ *over* S.

The known construction of stability conditions on smooth projective varieties is based on the operation of tilting bounded t-structures, starting from coherent sheaves, and uses weaker notions of stability similar to slope-stability. We first generalize Definition 1.1 to allow for a collection of weak stability conditions. Then we show that this tilting procedure extends to our setup. The main tool is the derived dual functor, which we show to provide a notion of double dual inside the hearts $\mathcal{A}_s$ for $s \in S$, and $\mathcal{A}_C$ for a Dedekind scheme C over S.

Relative moduli spaces and properness. Moduli spaces of semistable objects, and their wall-crossing, are what makes stability conditions useful to algebraic geometers. Given a class $\mathbf{v} \in \Lambda$, we denote by $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ the stack parameterizing $\underline{\sigma}$-semistable objects of class $\mathbf{v}$. Using results of Lieblich, Piyaratne, and Toda [Lie06a, Tod08, PT19], we show the following.

Theorem 1.4 (Theorem 21.24)**.** *Let* $\underline{\sigma}$ *be a stability condition on* $\mathrm{D}^{\mathrm{b}}(X)$ *over* S *and* $\mathbf{v} \in \Lambda$. *Then* $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ *is an algebraic stack of finite type over* S. *In characteristic 0 it admits a good moduli space* $M_{\underline{\sigma}}(\mathbf{v})$ *which is an algebraic space proper over* S.

The notion of good moduli space was introduced by Alper in [Alp13]. The existence of a good moduli space is a consequence of the very recent result [AHLH18].

When moduli stacks of Bridgeland stable objects are well-behaved, associated Donaldson–Thomas invariants have been defined in [PT19], based on [KS08, JS12]. When $M_\sigma(\mathbf{v}) =$

$M_\sigma^{\mathrm{st}}(\mathbf{v})$, namely there are no properly semistable objects, the definition is the same as in [Tho00, Beh09], via virtual classes or weighted Euler characteristic:

$$\mathrm{DT}_\sigma(\mathbf{v}) := \int_{[M_\sigma(\mathbf{v})]^{\mathrm{vir}}} 1 = \int_{M_\sigma(\mathbf{v})} \chi_B \in \mathbb{Z}.$$

As a first application of Theorem 1.3 and Theorem 1.4, we can prove the deformation-invariance of DT invariants for those CY threefolds on which Bridgeland stability conditions exist; in particular, by [Li19a], for quintic threefolds.

Corollary 1.5 (Theorem 28.1)**.** *Let $X \to S$ be a flat family of Calabi–Yau threefolds, defined over $\mathbb{C}$, with S connected. If the fibers X_s satisfy the generalized Bogomolov–Gieseker inequality of* [BMT14, BMS16]*, then the Donaldson–Thomas invariant $\mathrm{DT}_{\sigma_s}(\mathbf{v})$ is independent of s. In particular, this holds for smooth quintic threefolds.*

Another application, pointed out by Koseki [Kos20, Proposition 3.2], is a method to construct stability conditions on threefolds via degeneration, see Proposition 27.1.

Cubic fourfolds. In the case of higher-dimensional Fano varieties, moduli spaces of semistable objects often become more useful when we restrict our attention to objects lying in certain semiorthogonal components of $\mathrm{D}^{\mathrm{b}}(X_s)$, called *Kuznetsov components*.

This can also be done in families. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an admissible subcategory that is invariant under tensoring with perfect complexes on S (see Section 3 for the precise assumptions). Then Theorems 1.2 and 1.4 hold similarly with $\mathrm{D}^{\mathrm{b}}(X)$ replaced by $\mathcal{D}$. We also give conditions under which a weak stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ over S induces a stability condition on $\mathcal{D}$ over S, see Theorem 23.1 which extends the corresponding results in the absolute case from [BLMS17].

Our main application, treated in Part VI, concerns cubic fourfolds. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold over the complex numbers. We denote by $\mathcal{K}u(X)$ its *Kuznetsov component*

$$\mathcal{K}u(X) := \mathcal{O}_X^\perp \cap \mathcal{O}_X(H)^\perp \cap \mathcal{O}_X(2H)^\perp \subset \mathrm{D}^{\mathrm{b}}(X).$$

Over the moduli space of cubic fourfolds, these categories give a family of polarized non-commutative K3 surfaces [Kuz10, AT14] (see also [Huy17, MS19a]). There is no analogue of slope- or Gieseker-stability for such non-commutative K3 surfaces; instead, Bridgeland stability conditions have been constructed in [BLMS17]. The main result of Part VI is that our notion of stability over a base exists in this setup.

The first application concerns moduli spaces $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ of σ-stable objects in $\mathcal{K}u(X)$ with Mukai vector $\mathbf{v}$, for σ a Bridgeland stability condition in the distinguished connected component $\mathrm{Stab}^\dagger(\mathcal{K}u(X))$, generalizing a series of results for K3 surfaces [Bea83, Muk84, Muk87, O'G97, Huy97, Yos01, Tod08, BM14b].

Theorem 1.6 (Theorem 29.2)**.** *Let X be a cubic fourfold with Kuznetsov component $\mathcal{K}u(X)$. Let $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$ be its extended Mukai lattice, together with the Mukai Hodge structure. Then $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z}) = \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$. Moreover, assume that $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ is a non-zero primitive vector and let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ be a stability condition on $\mathcal{K}u(X)$ that is generic for $\mathbf{v}$. Then:*

(1) *$M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is non-empty if and only if $\mathbf{v}^2 \geqslant -2$. Moreover, it is a smooth projective irreducible holomorphic symplectic variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.*

(2) *If* $\mathbf{v}^2 \geqslant 0$*, then there exists a natural Hodge isometry*

$$\theta\colon H^2(M_\sigma(\mathcal{K}u(X),\mathbf{v}),\mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbf{v}^\perp & \text{if } \mathbf{v}^2 > 0 \\ \mathbf{v}^\perp/\mathbb{Z}\mathbf{v} & \text{if } \mathbf{v}^2 = 0, \end{cases}$$

where the orthogonal is taken in $\widetilde{H}(\mathcal{K}u(X),\mathbb{Z})$*.*

Theorem 1.6 has many interesting applications to cubic fourfolds and to families of polarized hyperkähler manifolds. We invite the reader to jump to Section 29 for a comprehensive summary of our related results, and give a shorter overview here. First of all, we can extend a result by Addington and Thomas, [AT14, Theorem 1.1]:

Corollary 1.7 (Corollary 29.7)**.** *Let* X *be a cubic fourfold. Then* X *has a Hodge-theoretically associated K3 if and only if there exists a smooth projective K3 surface* S *and an equivalence* $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$*.*

A version of the corollary also holds for K3 surfaces with a Brauer twist, extending a result of Huybrechts [Huy17, Theorem 1.4]; the corresponding Hodge-theoretic condition is the existence of a square-zero class in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z})$.

A second application is that relative moduli spaces give rise to unirational locally-complete families of polarized hyperkähler manifolds of arbitrarily large dimension and degree over the moduli space of cubic fourfolds:

Corollary 1.8 (Corollary 29.5)**.** *For any pair* (a,b) *of coprime integers, there is a unirational locally complete* 20*-dimensional family, over an open subset of the moduli space of cubic fourfolds, of polarized smooth projective irreducible holomorphic symplectic manifolds of dimension* $2n+2$*, where* $n = a^2 - ab + b^2$*. The polarization has either degree* $6n$ *and divisibility* $2n$ *if* 3 *does not divide* n*, or degree and divisibility* $\frac{2}{3}n$ *otherwise.*

When $a = 1$ and $b = 1$, this is nothing but the family of Fano varieties of lines in the cubic fourfold from [BD85] (with the Plücker polarization of degree 6), while when $a = 2$ and $b = 1$, we find the family of polarized eightfold from [LLSvS17] (with the Plücker polarization of degree 2). For the proof of these two examples, we refer to [LPZ18] (see also [LLMS18]).

Strategy of the proof: base change, Quot spaces, HN structures over a curve, and the support property. The theory of base change for semiorthogonal decompositions has been developed by Kuznetsov [Kuz11]; the corresponding theory for t-structures is originally due to Abramovich and Polishchuk [AP06, Pol07]. In Part I we recall and generalize these results to our context. In particular, Theorem 5.7 addresses an important technical point, base change of t-structures with respect to essentially perfect morphisms.

A second technical point of this paper is a rigorous treatment of Quot spaces for fiberwise t-structures. This is the main goal of Section 11, in Part II. The existence of Quot spaces allows us, among other things, to give a proof of Langton-Maruyama's semistable reduction without using completions, see Theorem 16.1. This result is fundamental for us, since it gives examples for HN structures over a curve, and is thus part of the proof of Theorem 1.3. It is worth mentioning that we use the algebraicity of the moduli stack to prove the algebraicity of the Quot functor. This implication is in the opposite direction from the construction of classical moduli spaces of sheaves, so our results depend heavily on the powerful machinery of [Lie06b].

The support property in Definition 1.1.(4) is studied in detail in Section 21. In combination with boundedness and universal closedness of relative moduli spaces it ensures, for example, that openness of stability is preserved by deformations, by tilting, or when inducing stability conditions on the admissible subcategory $\mathcal{D}$. It also allows us to avoid imposing the noetherianity of the heart of the t-structure.

Relation with existing works and open questions. The theory of deformations of Bridgeland stable objects and relative moduli spaces can be applied to other families of polarized non-commutative K3 surfaces. In fact, Theorem 1.6 and its Corollaries will hold in a polarized family of non-commutative K3 surfaces, as long as at least one fiber is an actual K3 surface, and stability conditions exist on the whole family. For example, the case of Gushel–Mukai fourfolds (see [IM11, DIM15, DK18, DK19, KP18, KP21]) has now been established in [PPZ19]. Another interesting example to study is the case of Debarre–Voisin manifolds [DV10].

Even in the case of cubic fourfolds, there are other examples of families of polarized hyperkähler manifolds associated to some geometric construction which should be possible to interpret as relative moduli spaces. For instance, in Corollary 1.8, when $a = 2$ and $b = 2$ the (singular) relative moduli space is birational to the construction in [Voi18] by [LPZ20].

Structure of the paper. Part I and Part II of the paper are mostly review sections; in these sections, we prove results about base change for triangulated categories and t-structures, and existence of moduli spaces in the generality needed in the paper. In particular, base change results for t-structures for essentially perfect morphisms in Section 5 are new, as well as the complete proof of representability of the Quot functor in Section 11.

Part III is the first instance of a (weak) stability condition over a base. We consider the case where the base is a Dedekind scheme and develop the theory of HN structures over it. We do not yet introduce the support property, but we do require stronger conditions in the one-dimensional case, which essentially translate into the valuative criterion of properness for relative moduli spaces later on. In Section 12 we briefly review Bridgeland stability conditions over a field, and prove a base change result in this setting. In Section 13 we present the main definition and results for HN structures. In order to construct examples, we first extend this to the less well-behaved, and more technical, notion of weak stability conditions in Section 15. Then, in Section 16, we prove the fundamental result of semistable reduction in our context. The remaining sections are more technical. Section 17 shows that a HN structure comes with a well-behaved notion of C-torsion, and Section 18 gives a criterion for a collection of stability conditions on the fibers to "integrate" to a HN structure. Finally, in Section 19, we give conditions under which a (weak) HN structure can be rotated.

Part IV includes the main sections in the paper. Definition 1.1 is presented there (in Section 20 and Section 21), together with the proofs of Theorem 1.2 (in Section 22) and Theorem 1.4 (in Section 21). In Section 23, we discuss how to induce stability conditions on semiorthogonal components, which will be important in applications, e.g. to cubic fourfolds.

In Part V we deal with the construction of stability conditions over a base, thus proving Theorem 1.3 and its immediate application to Donaldson–Thomas invariants (in Section 28). Finally, the applications to cubic fourfolds are contained in Part VI.

Acknowledgments. The paper benefited from many useful discussions with Nicolas Addington, Jarod Alper, Tom Bridgeland, François Charles, Olivier Debarre, Johan de Jong, Laure Flapan, Fabian Haiden, Daniel Halpern-Leistner, Brendan Hassett, Jochen Heinloth, Daniel Huybrechts, Alexander Kuznetsov, Chunyi Li, Kieran O'Grady, Arvid Perego, Laura Pertusi, Alexander Polishchuk, Giulia Saccà, Claire Voisin, and Xiaolei Zhao.

Parts of the paper were written while the authors were visiting several institutions. In particular, we would like to thank Institut des Hautes Études Scientifiques, Institut Henri Poincaré, Northeastern University, Università degli studi di Milano, University of Edinburgh, and the MSRI in Berkeley for the excellent working conditions.

We would also like to thank the referees for their careful reading of the manuscript and insightful comments.

2. Setup, notation, and some terminology

Main Setup. The main results in this paper work in the following setup:

- S is a noetherian *Nagata scheme* which is quasi-projective over a noetherian affine scheme (for the existence of good moduli spaces, we need S to have characteristic 0);
- if S is also integral, regular, and one-dimensional, we call it a *Dedekind scheme*, and often write C instead of S;
- X is a noetherian scheme of finite Krull dimension;
- $g\colon X \to S$ is a flat projective morphism (the notion of projective morphism we use is the one in [Sta21, Tag 01W8]); in our constructions of stability conditions we further assume g to be smooth;
- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an S-linear strong semiorthogonal component of finite cohomological amplitude (see Definitions 3.5 and 3.7).

We assume S to be Nagata since in Parts II and IV we need the valuative criterion for relative moduli spaces of stable objects, or for Quot spaces. However, we will only consider DVRs that are essentially of finite type over S; this is sufficient to deduce universal closedness or properness if S is locally given by a Nagata ring, see Section 11.4. We refer to [Sta21, Tag 032E] for the definition of Nagata rings. It is preserved by morphisms essentially of finite type, and it is implied by being excellent [Sta21, Tag 07QV]. Examples include localizations of finite type ring extensions of a field, of $\mathbb{Z}$, or more generally of a Dedekind domain with generic point of characteristic zero.

The general assumptions in each part of the paper are the following. Some results work is greater generality and we will explain this by recalling the precise assumptions in each section.

Part I:

(1) Sections 3–5 work with $g\colon X \to S$ a morphism of quasi-compact schemes with affine diagonal and X noetherian of finite Krull dimension;
(2) In Section 6, g is in addition flat;
(3) In Section 7, the key results hold when g is smooth and proper and S is noetherian with an ample line bundle.

Part II:

(a) g is a flat, proper, finitely presented morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension;

(b) $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an S-linear strong semiorthogonal component of finite cohomological amplitude.

Part III:

(a) $g\colon \mathcal{X} \to C$ is as in Part II and it is in addition projective, where C is a Dedekind scheme;

(b) $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ is as in Part II.

Part IV:

(a) $g\colon \mathcal{X} \to S$ is as in Part II and it is in addition projective, where S is a Nagata scheme which is quasi-projective over a noetherian affine scheme;

(b) In Section 23, we further assume g to be smooth;

(c) $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ is as in Part II.

Part V:

(a) $g\colon \mathcal{X} \to S$ is as in Part II and it is in addition smooth, projective of relative dimension $n \leqslant 3$, where S is a Nagata scheme which is quasi-projective over a noetherian affine scheme.

(b) $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ is as in Part II.

Part VI:

(a) S is a quasi-projective variety over $\mathbb{C}$;

(b) $g\colon \mathcal{X} \to S$ is a smooth, projective family of cubic fourfolds.

Derived categories. For a scheme X we consider various versions of the derived category:

- the category of perfect complexes $\mathrm{D}_{\mathrm{perf}}(X)$,
- the category of pseudo-coherent complexes with bounded cohomology $\mathrm{D}^{\mathrm{b}}(X)$,
- the category of pseudo-coherent complexes $\mathrm{D}_{\mathrm{pc}}(X)$, and
- the unbounded derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology $\mathrm{D}_{\mathrm{qc}}(X)$,
- the unbounded derived category of $\mathcal{O}_X$-modules $\mathrm{D}(X)$.

These categories are related by inclusions

$$\mathrm{D}_{\mathrm{perf}}(X) \subset \mathrm{D}^{\mathrm{b}}(X) \subset \mathrm{D}_{\mathrm{pc}}(X) \subset \mathrm{D}_{\mathrm{qc}}(X) \subset \mathrm{D}(X).$$

For background on pseudo-coherent complexes, see [Sta21, Tag 08E4]. We note that a complex $E \in \mathrm{D}(X)$ is pseudo-coherent if and only if there exists an open cover $X = \bigcup U_i$ such that E_{U_i} is quasi-isomorphic to a bounded above complex of finitely generated locally free sheaves on U_i; indeed, the forward direction follows directly from the definitions, and the converse follows from [Sta21, Tag 064U]. If X is noetherian, then pseudo-coherence boils down to a more classical notion: $\mathrm{D}_{\mathrm{pc}}(X)$ coincides with the bounded above derived category of $\mathcal{O}_X$-modules with coherent cohomologies [Sta21, Tag 08E8]. Moreover, in the noetherian case we have equivalences $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{Coh} X)$ and $\mathrm{D}_{\mathrm{qc}}(X) \simeq \mathrm{D}(\operatorname{Qcoh} X)$.

We use the "classical" language of triangulated categories, with the following exceptions that occur in a few places in Sections 3 and 5. The triangulated categories we consider arise as subcategories of the derived categories of schemes, and hence come with natural stable ∞-category enhancements. All colimits in such categories are meant in the ∞-categorical sense. Moreover, we sometimes use the enhanced structure to make natural constructions, e.g. tensor products, with such categories; see Remark 3.16 for further discussion.

We exclusively use the language of "classical" scheme theory. Using derived algebraic geometry, it would be possible to remove various flatness and transversality assumptions. We have not done so because our results already cover the situations of interest in applications.

Unless otherwise stated, all derived functors (e.g. pushforward, pullback, tensor products) will be denoted as if they are underived. We denote the S-relative derived sheaf Hom by

$$\mathcal{H}om_S(-,-) = g_*\mathcal{H}om(-,-).$$

Given $E \in \mathrm{D}_{\mathrm{qc}}(S)$ and a morphism $T \to S$, we let X_T denote the base change of X, and E_T the pullback of E. In particular, for a closed subset $W \subset S$, we write $i_W \colon X_W \to X$ for the embedding of the fiber over W, and $E_W = i_W^* E$ for the derived restriction. We write $s \in S$ for (closed or non-closed) points of S, and E_s for the pullback to the residue field $k(s)$. In the case of a Dedekind scheme C, we write $p \in C$ for a closed point, $c \in C$ for an arbitrary point, K for its fraction field, and E_p, E_C or E_K for the corresponding pullbacks.

In Section 3 we will consider an *S-linear strong semiorthogonal component $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ of finite cohomological amplitude*; we will write $\mathcal{D}_T$ for its base change to T when that exists, in particular $\mathcal{D}_s$ and $\mathcal{D}_W$ for its base change to points or closed subschemes of S, and similarly $\mathcal{D}_p$, $\mathcal{D}_K$, or $\mathcal{D}_c$ in the case of a Dedekind scheme C.

Part I. Semiorthogonal decompositions and t-structures in families

3. Semiorthogonal decompositions and base change

In this section, following [Kuz11], we recall some results on base change for semiorthogonal decompositions.

3.1. Semiorthogonal decompositions.

Definition 3.1. Let $\mathcal{D}$ be a triangulated category. A *semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$$

is a sequence of full triangulated subcategories $\mathcal{D}_1, \dots, \mathcal{D}_n$ of $\mathcal{D}$ — called the *components* of the decomposition — such that:

(1) $\mathrm{Hom}(F, G) = 0$ for all $F \in \mathcal{D}_i$, $G \in \mathcal{D}_j$ and $i > j$.
(2) For any $F \in \mathcal{D}$, there is a sequence of morphisms

$$0 = F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 = F,$$

such that $\mathrm{cone}(F_i \to F_{i-1}) \in \mathcal{D}_i$ for $1 \leqslant i \leqslant n$.

Remark 3.2. Condition (1) of the definition implies the "filtration" in (2) and its "factors" are unique and functorial. The functor $\mathrm{pr}_i \colon \mathcal{D} \to \mathcal{D}$ given by the i-th "factor", i.e.,

$$\mathrm{pr}_i(F) = \mathrm{cone}(F_i \to F_{i-1}),$$

is called the *projection functor* onto $\mathcal{D}_i$.

A full triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is called *right admissible* if the embedding functor $\gamma \colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint $\gamma^!$, *left admissible* if γ admits a left adjoint γ^*, and *admissible* if γ admits both right and left adjoints. If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semiorthogonal decomposition, then $\mathcal{D}_1$ is left admissible and $\mathcal{D}_2$ is right admissible. Vice versa, if $\mathcal{C} \subset \mathcal{D}$ is left admissible then there is a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{C}, {}^{\perp}\mathcal{C} \rangle$, and if $\mathcal{C} \subset \mathcal{D}$ is right admissible then there is a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{C}^{\perp}, \mathcal{C} \rangle$. Here,

$$^{\perp}\mathcal{C} = \{\, G \in \mathcal{D} \mid \mathrm{Hom}(G, F) = 0 \text{ for all } F \in \mathcal{C} \,\},$$
$$\mathcal{C}^{\perp} = \{\, G \in \mathcal{D} \mid \mathrm{Hom}(F, G) = 0 \text{ for all } F \in \mathcal{C} \,\},$$

denote respectively the *left* and *right orthogonals* to $\mathcal{C} \subset \mathcal{D}$. These remarks lead to the notion of mutation functors.

Definition 3.3. Let $\mathcal{C} \subset \mathcal{D}$ be an inclusion of triangulated categories. If $\mathcal{C}$ is right admissible, then the inclusion $\alpha \colon \mathcal{C}^{\perp} \to \mathcal{D}$ admits a left adjoint α^*, and the functor

$$\mathrm{L}_{\mathcal{C}} = \alpha \circ \alpha^* \colon \mathcal{D} \to \mathcal{D}$$

is called the *left mutation functor* through $\mathcal{C}$. Similarly, if $\mathcal{C} \subset \mathcal{D}$ is left admissible, then the inclusion $\beta \colon {}^{\perp}\mathcal{C} \to \mathcal{D}$ admits a right adjoint $\beta^!$, and the functor

$$\mathrm{R}_{\mathcal{C}} = \beta \circ \beta^! \colon \mathcal{D} \to \mathcal{D}$$

is called the *right mutation functor* through $\mathcal{C}$.

Remark 3.4. When they exist, the mutation functors fit into exact triangles

$$\gamma \circ \gamma^! \to \mathrm{id}_{\mathcal{D}} \to \mathrm{L}_{\mathcal{C}} \quad \text{and} \quad \mathrm{R}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{D}} \to \gamma \circ \gamma^*.$$

In our discussion of base change below, we need the following technical notions.

Definition 3.5. A semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$ is called *strong* if for each i the category $\mathcal{D}_i$ is right admissible in $\mathcal{D}$. We will call a subcategory $\mathcal{D}' \subset \mathcal{D}$ a *strong semiorthogonal component* if it is part of a strong semiorthogonal decomposition.

Remark 3.6. Definition 3.5 is slightly different than the one given in [Kuz11, Definition 2.6], but is easily seen to be equivalent.

Definition 3.7. Let X be a scheme and let $\mathcal{D} \subset \mathrm{D}_{\mathrm{qc}}(X)$ be a triangulated subcategory. If Y is a scheme and $\Phi\colon \mathcal{D} \to \mathrm{D}_{\mathrm{qc}}(Y)$ is a triangulated functor, we say that Φ has *cohomological amplitude* $[a, b]$ if

$$\Phi(\mathcal{D} \cap \mathrm{D}_{\mathrm{qc}}^{[p,q]}(X)) \subset \mathrm{D}_{\mathrm{qc}}^{[p+a,q+b]}(Y)$$

for all $p, q \in \mathbb{Z}$. We say Φ has *left finite cohomological amplitude* if a can be chosen finite, *right finite cohomological amplitude* if b can be chosen finite, and *finite cohomological amplitude* if a and b can be chosen finite. We say that a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$ is of *(right or left) finite cohomological amplitude* if its projection functors have (right or left) finite cohomological amplitude.

3.2. **Linear categories and base change.** We will be interested in triangulated subcategories that occur as subcategories of the derived category of a scheme. A crucial point is that there is a good notion of a "family" of such categories, made precise by the notion of a linear category.

Definition 3.8. Let $g\colon X \to S$ be a morphism of schemes. A triangulated subcategory $\mathcal{D} \subset \mathrm{D}_{\mathrm{qc}}(X)$ is called *S-linear* if it is stable with respect to tensoring by pullbacks of perfect complexes on S, i.e., if for every $F \in \mathcal{D}$ and $G \in \mathrm{D}_{\mathrm{perf}}(S)$ we have $F \otimes g^*G \in \mathcal{D}$. A semiorthogonal decomposition of $\mathcal{D}$ is called *S-linear* if all of its components are S-linear.

Remark 3.9. By [Kuz11, Lemma 2.7], for an S-linear semiorthogonal decomposition the condition (1) of Definition 3.1 is equivalent to the following: $\mathcal{H}om_S(F, G) = 0$ for all $F \in \mathcal{D}_i$, $G \in \mathcal{D}_j$ and $i > j$.

In the rest of this subsection, we review a formalism of base change for linear categories from [Kuz11]. Along the way we explain how to upgrade results proved in [Kuz11] for quasi-projective varieties over a field to more general settings. Another generalization is discussed in [BOR20, Section 3]; we also refer to [AE21, Proposition 3.13] for a statement in derived algebraic geometry language.

Our main interest is in linear categories that occur as semiorthogonal components of $\mathrm{D}^{\mathrm{b}}(X)$ for a scheme X, but it is convenient to consider the whole chain of derived categories

$$\mathrm{D}_{\mathrm{perf}}(X) \subset \mathrm{D}^{\mathrm{b}}(X) \subset \mathrm{D}_{\mathrm{pc}}(X) \subset \mathrm{D}_{\mathrm{qc}}(X).$$

Namely, we will see that base change for semiorthogonal decompositions of $\mathrm{D}^{\mathrm{b}}(X)$ can be obtained by a combination of inducing and restricting semiorthogonal decompositions along the above inclusions and base change for $\mathrm{D}_{\mathrm{perf}}(X)$.

Lemma 3.10. *Let $X \to S$ be a morphism of schemes where X is noetherian of finite Krull dimension. Let*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$$

be a strong S-linear semiorthogonal decomposition. Define $\mathcal{D}_{i,\mathrm{perf}} = \mathcal{D}_i \cap \mathrm{D}_{\mathrm{perf}}(X)$. Then there is an S-linear semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{perf}}(X) = \langle \mathcal{D}_{1,\mathrm{perf}}, \dots, \mathcal{D}_{n,\mathrm{perf}} \rangle.$$

Proof. An object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is called *homologically finite* if for all $F \in \mathrm{D}^{\mathrm{b}}(X)$ we have $\mathrm{Hom}(E, F[i]) = 0$ for all but finitely many $i \in \mathbb{Z}$. Let $\mathrm{D}^{\mathrm{b}}(X)_{\mathrm{hf}} \subset \mathrm{D}^{\mathrm{b}}(X)$ denote the full triangulated subcategory of homologically finite objects, and let $\mathcal{D}_{i,\mathrm{hf}} = \mathcal{D}_i \cap \mathrm{D}^{\mathrm{b}}(X)_{\mathrm{hf}}$. Then by [Orl06, Proposition 1.10] we have a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X)_{\mathrm{hf}} = \langle \mathcal{D}_{1,\mathrm{hf}}, \dots, \mathcal{D}_{n,\mathrm{hf}} \rangle.$$

So it suffices to show that $\mathrm{D}^{\mathrm{b}}(X)_{\mathrm{hf}} = \mathrm{D}_{\mathrm{perf}}(X)$. This follows by an easy modification of the argument of [Orl06, Proposition 1.11]. □

We will often work with quasi-compact, quasi-separated schemes. The relevance of these conditions for us is the following result.

Proposition 3.11 ([Nee96, TT90])**.** *If X is a quasi-compact and quasi-separated scheme, then $\mathrm{D}_{\mathrm{qc}}(X)$ is compactly generated with compact objects the perfect complexes.*

Remark 3.12. By [BZFN10, Propositions 3.6 and 3.9], for a scheme X we have $\mathrm{D}_{\mathrm{qc}}(X) = \mathrm{Ind}(\mathrm{D}_{\mathrm{perf}}(X))$ if and only if $\mathrm{D}_{\mathrm{qc}}(X)$ is compactly generated with compact objects the perfect complexes; in particular, by Proposition 3.11 this holds if X is quasi-compact and quasi-separated. Here, $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}_{\mathrm{perf}}(X)$ are regarded as ∞-categories, and $\mathrm{Ind}(\mathrm{D}_{\mathrm{perf}}(X))$ denotes the category of Ind-objects in the sense of [Lur09, Section 5.3]. This recipe for recovering $\mathrm{D}_{\mathrm{qc}}(X)$ from $\mathrm{D}_{\mathrm{perf}}(X)$ will be used several times below.

We also note that in [BZFN10], X is called *perfect* if it has affine diagonal and $\mathrm{D}_{\mathrm{qc}}(X) = \mathrm{Ind}(\mathrm{D}_{\mathrm{perf}}(X))$. In particular, a quasi-compact scheme with affine diagonal is perfect. Below we will often consider such schemes in order to take advantage of the results of [BZFN10].

Next we show that a semiorthogonal decomposition of $\mathrm{D}_{\mathrm{perf}}(X)$ induces one of $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}_{\mathrm{pc}}(X)$. We use the following terminology. Let $\Phi\colon \mathcal{D} \to \mathcal{D}'$ be a functor between triangulated categories. We say that Φ is *compatible* with semiorthogonal decompositions $\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$ and $\mathcal{D}' = \langle \mathcal{D}'_1, \dots, \mathcal{D}'_n \rangle$ if $\Phi(\mathcal{D}_i) \subset \mathcal{D}'_i$ for all i. If Φ is fully faithful, then Φ is compatible if and only if $\mathcal{D}_i = \Phi^{-1}(\mathcal{D}'_i)$, see [Kuz11, Lemma 3.3]. Moreover, if Φ is fully faithful and $\mathcal{D}' = \langle \mathcal{D}'_1, \dots, \mathcal{D}'_n \rangle$ is given, then $\mathcal{D}_i = \Phi^{-1}(\mathcal{D}'_i)$ defines a semiorthogonal decomposition of $\mathcal{D}$ if and only if the image of Φ is preserved by the projection functors of the decomposition of $\mathcal{D}'$, see [Kuz11, Lemma 3.4].

Lemma 3.13. *Let $X \to S$ be a morphism of schemes where X is quasi-compact and quasi-separated. Let*

$$\mathrm{D}_{\mathrm{perf}}(X) = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle \tag{3.1}$$

be an S-linear semiorthogonal decomposition.

(1) *Define $\mathcal{D}_{i,\mathrm{qc}} \subset \mathrm{D}_{\mathrm{qc}}(X)$ to be the minimal triangulated subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$ which is closed under arbitrary direct sums and contains $\mathcal{D}_i$. Then there is an S-linear semiorthogonal decomposition*

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathcal{D}_{1,\mathrm{qc}}, \dots, \mathcal{D}_{n,\mathrm{qc}} \rangle \tag{3.2}$$

whose projection functors are cocontinuous.

(2) *Assume that the decomposition* (3.2) *has right finite cohomological amplitude. Define* $\mathcal{D}_{i,\mathrm{pc}} = \mathcal{D}_{i,\mathrm{qc}} \cap \mathrm{D}_{\mathrm{pc}}(X)$. *Then there is an* S*-linear semiorthogonal decomposition*

$$\mathrm{D}_{\mathrm{pc}}(X) = \langle \mathcal{D}_{1,\mathrm{pc}}, \dots, \mathcal{D}_{n,\mathrm{pc}} \rangle. \tag{3.3}$$

(3) *Assume that* X *is noetherian of finite Krull dimension, and that the decomposition* (3.1) *is induced via Lemma 3.10 by a strong* S*-linear decomposition*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}_1^{\mathrm{b}}, \dots, \mathcal{D}_n^{\mathrm{b}} \rangle$$

of right finite cohomological amplitude. Then the inclusion $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}_{\mathrm{qc}}(X)$ *is compatible with the given decomposition of* $\mathrm{D}^{\mathrm{b}}(X)$ *and the decomposition* (3.2) *of* $\mathrm{D}_{\mathrm{qc}}(X)$*, and the projection functors of* $\mathrm{D}_{\mathrm{qc}}(X)$ *have the same cohomological amplitude as those of* $\mathrm{D}^{\mathrm{b}}(X)$.

Proof. Using Proposition 3.11, the argument of [Kuz11, Proposition 4.2] goes through to prove (1) in the stated generality. Alternatively, $\mathrm{D}_{\mathrm{qc}}(X) = \mathrm{Ind}(\mathrm{D}_{\mathrm{perf}}(X))$ by Remark 3.12, so the result follows from the more general [Per19, Lemma 3.12], which also shows $\mathcal{D}_{i,\mathrm{qc}} = \mathrm{Ind}(\mathcal{D}_i)$.

For (2) we must show that the projection functors $\hat{\mathrm{pr}}_i \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(X)$ of the decomposition (3.2) preserve $\mathrm{D}_{\mathrm{pc}}(X)$. We use the following characterization of pseudo-coherent objects on a quasi-compact and quasi-separated scheme, see [Sta21, Tag 0DJN]: an object $F \in \mathrm{D}_{\mathrm{qc}}(X)$ is pseudo-coherent if and only if $F = \operatorname{colim}_{k \in \mathbb{Z}} F_k$ where F_k is perfect and for any $n \in \mathbb{Z}$ the map $\tau^{\geqslant n} F_k \to \tau^{\geqslant n} F$ is an isomorphism for $k \gg 0$. Given such an F, by the cocontinuity of $\hat{\mathrm{pr}}_i$ we have

$$\hat{\mathrm{pr}}_i(F) \simeq \operatorname{colim} \hat{\mathrm{pr}}_i(F_k). \tag{3.4}$$

By construction, $\hat{\mathrm{pr}}_i$ restricts to the projection functor of the decomposition (3.1) of $\mathrm{D}_{\mathrm{perf}}(X)$, so $\hat{\mathrm{pr}}_i(F_k)$ is perfect. Defining C_k as the cone of $F_k \to F$, the above condition on the truncations of $F_k \to F$ can be rephrased as follows: for any $n \in \mathbb{Z}$ we have $C_k \in \mathrm{D}_{\mathrm{qc}}^{\leqslant n}(X)$ for $k \gg 0$. By the assumption that $\hat{\mathrm{pr}}_i$ has right finite cohomological amplitude, it follows that for any $n \in \mathbb{Z}$ we have $\hat{\mathrm{pr}}_i(C_k) \in \mathrm{D}_{\mathrm{qc}}^{\leqslant n}(X)$ for $k \gg 0$. In other words, for any $n \in \mathbb{Z}$ the map $\tau^{\geqslant n} \hat{\mathrm{pr}}_i(F_k) \to \tau^{\geqslant n} \hat{\mathrm{pr}}_i(F)$ is an isomorphism for $k \gg 0$. All together, this proves that $\hat{\mathrm{pr}}_i(F)$ satisfies the above criterion for pseudo-coherence.

Now we turn to (3). To show the compatibility of $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}_{\mathrm{qc}}(X)$, we must show that if $F \in \mathcal{D}_i^{\mathrm{b}}$ then $\hat{\mathrm{pr}}_i(F) \simeq F$. Since F is in particular pseudo-coherent, the description (3.4) above applies, so we need to show $\operatorname{colim} \hat{\mathrm{pr}}_i(F_k) \simeq F$. By construction both $\hat{\mathrm{pr}}_i$ and the projection functor $\mathrm{pr}_i^{\mathrm{b}}$ of (3) restrict to the same projection functor of the decomposition (3.1), hence $\operatorname{colim} \hat{\mathrm{pr}}_i(F_k) \simeq \operatorname{colim} \mathrm{pr}_i^{\mathrm{b}}(F_k)$. As above, we consider the cone C_k of $F_k \to F$, which we note lies in $\mathrm{D}^{\mathrm{b}}(X)$ in our current situation. Since $\mathrm{pr}_i^{\mathrm{b}}(F) \simeq F$, we obtain an exact triangle

$$\mathrm{pr}_i^{\mathrm{b}}(F_k) \to F \to \mathrm{pr}_i^{\mathrm{b}}(C_k).$$

Using the assumption that $\mathrm{pr}_i^{\mathrm{b}}$ has right finite cohomological amplitude, the argument from the previous paragraph shows that for any $n \in \mathbb{Z}$ the map $\tau^{\geqslant n} \mathrm{pr}_i^{\mathrm{b}}(F_k) \to \tau^{\geqslant n} F$ is an isomorphism for $k \gg 0$. Applying [Sta21, Tag 0CRK] (taking H there to be the cohomology functors on $\mathrm{D}(X)$), it follows that $\operatorname{colim} \mathrm{pr}_i^{\mathrm{b}}(F_k) \simeq F$, as required.

Finally, we show that if $\mathrm{pr}_i^{\mathrm{b}}$ has cohomological amplitude $[a_i, b_i]$, then so does $\hat{\mathrm{pr}}_i$. In other words, we claim that for any object $F \in \mathrm{D}_{\mathrm{qc}}^{[p,q]}(X)$ we have $\hat{\mathrm{pr}}_i(F) \in \mathrm{D}_{\mathrm{qc}}^{[p+a_i, q+b_i]}(X)$.

By Lemma 3.14 below, we can write such an object as filtered colimit $F = \operatorname{colim} F_\alpha$ where $F_\alpha \in \mathrm{D}^{\mathrm{b}}(X)^{[p,q]}$. Then we have

$$\hat{\mathrm{pr}}_i(F) \simeq \operatorname{colim} \hat{\mathrm{pr}}_i(F_\alpha) \simeq \operatorname{colim} \mathrm{pr}_i^{\mathrm{b}}(F_\alpha),$$

where the second equivalence holds by the compatibility of $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}_{\mathrm{qc}}(X)$ shown above. But $\mathrm{pr}_i^{\mathrm{b}}(F_\alpha) \in \mathrm{D}^{\mathrm{b}}(X)^{[p+a_i,q+b_i]}$, so Lemma 3.14 gives $\hat{\mathrm{pr}}_i(F) \in \mathrm{D}_{\mathrm{qc}}^{[p+a_i,q+b_i]}(X)$. □

Lemma 3.14. *Let X be a noetherian scheme. Let $F \in \mathrm{D}_{\mathrm{qc}}(X)$. Then $F \in \mathrm{D}_{\mathrm{qc}}^{[a,b]}(X)$ if and only if $F = \operatorname{colim} F_\alpha$ for a filtered system of $F_\alpha \in \mathrm{D}^{\mathrm{b}}(X)^{[a,b]}$.*

Proof. By [Sta21, Tag 09T4] we have $\mathrm{D}_{\mathrm{qc}}(X) \simeq \mathrm{D}(\mathrm{QCoh}X)$, so by [Lur17, Propositions 1.3.5.21 and 1.4.4.13] the truncation functors for the standard t-structure on $\mathrm{D}_{\mathrm{qc}}(X)$ commute with filtered colimits. This implies that if $F = \operatorname{colim} F_\alpha$ with $F_\alpha \in \mathrm{D}^{\mathrm{b}}(X)^{[a,b]}$, then $F \in \mathrm{D}_{\mathrm{qc}}^{[a,b]}(X)$. Conversely, since $\mathrm{D}_{\mathrm{qc}}(X) = \operatorname{Ind}(\mathrm{D}_{\mathrm{perf}}(X))$ by Remark 3.12, any $F \in \mathrm{D}_{\mathrm{qc}}(X)$ can be expressed as a filtered colimit $F = \operatorname{colim} G_\alpha$ where $G_\alpha \in \mathrm{D}_{\mathrm{perf}}(X)$. If $F \in \mathrm{D}_{\mathrm{qc}}^{[a,b]}(X)$, then we have

$$F \simeq \tau^{\geqslant a}\tau^{\leqslant b}F \simeq \operatorname{colim} \tau^{\geqslant a}\tau^{\leqslant b}G_\alpha.$$

Hence we may take $F_\alpha = \tau^{\geqslant a}\tau^{\leqslant b}G_\alpha \in \mathrm{D}^{\mathrm{b}}(X)^{[a,b]}$. □

Now we discuss base change for semiorthogonal decompositions. Given morphisms of schemes $g\colon X \to S$ and $\phi\colon T \to S$, we use the following notation for the base change diagram:

$$\begin{tikzcd} X_T \arrow[r, "\phi'"] \arrow[d, "g'"'] & X \arrow[d, "g"] \\ T \arrow[r, "\phi"] & S \end{tikzcd}$$

Following [Kuz11], we say that $\phi\colon T \to S$ is *faithful* with respect to $g\colon X \to S$ if the canonical morphism of functors $\phi^* g_* \to g'_* \phi'^*$ is an isomorphism; this holds, for instance, if either $\phi\colon T \to S$ or $g\colon X \to S$ is flat.

Proposition 3.15. *Let $g\colon X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal. Let*

$$\mathrm{D}_{\mathrm{perf}}(X) = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$$

be an S-linear semiorthogonal decomposition. Let $\phi\colon T \to S$ be a morphism from a scheme T which is quasi-compact with affine diagonal, such that ϕ is faithful with respect to g. Define the following categories:

(1) *$\mathcal{D}_{iT} \subset \mathrm{D}_{\mathrm{perf}}(X_T)$ is the minimal triangulated category subcategory which is idempotent complete*[1] *and contains $\phi'^* F \otimes g'^* G$ for all $F \in \mathcal{D}_i$ and $G \in \mathrm{D}_{\mathrm{perf}}(T)$.*
(2) *$(\mathcal{D}_{i,\mathrm{qc}})_T \subset \mathrm{D}_{\mathrm{qc}}(X_T)$ is the minimal triangulated subcategory which is closed under arbitrary direct sums and contains $\mathcal{D}_{iT}$.*

Then there are T-linear semiorthogonal decompositions

$$\mathrm{D}_{\mathrm{perf}}(X_T) = \langle \mathcal{D}_{1T}, \dots, \mathcal{D}_{nT} \rangle, \tag{3.5}$$

$$\mathrm{D}_{\mathrm{qc}}(X_T) = \langle (\mathcal{D}_{1,\mathrm{qc}})_T, \dots, (\mathcal{D}_{n,\mathrm{qc}})_T \rangle, \tag{3.6}$$

[1] Recall that an additive category is idempotent complete if any idempotent $e\colon F \to F$ arises from a splitting $A = \operatorname{Im}(e) \oplus \operatorname{Ker}(e)$.

where the projection functors of (3.6) *are cocontinuous. Moreover, the functors* ϕ'_* *and* ϕ'^* *are compatible with all of the above decompositions.*

Proof. In the setting of quasi-projective varieties over a field, this is a combination of [Kuz11, Proposition 5.1 and 5.3]. The proof there goes through once we know the following claim: the category $\mathrm{D}_{\mathrm{perf}}(X_T)$ coincides with the minimal idempotent complete triangulated subcategory of $\mathrm{D}_{\mathrm{qc}}(X_T)$ which is idempotent complete and contains $\phi'^*F \otimes g'^*G$ for all $F \in \mathcal{D}_i$ and $G \in \mathrm{D}_{\mathrm{perf}}(T)$. Note that X_T agrees with the derived fiber product of X and T over S, since ϕ is faithful with respect to g. Hence by [BZFN10, Theorem 1.2] we have an equivalence

$$\mathrm{D}_{\mathrm{perf}}(X_T) \simeq \mathrm{D}_{\mathrm{perf}}(X) \otimes_{\mathrm{D}_{\mathrm{perf}}(S)} \mathrm{D}_{\mathrm{perf}}(T),$$

so the above claim follows from [Per19, Lemma 2.7]. Alternatively, we note that the proposition follows by combining the above equivalence, the fact that X_T is perfect (see [BZFN10, Proposition 3.24]), and [Per19, Lemma 3.15 and 3.12]. □

Remark 3.16. As hinted in the proof, Proposition 3.15 can be formulated in a much more general setting, which is sometimes useful. We briefly summarize the situation; see [Per19, Section 2.3] for details. Namely, there is an abstract notion of an *S-linear ∞-category* (called simply an "S-linear category" in [Per19]), which is a small idempotent-complete stable ∞-category $\mathcal{C}$ equipped with a module structure over $\mathrm{D}_{\mathrm{perf}}(S)$. These categories are organized into an ∞-category Cat_S. As an example, the semiorthogonal components $\mathcal{D}_i$ in Proposition 3.15 have a canonical S-linear ∞-category structure, and hence may be regarded as objects of Cat_S.

Further, there is a notion of a *presentable S-linear ∞-category* (called simply a "presentable S-linear category" in [Per19]), which is a presentable ∞-category equipped with a module structure over $\mathrm{D}_{\mathrm{qc}}(S)$. These categories also organize into an ∞-category PrCat_S. The operation of taking Ind-categories gives a functor $\mathrm{Ind}\colon \mathrm{Cat}_S \to \mathrm{PrCat}_S$. For instance, $\mathcal{D}_{i,\mathrm{qc}} = \mathrm{Ind}(\mathcal{D}_i)$ is an object of PrCat_S.

There are several advantages of working in this setting. For instance, for any S-scheme T we get categories $\mathrm{D}_{\mathrm{perf}}(T) \in \mathrm{Cat}_S$ and $\mathrm{D}_{\mathrm{qc}}(T) \in \mathrm{PrCat}_S$, and using the monoidal structures on Cat_S and PrCat_S we can form for any $\mathcal{C} \in \mathrm{Cat}_S$ and $\mathcal{C}' \in \mathrm{PrCat}_S$ new categories

$$\mathcal{C}_T = \mathcal{C} \otimes_{\mathrm{D}_{\mathrm{perf}}(S)} \mathrm{D}_{\mathrm{perf}}(T) \in \mathrm{Cat}_T \quad \text{and} \quad \mathcal{C}'_T = \mathcal{C}' \otimes_{\mathrm{D}_{\mathrm{qc}}(S)} \mathrm{D}_{\mathrm{qc}}(T) \in \mathrm{PrCat}_T.$$

This construction makes sense for *any* S-scheme T, and plays well with semiorthogonal decompositions by [Per19, Lemma 3.15 and 3.12]. In case $\mathcal{C} = \mathcal{D}_i$ and $\mathcal{C}' = \mathcal{D}_{i,\mathrm{qc}}$, this construction recovers the categories from Proposition 3.15 *if* $T \to S$ is faithful with respect to g and T is quasi-compact with affine diagonal.

Besides the use of this technology in the proof of Proposition 3.15, the only other time we will need it in this paper is in the proof of Theorem 5.3 below.

Theorem 3.17. *Let $g\colon X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle \tag{3.7}$$

be a strong S-linear semiorthogonal decomposition. Let $\phi\colon T \to S$ be a morphism from a scheme T which is quasi-compact with affine diagonal, such that ϕ is faithful with respect to g. Let

$$(\mathcal{D}_{i,\mathrm{perf}})_T \subset \mathrm{D}_{\mathrm{perf}}(X_T) \quad \textit{and} \quad (\mathcal{D}_{i,\mathrm{qc}})_T \subset \mathrm{D}_{\mathrm{qc}}(X_T)$$

be the categories obtained by combining Lemma 3.10 and Proposition 3.15, and define

$$(\mathcal{D}_{i,\mathrm{pc}})_T = (\mathcal{D}_{i,\mathrm{qc}})_T \cap \mathrm{D}_{\mathrm{pc}}(X_T) \quad \textit{and} \quad \mathcal{D}_{iT} = (\mathcal{D}_{i,\mathrm{qc}})_T \cap \mathrm{D}^{\mathrm{b}}(X_T).$$

Then there are T-linear semiorthogonal decompositions

$$\mathrm{D}_{\mathrm{perf}}(X_T) = \langle (\mathcal{D}_{1,\mathrm{perf}})_T, \dots, (\mathcal{D}_{n,\mathrm{perf}})_T \rangle, \tag{3.8}$$

$$\mathrm{D}_{\mathrm{qc}}(X_T) = \langle (\mathcal{D}_{1,\mathrm{qc}})_T, \dots, (\mathcal{D}_{n,\mathrm{qc}})_T \rangle, \tag{3.9}$$

where the projection functors of (3.9) are cocontinuous. If the decomposition (3.7) has right finite cohomological amplitude, then the projection functors of (3.9) have the same cohomological amplitude as those of (3.7). In this case, we have a T-linear semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{pc}}(X_T) = \langle (\mathcal{D}_{1,\mathrm{pc}})_T, \dots, (\mathcal{D}_{n,\mathrm{pc}})_T \rangle, \tag{3.10}$$

and if (3.7) has finite cohomological amplitude we have a T-linear semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X_T) = \langle \mathcal{D}_{1T}, \dots, \mathcal{D}_{nT} \rangle. \tag{3.11}$$

Moreover, the functors ϕ'_* and ϕ'^* are compatible with all of the above decompositions.

Proof. Combining Lemma 3.10 and Proposition 3.15 gives the decompositions (3.8) and (3.9), and guarantees that the projection functors of (3.9) are cocontinuous. To prove the theorem, it suffices to show that if the decomposition (3.7) has right finite cohomological amplitude, then the projection functors of (3.9) have the same cohomological amplitude as those of (3.7). Indeed, then the decomposition (3.10) holds by Lemma 3.13.(2). Moreover, if (3.7) has finite cohomological amplitude, then the projection functors of (3.10) preserve $\mathrm{D}^{\mathrm{b}}(X_T) \subset \mathrm{D}_{\mathrm{pc}}(X_T)$ because they are of finite cohomological amplitude, so (3.11) holds.

Let pr_i, $\hat{\mathrm{pr}}_i$, and $\hat{\mathrm{pr}}_{iT}$ denote the projection functors for the semiorthogonal decompositions of $\mathrm{D}^{\mathrm{b}}(X), \mathrm{D}_{\mathrm{qc}}(X)$, and $\mathrm{D}_{\mathrm{qc}}(X_T)$. We want to show that if $[a_i, b_i]$ is the cohomological amplitude of pr_i, then it is also the amplitude of $\hat{\mathrm{pr}}_{iT}$, i.e., for any $F \in \mathrm{D}_{\mathrm{qc}}^{[p,q]}(X_T)$ we have $\hat{\mathrm{pr}}_{iT}(F) \in \mathrm{D}_{\mathrm{qc}}^{[p+a_i,q+b_i]}(X_T)$.

If $U \subset T$ is an affine open subset, then Proposition 3.15 also gives a semiorthogonal decomposition of $\mathrm{D}_{\mathrm{qc}}(X_U)$, whose projection functors we denote by $\hat{\mathrm{pr}}_{iU}$. By the compatibility of the restriction functor $\mathrm{D}_{\mathrm{qc}}(X_T) \to \mathrm{D}_{\mathrm{qc}}(X_U)$ with the semiorthogonal decompositions, we have

$$\hat{\mathrm{pr}}_{iT}(F)_U \simeq \hat{\mathrm{pr}}_{iU}(F_U).$$

It follows that the claim on the amplitude of $\hat{\mathrm{pr}}_{iT}$ can be checked on an affine open cover of T, so we may assume T is affine.

Since T is affine and S has affine diagonal, the morphism $\phi\colon T \to S$ is affine. Therefore so is the base change $\phi'\colon X_T \to X$. In particular, $\phi'_*\colon \mathrm{D}_{\mathrm{qc}}(X_T) \to \mathrm{D}_{\mathrm{qc}}(X)$ is conservative and t-exact, which implies that an object $F \in \mathrm{D}_{\mathrm{qc}}(X_T)$ satisfies $F \in \mathrm{D}_{\mathrm{qc}}^{[p,q]}(X_T)$ if and only if $\phi'_*(F) \in \mathrm{D}_{\mathrm{qc}}^{[p,q]}(X)$. Further, by compatibility of ϕ'_* with the semiorthogonal decompositions, we have

$$\phi'_*(\hat{\mathrm{pr}}_{iT}(F)) \simeq \hat{\mathrm{pr}}_i(\phi'_*(F)).$$

It follows that to prove the result, we just need to show that $\hat{\mathrm{pr}}_i$ has the same cohomological amplitude as pr_i. But this holds by Lemma 3.13.(3). □

We will often apply the above results in the following way. Let $X \to S$ be a morphism of schemes with X noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an admissible S-linear subcategory. Then it follows directly from the definitions that

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}, {}^{\perp}\mathcal{D} \rangle \tag{3.12}$$

is a strong S-linear semiorthogonal decomposition. Hence by Lemmas 3.10 and 3.13 we obtain S-linear left admissible subcategories $\mathcal{D}_{\mathrm{perf}} \subset \mathrm{D}_{\mathrm{perf}}(X)$ and $\mathcal{D}_{\mathrm{qc}} \subset \mathrm{D}_{\mathrm{qc}}(X)$.

Moreover, if $\phi\colon T \to S$ is a morphism which is faithful with respect to $X \to S$ and all of X, S, and T are quasi-compact with affine diagonal, then by Theorem 3.17 we obtain T-linear left admissible subcategories $(\mathcal{D}_{\mathrm{perf}})_T \subset \mathrm{D}_{\mathrm{perf}}(X_T)$ and $(\mathcal{D}_{\mathrm{qc}})_T \subset \mathrm{D}_{\mathrm{qc}}(X_T)$. We call the projection functor onto $\mathcal{D}$ in the decomposition (3.12) the *left projection functor of* $\mathcal{D}$. The decomposition (3.12) has finite cohomological amplitude if this projection functor does. Hence in this case we also obtain by Theorem 3.17 a T-linear left admissible subcategory $\mathcal{D}_T \subset \mathrm{D}^{\mathrm{b}}(X_T)$.

If $T \to S$ and $T' \to S$ are morphisms of quasi-compact schemes with affine diagonal which are faithful with respect to $X \to S$, and if $f\colon T' \to T$ is a morphism over S, then we consider the base change morphism $f'\colon X_{T'} \to X_T$. It follows from the definitions that pushforward and pullback along f' induce functors

$$f'_*\colon (\mathcal{D}_{\mathrm{qc}})_{T'} \to (\mathcal{D}_{\mathrm{qc}})_T \quad \text{and} \quad f'^*\colon (\mathcal{D}_{\mathrm{qc}})_T \to (\mathcal{D}_{\mathrm{qc}})_{T'}.$$

Further, if the projection functor of $\mathcal{D}$ has finite cohomological amplitude, then we get a functor $f'_*\colon \mathcal{D}_{T'} \to \mathcal{D}_T$ if f is proper, and a functor $f'^*\colon \mathcal{D}_T \to \mathcal{D}_{T'}$ if f has finite Tor-dimension. We will often refer to pullback as restriction and use the notation $F_T = f'^*F$.

Below we use the following observations. The inclusion $\phi\colon U \to S$ of an open subscheme is always faithful with respect to $X \to S$ and of finite Tor-dimension; moreover, if S has affine diagonal then so does U. Similarly, if $S = \mathrm{Spec}(A)$ and B is a localization of A, then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is faithful with respect to $X \to S$ and of finite Tor-dimension.

Lemma 3.18. *Let $X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component. If $T \to S$ is either:*

(1) *the inclusion of a quasi-compact open subset, or*
(2) *a morphism of affine schemes corresponding to a localization of rings $A \to B$,*

then the pullback functors

$$\mathcal{D}_{\mathrm{qc}} \to (\mathcal{D}_{\mathrm{qc}})_T \quad \textit{and} \quad \mathcal{D} \to \mathcal{D}_T$$

are essentially surjective. For the statement for $\mathcal{D} \to \mathcal{D}_T$, we assume the projection functor of $\mathcal{D}$ has finite cohomological amplitude so that $\mathcal{D}_T$ is defined.

Proof. In both cases (1) and (2), the pullback functors

$$\mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(X_T) \quad \text{and} \quad \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X_T) \tag{3.13}$$

are essentially surjective. In case (1) this holds for $\mathrm{D}_{\mathrm{qc}}(-)$ by [Sta21, Tag 08ED] and for $\mathrm{D}^{\mathrm{b}}(-)$ by [Pol07, Lemma 2.3.1]. In case (2) essential surjectivity holds by the same arguments. The result follows since pullback along $T \to S$ is compatible with the semiorthogonal decompositions of the source and target categories in (3.13). □

3.3. **Relative exceptional collections.** In the applications later in this paper, we will consider S-linear semiorthogonal decompositions induced by relative exceptional collections, which are defined and studied below.

Definition 3.19. Let $X \to S$ be a morphism of schemes, and let $\mathcal{D} \subset \mathrm{D}_{\mathrm{qc}}(X)$ be an S-linear subcategory. A *relative exceptional object in* $\mathcal{D}$ is an object $E \in \mathcal{D}$ such that E is perfect and $\mathcal{H}om_S(E, E) = \mathcal{O}_S$. A *relative exceptional collection in* $\mathcal{D}$ is a sequence $E_1, \dots, E_m$ of relative exceptional objects in $\mathcal{D}$ such that $\mathcal{H}om_S(E_i, E_j) = 0$ for all $i > j$.

Remark 3.20. If $S = \mathrm{Spec}(k)$ is a point, then $\mathcal{H}om_S(-,-)$ is simply $\mathrm{RHom}(-,-)$ regarded as a k-complex. Hence the above definition reduces to the usual one in this case.

We note that the relative Hom functor behaves well under base change.

Lemma 3.21. *Let $g\colon X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in \mathrm{D}_{\mathrm{perf}}(X)$ and $F \in \mathrm{D}_{\mathrm{qc}}(X)$. Let $\phi\colon T \to S$ be a morphism which is faithful with respect to g. Then we have*

$$\mathcal{H}om_S(E, F)_T \simeq \mathcal{H}om_T(E_T, F_T).$$

Proof. Indeed, using the notation of (3.2), we have

$$\mathcal{H}om_S(E, F)_T \simeq g'_* \phi'^* \mathcal{H}om_X(E, F) \simeq g'_* \mathcal{H}om_{X_T}(E_T, F_T) = \mathcal{H}om_T(E_T, F_T),$$

where the first isomorphism holds by base change [Sta21, Tag 08IB] and the second since E is perfect. □

The property that a set of perfect objects is a relative exceptional collection can be checked fiberwise:

Lemma 3.22. *Let $X \to S$ be a flat quasi-compact and quasi-separated morphism of schemes. Then an object $E \in \mathrm{D}_{\mathrm{perf}}(X)$ is relative exceptional if and only if $E_s \in \mathrm{D}_{\mathrm{perf}}(X_s)$ is exceptional for all points $s \in S$, if and only if $E_s \in \mathrm{D}_{\mathrm{perf}}(X_s)$ is exceptional for all closed points $s \in S$. Similarly, a sequence $E_1, \dots, E_m \in \mathrm{D}_{\mathrm{perf}}(X)$ is a relative exceptional collection if and only its restriction to X_s is an exceptional collection for all points $s \in S$, or equivalently for all closed points $s \in S$.*

Proof. Let $E \in \mathrm{D}_{\mathrm{perf}}(X)$. The canonical morphism $\mathcal{O}_S \to \mathcal{H}om_S(E, E)$ is an isomorphism if and only if its restriction $\kappa(s) \to \mathcal{H}om_S(E, E)_s$ is an isomorphism for every point $s \in S$, or equivalently for every closed point $s \in S$. But by Lemma 3.21 and the flatness of g, this restriction is identified with $\kappa(s) \to \mathcal{H}om_s(E_s, E_s)$, which is an isomorphism if and only if E_s is exceptional. This proves the first claim of the lemma, and the second follows similarly. □

In the next lemma, we will use the following observation. If $g\colon X \to S$ is a proper morphism of finite Tor-dimension between noetherian schemes, then pushforward and pullback give functors $g_*\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(S)$ and $g^*\colon \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(X)$. The functor g_* is right adjoint to g^*. If the relative dualizing complex ω_g of g is a shift of a line bundle on X, then g_* also admits a right adjoint $g^!\colon \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(X)$ and g^* admits a left adjoint $g_!\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(S)$, given by

$$g^!(-) = g^*(-) \otimes \omega_g \quad \text{and} \quad g_!(-) = g_*(- \otimes \omega_g).$$

Indeed, the first formula holds by Grothendieck duality. More precisely, [Nee96, Theorem 5.4] applies because S is separated and $g^!$ preserves arbitrary coproducts since g_* sends perfect

complexes to perfect complexes by our assumption. Moreover, since ω_g is the shift of a line bundle, $g^!$ preserves bounded derived categories. The formula for $g_!$ follows from the one for $g^!$. The condition that ω_g is the shift of a line bundle holds for instance if $g\colon X \to S$ is smooth, or, more generally, if X and S are smooth over some common base.

Lemma 3.23. *Let $g\colon X \to S$ be a proper morphism of finite Tor-dimension between noetherian schemes. Let $E \in \mathrm{D}_{\mathrm{perf}}(X)$ be a relative exceptional object. Then the functor*

$$\alpha_E\colon \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(X),\ F \mapsto g^*(F) \otimes E.$$

is fully faithful, and admits a right adjoint given by

$$\alpha_E^!\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(S),\ G \mapsto \mathcal{H}om_S(E, G).$$

Moreover, if ω_g is a shift of a line bundle, then α_E admits a left adjoint given by

$$\alpha_E^*\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(S),\ G \mapsto g_!\mathcal{H}om_X(E, G).$$

Proof. For $F \in \mathrm{D}^{\mathrm{b}}(S)$ and $G \in \mathrm{D}^{\mathrm{b}}(X)$, we compute

$$\begin{aligned}\mathrm{Hom}(\alpha_E(F), G) &= \mathrm{Hom}(g^*(F) \otimes E, G)\\ &\simeq \mathrm{Hom}(g^*(F), \mathcal{H}om_X(E, G))\\ &= \mathrm{Hom}(F, \mathcal{H}om_S(E, G)),\end{aligned}$$

which proves the formula for $\alpha_E^!$. Moreover, we have

$$\begin{aligned}\alpha_E^!\alpha_E(F) &= \mathcal{H}om_S(E, g^*(F) \otimes E)\\ &= g_*\mathcal{H}om_X(E, g^*(F) \otimes E)\\ &\simeq g_*(g^*(F) \otimes \mathcal{H}om_X(E, E))\\ &= F \otimes \mathcal{H}om_S(E, E) \simeq F,\end{aligned}$$

where the final line holds because E is relative exceptional. This proves that α_E is fully faithful. Finally, an argument similar to the one for $\alpha_E^!$ shows the existence and claimed formula for α_E^* in the presence of a relative dualizing complex which is a shift of a line bundle. □

Remark 3.24. If the assumption in Lemma 3.23 guaranteeing the existence of α_E^* holds and X is in addition regular of finite Krull dimension, then in fact $\alpha_E^*(G) \simeq \mathcal{H}om_S(G, E)^\vee$. Indeed, the regularity of X guarantees that $\mathrm{D}^{\mathrm{b}}(X) = \mathrm{D}_{\mathrm{perf}}(X)$, and then the claim follows easily from Grothendieck duality.

In the situation of Lemma 3.23, we write $\mathrm{L}_{E/S}$ and $\mathrm{R}_{E/S}$ for the right and left mutation functors through $\alpha_E(\mathrm{D}^{\mathrm{b}}(S)) \subset \mathrm{D}^{\mathrm{b}}(X)$, see Definition 3.3. Note that these can be computed via exact triangles

$$g^*\mathcal{H}om_S(E, F) \otimes E \to F \to \mathrm{L}_{E/S}(F) \quad \text{and} \quad \mathrm{R}_{E/S}(F) \to F \to g^*g_!\mathcal{H}om_X(E, F).$$

Lemma 3.25. *Let $g\colon X \to S$ be a proper morphism of finite Tor-dimension between noetherian schemes. Let $E_1, \dots, E_m$ be a relative exceptional collection in $\mathrm{D}_{\mathrm{perf}}(X)$. Then there is an S-linear semiorthogonal decomposition of finite cohomological amplitude*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}, \alpha_{E_1}(\mathrm{D}^{\mathrm{b}}(S)), \dots, \alpha_{E_m}(\mathrm{D}^{\mathrm{b}}(S))\rangle, \tag{3.14}$$

where the left adjoint to the inclusion $\mathcal{D} \to \mathrm{D}^{\mathrm{b}}(X)$ is given by

$$\mathrm{L}_{E_1/S} \circ \mathrm{L}_{E_2/S} \cdots \circ \mathrm{L}_{E_m/S}. \tag{3.15}$$

If further $g\colon X \to S$ is smooth, then the components appearing in (3.14) are all admissible; in particular, (3.14) is a strong semiorthogonal decomposition.

Proof. First we claim the sequence $\alpha_{E_1}(\mathrm{D}^{\mathrm{b}}(S)), \dots, \alpha_{E_m}(\mathrm{D}^{\mathrm{b}}(S))$ is semiorthogonal. For this, it suffices to show that the composition $\alpha^!_{E_i} \circ \alpha_{E_j}$ vanishes for $i > j$. For $F \in \mathrm{D}^{\mathrm{b}}(S)$, we compute

$$\begin{aligned}
\alpha^!_{E_i}\alpha_{E_j}(F) &= \mathcal{H}om_S(E_i, g^*(F) \otimes E_j) \\
&= g_*\mathcal{H}om_X(E_i, g^*(F) \otimes E_j) \\
&\simeq g_*(g^*(F) \otimes \mathcal{H}om_X(E_i, E_j)) \\
&\simeq F \otimes \mathcal{H}om_S(E_i, E_j) = 0,
\end{aligned}$$

where in the third line we used that E_i is perfect and in the fourth we used the projection formula. Now the decomposition (3.14) and the formula (3.15) follow from [Per19, Lemma 3.10], since the subcategory $\alpha_{E_i}(\mathrm{D}^{\mathrm{b}}(S)) \subset \mathrm{D}^{\mathrm{b}}(X)$ is right admissible by Lemma 3.23.

If further the morphism $g\colon X \to S$ is smooth, then it follows from Lemma 3.23 that the $\alpha_{E_i}(\mathrm{D}^{\mathrm{b}}(S))$ are in fact admissible in $\mathrm{D}^{\mathrm{b}}(X)$. Then again by [Per19, Lemma 3.10] it follows that the subcategory $\langle \alpha_{E_1}(\mathrm{D}^{\mathrm{b}}(S)), \dots, \alpha_{E_m}(\mathrm{D}^{\mathrm{b}}(S))\rangle$ they generate is admissible. Thus $\mathcal{D}$ is also admissible by Lemma 3.26 below. □

Lemma 3.26. *Let $X \to S$ be a smooth and proper morphism of noetherian schemes. Let*

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$$

be an S-linear semiorthogonal decomposition. Then $\mathcal{D}_1$ is admissible if and only if $\mathcal{D}_2$ is admissible.

Proof. By our assumptions, $\mathrm{D}^{\mathrm{b}}(X)$ admits a relative Serre functor $\mathrm{S}_{\mathrm{D}^{\mathrm{b}}(X)/S}$; see the discussion preceding Corollary 7.5 for a review of this notion. Assume that $\mathcal{D}_1$ is admissible. Then there is a semiorthogonal decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{D}_2', \mathcal{D}_1 \rangle$. Moreover, it follows directly from the definition of the Serre functor that the autoequivalence $\mathrm{S}_{\mathrm{D}^{\mathrm{b}}(X)/S}$ takes $\mathcal{D}_2$ to $\mathcal{D}_2'$. Since $\mathcal{D}_2'$ admits a left adjoint, it follows that $\mathcal{D}_2$ does too. Hence $\mathcal{D}_2$ is admissible. The proof that admissibility of $\mathcal{D}_2$ implies that of $\mathcal{D}_1$ is similar. □

4. Local t-structures

In this section, we discuss basic definitions and results about local t-structures. We start by recalling the absolute case.

4.1. t-structures.

Definition 4.1. A *t-structure* τ on a triangulated category $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ satisfying the following conditions:

(1) $\mathcal{D}^{\leqslant 0}[1] \subset \mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 0}[-1] \subset \mathcal{D}^{\geqslant 0}$;
(2) $\mathrm{Hom}(F, G) = 0$, for every $F \in \mathcal{D}^{\leqslant 0}$ and $G \in \mathcal{D}^{\geqslant 1}$;

(3) every object $E \in \mathcal{D}$ fits into an exact triangle

$$\tau^{\leqslant 0}E \to E \to \tau^{\geqslant 1}E \to \tau^{\leqslant 0}E[1]$$

with $\tau^{\leqslant 0}E \in \mathcal{D}^{\leqslant 0}$ and $\tau^{\geqslant 1}E \in \mathcal{D}^{\geqslant 1}$.

Here we used the notation $\mathcal{D}^{\leqslant n} := \mathcal{D}^{\leqslant 0}[-n]$ and $\mathcal{D}^{\geqslant n} := \mathcal{D}^{\geqslant 0}[-n]$, for any $n \in \mathbb{Z}$. Similarly, for the truncation functors $\tau^{\leqslant n}, \tau^{\geqslant n}$. Moreover, we let $\mathcal{D}^{[a,b]} = \mathcal{D}^{\leqslant b} \cap \mathcal{D}^{\geqslant a}$, for all $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a \leqslant b$.

Definition 4.2. The *heart* of a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is the abelian category $\mathcal{A} = \mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$.

The cohomology objects of an object $E \in \mathcal{D}$ with respect to the heart of a t-structure $\mathcal{A}$ will be denoted by $\mathrm{H}^\bullet_{\mathcal{A}}(E)$. When $\mathcal{D} = \mathrm{D^b}(X)$ and $\mathcal{A} = \operatorname{Coh} X$ we simply write $\mathrm{H}^\bullet(E)$.

Definition 4.3. A t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is *bounded* if $\mathcal{D} = \bigcup_{n,m \in \mathbb{Z}} \mathcal{D}^{\leqslant n} \cap \mathcal{D}^{\geqslant m}$.

Remark 4.4. Our terminology for bounded t-structures follows [Bri07]. Such a t-structure is called nondegenerate in [AP06, Pol07], and bounded and nondegenerate in [BBD82].

A bounded t-structure is uniquely determined by its heart.

Proposition 4.5 ([Bri08, Lemma 3.1])**.** *If $\mathcal{A} \subset \mathcal{D}$ is a full additive subcategory of a triangulated category $\mathcal{D}$, then $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ if and only if*

(1) *for $F, G \in \mathcal{A}$, we have $\operatorname{Hom}_{\mathcal{D}}(F, G[k]) = 0$, for all $k < 0$; and*
(2) *for all $E \in \mathcal{D}$, there are integers $m < n$ and a collection of triangles*

$$\begin{tikzcd}
0 = E_{m-1} \arrow[rr] & & E_m \arrow[rr] \arrow[dl] & & E_{m+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_m \arrow[ul, dashed] & & A_{m+1} \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_i[i] \in \mathcal{A}$ for all i.

Definition 4.6. Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a t-structure. A pair of additive subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is called a *torsion pair* if

- for all $T \in \mathcal{T}$ and for all $F \in \mathcal{F}$, we have $\operatorname{Hom}(T, F) = 0$,
- for all $E \in \mathcal{A}$, there exist $T_E \in \mathcal{T}$, $F_E \in \mathcal{F}$, and an exact sequence

$$0 \to T_E \to E \to F_E \to 0.$$

Note that if $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{F} = \mathcal{T}^\perp$ is the right orthogonal to $\mathcal{T}$ in $\mathcal{A}$; we will call a subcategory $\mathcal{T} \subset \mathcal{A}$ a *torsion subcategory* if $(\mathcal{T}, \mathcal{T}^\perp)$ forms a torsion pair.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, we can *tilt* to obtain a new t-structure $(\mathcal{D}^{\sharp,\leqslant 0}, \mathcal{D}^{\sharp,\geqslant 0})$ with

$$\mathcal{D}^{\sharp,\leqslant 0} := \left\{ E \in \mathcal{D}^{\leqslant 0} \,\middle|\, \mathrm{H}^0_{\mathcal{A}}(E) \in \mathcal{T} \right\}, \quad \mathcal{D}^{\sharp,\geqslant 0} := \left\{ E \in \mathcal{D}^{\geqslant -1} \,\middle|\, \mathrm{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \right\},$$

see [HRS96]. Its heart $\mathcal{A}^\sharp$ can be described as the extension-closure

$$\mathcal{A}^\sharp := \langle \mathcal{F}[1], \mathcal{T} \rangle.$$

If $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is bounded, then so is $(\mathcal{D}^{\sharp,\leqslant 0}, \mathcal{D}^{\sharp,\geqslant 0})$. By [Pol07, Lemma 1.1.2], $(\mathcal{D}^{\sharp,\leqslant 0}, \mathcal{D}^{\sharp,\geqslant 0})$ can be obtained by tilting from $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ if and only if $\mathcal{D}^{\leqslant -1} \subset \mathcal{D}^{\sharp,\leqslant 0} \subset \mathcal{D}^{\leqslant 0}$.

Definition 4.7. A t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is

(1) *noetherian* if its heart is noetherian, and
(2) *tilted-noetherian* if it can be obtained from a noetherian t-structure $(\mathcal{D}_0^{\leqslant 0}, \mathcal{D}_0^{\geqslant 0})$ on $\mathcal{D}$ by tilting, i.e., if $\mathcal{D}_0^{\leqslant -1} \subset \mathcal{D}^{\leqslant 0} \subset \mathcal{D}_0^{\leqslant 0}$.

In [Pol07] a tilted-noetherian t-structure is called close to noetherian.

Definition 4.8. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be a pair of triangulated categories equipped with t-structures. An exact functor $\Phi\colon \mathcal{D}_1 \to \mathcal{D}_2$ is called *left* (resp. *right*) *t-exact* if $\Phi(\mathcal{D}_1^{\geqslant 0}) \subseteq \mathcal{D}_2^{\geqslant 0}$ (resp. $\Phi(\mathcal{D}_1^{\leqslant 0}) \subseteq \mathcal{D}_2^{\leqslant 0}$). A *t-exact functor* is a functor which is both left and right t-exact.

Remark 4.9. Recall that a functor between triangulated categories is *conservative* if $\Phi(E) \cong 0$ implies $E \cong 0$. Now assume $\Phi\colon \mathcal{D}_1 \to \mathcal{D}_2$ is both conservative and t-exact, and write $\mathcal{A}_i \subset \mathcal{D}_i$ for the corresponding hearts. Then $\Phi(E) \in \mathcal{A}_2$ if and only if $E \in \mathcal{A}_1$. Similarly, in this case two morphisms $A \to B \to C$ in $\mathcal{D}_1$ form a short exact sequence in $\mathcal{A}_1$ if and only if $\Phi(A) \to \Phi(B) \to \Phi(C)$ is a short exact sequence in $\mathcal{A}_2$.

4.2. **Local t-structures.** In the case of a linear category, we will be interested in t-structures that are local over the base scheme in the following sense.

Definition 4.10. Let $X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component.

(1) A t-structure on $\mathcal{D}_{\mathrm{qc}}$ is called *S-local* if for every quasi-compact open $U \subset S$, there exists a t-structure on $(\mathcal{D}_{\mathrm{qc}})_U$ such that the restriction functor $\mathcal{D}_{\mathrm{qc}} \to (\mathcal{D}_{\mathrm{qc}})_U$ is t-exact.
(2) Assume the projection functor of $\mathcal{D}$ has finite cohomological amplitude. Then a t-structure on $\mathcal{D}$ is called *S-local* if for every quasi-compact open $U \subset S$, there exists a t-structure on $\mathcal{D}_U$ such that the restriction functor $\mathcal{D} \to \mathcal{D}_U$ is t-exact.

In Definition 4.10 we require $U \subset S$ to be quasi-compact so that the base change categories $(\mathcal{D}_{\mathrm{qc}})_U$ and (if the projection functor has finite cohomological amplitude) $\mathcal{D}_U$ are defined; see the discussion preceding Lemma 3.18. In particular, U may be any affine open in S, or an arbitrary open if S is noetherian.

Lemma 3.18 implies that, given an S-local t-structure on $\mathcal{D}$ or $\mathcal{D}_{\mathrm{qc}}$, for every quasi-compact open $U \subset S$ the t-structure on $\mathcal{D}_U$ or $(\mathcal{D}_{\mathrm{qc}})_U$ is uniquely determined. We shall sometimes refer to this as the induced t-structure on $\mathcal{D}_U$ or $(\mathcal{D}_{\mathrm{qc}})_U$. We denote by $\mathcal{A}_U \subset \mathcal{D}_U$ or $(\mathcal{A}_{\mathrm{qc}})_U \subset (\mathcal{D}_{\mathrm{qc}})_U$ the heart of the corresponding t-structure.

Our primary interest is local t-structures on $\mathcal{D}$, but parallel to the situation for base change of linear categories from Section 3.2, when we discuss base change of t-structures in Section 5 we will also need to consider $\mathcal{D}_{\mathrm{qc}}$. In the rest of this section, we focus on results that do not require the use of $\mathcal{D}_{\mathrm{qc}}$.

Note that for $F \in \mathcal{D}$ the condition $F \in \mathcal{D}^{[a,b]}$ can be checked locally on S, since the cohomology functors $\mathrm{H}^\bullet_{\mathcal{A}_U}$ commute with restriction and the condition that an object in $\mathcal{D}$ vanishes can be checked locally. This observation has the following consequence.

Lemma 4.11. *Let $g\colon X \to S$ be a morphism schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude, and which is equipped with an S-local t-structure. Then for any vector bundle V on S, the functor $(g^*V \otimes -)\colon \mathcal{D} \to \mathcal{D}$ is t-exact.*

The following gives a relative analogue of condition (2) in Definition 4.1; recall from Section 2 our notation $\mathcal{H}om_S(-,-)$ for the S-relative derived sheaf Hom.

Lemma 4.12. *Let $X \to S$ be a proper morphism of noetherian schemes with affine diagonals, where X has finite Krull dimension and S admits an ample line bundle. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude, and which is equipped with an S-local t-structure. Let $F \in \mathcal{D}^{\leqslant a} \cap \mathrm{D}_{\mathrm{perf}}(X)$ and $G \in \mathcal{D}^{\geqslant b}$. Then*

$$\mathcal{H}om_S(F,G) \in \mathrm{D}^{\mathrm{b}}(S)^{\geqslant b-a}.$$

Proof. Note that since g is a proper morphism between noetherian schemes and F is perfect, the object $\mathcal{H}om_S(F,G)$ indeed lies in $\mathrm{D}^{\mathrm{b}}(S)$. Let $q \in \mathbb{Z}$ be the smallest integer so that $\mathrm{H}^q_{\mathrm{Coh}\,S}\mathcal{H}om_S(F,G) \neq 0$. We must show $q \geqslant b-a$. Note that if L is a line bundle on S, then by the projection formula and perfectness of F we have

$$\mathcal{H}om_S(F,G) \otimes L \simeq \mathcal{H}om_S(F, G \otimes g^*L).$$

Hence there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(S, \mathrm{H}^q_{\mathrm{Coh}\,S}\mathcal{H}om_S(F,G) \otimes L) \Rightarrow \mathrm{Hom}(F, G \otimes g^*L[p+q]).$$

For degree reasons, the term $\mathrm{H}^0(S, \mathrm{H}^q_{\mathrm{Coh}\,S}\mathcal{H}om_S(F,G) \otimes L)$ must survive in the spectral sequence. By choosing a suitably ample L we can ensure this term, and hence also $\mathrm{Hom}(F, G \otimes g^*L[q])$, is nonzero. But by Lemma 4.11 we have $G \otimes g^*L \in \mathcal{D}^{\geqslant b}$, so it follows that $q \geqslant b-a$. □

Theorem 4.13. *Let $X \to S$ be a morphism schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension and S admits an ample line bundle L. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude. Then a bounded t-structure on $\mathcal{D}$ is S-local if and only if tensoring with g^*L is left t-exact, or equivalently, if and only if tensoring with g^*L is t-exact. In particular, if S is affine then any bounded t-structure on $\mathcal{D}$ is automatically S-local.*

Proof. This is the analogue of [Pol07, Theorem 2.3.2] (that builds on [AP06, Theorem 2.1.4]) in our setup, and follows by the same argument. □

Definition 4.14. Let $X \to S$ be a morphism schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude. An S-local t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ on $\mathcal{D}$ is called:

(1) *noetherian locally on S* if for every quasi-compact open subscheme $U \subset S$, the induced t-structure on $\mathcal{D}_U$ is noetherian.
(2) *tilted-noetherian locally on S* if there exists an S-local t-structure $(\mathcal{D}_0^{\leqslant 0}, \mathcal{D}_0^{\geqslant 0})$ on $\mathcal{D}$ which is noetherian locally on S, such that $\mathcal{D}_0^{\leqslant -1} \subset \mathcal{D}^{\leqslant 0} \subset \mathcal{D}_0^{\leqslant 0}$.

An S-local t-structure is tilted-noetherian locally on S if and only if it is obtained by tilting (on each quasi-compact open) from an S-local t-structure which is noetherian locally on S.

Lemma 4.15. *Let $X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude, and which is equipped with an S-local t-structure. The following are equivalent:*

(1) *The t-structure on $\mathcal{D}$ is noetherian locally on S.*
(2) *For every quasi-compact open $U \subset S$ in a basis for the topology of S, the induced t-structure on $\mathcal{D}_U$ is noetherian.*
(3) *There is an open cover $S = \cup_i U_i$ by quasi-compact opens $U_i \subset S$ such that the induced t-structure on $\mathcal{D}_{U_i}$ is noetherian locally on U_i for each i.*
(4) *There is an affine open cover $S = \cup_i U_i$ such that the induced t-structure on $\mathcal{D}_{U_i}$ is noetherian for each i.*

Moreover, if S admits an ample line bundle, then the above conditions are further equivalent to the following:

(5) *The t-structure on $\mathcal{D}$ is noetherian.*

Proof. Clearly (1) implies (2), (3), and (4).

To see (2) implies (1), let $U \subset S$ be an arbitrary quasi-compact open. Choose a finite cover $U = \cup U_i$ by quasi-compact opens $U_i \subset S$ in the given basis for the topology of S. Now observe that an increasing sequence $F_1 \subset F_2 \subset \cdots \subset F$ in $\mathcal{A}_U \subset \mathcal{D}_U$ stabilizes at the N-th term if and only if its restriction to U_i stabilizes at the N-th term for all i. Since there are finitely many U_i, it follows that (2) implies (1).

Similarly, if (3) holds and $U \subset S$ is a quasi-compact open, then we can write $U = \cup_j V_j$ as a union of finitely many $V_j = U \cap U_j$. Since the t-structure on $\mathcal{D}_{V_j}$ is noetherian by assumption (note that each V_j is quasi-compact since S has affine diagonal), we conclude as above that the t-structure on $\mathcal{D}_U$ is noetherian, i.e., (1) holds.

If S admits an ample line bundle, then S is in particular quasi-compact, so (1) implies (5). Conversely, we show that if S admits an ample line bundle, then (5) implies (2). Note that S has a basis for its topology given by affine open sets of the form $U = \{x \in S \mid f(x) \neq 0\}$, where f is a global section of a line bundle L on S. Hence it suffices to show that for such a U, any sequence $F_1 \subset F_2 \subset \cdots \subset F$ of inclusions in the induced heart $\mathcal{A}_U \subset \mathcal{D}_U$ stabilizes. By Lemma 4.16.(3) below, there is a sequence $\tilde{F}_1 \subset \tilde{F}_2 \subset \cdots \subset \tilde{F}$ in $\mathcal{A}_S$ which restricts to the given sequence $F_1 \subset F_2 \subset \cdots \subset F$ in $\mathcal{A}_U$. Since the sequence in $\mathcal{A}_S$ must stabilize by assumption, this claim implies the result.

So far we have shown (1), (2), and (3) are equivalent, imply (4), and if S admits an ample line bundle then they are further equivalent to (5). Since any affine scheme admits an ample line bundle, we thus conclude that (4) implies (3), finishing the proof. □

The proof relied on the following Lemma, which we state in a more general form that will become useful in Section 5.

Lemma 4.16. *Let $g\colon X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude. Let $f\colon T \to S$ be one, or the composition, of*

(a) *the inclusion of an open subset given by the non-vanishing locus of a section $h \in \Gamma(S, L)$ of a line bundle L on S, and*
(b) *a morphism between affine schemes given by a localization of rings.*

(1) *Let $\tilde{F} \in \mathcal{D}$ be an object with pullback $F = \tilde{F}_T \in \mathcal{D}_T$. Let $\beta\colon G \to F$ be a morphism in $\mathcal{D}_T$. Then there exists a morphism $\tilde{\beta}\colon \tilde{G} \to \tilde{F}$ in $\mathcal{D}$ such that $g^*\tilde{\beta} = \beta$.*

(2) *Assume that $\mathcal{D}$ and $\mathcal{D}_T$ have t-structures such that f^* is t-exact. Let $\tilde{F} \in \mathcal{A}_S$ and $F = \tilde{F}_T$, and let $\beta\colon G \to F$ be a morphism in $\mathcal{A}_T$. Then there exists a morphism $\tilde{\beta}\colon \tilde{G} \to \tilde{F}$ in $\mathcal{A}_S$ with $f^*\tilde{\beta} = \beta$. If β is injective, we can choose $\tilde{\beta}$ to be injective; if β is instead surjective, we can replace $\tilde{F}$ by another lift $\tilde{F}'$ to make $\tilde{\beta}$ surjective.*

(3) *More generally, any (not necessarily finite) filtration in $\mathcal{A}_T$*

$$F_1 \subset F_2 \subset \ldots \subset F,$$

can be lifted to a filtration in $\mathcal{A}_S$

$$\tilde{F}_1 \subset \tilde{F}_2 \subset \ldots \subset \tilde{F}.$$

Proof. To prove (1), we first use Lemma 3.18 and choose an arbitrary lift $\tilde{G}_0 \in \mathcal{D}$ of G. In case (a), [AP06, Lemma 2.1.8] shows that $h^k\beta$ extends to a morphism $G_0 \otimes L^{-k} \to F$, which restricts to β after the identification $L_U^{-1} \cong \mathcal{O}_U$ induced by h. In case (b), flat base change implies $\operatorname{Hom}(G, F) = \operatorname{Hom}(\tilde{G}_0, \tilde{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_T$; thus a similar statement holds with L trivial, and $h \in H^0(\mathcal{O}_S)$ in the localizing subset; replacing $\tilde{G}_0$ by the isomorphic object $\tilde{G}_0 \otimes (h)$ then yields a morphism as claimed.

To prove (2), we first choose a lift $\tilde{G}_1 \in \mathcal{D}$ as in (1). Since f^* is exact, we can replace $\tilde{G}_1$ by $\tilde{G}_2 = \mathrm{H}^0_{\mathcal{A}_S}(\tilde{G}_1)$ and obtain a lift of G and β in $\mathcal{A}_S$. Finally, since f^* is t-exact, the morphism β is injective if and only if $f^* \operatorname{Ker} \tilde{\beta} = 0$; in this case, we can thus replace $\tilde{G}_2$ by the image of $\tilde{\beta}$ to obtain an injective lift of β in $\mathcal{A}_T$. The case of β surjective follows similarly.

Finally, to prove (3), we first choose lifts $\tilde{\beta}_i\colon \tilde{F}'_i \hookrightarrow \tilde{F}$ given by the previous steps. Then we obtain a desired filtration of $\tilde{F}$ by replacing $\tilde{F}'_i$ with

$$\tilde{F}_i := \bigoplus_{j \leqslant i} \operatorname{Im} \tilde{\beta}_i. \qquad \square$$

Remark 4.17. We will see in Theorem 5.7 that such a t-structure on $\mathcal{D}_T$ always exists.

5. Base change of local t-structures

In this section we prove results on base changes of local t-structures. Namely, given an S-linear category $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ with an S-local t-structure, we construct under suitable hypotheses induced t-structures on the base changes of $\mathcal{D}_{\mathrm{qc}}$ and $\mathcal{D}$ along a morphism $T \to S$ (Theorems 5.3 and 5.7). The result for base changes of $\mathcal{D}$ generalizes the results on "constant t-structures" from [AP06, Section 2] and [Pol07, Theorem 3.3.6], which correspond to the case where $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ and S is a point. In fact, many of the ingredients in our proof come from [AP06, Pol07].

5.1. The unbounded quasi-coherent case.

In this subsection we focus on base change of t-structures in the setting of unbounded derived categories of quasi-coherent sheaves. In this setting it is possible to prove results with very few hypotheses, because it is very easy to construct t-structures, as the following lemma illustrates.

Lemma 5.1. *Let $X \to S$ be a morphism of schemes where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component endowed with a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. Then there is a t-structure on $(\mathcal{D}^{\leqslant 0}_{\mathrm{qc}}, \mathcal{D}^{\geqslant 0}_{\mathrm{qc}})$ on $\mathcal{D}_{\mathrm{qc}}$ where:*

- *$\mathcal{D}^{\leqslant 0}_{\mathrm{qc}}$ is the smallest full subcategory of $\mathcal{D}_{\mathrm{qc}}$ which contains $\mathcal{D}^{\leqslant 0}$ and is closed under extensions and small colimits.*

- $\mathcal{D}_{\mathrm{qc}}^{\geqslant 0} = \{F \in \mathcal{D}_{\mathrm{qc}} \mid \mathrm{Hom}(G, F) = 0 \textit{ for all } G \in \mathcal{D}^{\leqslant -1}\}$.

This t-structure has the following properties:

(1) *The truncation functors commute with filtered colimits.*
(2) *The inclusion $\mathcal{D} \to \mathcal{D}_{\mathrm{qc}}$ is t-exact.*
(3) *For $F \in \mathcal{D}_{\mathrm{qc}}$, we have $F \in \mathcal{D}_{\mathrm{qc}}^{[a,b]}$ if and only if $F = \operatorname{colim} F_\alpha$ for a filtered system of objects $F_\alpha \in \mathcal{D}^{[a,b]}$.*
(4) *Assume X and S are quasi-compact with affine diagonal, the projection functor of $\mathcal{D}$ has finite cohomological amplitude, and the t-structure on $\mathcal{D}$ is S-local. Then the above t-structure on $\mathcal{D}_{\mathrm{qc}}$ is S-local. More precisely, if $U \subset S$ is a quasi-compact open, then the t-structure on $(\mathcal{D}_{\mathrm{qc}})_U$ making $\mathcal{D}_{\mathrm{qc}} \to (\mathcal{D}_{\mathrm{qc}})_U = (\mathcal{D}_U)_{\mathrm{qc}}$ t-exact is obtained by applying the above construction to the t-structure on $\mathcal{D}_U$.*

Proof. By [Lur17, Propositions 1.4.4.11 and 1.4.4.13] there is a t-structure on $\mathcal{D}_{\mathrm{qc}}$ with $\mathcal{D}_{\mathrm{qc}}^{\leqslant 0}$ as described, whose truncation functors commute with filtered colimits. Then $\mathcal{D}_{\mathrm{qc}}^{\geqslant 0}$ is the right orthogonal to $\mathcal{D}_{\mathrm{qc}}^{\leqslant -1}$, which is easily seen to be given by the stated formula. The t-exactness of $\mathcal{D} \to \mathcal{D}_{\mathrm{qc}}$ then follows directly. We note that the above argument is essentially the same as [Pol07, Lemma 2.1.1], but we use [Lur17] for the statement about truncation functors.

By construction $\mathcal{D}_{\mathrm{qc}} = \mathrm{Ind}(\mathcal{D}_{\mathrm{perf}})$, so using parts (1) and (2) of the lemma, the argument of Lemma 3.14 proves part (3).

Now assume we are in the situation of part (4). Let $U \subset S$ be a quasi-compact open subset. By what we have already shown, the t-structure on $\mathcal{D}_U$ induces one on $(\mathcal{D}_U)_{\mathrm{qc}}$. Note that $(\mathcal{D}_U)_{\mathrm{qc}} = (\mathcal{D}_{\mathrm{qc}})_U$. We claim that the restriction functor $\mathcal{D}_{\mathrm{qc}} \to (\mathcal{D}_U)_{\mathrm{qc}}$ is t-exact, which will prove that the t-structure on $\mathcal{D}_{\mathrm{qc}}$ is S-local. This follows from part (3) of the lemma, the fact that the restriction functor $\mathcal{D}_{\mathrm{qc}} \to (\mathcal{D}_U)_{\mathrm{qc}}$ commutes with colimits, and the S-locality of the t-structure on $\mathcal{D}$. □

Remark 5.2. In Lemma 5.1, if $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ with the standard t-structure, then the induced t-structure on $\mathcal{D}_{\mathrm{qc}} = \mathrm{D}_{\mathrm{qc}}(X)$ is the standard one.

In [Pol07], the focus is on inducing t-structures in the setting of bounded derived categories of coherent sheaves, but the idea of first constructing t-structures on unbounded derived categories of quasi-coherent sheaves is used extensively. The following theorem can be thought of as an elaboration on this idea; see also [HL14, Section 5] for similar results in the setting of varieties over a field.

Theorem 5.3. *Let $g\colon X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component with a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. Let $\phi\colon T \to S$ be a morphism from a scheme T which is quasi-compact with affine diagonal, such that ϕ is faithful with respect to g. Let g' and ϕ' be the morphisms in the following base change diagram:*

$$\begin{tikzcd} X_T \arrow[r, "\phi'"] \arrow[d, "g'"'] & X \arrow[d, "g"] \\ T \arrow[r, "\phi"] & S \end{tikzcd}$$

If T is affine, then there is a t-structure $((\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}, (\mathcal{D}_{\mathrm{qc}})_T^{\geqslant 0})$ on $(\mathcal{D}_{\mathrm{qc}})_T$ where:

- *$(\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}$ is the smallest full subcategory of $(\mathcal{D}_{\mathrm{qc}})_T$ which contains $\phi'^*(\mathcal{D}^{\leqslant 0})$ and is closed under extensions and small colimits.*
- $(\mathcal{D}_{\mathrm{qc}})_T^{\geqslant 0} = \{F \in (\mathcal{D}_{\mathrm{qc}})_T \mid \mathrm{Hom}(G, F) = 0$ *for all* $G \in \phi'^*(\mathcal{D}^{\leqslant -1})\}$.

In general, there is a t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ given by

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \;\middle|\; \begin{array}{c} F_U \in (\mathcal{D}_{\mathrm{qc}})_U^{[a,b]} \text{ for any flat morphism} \\ U \to T \text{ with } U \text{ affine} \end{array} \right\}. \tag{5.1}$$

This t-structure has the following properties:

(1) *The truncation functors of $(\mathcal{D}_{\mathrm{qc}})_T$ commute with filtered colimits.*
(2) *For any fpqc cover $\{U_i \to T\}$ of T by affine schemes U_i, we have*

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \;\middle|\; F_{U_i} \in (\mathcal{D}_{\mathrm{qc}})_{U_i}^{[a,b]} \text{ for all } i \right\}. \tag{5.2}$$

In particular, for T affine the above prescriptions for t-structures on $(\mathcal{D}_{\mathrm{qc}})_T$ agree.

(3) *For any $G \in \mathrm{D}_{\mathrm{qc}}^{\leqslant 0}(T)$, the functor $(g_T^*(G) \otimes -) \colon (\mathcal{D}_{\mathrm{qc}})_T \to (\mathcal{D}_{\mathrm{qc}})_T$ is right t-exact, where $g_T \colon X_T \to T$ denotes the projection.*
(4) *Let $T' \to S$ be another morphism with the same assumptions as ϕ, let $f \colon T' \to T$ be a morphism of schemes over S, and let $f' \colon X_{T'} \to X_T$ be the induced morphism.*
   (a) *$f'^* \colon (\mathcal{D}_{\mathrm{qc}})_T \to (\mathcal{D}_{\mathrm{qc}})_{T'}$ is right t-exact.*
   (b) *$f'_* \colon (\mathcal{D}_{\mathrm{qc}})_{T'} \to (\mathcal{D}_{\mathrm{qc}})_T$ is left t-exact.*
   (c) *If f is flat, then $f'^* \colon (\mathcal{D}_{\mathrm{qc}})_T \to (\mathcal{D}_{\mathrm{qc}})_{T'}$ is t-exact.*
   (d) *If f is affine, then $f'_* \colon (\mathcal{D}_{\mathrm{qc}})_{T'} \to (\mathcal{D}_{\mathrm{qc}})_T$ is t-exact.*
(5) *Assume the projection functor of $\mathcal{D}$ has finite cohomological amplitude and the t-structure on $\mathcal{D}$ is S-local. Then for $T = S$ the above t-structure agrees with the one on $\mathcal{D}_{\mathrm{qc}}$ from Lemma 5.1. Moreover, for general T we have*

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \;\middle|\; \begin{array}{c} \phi'_{U*}(F_U) \in \mathcal{D}_{\mathrm{qc}}^{[a,b]} \text{ for any flat morphism} \\ U \to T \text{ with } U \text{ affine} \end{array} \right\} \tag{5.3}$$

where $\phi'_U \colon X_U \to X$ is the morphism induced by $U \to T \xrightarrow{\phi} S$, and for any fpqc cover $\{U_i \to T\}$ by affine schemes U_i, we have

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \;\middle|\; \phi'_{U_i*}(F_{U_i}) \in \mathcal{D}_{\mathrm{qc}}^{[a,b]} \text{ for all } i \right\}. \tag{5.4}$$

Proof. We prove the theorem in several steps.

Step 1. If T is affine, the prescription for $((\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}, (\mathcal{D}_{\mathrm{qc}})_T^{\geqslant 0})$ defines a t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ such that parts (1) and (3) of the theorem hold.

This argument of Lemma 5.1 shows the prescription defines a t-structure satisfying (1). Since T is affine, the object $\mathcal{O}_T$ generates $\mathrm{D}_{\mathrm{qc}}(T)^{\leqslant 0}$ under colimits. But g_T^* and tensor products commute with colimits, and tensoring with $\mathcal{O}_{X_T} = g_T^*(\mathcal{O}_T)$ preserves $\phi'^*(\mathcal{D}^{\leqslant 0})$. So it follows that $g_T^*(G) \otimes (\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0} \subset (\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}$ for any $G \in \mathrm{D}_{\mathrm{qc}}(T)^{\leqslant 0}$, i.e., (3) holds.

Step 2. If T and T' are affine, part (4) of the theorem holds for the t-structures from Step 1.

Since f'^* admits a right adjoint, it commutes with colimits. It follows that $f'^*((\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}) \subset (\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}$, i.e., f'^* is right t-exact.

Since f'_* is right adjoint to f'^*, it follows formally that f'_* is left t-exact. Note that for any object $F \in (\mathcal{D}_{\mathrm{qc}})_T$ we have $f'_* f'^*(F) \simeq F \otimes f'_*(\mathcal{O}_{T'})$. The morphism f' is affine, being

the base change of f, and hence $f'_*(\mathcal{O}_{T'})$ is a sheaf. Therefore by the property (3) proved in Step 1, it follows that $f'_*(f'^*\phi'^*(\mathcal{D}^{\leqslant 0})) \subset (\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}$. But since f' is affine f'_* preserves colimits (see for instance [Lur18, Proposition 2.5.1.1]), so it follows that $f'_*((\mathcal{D}_{\mathrm{qc}})_{T'}^{\leqslant 0}) \subset (\mathcal{D}_{\mathrm{qc}})_T^{\leqslant 0}$. This proves f'_* is also right t-exact, and hence t-exact.

Finally, assume f is flat. Since the functor f'_* is conservative and by the above t-exact, the t-exactness of f'^* is equivalent to t-exactness of the functor

$$f'_* \circ f'^* \simeq (f'_*(\mathcal{O}_{T'}) \otimes -) \colon \mathcal{D}_T \to \mathcal{D}_T.$$

Note that $f'_*(\mathcal{O}_{T'}) \simeq g_T^*(f_*\mathcal{O}_T)$. Since $f \colon T' \to T$ is a flat morphism of affine schemes, by Lazard's theorem $f_*\mathcal{O}_T$ is a filtered colimit of finite free $\mathcal{O}_T$-modules. Hence $g_T^*(f_*\mathcal{O}_T)$ is a filtered colimit of finite free $\mathcal{O}_{X_T}$-modules. Now it follows from the property (1) proved in Step 1 that the above functor is t-exact.

Step 3. For any fpqc cover $\{U_i \to T\}$ of T by affine schemes U_i, the prescription (5.2) defines a t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ such that part (1) of the theorem holds.

Let $U^\bullet \to T$ be the Čech nerve of the map $U = \bigsqcup U_i \to T$, i.e., the simplicial scheme which in degree n is given by the $(n+1)$-fold fiber product of U over T. Then by fpqc descent for $\mathrm{D}_{\mathrm{qc}}(-)$, pullback induces an equivalence

$$\mathrm{D}_{\mathrm{qc}}(T) \simeq \mathrm{Tot}(\mathrm{D}_{\mathrm{qc}}(U^\bullet))$$

where the right side denotes the totalization, i.e., the limit, of the cosimplicial diagram of ∞-categories $\mathrm{D}_{\mathrm{qc}}(U^\bullet)$. Note that $\mathcal{D}_{\mathrm{qc}}$ is compactly generated (by $\mathcal{D}_{\mathrm{perf}}$). Hence by [Per19, Lemma 4.3] the category $\mathcal{D}_{\mathrm{qc}}$ is dualizable as an object of PrCat_S (see Remark 3.16). It follows that the functor from PrCat_S to itself given by tensoring with $\mathcal{D}_{\mathrm{qc}}$ admits a right adjoint (given by tensoring with the dual of $\mathcal{D}_{\mathrm{qc}}$), and thus commutes with limits. In particular, the above equivalence implies that pullback induces an equivalence

$$(\mathcal{D}_{\mathrm{qc}})_T \simeq \mathrm{Tot}((\mathcal{D}_{\mathrm{qc}})_{U^\bullet}). \tag{5.5}$$

For $i_0, i_1, \ldots, i_n$, we write

$$U_{i_0,\ldots,i_n} = U_{i_0} \times_T U_{i_1} \cdots \times_T U_{i_n}.$$

Then U^n is the coproduct of the $U_{i_0,\ldots,i_n}$ over all $i_0, \ldots, i_n$. Hence pullback induces an equivalence

$$\mathrm{D}_{\mathrm{qc}}(U^n) \simeq \prod_{i_0,\ldots,i_n} \mathrm{D}_{\mathrm{qc}}(U_{i_0,\ldots,i_n}).$$

By the observation above, this implies that pullback induces an equivalence

$$(\mathcal{D}_{\mathrm{qc}})_{U^n} \simeq \prod_{i_0,\ldots,i_n} (\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\ldots,i_n}}. \tag{5.6}$$

Combining (5.5) and (5.6), we see that $(\mathcal{D}_{\mathrm{qc}})_T$ is expressed as a limit of the diagram of categories $(\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\ldots,i_n}}$.

Note that each $U_{i_0,\ldots,i_n}$ is affine since T has affine diagonal. Hence by Step 1 each term $(\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\ldots,i_n}}$ carries a t-structure whose truncation functors preserve filtered colimits. Moreover, the projection morphisms between the $U_{i_0,\ldots,i_n}$ are flat. Hence by property (4c) verified

in Step 2, the pullback functors between the $(\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\dots,i_n}}$ are t-exact. By [GR17, Chapter I.3, Lemma 1.5.8] their limit $(\mathcal{D}_{\mathrm{qc}})_T$ thus carries a t-structure given by

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \,\middle|\, F_{U_{i_0,\dots,i_n}} \in (\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\dots,i_n}}^{[a,b]} \text{ for all } i_0,\dots,i_n \right\},$$

whose truncation functors preserve filtered colimits. Since the pullback functors between the $(\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\dots,i_n}}$ are t-exact, the condition $F_{U_{i_0,\dots,i_n}} \in (\mathcal{D}_{\mathrm{qc}})_{U_{i_0,\dots,i_n}}^{[a,b]}$ for all $i_0,\dots,i_n$ is equivalent to $F_{U_i} \in (\mathcal{D}_{\mathrm{qc}})_{U_i}^{[a,b]}$ for all i.

Step 4. The prescription (5.1) *defines a t-structure on* $(\mathcal{D}_{\mathrm{qc}})_T$ *such that parts* (1) *and* (2) *of the theorem hold.*

Take any fpqc cover $\{U_i \to T\}$ of T by affine schemes U_i, and let $U \to T$ be a flat morphism with U affine. Then $\{U_i \to T\} \cup \{U \to T\}$ is also an fpqc cover. By Step 3, both the formula

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \,\middle|\, F_U \in (\mathcal{D}_{\mathrm{qc}})_U^{[a,b]} \text{ and } F_{U_i} \in (\mathcal{D}_{\mathrm{qc}})_{U_i}^{[a,b]} \text{ for all } i \right\} \tag{5.7}$$

and (5.2) define t-structures on $(\mathcal{D}_{\mathrm{qc}})_T$ such that part (1) of the theorem holds. Since the right side of (5.7) is contained in the right side of (5.2) and both define t-structures, they coincide. It follows that (5.1) defines a t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ such that parts (1) and (2) of the theorem hold.

Step 5. Parts (3) *and* (4) *of the theorem hold.*

Using (5.2), parts (3), (4a), (4c), and (4d) reduce to the case where T and T' are affine, which were handled in Step 1. Part (4b) follows from (4a) since f'_* is right adjoint to f'^*.

Step 6. Part (5) *of the theorem holds.*

To show the two t-structures on $\mathcal{D}_{\mathrm{qc}}$ agree, by part (2) and Lemma 5.1.(4) it suffices to show that for any affine open $U \subset S$ the t-structure on $(\mathcal{D}_{\mathrm{qc}})_U$ constructed in this theorem agrees with the one induced by the local t-structure on $\mathcal{D}_{\mathrm{qc}}$ from Lemma 5.1.(4). This follows from the description of the induced t-structure on $(\mathcal{D}_{\mathrm{qc}})_U$ in Lemma 5.1.(4).

Finally, by (5.1) and (5.2), the formulas (5.3) and (5.4) reduce to showing that for $\phi\colon T \to S$ a morphism from an affine scheme T, we have

$$(\mathcal{D}_{\mathrm{qc}})_T^{[a,b]} = \left\{ F \in (\mathcal{D}_{\mathrm{qc}})_T \,\middle|\, \phi'_*(F) \in \mathcal{D}_{\mathrm{qc}}^{[a,b]} \right\}.$$

The functor ϕ'_* is conservative since ϕ' is affine, so this follows from (4d). □

5.2. **The bounded coherent case.** Our next goal is to show that under suitable hypotheses, the base changed t-structures constructed in Theorem 5.3 induce t-structures at the level of bounded derived categories of coherent sheaves.

A map $A \to B$ of rings is called *perfect* if it is pseudo-coherent and B has finite Tor-dimension over A, see [Sta21, Tag 067G]. A morphism of schemes $X \to Y$ is called *perfect* if if there exists an affine open cover $Y = \bigcup_{j\in J} V_j$ and affine open covers $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that the ring map $\mathcal{O}_Y(V_j) \to \mathcal{O}_X(U_i)$ is perfect for all $j \in J$, $i \in I_j$. For a discussion of this notion, see [Sta21, Tag 0685]. In other words, a morphism $f\colon X \to Y$ is perfect if and only if it is pseudo-coherent and of finite Tor-dimension. Note that if Y is locally noetherian, then f is pseudo-coherent if and only if f is locally of finite type [Sta21, Tag 0684]. In particular, if Y is regular of finite Krull dimension, then f is perfect if and only if f is locally

of finite type. We will consider base changes along the following mild generalization of the class of perfect morphisms.

Definition 5.4. A map $A \to B$ of rings is called *essentially perfect* if it is a localization of a perfect A-algebra. A morphism of schemes $X \to Y$ is called *essentially perfect* if there exists an affine open cover $Y = \bigcup_{j \in J} V_j$ and affine open covers $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_Y(V_j) \to \mathcal{O}_X(U_i)$ is essentially perfect for all $j \in J$, $i \in I_j$.

Remark 5.5. As a warning, if $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an essentially perfect morphism of affine schemes, then $A \to B$ may not be essentially perfect.

If $\tilde{\mathcal{D}}$ is a triangulated category with a t-structure $(\tilde{\mathcal{D}}^{\leqslant 0}, \tilde{\mathcal{D}}^{\geqslant 0})$ and $\mathcal{D} \subset \tilde{\mathcal{D}}$ is a triangulated subcategory, we say that $(\tilde{\mathcal{D}}^{\leqslant 0}, \tilde{\mathcal{D}}^{\geqslant 0})$ induces a t-structure on $\mathcal{D}$ if setting $\mathcal{D}^{\leqslant 0} = \tilde{\mathcal{D}}^{\leqslant 0} \cap \mathcal{D}$ and $\mathcal{D}^{\geqslant 0} = \tilde{\mathcal{D}}^{\geqslant 0} \cap \mathcal{D}$ defines a t-structure. This is equivalent to the truncation functors of $\tilde{\mathcal{D}}$ preserving $\mathcal{D}$. The following analog of [Pol07, Theorem 2.3.5] in our setting, which holds by the same argument, is a key ingredient for the general base change result below.

Theorem 5.6. *Let $g\colon X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude, and which is equipped with an S-local t-structure. Let $\phi\colon T \to S$ be a finite perfect morphism from a scheme T which is quasi-compact with affine diagonal, such that ϕ is faithful with respect to g. Then the t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ from Theorem 5.3 induces a t-structure on $\mathcal{D}_T$, which is bounded or tilted-noetherian or noetherian if the given t-structure on $\mathcal{D}$ is.*

Now we can prove our main base change result.

Theorem 5.7. *Let $g\colon X \to S$ be a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude. Let $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ be a bounded S-local t-structure on $\mathcal{D}$ which is tilted-noetherian locally on S. Let $\phi\colon T \to S$ be an essentially perfect morphism from a scheme T which is quasi-compact with affine diagonal, such that ϕ is faithful with respect to g. Then:*

(1) *The t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ from Theorem 5.3 induces a bounded T-local t-structure on $\mathcal{D}_T$ which is tilted-noetherian locally on T.*
(2) *If the t-structure on $\mathcal{D}$ is noetherian locally on S, so is the induced t-structure on $\mathcal{D}_T$.*
(3) *If $\phi\colon T \to S$ is projective, L is a ϕ-relatively ample line bundle on T, and L_{X_T} denotes its pullback to X_T, then the t-structure on $\mathcal{D}_T$ satisfies*

$$\mathcal{D}_T^{[a,b]} = \left\{ F \in \mathcal{D}_T \,\middle|\, \phi'_*(F \otimes L_{X_T}^n) \in \mathcal{D}^{[a,b]} \text{ for all } n \gg 0 \right\}.$$

(4) *Let $T' \to S$ be another morphism with the same assumptions as ϕ, let $f\colon T' \to T$ be a morphism of schemes over S, and let $f'\colon X_{T'} \to X_T$ be the induced morphism.*
   (a) *$f'^*\colon \mathcal{D}_T \to \mathcal{D}_{T'}$ is right t-exact.*
   (b) *If f is flat, then $f'^*\colon \mathcal{D}_T \to \mathcal{D}_{T'}$ is t-exact.*
   (c) *If f is finite, then $f'_*\colon \mathcal{D}_{T'} \to \mathcal{D}_T$ is t-exact.*

Proof. We prove the theorem in several steps.

Step 1. If S is affine and $\phi\colon T = \mathbb{P}^r_S \to S$ is a projective space over S, then the formula

$$\mathcal{D}_T^{[a,b]} = \left\{ F \in \mathcal{D}_T \,\middle|\, \phi'_*(F(n)) \in \mathcal{D}^{[a,b]} \text{ for all } n \gg 0 \right\} \tag{5.8}$$

defines a bounded T-local t-structure on $\mathcal{D}_T$ which is tilted-noetherian locally on T, and noetherian locally on T if the given t-structure on $\mathcal{D}$ is.

In [AP06, Theorem 2.3.6] it is shown that if $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$, X is smooth and projective over the spectrum S of a field, and the t-structure on $\mathcal{D}$ is noetherian, then (5.8) defines a bounded noetherian T-local t-structure on $\mathcal{D}_T$. We leave it to the reader to verify that their proof goes through in greater generality: in our setup if S is affine and the t-structure on $\mathcal{D}$ is noetherian, then (5.8) defines a bounded noetherian T-local t-structure on $\mathcal{D}_T$. Since by Lemma 4.15 a t-structure on $\mathcal{D}$ or $\mathcal{D}_T$ is noetherian if and only if it is noetherian locally on the base, this proves the desired claim in case the given t-structure on $\mathcal{D}$ is noetherian locally on S. From this, the tilted-noetherian case follows as in the proof of [Pol07, Lemma 3.3.2].

Step 2. If S is affine and $\phi\colon T = \mathbb{A}^r_S \to S$ is an affine space over S, then the bounded T-local t-structure on $\mathcal{D}_T$ induced by the $\mathbb{P}^r_S$-local t-structure on $\mathcal{D}_{\mathbb{P}^r_S}$ from Step 1 is given by

$$\mathcal{D}_T^{[a,b]} = \left\{ F \in \mathcal{D}_T \,\middle|\, \phi'_*(F) \in \mathcal{D}_{\mathrm{qc}}^{[a,b]} \right\}.$$

This holds by the same argument as in [Pol07, Lemma 3.3.4].

Step 3. If S and T are affine and $\phi\colon T \to S$ is perfect, then the t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ from Theorem 5.3 induces a bounded t-structure on $\mathcal{D}_T$, which is noetherian if the t-structure on $\mathcal{D}$ is.

The morphism ϕ is of finite type (see [Sta21, Tag 0682]), so it factors through a closed immersion $T \hookrightarrow \mathbb{A}^r_S$. The morphism $T \hookrightarrow \mathbb{A}^r_S$ is pseudo-coherent by [Sta21, Tag 0683], of finite Tor-dimension by [Sta21, Tag 068X], and hence perfect. Thus the claim follows from Theorem 5.6 combined with Step 2.

Step 4. If $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ are affine and $\phi\colon T \to S$ corresponds to an essentially perfect map of rings $A \to B$, then the t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ from Theorem 5.3 induces a bounded t-structure on $\mathcal{D}_T$, which is noetherian if the t-structure on $\mathcal{D}$ is.

By Step 3 we reduce to the case where B is a localization of A. Then $\phi\colon T \to S$ is flat, so by Theorem 5.3.(4c) the functor $\phi'^*\colon \mathcal{D}_{\mathrm{qc}} \to (\mathcal{D}_{\mathrm{qc}})_T$ is t-exact. Since the functor $\mathcal{D} \to \mathcal{D}_T$ is essentially surjective by Lemma 3.18, it follows that the t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$ induces a bounded t-structure on $\mathcal{D}_T$. To prove noetherianity, we first note that any subobject $F_i \subset F$ of a fixed object F is already defined over the complement $U_i \subset S$ of the zero locus of some $f_i \in A$. Thus we can proceed exactly as in Lemma 4.16.(3) to lift any possibly infinite filtration of F in $\mathcal{A}_T$ to a filtration of $\tilde{F}$ in $\mathcal{A}_S$, which proves that $\mathcal{A}_T$ is noetherian if $\mathcal{A}_S$ is.

Step 5. Parts (1) and (2) of the theorem hold in general.

First we show that $(\mathcal{D}_{\mathrm{qc}})_T$ induces a t-structure on $\mathcal{D}_T$. For any $a \in \mathbb{Z}$, let $\hat{\tau}_T^{\geqslant a}$ denote the truncation functor for the t-structure on $(\mathcal{D}_{\mathrm{qc}})_T$. Then we must show that for any $F \in \mathcal{D}_T$ we have $\hat{\tau}_T^{\geqslant a}(F) \in \mathcal{D}_T$. Since T is quasi-compact and ϕ is essentially perfect, we may choose a finite affine open cover $T = \cup U_i$ and affine opens $V_i \subset S$ such that for each i the morphism $U_i \hookrightarrow T \xrightarrow{\phi} S$ factors through V_i and the corresponding ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_T(U_i)$ is essentially perfect. To show $\hat{\tau}_T^{\geqslant a}(F) \in \mathcal{D}_T$ it suffices to show $\hat{\tau}_T^{\geqslant a}(F)_{U_i} \in \mathcal{D}_{U_i}$ for each i because pseudo-coherence is a local property and boundedness can be checked on a finite open

cover. We have $\hat{\tau}_T^{\geqslant a}(F)_{U_i} \simeq \hat{\tau}_{U_i}^{\geqslant a}(F_{U_i})$ because the restriction functor $(\mathcal{D}_{\mathrm{qc}})_T \to (\mathcal{D}_{\mathrm{qc}})_{U_i}$ is t-exact. In view of Theorem 5.3.(5), we thus reduce to the case of the affine morphism $U_i \to V_i$, which was handled in Step 4. This shows that $(\mathcal{D}_{\mathrm{qc}})_T$ induces a t-structure on $\mathcal{D}_T$, and a similar argument shows that this t-structure on $\mathcal{D}_T$ is bounded.

Moreover, the t-structure on $\mathcal{D}_T$ is T-local. Indeed, if $U \subset T$ is a quasi-compact open subset, then U also has affine diagonal and the morphism $U \to T$ is perfect. Hence by what we have already shown, we conclude that $(\mathcal{D}_{\mathrm{qc}})_U$ induces a t-structure on $\mathcal{D}_U$. Since the restriction functor $(\mathcal{D}_{\mathrm{qc}})_T \to (\mathcal{D}_{\mathrm{qc}})_U$ is t-exact by Theorem 5.3.(4c), so is the functor $\mathcal{D}_T \to \mathcal{D}_U$. Hence the t-structure on $\mathcal{D}_T$ is T-local.

Further, the t-structure on $\mathcal{D}_T$ is noetherian locally on T if the t-structure on $\mathcal{D}$ is noetherian locally on S. Indeed, then for each affine $U_i \subset T$ in the affine cover considered above, the induced t-structure on $\mathcal{D}_{U_i}$ is noetherian by Step 4. So by Lemma 4.15 the claim holds.

Finally, it follows directly from the definitions and the result of the previous paragraph that the t-structure on $\mathcal{D}_T$ is tilted-noetherian locally on T.

Step 6. Part (3) *of the theorem holds.*

The proof of [Pol07, Theorem 3.3.6(ii)] goes through in our setup.

Step 7. Part (4) *of the theorem holds.*

These claims follow immediately from the corresponding statements in Theorem 5.3.(4). □

We make explicit the following immediate consequence of the theorem:

Corollary 5.8. *In the assumptions of Theorem 5.7, assume also that g is flat, that S is regular of finite Krull dimension. Let s be a point of S and let $\mathcal{D}_s$ be the base change category to* $\operatorname{Spec} \kappa(s)$.

(1) *Then $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ induces a bounded t-structure on $\mathcal{D}_s$.*
(2) *Moreover, if S is irreducible and $s \in S$ the generic point, then base change to $\mathcal{D}_s$ is t-exact.*

Proof. Let $f\colon \bar{s} \subset S$ be the inclusion of the closure of s. Since f is finite and S is regular, it is automatically perfect [Sta21, Tag 068B]. Hence the composition $\operatorname{Spec} \kappa(s) \to \bar{s} \to S$ is essentially perfect, and part (1) is a special case of Theorem 5.7. Then part (2) follows from flatness, i.e., Theorem 5.7.(4b). □

Another important consequence of the theorem is base change for t-structures along field extensions. By a common abuse of notation, when we base change to an affine scheme $\operatorname{Spec}(A)$, we denote $\mathcal{D}_{\operatorname{Spec}(A)}$ by $\mathcal{D}_A$.

Proposition 5.9. *Let X be a noetherian scheme of finite Krull dimension over a field k. Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be a k-linear strong semiorthogonal component whose projection is of finite cohomological amplitude, and which is equipped with a bounded tilted-noetherian t-structure. Let $k \subset \ell$ be a (not necessarily finitely generated) field extension. Then:*

(1) *The t-structure on $(\mathcal{D}_{\mathrm{qc}})_\ell$ from Theorem 5.3 induces a bounded t-structure on $\mathcal{D}_\ell$.*
(2) *Base change $\mathcal{D} \to \mathcal{D}_\ell$ is t-exact.*
(3) *For every object $E \in \mathcal{A}_\ell$ there exists a subfield $\ell' \subset \ell$, finitely generated over k, and an object $F \in \mathcal{A}_{\ell'}$ such that E is the base change of F.*

(4) *If the field extension $k \subset \ell$ is finitely generated, then the induced t-structure on $\mathcal{D}_\ell$ is tilted-noetherian. If further the t-structure on $\mathcal{D}$ is noetherian, then so is the induced t-structure on $\mathcal{D}_\ell$.*

Proof. Let $F \in \mathcal{D}_\ell$. Since $F \in \mathrm{D}^{\mathrm{b}}(X_\ell)$ it descends to an object $F_A \in \mathrm{D}^{\mathrm{b}}(X_A)$ for some finitely generated k-subalgebra $A \subset \ell$. By projecting into $\mathcal{D}_A \subset \mathrm{D}^{\mathrm{b}}(X_A)$, we may assume $F_A \in \mathcal{D}_A$. But by Theorem 5.7 the t-structure on $(\mathcal{D}_{\mathrm{qc}})_A$ induces a bounded t-structure on $\mathcal{D}_A$, and by its statement (4b) the pullback functor $(\mathcal{D}_{\mathrm{qc}})_A \to (\mathcal{D}_{\mathrm{qc}})_\ell$ is t-exact. It follows that the t-structure on $(\mathcal{D}_{\mathrm{qc}})_\ell$ induces a bounded t-structure on $\mathcal{D}_\ell$, and that it satisfies (2).

To prove (3), we can let ℓ' be the fraction field of A; then $F_{\ell'} \in \mathcal{A}_{\ell'}$ as pullback to ℓ is t-exact and conservative.

If $k \subset \ell$ is finitely generated, then ℓ can be written as the fraction field of a finitely generated k-algebra. Hence $\mathrm{Spec}(\ell) \to \mathrm{Spec}(k)$ is an essentially perfect morphism, so (4) holds by Theorem 5.7. □

6. Flat, torsion, and torsion free objects

In this section, we consider the following situation:

- $g\colon X \to S$ is a flat morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension.
- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude.
- $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is an S-local t-structure on $\mathcal{D}$ with heart $\mathcal{A}_S$.

The significance of $g\colon X \to S$ being flat is that any morphism $\phi\colon T \to S$ is faithful with respect to g. Following [AP06], we adapt to our setup the relative notions of flat, torsion, and torsion free objects. In Section 6.1 and Section 6.2 we discuss these notions in a general setting, while in Section 6.3 we study in more detail the case where $S = C$ is a Dedekind scheme, which will be particularly important later in the paper.

6.1. Flat objects.

For any (not necessarily closed) point $s \in S$, we write $\mathcal{D}_s$, $(\mathcal{D}_{\mathrm{perf}})_s$, and $(\mathcal{D}_{\mathrm{qc}})_s$ for the base change categories along the canonical morphism $\phi\colon \mathrm{Spec}(\kappa(s)) \to S$.

Definition 6.1. Let $\phi\colon T \to S$ be a morphism, and let $E \in \mathrm{D}_{\mathrm{qc}}(X_T)$. Then E is *T-flat* if $E_t \in (\mathcal{A}_{\mathrm{qc}})_t$ for every point $t \in T$, where $(\mathcal{A}_{\mathrm{qc}})_t \subset (\mathcal{D}_{\mathrm{qc}})_t$ is the heart of the t-structure given by Theorem 5.3 applied to the composition $\mathrm{Spec}\,\kappa(t) \to T \to S$.

Remark 6.2. If $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ with the standard t-structure, then Definition 6.1 agrees with the usual notion of flatness for an object $F \in \mathrm{Coh}\,X$ under either of the following hypotheses: T is noetherian or $X_T \to T$ is of finite presentation. Indeed, the statement reduces to the local affine case, where for T noetherian it holds by the local criterion for flatness [Sta21, Tag 00MK], and for $X_T \to T$ of finite presentation we can reduce to the case where S is noetherian using [Sta21, Tag 00QX].

In Section 6.3.2 below we prove some results about flat objects in the case of a Dedekind base.

6.2. **Torsion and torsion free objects.** In this subsection we assume S is integral. Recall that, in accordance with Section 2, if $W \subset S$ is a closed subscheme, then $i_W \colon X_W \to X$ denotes the embedding of the fiber over W.

Definition 6.3. An object $E \in \mathcal{D}$ is called *S-torsion* if it is the pushforward of an object in $\mathcal{D}_Z$ for some proper closed subscheme $Z \subset S$. We denote by $\mathcal{D}_{S\text{-tor}}$ the subcategory of S-torsion objects in $\mathcal{D}$.

Lemma 6.4. *Let K be the function field of S. For $E \in \mathcal{D}$, we have $E \in \mathcal{D}_{S\text{-tor}}$ if and only if $E_K = 0$, namely there is an exact sequence of triangulated categories*

$$\mathcal{D}_{S\text{-tor}} \to \mathcal{D} \to \mathcal{D}_K. \tag{6.1}$$

Proof. The statement follows directly from the corresponding statement for $\mathrm{D}^{\mathrm{b}}(X)$. □

Definition 6.5. An object $E \in \mathcal{A}_S$ is called *S-torsion free* if it contains no nonzero S-torsion subobject. We denote by $\mathcal{A}_{S\text{-tor}} \subset \mathcal{A}_S$ the subcategory of S-torsion objects, and by $\mathcal{A}_{S\text{-tf}} \subset \mathcal{A}_S$ the subcategory of S-torsion free objects.

Lemma 6.6. *The following hold:*

(1) *The subcategory $\mathcal{A}_{S\text{-tf}} \subset \mathcal{A}_S$ is closed under subobjects and extensions.*
(2) *The subcategory $\mathcal{A}_{S\text{-tor}} \subset \mathcal{A}_S$ is closed under subobjects, quotients, and extensions.*
(3) *The t-structure on $\mathcal{D}$ induces one on $\mathcal{D}_{S\text{-tor}}$, which is bounded if the given t-structure on $\mathcal{D}$ is.*
(4) *If L is a line bundle on S, then tensoring by g^*L preserves $\mathcal{A}_{S\text{-tor}}$ and $\mathcal{A}_{S\text{-tf}}$.*

Proof. Part (1) follows immediately from the definitions. Let K be the function field of S. Then $\mathrm{Spec}(K) \to S$ is flat, so by Theorem 5.7.(4b) the pullback functor $\mathcal{D} \to \mathcal{D}_K$ is t-exact. Using this and Lemma 6.4, parts (2) and (3) follow easily. Part (4) is immediate as $_ \otimes g^*L$ preserves $\mathcal{A}_S$, see Lemma 4.11, and $\mathcal{D}_{S\text{-tor}}$. □

Lemma 6.7. *Let $T \subset S$ be the zero locus of a regular section of a vector bundle, and let $\mathcal{N}$ be the normal bundle. Then for any $E \in \mathcal{A}_T$, there are isomorphisms*

$$\mathrm{H}^i_{\mathcal{A}_T}(i_T^* i_{T*} E) \cong \bigwedge^{-i} g_T^* \mathcal{N}^\vee \otimes E, \tag{6.2}$$

for all $i \in \mathbb{Z}$. In particular, we always have $\mathrm{H}^0_{\mathcal{A}_T}(i_T^* i_{T*} E) \cong E$, and if T is zero-dimensional, so that $\mathcal{N} \cong \mathcal{O}_T^{\oplus \dim S}$, then we also have $\mathrm{H}^{-\dim S}_{\mathcal{A}_T}(i_T^* i_{T*} E) \cong E$. Finally, $i_{T*} \colon \mathcal{A}_T \to \mathcal{A}_S$ is fully faithful.

Proof. The isomorphism on cohomology objects follows by the t-exactness of i_{T*} and the Koszul complex as in [Huy06, Corollary 11.2].

To see the claim about the fully faithfulness of i_{T*}, we let $E, F \in \mathcal{A}_T$, and observe that $i_T^* i_{T*} E \in \mathcal{D}_T^{\leqslant 0}$ by (6.2). It then follows by adjunction and the $i = 0$ case of (6.2) that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(i_{T*}E, i_{T*}F) &\cong \mathrm{Hom}_{\mathcal{D}}(i_{T*}E, i_{T*}F) \cong \mathrm{Hom}_{\mathcal{D}_T}(i_T^* i_{T*} E, F) \\ &\cong \mathrm{Hom}_{\mathcal{D}_T}(\mathrm{H}^0_{\mathcal{A}_T}(i_T^* i_{T*} E), F) \cong \mathrm{Hom}_{\mathcal{D}_T}(E, F) \cong \mathrm{Hom}_{\mathcal{A}_T}(E, F), \end{aligned}$$

as required. □

The following is a version of Lemma 4.16.(2) for S-torsion free objects.

Lemma 6.8. *Assume that $g\colon X \to S$ is flat and S is integral with function field K. Let L be an ample line bundle on S, and let $A, B \in \mathcal{A}_S$ be objects with A being S-torsion free. Given any isomorphism $A_K \cong B_K$, there exists $k \in \mathbb{Z}$ and an injective map $A \otimes g^*L^{-k} \hookrightarrow B$ inducing the given isomorphism over K.*

Proof. Since A, B are bounded complexes, the isomorphism is defined over some open subset $U \subset S$, so by [AP06, Lemma 2.1.8], there exists a map $A \otimes g^*L^{-k} \to B$ that induces the isomorphism over K. Let $Q \in \langle \mathcal{A}_S, \mathcal{A}_S[1] \rangle$ be the cone of this map; note $Q_K = 0$. Hence $Q \in \mathcal{D}_{S\text{-tor}}$ and all cohomology objects $\mathrm{H}^i_{\mathcal{A}_S}(Q)$ are S-torsion. Since A is S-torsion free, the long exact cohomology sequence shows $\mathrm{H}^{-1}_{\mathcal{A}_S}(Q) = 0$, proving the injectivity as claimed. □

6.3. **Dedekind bases.** In this subsection we assume $S = C$ is a Dedekind scheme.

6.3.1. *Torsion objects.* For any non-trivial closed subscheme $W \subset C$, the ideal sheaf I_W is a line bundle. We will abuse notation by writing $I_W \otimes (-)$ for the tensor product $g^*(I_W) \otimes (-)$, which preserves $\mathcal{A}_C$. Note that by Theorem 5.6 there is an induced t-structure on $\mathcal{D}_W$ whose heart we denote by $\mathcal{A}_W$.

Lemma 6.9. *Let $W \subset C$ be a 0-dimensional subscheme with ideal sheaf I_W. Let $\mathcal{A}_W$ be the heart of the induced t-structure on $\mathcal{D}_W$ given by Theorem 5.6. Then for any $E \in \mathcal{A}_C$ we have short exact sequences*

$$0 \to \mathrm{Ann}(I_W; E) = i_{W*}\mathrm{H}^{-1}_{\mathcal{A}_W}(E_W) \hookrightarrow I_W \otimes E \twoheadrightarrow I_W \cdot E \to 0 \tag{6.3}$$

and

$$0 \to I_W \cdot E \hookrightarrow E \twoheadrightarrow E/I_W \cdot E = i_{W*}\mathrm{H}^0_{\mathcal{A}_W}(E_W) \to 0,$$

where $\mathrm{Ann}(I_W; E)$ and $I_W \cdot E$ denote the kernel and the image of the canonical map

$$I_W \otimes E \to E.$$

Moreover, $\mathrm{H}^i_{\mathcal{A}_W}(E_W) = 0$ for $i \neq -1, 0$.

Proof. We take cohomology of the exact triangle $I_W \otimes E \to E \to i_{W*}i_W^*E$ with respect to $\mathcal{A}_S$, using that i_{W*} is t-exact by Theorem 5.7.(4c). □

In the situation of Lemma 6.9, we call the essential image of $i_{W*}\colon \mathcal{A}_W \to \mathcal{A}_C$ the subcategory of objects *schematically supported over* W; it is equivalent to $\mathcal{A}_W$ by the last claim of Lemma 6.7.

Corollary 6.10. *Let $W \subset C$ be a 0-dimensional subscheme. Then the subcategory of $\mathcal{A}_C$ of objects schematically supported over W is closed under subobjects and quotients.*

Proof. By Lemma 6.9, an object $E \in \mathcal{A}_C$ is schematically supported on W if and only if the map $I_W \otimes E \to E$ vanishes. Given a subobject $A \hookrightarrow E$, it follows that the composition $I_W \otimes A \to A \hookrightarrow E$ vanishes (as it also factors via $I_W \otimes E$), and therefore A is also schematically supported on W. The case of quotients follows similarly. □

We say that $E \in \mathcal{A}_{C\text{-tor}}$ is set-theoretically supported over a closed point $p \in C$ if it is scheme-theoretically supported over some infinitesimal neighborhood of p.

Lemma 6.11. *Let $p \in C$ be a closed point, π be a local generator of I_p and $E \in \mathcal{A}_{C\text{-tor}}$ be an object set-theoretically supported over p. Then E admits a filtration*

$$0 = \pi^{m+1} \cdot E \subset \pi^m \cdot E \subset \cdots \subset \pi \cdot E \subset E$$

where all filtration quotients $\pi^i \cdot E/\pi^{i+1} \cdot E$ are quotients of $E/\pi \cdot E$ in $\mathcal{A}_p$.

Proof. By assumption, there exists m such that $I_p^{m+1} \cdot E = 0$. The local isomorphism between $\mathcal{O}_C$ and I_p induced by π identifies $I_p^i \cdot E$ with $\pi^i \cdot E$, i.e., the image of the endomorphism of E induced by π^i. Finally, by definition, π^i induces a surjection $E/\pi \cdot E \twoheadrightarrow \pi^i \cdot E/\pi^{i+1} \cdot E$. □

6.3.2. *Flat objects.* For any point $c \in C$, there is an induced heart $\mathcal{A}_c$ on $\mathcal{D}_c$. Indeed, if $c \in C$ is closed then this holds by Theorem 5.6, while if $c \in C$ is the generic point this holds by Theorem 5.7. In particular, suppose $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is an object such that $E_c \in \mathcal{D}_c$ for every $c \in C$; this holds for instance if $E \in \mathcal{D}$ (see Lemma 8.3). Then E is C-flat in the sense of Definition 6.1 if and only if $E_c \in \mathcal{A}_c$ for every point $c \in C$. This observation will be used without mention below.

Lemma 6.12. *Let $E \in \mathcal{A}_C$. Then E is C-flat if and only if E is C-torsion free.*

Proof. If $c \in C$ is the generic point, then by Corollary 5.8.(2) the pullback $\mathcal{D} \to \mathcal{D}_c$ is t-exact, i.e., $E_c \in \mathcal{A}_c$ is automatic. Thus, E is C-flat if and only if $E_c \in \mathcal{A}_c$ for all closed points $c \in C$.

Now it follows from Lemma 6.9 that E is C-flat if and only if for every closed point $c \in C$ the map $I_c \otimes E \to E$ is injective in $\mathcal{A}_C$. Moreover, Lemma 6.9 also shows that if this map is not injective for some c, then E is not C-torsion free. Conversely, assume E has a torsion subobject $A \hookrightarrow E$. We may assume that A is set-theoretically supported over a closed point $c \in C$. Then there exists a positive integer m so that the natural map $I_c^{\otimes m} \otimes A \to A$ vanishes. In particular this map $I_c^{\otimes m} \otimes E \to E$ is non-injective as a map in $\mathcal{A}_C$, which implies the same for $I_c \otimes E \to E$. □

Lemma 6.13. *Let $E \in \mathcal{D}$ be a C-flat object. Then $E \in \mathcal{A}_C$.*

Proof. Since $E_K \in \mathcal{A}_K$ by assumption, and since pullback to $\mathcal{D}_K$ is t-exact by Corollary 5.8.(2), we have $\mathrm{H}^i_{\mathcal{A}_C}(E) \in \mathcal{A}_{C\text{-tor}}$ for all $i \neq 0$. Consider the maximal i such that $0 \neq \mathrm{H}^i_{\mathcal{A}_C}(E) \in \mathcal{A}_{C\text{-tor}}$ and assume $i > 0$; by the previous lemmas, it is of the form $i_{W*}(F)$ for some 0-dimensional closed subscheme $W \subset C$. Since the question is local on C, we may assume it is supported over a single closed point $p \in C$. Then as $\mathcal{O}_W$ is an iterated extension of copies of $\mathcal{O}_p$, it follows that $i_{W*}E_W = E \otimes i_{W*}\mathcal{O}_W$ is a self-extension of a number of copies of $i_{p*}E_p = E \otimes i_{p*}\mathcal{O}_p$. Therefore $i_{W*}E_W \in \mathcal{A}_C$, and as i_{W*} is t-exact and conservative, we have $E_W \in \mathcal{A}_W$. Therefore, by adjunction

$$\mathrm{Hom}\left(E, \mathrm{H}^i_{\mathcal{A}_C}(E)[-i]\right) = \mathrm{Hom}\left(E_W, F[-i]\right) = 0,$$

a contradiction. Similarly, if $i < 0$ is minimal with $\mathrm{H}^i_{\mathcal{A}_C}(E) \neq 0$, and hence of the form $i_{W*}(F)$ for $F \in \mathcal{A}_W$, we observe that since $X_W \subset X$ is the inclusion of a Cartier divisor with trivial restriction to itself, we have $i_W^! E = E_W[-1]$ (see e.g. [Sta21, Tag 0AA4]). We therefore obtain an analogous contradiction from

$$\mathrm{Hom}\left(\mathrm{H}^i_{\mathcal{A}_C}(E)[-i], E\right) = \mathrm{Hom}(F[-i], i_W^! E) = \mathrm{Hom}(F, E_W[-1+i]) = 0. \qquad \square$$

We say the heart $\mathcal{A}_C$ satisfies *openness of flatness* if for every $E \in \mathcal{D}$, the set

$$\{c \in C \colon E_c \in (\mathcal{A}_{\mathrm{qc}})_c\}$$

is open. Later in Section 10.1 we will study this property in a more general setting (using Lemma 8.3, our definition there — Definition 10.4 — is easily seen to be equivalent to the one above under the assumption that the base C is Dedekind). As a consequence of Lemma 6.13, openness of flatness implies *openness of the heart*:

Corollary 6.14. *Assume that $\mathcal{A}_C$ satisfies openness of flatness. Let $E \in \mathcal{D}$ and assume $E_c \in \mathcal{A}_c$ for some $c \in C$. Then there exists an open neighborhood $U \subset C$ of c such that $E_U \in \mathcal{A}_U$.*

6.3.3. *Torsion theories.* The following property will play an important role later in the paper.

Definition 6.15. We say $\mathcal{A}_C$ has a *C-torsion theory* if the pair of subcategories $(\mathcal{A}_{C\text{-tor}}, \mathcal{A}_{C\text{-tf}})$ forms a torsion pair.

Remark 6.16. It follows from Lemma 6.6.(2) that the heart $\mathcal{A}_C$ has a C-torsion theory if and only if every object $E \in \mathcal{A}_C$ contains a unique maximal C-torsion subobject $E_{C\text{-tor}} \subset E$. In this case, we denote by $E_{C\text{-tf}} = E/E_{C\text{-tor}}$ the C-torsion free quotient in $\mathcal{A}_C$. We also note that $\mathcal{A}_C$ being noetherian implies the existence of a C-torsion theory.

The following lemma can be helpful in proving the existence of a C-torsion theory.

Lemma 6.17. *Let $A \hookrightarrow E \twoheadrightarrow B$ be a short exact sequence in a C-local heart, and assume that A and B admit maximal C-torsion subobjects. Then the same holds for E.*

Proof. We may assume that B is C-torsion, as any C-torsion subobject of E will factor via the preimage of the maximal C-torsion subobject of B; and evidently we may assume that A is C-torsion free. Let W be the schematic support of B, in the sense of Lemma 6.9, and consider the short exact sequence $\mathrm{Ann}(I_W; E) \hookrightarrow I_W \otimes E \twoheadrightarrow I_W \cdot E$; recall that $I_W \cdot E$ is the image of the natural map $I_W \otimes E \to E$. Since $I_W \cdot B = 0$, this map factors via $A \hookrightarrow E$; therefore, $I_W \cdot E \subset A$ is C-torsion free. Replacing E by $E \otimes I_W^{-1}$ in this argument, we have found a torsion free quotient of E by a C-torsion subobject. □

In fact, $E_{C\text{-tor}}$ can be identified somewhat more explicitly as follows.

Lemma 6.18. *Assume $E \in \mathcal{A}_C$ admits a maximal C-torsion subobject $E_{C\text{-tor}} \subset E$. Let $W \subset C$ be the schematic support of $E_{C\text{-tor}}$. Then $E_{C\text{-tor}} = I_W^{-1} \otimes i_{W*}\mathrm{H}^{-1}_{\mathcal{A}_W}(E_W)$.*

Proof. By the choice of W we have $I_W \cdot E_{C\text{-tor}} = 0$, so by Lemma 6.9 we find

$$E_{C\text{-tor}} = I_W^{-1} \otimes i_{W*}\mathrm{H}^{-1}_{\mathcal{A}_W}\left((E_{C\text{-tor}})_W\right).$$

Moreover, upon restricting the short exact sequence $E_{C\text{-tor}} \hookrightarrow E \twoheadrightarrow E_{C\text{-tf}}$ to W, we see that $\mathrm{H}^{-1}_{\mathcal{A}_W}\left((E_{C\text{-tor}})_W\right) = \mathrm{H}^{-1}_{\mathcal{A}_W}(E_W)$, giving the claimed equality. Indeed, $E_{C\text{-tf}}$ is C-flat by Lemma 6.12 so we also have $\mathrm{H}^{i}_{\mathcal{A}_W}\left((E_{C\text{-tf}})_W\right) = 0$ for $i \neq 0$ by the proof of Lemma 6.13. □

Proposition 6.19. *Assume that $\mathcal{A}_C$ has a C-torsion theory.*

(1) *(Nakayama's Lemma) If $E \in \mathcal{A}_C$ satisfies $E/I_c \cdot E = 0$ for some $c \in C$, then there exists an open neighborhood $c \in U \subset C$ such that $E_U = 0$. In particular, if E is also C-torsion free, then $E = 0$.*

(2) *The heart $\mathcal{A}_C$ satisfies openness of flatness.*

Proof. By Lemmas 6.9 and 6.12, there is a short exact sequence

$$\mathrm{H}^0_{\mathcal{A}_c}((E_{C\text{-tor}})_c) \hookrightarrow \mathrm{H}^0_{\mathcal{A}_c}(E_c) \twoheadrightarrow \mathrm{H}^0_{\mathcal{A}_c}((E_{C\text{-tf}})_c) = (E_{C\text{-tf}})_c\,.$$

Hence the assumption in (1) implies $(E_{C\text{-tf}})_c = 0$, and so $E_{C\text{-tf}} = 0$. So E is C-torsion. If its support contains c, then c is a closed point, and we proceed as in the proof of Lemma 6.12 to show that the canonical map $I_c \otimes E \to E$ cannot be surjective, so that its cokernel $\mathrm{H}^0_{\mathcal{A}_c}(E_c)$ cannot vanish. This contradiction completes the proof of (1).

To prove (2), assume that $E_c \in \mathcal{A}_c$. Let i be maximal with $F^i := \mathrm{H}^i_{\mathcal{A}_C}(E) \neq 0$, and assume $i > 0$. Since i_c^* is right t-exact by Theorem 5.7.(4b), one deduces $F^i/I_c \cdot F^i = i_{c*}\mathrm{H}^0_{\mathcal{A}_c}(F^i_c) = 0$. By (1) we can replace C by an open neighborhood $c \in U \subset C$ such that F^i_U vanishes; repeating this process, we obtain $\mathrm{H}^i_{\mathcal{A}_U}(E_U) = 0$ for $i > 0$. Similarly, if $i \leqslant 0$ is minimal with $F^i \neq 0$, then we can conclude from Lemma 6.9 that $\mathrm{H}^{-1}_{\mathcal{A}_c}((F^i)_c) = 0$ and thus that F^i has no torsion supported at c. Indeed, we first observe that for any j and $E \in \mathcal{D}$, we can prove by induction and the final statement of Lemma 6.9 that $\big(\tau^{\geqslant j+1}(E)\big)_c \in \mathcal{D}^{\geqslant j}_c$. Then the vanishing of $\mathrm{H}^{-1}_{\mathcal{A}_c}((F^i)_c)$ follows from the vanishing of both $\mathrm{H}^{i-1}_{\mathcal{A}_c}(E_c)$ and $\mathrm{H}^{i-2}_{\mathcal{A}_c}\big(\big(\tau^{\geqslant i+1}(E)\big)_c\big)$, and it follows from Lemma 6.9 that F_i has no torsion supported at c. After restricting to a smaller open neighborhood of c, we may even assume that F^i is C-torsion free. If $i < 0$, then $\big(\tau^{\geqslant i+1}(E)\big)_c \in \mathcal{D}^{\geqslant i}_c$ implies that $F^i_c = \mathrm{H}^i_{\mathcal{A}_c}(E_c) = 0$, so that $F^i_U = 0$ by (1). Repeating this process, we obtain an open neighborhood U such that $E_U \in \mathcal{A}_U$ and is U-torsion free, and thus flat over U by Lemma 6.12. □

Remark 6.20. Even if $\mathcal{A}_C$ does not have a C-torsion theory, if $E \in \mathcal{A}_C$ is any particular object that *does* contain a maximal C-torsion subobject, then the conclusion of Proposition 6.19.(1) holds by the same argument.

Note that in [AP06, Section 3] some of the previous results are proven in a more general setting when S is smooth but not necessarily Dedekind, under the assumption that $\mathcal{A}_S$ is noetherian.

7. Inducing local t-structures on semiorthogonal components

In this section, we consider the following situation:

- $g\colon X \to S$ is a morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension.
- $\mathcal{D} \subset \mathrm{D^b}(X)$ is an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude.

Our goal is to generalize to the relative case the results of [BLMS17, Section 4] on inducing t-structures on semiorthogonal components.

Definition 7.1. A *relative spanning class* of $\mathcal{D}$ is a set of objects $\mathcal{G}$ of $\mathcal{D}$ such that if $F \in \mathcal{D}$ satisfies $\mathcal{H}om_S(G, F) = 0$ for all $G \in \mathcal{G}$, then $F = 0$.

This property can be checked fiberwise:

Lemma 7.2. *Assume $g\colon X \to S$ is flat. Let $\mathcal{G}$ be a set of perfect objects in $\mathcal{D}$. Then $\mathcal{G}$ is a relative spanning class if and only if for all closed points $s \in S$ the restriction $\mathcal{G}_s := i_s^*\mathcal{G}$ is a spanning class of $\mathcal{D}_s$.*

Proof. For the forward direction, assume $\mathcal{G}$ is a relative spanning class, and let $F \in \mathcal{D}_s$ be an object which satisfies $\mathcal{H}om_s(G_s, F) = 0$ for all $G \in \mathcal{G}$. Note that $\mathcal{H}om_s(-,-)$ is simply $\mathrm{RHom}(-,-)$ regarded as a $\kappa(s)$-complex. We must show that $F = 0$. Consider the base change diagram

$$\begin{tikzcd} X_s \arrow[r, "i_s"] \arrow[d, "g_s"'] & X \arrow[d, "g"] \\ s \arrow[r, "j_s"] & S \end{tikzcd}$$

We have isomorphisms

$$\begin{aligned} \mathcal{H}om_S(G, i_{s*}F) &\simeq g_*\mathcal{H}om(G, i_{s*}F) \\ &\simeq g_* i_{s*}\mathcal{H}om(G_s, F) \\ &\simeq j_{s*} g_{s*}\mathcal{H}om(G_s, F) \\ &\simeq j_{s*}\mathcal{H}om_s(G_s, F), \end{aligned}$$

where the second holds by the local adjunction between i_s^* and i_{s*} and the others are evident. By assumption this vanishes for any $G \in \mathcal{G}$, so $i_{s*}F = 0$ since $\mathcal{G}$ is a relative spanning class, and thus $F = 0$.

Conversely, assume $\mathcal{G}_s$ is a spanning class of $\mathcal{D}_s$ for all s, and let $F \in \mathcal{D}$ be an object which satisfies $\mathcal{H}om_S(G, F) = 0$ for all $G \in \mathcal{G}$. We must show that $F = 0$. For any $s \in S$, we have $\mathcal{H}om_S(G, F)_s \simeq \mathcal{H}om_s(G_s, F_s)$ by Lemma 3.21 and the flatness of g. By assumption this vanishes for any $G \in \mathcal{G}$, so $F_s = 0$ since $\mathcal{G}_s$ is a relative spanning class, and thus $F = 0$. □

The following is an easy consequence of the definitions.

Lemma 7.3. *Let $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be an S-linear semiorthogonal decomposition. Let $\mathcal{G}$ be a spanning class of $\mathcal{D}_2$. Then for an object $F \in \mathcal{D}$, we have $F \in \mathcal{D}_1$ if and only if $\mathcal{H}om_S(G, F) = 0$ for all $G \in \mathcal{G}$.*

The following is a relative version of [BLMS17, Lemma 4.3].

Lemma 7.4. *Assume $g\colon X \to S$ is proper, and S is noetherian and admits an ample line bundle. Let $\mathcal{A}_S \subset \mathcal{D}$ be the heart of a bounded S-local t-structure. Let $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be an S-linear semiorthogonal decomposition. Let $\mathcal{G}$ be a relative spanning class of $\mathcal{D}_2$ such that $\mathcal{G} \subset \mathcal{A}_S \cap \mathcal{D}_2 \cap \mathrm{D}_{\mathrm{perf}}(X)$ and*

$$\mathcal{H}om_S(G, F) \in \mathrm{D}^{\mathrm{b}}(S)^{\leqslant 1} \tag{7.1}$$

for all $G \in \mathcal{G}$ and $F \in \mathcal{A}_S$. Then

$$(\mathcal{A}_S)_1 = \mathcal{A}_S \cap \mathcal{D}_1 \subset \mathcal{D}_1$$

is the heart of a bounded S-local t-structure on $\mathcal{D}_1$, such that the inclusion $\mathcal{D}_1 \to \mathcal{D}$ is t-exact.

Proof. To show $(\mathcal{A}_S)_1$ is the heart of a bounded t-structure on $\mathcal{D}_1$, we check the conditions of Proposition 4.5. Condition (1) is clearly satisfied.

For condition (2) it suffices to show that for $F \in \mathcal{D}_1$ we have $\mathrm{H}^q_{\mathcal{A}_S}(F) \in (\mathcal{A}_S)_1$ for all $q \in \mathbb{Z}$. Note that this will also show $\mathrm{H}^q_{(\mathcal{A}_S)_1}(F) = \mathrm{H}^q_{\mathcal{A}_S}(F)$, so that $\mathcal{D}_1 \to \mathcal{D}$ is t-exact. Let q be the smallest integer such that $\mathrm{H}^q_{\mathcal{A}_S}(F) \neq 0$, so that there is an exact triangle

$$\mathrm{H}^q_{\mathcal{A}_S}(F)[-q] \to F \to \tau^{>q}(F).$$

It suffices to show $\mathrm{H}^q_{\mathcal{A}_S}(F) \in (\mathcal{A}_S)_1$ for this particular q, because then by induction the statement follows for all q. By Lemma 7.3 we have $\mathrm{H}^q_{\mathcal{A}_S}(F) \in (\mathcal{A}_S)_1$ if and only if

$$\mathcal{H}om_S(G, \mathrm{H}^q_{\mathcal{A}_S}(F)) = 0 \tag{7.2}$$

for all $G \in \mathcal{G}$. Since $G \in \mathcal{D}_2$ and $F \in \mathcal{D}_1$ we have $\mathcal{H}om_S(G, F) = 0$ (see Remark 3.9), so by the above exact triangle we have an isomorphism

$$\mathcal{H}om_S(G, \mathrm{H}^q_{\mathcal{A}_S}(F)) \simeq \mathcal{H}om_S(G, \tau^{>q}(F))[q-1].$$

By the assumption (7.1) the left side lies in $\mathrm{D}^{\mathrm{b}}(S)^{\leqslant 1}$, while by Lemma 4.12 the right side lies in $\mathrm{D}^{\mathrm{b}}(S)^{>1}$. Hence both sides vanish, which proves (7.2) for all $G \in \mathcal{G}$.

Finally, it follows from Theorem 4.13 that the t-structure $(\mathcal{A}_S)_1 \subset \mathcal{D}_1$ is S-local. □

To formulate a useful situation in which Lemma 7.4 applies, we need the notion of a relative Serre functor. An S-linear functor $\mathrm{S}_{\mathcal{D}/S} \colon \mathcal{D} \to \mathcal{D}$ is called a relative Serre functor for $\mathcal{D}$ over S if there are functorial isomorphisms

$$\mathcal{H}om_S(F, \mathrm{S}_{\mathcal{D}/S}(G)) \simeq \mathcal{H}om_S(G, F)^\vee$$

for all $F, G \in \mathcal{D}$. The case where S is a point reduces to the usual notion of a Serre functor. For instance, if $g \colon X \to S$ is a smooth and proper morphism of noetherian schemes, then the functor

$$(- \otimes \omega_g[\dim(g)]) \colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$$

is a relative Serre functor by Grothendieck duality. Moreover, if $\gamma \colon \mathcal{C} \to \mathcal{D}$ is a right admissible subcategory and $\mathrm{S}_{\mathcal{D}/S}$ is a relative Serre functor for $\mathcal{D}$, then it follows easily from the definitions that $\mathcal{C}$ has a relative Serre functor given by $\mathrm{S}_{\mathcal{C}/S} = \gamma^! \circ \mathrm{S}_{\mathcal{D}/S} \circ \gamma$. In particular, putting the previous two remarks together, we see that if $g \colon X \to S$ is a smooth and proper morphism of noetherian schemes, then any right admissible subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ admits a relative Serre functor.

Now we can give the relative version of [BLMS17, Corollary 4.4].

Corollary 7.5. *Assume $g \colon X \to S$ is smooth and proper, and S is noetherian and admits an ample line bundle. Let $\mathcal{A}_S \subset \mathcal{D}$ be the heart of a bounded S-local t-structure. Let $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be an S-linear semiorthogonal decomposition. Let $\mathcal{G}$ be a relative spanning class of $\mathcal{D}_2$ such that $\mathcal{G} \subset \mathcal{A}_S \cap \mathcal{D}_2 \cap \mathrm{D}_{\mathrm{perf}}(X)$ and every $G \in \mathcal{G}$ satisfies $\mathrm{S}_{\mathcal{D}/S}(G) \in \mathcal{A}_S[1]$, where $\mathrm{S}_{\mathcal{D}/S}$ denotes the relative Serre functor of $\mathcal{D}$. Then*

$$(\mathcal{A}_S)_1 = \mathcal{A}_S \cap \mathcal{D}_1 \subset \mathcal{D}_1$$

is the heart of a bounded S-local t-structure on $\mathcal{D}_1$, such that the inclusion $\mathcal{D}_1 \to \mathcal{D}$ is t-exact.

Proof. Let $G \in \mathcal{G}$ and $F \in \mathcal{A}_S$. We have

$$\mathcal{H}om_S(G, F) \simeq \mathcal{H}om_S(F, \mathrm{S}_{\mathcal{D}/S}(G))^\vee.$$

Since $\mathrm{S}_{\mathcal{D}/S}(G) \in \mathcal{A}_S[1]$, Lemma 4.12 gives

$$\mathcal{H}om_S(F, \mathrm{S}_{\mathcal{D}/S}(G)) \in \mathrm{D}^{\mathrm{b}}(S)^{\geqslant -1},$$

and hence

$$\mathcal{H}om_S(F, \mathrm{S}_{\mathcal{D}/S}(G))^\vee \in \mathrm{D}^{\mathrm{b}}(S)^{\leqslant 1}.$$

Therefore the assumptions of Lemma 7.4 are satisfied. □

We also have the following fibral variant.

Corollary 7.6. *Assume $g\colon X \to S$ is smooth and proper, and S is noetherian, regular, and admits an ample line bundle. Let $\mathcal{A}_S \subset \mathcal{D}$ be the heart of a bounded S-local t-structure. Let $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be an S-linear semiorthogonal decomposition. Let $\mathcal{G}$ be a relative spanning class of $\mathcal{D}_2$ such that $\mathcal{G} \subset \mathcal{A}_S$ and every $G \in \mathcal{G}$ satisfies $\mathrm{S}_{\mathcal{D}_s}(G_s) \in \mathcal{A}_s[1]$ for all closed points $s \in S$, where $\mathrm{S}_{\mathcal{D}_s}$ denotes the Serre functor of $\mathcal{D}_s$ and $\mathcal{A}_s \subset \mathcal{D}_s$ is the induced heart given by Theorem 5.6. Then*

$$(\mathcal{A}_S)_1 = \mathcal{A}_S \cap \mathcal{D}_1 \subset \mathcal{D}_1$$

is the heart of a bounded S-local t-structure on $\mathcal{D}_1$, such that the inclusion $\mathcal{D}_1 \to \mathcal{D}$ is t-exact.

Proof. Note that our assumptions imply that X is regular of finite Krull dimension, and thus $\mathrm{D}^{\mathrm{b}}(X) = \mathrm{D}_{\mathrm{perf}}(X)$. Let $G \in \mathcal{G}$ and $F \in \mathcal{A}_S$. For every closed point $s \in S$, we have

$$\mathcal{H}om_S(G, F)_s \simeq \mathcal{H}om_s(G_s, F_s) \simeq \mathcal{H}om_s(F_s, \mathrm{S}_{\mathcal{D}_s}(G_s))^\vee,$$

where the first isomorphism holds by Lemma 3.21 and the flatness of g, and for the second we used that $F_s, G_s \in \mathcal{D}_s$ since F and G are perfect. This object lies in $\mathrm{D}^{\mathrm{b}}(\kappa(s))^{\leqslant 1}$ since $F_s \in \mathcal{D}_s^{\leqslant 0}$ and $\mathrm{S}_{\mathcal{D}_s}(G_s) \in \mathcal{A}_s[1]$. Because this holds for all closed points s, it follows that $\mathcal{H}om_S(G, F) \in \mathrm{D}^{\mathrm{b}}(S)^{\leqslant 1}$. Therefore the assumptions of Lemma 7.4 are satisfied. □

Remark 7.7. The assumptions of Corollary 7.6 imply those of Corollary 7.5 if the heart $\mathcal{A}_S$ is noetherian or if the base is Dedekind. Indeed, it follows easily from the definitions that there is an isomorphism of functors $i_{s*} \circ i_s^* \circ \mathrm{S}_{\mathcal{D}/S} \simeq i_{s*} \circ \mathrm{S}_{\mathcal{D}_s} \circ i_s^*$. From this we see that $\mathrm{S}_{\mathcal{D}_s}(G_s) \in \mathcal{A}_s[1]$ is equivalent to $i_s^* \mathrm{S}_{\mathcal{D}/S}(G) \in \mathcal{A}_s[1]$. Hence [AP06, Proposition 3.3.2] (in the case $\mathcal{A}_S$ noetherian) or Lemma 6.13 (in case the base is Dedekind) implies $\mathrm{S}_{\mathcal{D}/S}(G) \in \mathcal{A}_S[1]$ for all $G \in \mathcal{G}$ under the assumptions of Corollary 7.6.

Remark 7.8. In the situation of Corollary 7.6, the formation of the induced heart commutes with restriction to fibers. In symbols, if $s \in S$ is a closed point, then $((\mathcal{A}_S)_1)_s = (\mathcal{A}_s)_1$. This follows easily from the definitions.

We will also need the following observation.

Lemma 7.9. *Assume in the setting of Lemma 7.4 (or Corollary 7.5 or 7.6) that $S = C$ is a Dedekind scheme C and $\mathcal{A}_C$ has a C-torsion theory. Then $(\mathcal{A}_C)_1$ also has a C-torsion theory.*

Proof. It suffices to show that if $E \in (\mathcal{A}_C)_1$ and $E_{C\text{-tor}} \in \mathcal{A}_C$ is its maximal C-torsion subobject in $\mathcal{A}_C$, then $E_{C\text{-tor}}$ lies in $(\mathcal{A}_C)_1$. Let $W \subset C$ be the support of $E_{C\text{-tor}}$. Then by Lemma 6.18 we have $E_{C\text{-tor}} = g^*(I_W^{-1}) \otimes i_{W*}\mathrm{H}^{-1}_{\mathcal{A}_W}(E_W)$. Note that

$$i_{W*}\mathrm{H}^{-1}_{\mathcal{A}_W}(E_W) = \mathrm{H}^{-1}_{\mathcal{A}_S}(i_{W*}E_W) = \mathrm{H}^{-1}_{(\mathcal{A}_S)_1}(i_{W*}E_W),$$

where the first equality holds by t-exactness of i_{W*} and the second by t-exactness of the inclusion $\mathcal{D}_1 \to \mathcal{D}$ and the fact that $i_{W*}E_W \in \mathcal{D}_1$. Since tensoring with the line bundle $g^*(I_W^{-1})$ is t-exact as an endofunctor of $\mathcal{D}_1$, we conclude $E_{C\text{-tor}} \in (\mathcal{A}_C)_1$. □

Part II. Moduli spaces

8. Moduli of complexes

In [Lie06a], Lieblich showed that for $g\colon X \to S$ a proper, flat, finitely presented morphism of schemes, there is an algebraic stack parametrizing "families" of objects in the bounded derived categories of the fibers of g with vanishing negative self-Exts. In fact, [Lie06a] handles more generally the case where X and S are algebraic spaces, but we will not need this. In this section we review Lieblich's results; we often give references to [Sta21], which contains a slightly different exposition that we find convenient.

8.1. **Relatively perfect objects.** The meaning of a family of objects in the bounded derived categories of the fibers is made precise by the notion of a relatively perfect object. We use the definition from [Sta21, Tag 0DI0], but it agrees with Lieblich's original definition, see [Sta21, Tag 0DI9]. Recall from Section 2 that a pseudo-coherent complex is one that is locally quasi-isomorphic to a bounded above complex of finitely generated locally free sheaves.

Definition 8.1. Let $g\colon X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Then an object $E \in \mathrm{D}(X)$ is *S-perfect* if E is pseudo-coherent and locally of finite Tor-dimension over $g^{-1}\mathcal{O}_S$.

Remark 8.2. If $X \to S$ is a morphism of schemes, then E is locally of finite Tor-dimension over $g^{-1}\mathcal{O}_S$ if and only if for any affine open $U \subset X$ mapping into an affine open $V \subset S$, the complex $\mathrm{R}\Gamma(U, E)$ is of finite Tor-dimension over $\mathcal{O}_S(V)$.

The following summarizes the relations between the notions of S-perfect, bounded coherent, and perfect complexes.

Lemma 8.3. *Let $g\colon X \to S$ be a morphism of schemes which is flat and locally of finite presentation.*

(1) *Assume X is quasi-compact. If $E \in \mathrm{D}(X)$ is S-perfect, then $E \in \mathrm{D}^{\mathrm{b}}(X)$.*
(2) *Assume S is regular of finite Krull dimension. If $E \in \mathrm{D}^{\mathrm{b}}(X)$, then E is S-perfect.*
(3) *If $E \in \mathrm{D}_{\mathrm{perf}}(X)$, then E is S-perfect.*

Proof. Recall from Section 2 that $\mathrm{D}^{\mathrm{b}}(X)$ is defined as the category of pseudo-coherent complexes with bounded cohomology. Hence in the situation of (1), it suffices to show that E has bounded cohomology. Since X is quasi-compact this can be checked locally, where it holds by Remark 8.2.

In the situation of (2), to check E is locally of finite Tor-dimension over $g^{-1}\mathcal{O}_S$ we may assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$. Then we must show $\mathrm{R}\Gamma(X, E) \in \mathrm{D}^{\mathrm{b}}(A)$ has finite Tor-dimension over R. But by regularity R has finite global dimension [Sta21, Tag 00OE], so we conclude by [Sta21, Tag 066P].

Finally, for (3) note that $E \in \mathrm{D}(X)$ is perfect if and only if E is pseudo-coherent and locally has finite Tor-dimension [Sta21, Tag 08CQ]. Since $g\colon X \to S$ is flat, E is then also locally of finite Tor-dimension over $g^{-1}\mathcal{O}_S$. □

Lemma 8.4 ([Sta21, Tag 0DI5])**.** *Let $g\colon X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $\phi\colon T \to S$ be a morphism of schemes. If $E \in \mathrm{D}(X)$ is S-perfect, then $E_T \in \mathrm{D}(X_T)$ is T-perfect.*

In particular, if $E \in \mathrm{D}(X)$ is S-perfect, and we let $T = \mathrm{Spec}(\kappa(s))$ for a point $s \in S$, then Lemma 8.3 and Lemma 8.4 show that the restriction E_s of E to the fiber of g over s is a bounded coherent complex. Therefore we may think of an S-perfect object as a family of bounded coherent complexes. However, these restrictions E_s are not necessarily perfect complexes, so one should not mistake "S-perfect" to mean a family of perfect complexes.

8.2. **Moduli of objects on a proper morphism.** We will consider relatively perfect objects satisfying the following condition introduced in [Lie06a, Definition 2.1.8, Proposition 2.1.9].

Definition 8.5. Let $g\colon X \to S$ be a flat, proper, finitely presented morphism of schemes. An S-perfect object $E \in \mathrm{D}(X)$ is *universally gluable* if for every point $s \in S$ we have

$$\mathrm{Ext}^i(E_s, E_s) = 0 \quad \text{for } i < 0.$$

We denote by $\mathrm{D}_{\mathrm{pug}}(X/S) \subset \mathrm{D}(X)$ the full subcategory of universally gluable S-perfect objects.

Now we can define the moduli stack of interest. In this paper, we will regard stacks as groupoid-valued (pseudo)functors (instead of fibered categories), but otherwise we follow the conventions of [Sta21]. Let $(\mathrm{Sch}\,/S)$ denote the category of all S-schemes, and let Gpds denote the category of groupoids.

Definition 8.6. Let $g\colon X \to S$ be a flat, proper, finitely presented morphism of schemes. We denote by

$$\mathcal{M}_{\mathrm{pug}}(X/S)\colon (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Gpds}$$

the functor whose value on $T \in (\mathrm{Sch}\,/S)$ consists of all $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$. On morphisms, $\mathcal{M}_{\mathrm{pug}}(X/S)$ is given by pullback.

The main theorem of [Lie06a] is as follows.

Theorem 8.7 ([Lie06a, Theorem 4.2.1])**.** *Let $g\colon X \to S$ be a flat, proper, finitely presented morphism of schemes. Then $\mathcal{M}_{\mathrm{pug}}(X/S)$ is an algebraic stack locally of finite presentation and locally quasi-separated over S, with separated diagonal.*

Later we will also need an auxiliary result on representability of Hom functors.

Definition 8.8. Let $g\colon X \to S$ be a morphism of schemes. For $E, F \in \mathrm{D}(X)$, we denote by

$$\underline{\mathrm{Hom}}_S(E, F)\colon (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Sets}$$

the functor given by $T \mapsto \mathrm{Hom}_{\mathrm{D}(X_T)}(E_T, F_T)$.

The following criterion can be proved using the methods of [Lie06a].

Lemma 8.9. *Let $g\colon X \to S$ be a flat, proper, finitely presented morphism of schemes. Let $E, F \in \mathrm{D}(X)$. Assume E is pseudo-coherent, F is S-perfect. Then there exist a scheme B, which is affine and of finite presentation over S, and $\xi \in \mathrm{Hom}_{\mathrm{D}(X_B)}(E_B, F_B)$ such that, for every geometric point $\bar{s}$ over S and $\varphi \in \mathrm{Hom}_{\mathrm{D}(X_{\bar{s}})}(E_{\bar{s}}, F_{\bar{s}})$, there exists a $\kappa(\bar{s})$-rational point b of $B \times_S \mathrm{Spec}(\kappa(\bar{s}))$ such that $\xi_b = \varphi$.*

Furthermore, if for every point s of S we have

$$\mathrm{Ext}^i(E_s, F_s) = 0 \quad \textit{for } i < 0,$$

then $\underline{\mathrm{Hom}}_S(E, F)$ is representable by a scheme which is affine and of finite presentation over S.

Proof. The second part of the statement is [Sta21, Tag 0DLC]. To prove the first statement we use a similar argument as in *loc. cit.*, which we now briefly recall.

We can reduce to the case where S is affine. By [Sta21, Tag 0DKY], there exists $L \in \mathrm{D}^{\mathrm{b}}(S)$ such that for every geometric point $\bar{s}$ over S and for all i we have that

$$\mathrm{Ext}^i(E_{\bar{s}}, F_{\bar{s}}) = \mathrm{Ext}^i(L_{\bar{s}}, \kappa(\bar{s})).$$

We can now consider the exact triangle given by the "stupid" truncation

$$\sigma_{>0}L \to L \to \sigma_{\leqslant 0}L$$

where, we can assume that $\sigma_{>0}L$ is a perfect complex sitting in strictly positive degrees. In particular, $\mathrm{Hom}((\sigma_{>0}L)_{\bar{s}}, \kappa(\bar{s})) = 0$. Hence we get a surjective morphism

$$\mathrm{Hom}((\sigma_{\leqslant 0}L)_{\bar{s}}, \kappa(\bar{s})) \twoheadrightarrow \mathrm{Hom}(L_{\bar{s}}, \kappa(\bar{s})).$$

Now we can apply the argument in the proof of [Sta21, Tag 0DLC] to the complex $\sigma_{\leqslant 0}L$. □

9. Moduli of objects in a subcategory

In this section we explain how the results from Section 8.2 extend to moduli of objects in an admissible subcategory, and prove some general results on boundedness of sub-moduli problems. We consider the following situation:

- $g\colon X \to S$ is a flat, proper, finitely presented morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension.
- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude.

9.1. Moduli of objects in $\mathcal{D}$.

Definition 9.1. We denote by

$$\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)\colon (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Gpds}$$

the functor whose value on $T \in (\mathrm{Sch}\,/S)$ consists of all $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$ such that $E_t \in \mathcal{D}_t$ for all $t \in T$.

We prove the following result in this section.

Proposition 9.2. *The functor $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is an algebraic stack locally of finite presentation over S, and the canonical morphism $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S) \to \mathcal{M}_{\mathrm{pug}}(X/S)$ is an open immersion.*

The key to showing that $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S) \to \mathcal{M}_{\mathrm{pug}}(X/S)$ is an open immersion is the following lemma.

Lemma 9.3. *Let $T \to S$ be a morphism from a quasi-compact scheme with affine diagonal, and let $E \in \mathrm{D}(X_T)$ be T-perfect.*

(1) *$E \in \mathcal{D}_T$ if and only if $E_t \in \mathcal{D}_t$ for all $t \in T$.*
(2) *The set*

$$\{t \in T \mid E_t \in \mathcal{D}_t\}$$

is open in T.

Proof. Note that since $X_T \to T$ is quasi-compact and T is quasi-compact, the scheme X_T is quasi-compact. Thus by Lemma 8.3.(1) we have $E \in \mathrm{D}^{\mathrm{b}}(X_T)$. Decomposing E with respect to the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X_T) = \langle \mathcal{D}_T, ({}^{\perp}\mathcal{D})_T \rangle$$

gives an exact triangle

$$G \to E \to F$$

where $F \in \mathcal{D}_T$ and $G \in ({}^{\perp}\mathcal{D})_T$. By Theorem 3.17, for any $t \in T$ the triangle

$$G_t \to E_t \to F_t$$

obtained by restriction gives the decomposition of E_t with respect to the semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X_t) = \langle \mathcal{D}_{t,\mathrm{qc}}, {}^{\perp}\mathcal{D}_{t,\mathrm{qc}} \rangle.$$

In the above terms, we have $E \in \mathcal{D}_T$ if and only if $G = 0$ and $E_t \in \mathcal{D}_t$ if and only if $G_t = 0$. Hence part (1) holds since $G = 0$ if and only if $G_t = 0$ for all $t \in T$, and part (2) holds since the set of $t \in T$ where $G_t = 0$ is open. Indeed, both statements follow easily from Nakayama's Lemma and from the fact [Sta21, Tag 064U] that the maximal cohomology sheaf of a pseudo-coherent complex is a quasi-coherent sheaf of finite type. □

With this preparation, we can prove the proposition.

Proof of Proposition 9.2. We show that the functor $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)\colon (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Gpds}$ is a stack in the fppf topology and that the canonical morphism $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S) \to \mathcal{M}_{\mathrm{pug}}(X/S)$ is representable by open immersions. The result then follows from the fact that $\mathcal{M}_{\mathrm{pug}}(X/S)$ is an algebraic stack locally of finite presentation over S by Theorem 8.7.

As $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is a subfunctor of the stack $\mathcal{M}_{\mathrm{pug}}(X/S)$, it suffices to show that for $T \in (\mathrm{Sch}\,/S)$ and $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$, the condition $E \in \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)(T)$ can be checked fppf locally on T. To see that this is true, let $k \subset \ell$ be a field extension for field k with a morphism $\mathrm{Spec}(k) \to S$. Denote by $\mathcal{D}_k$ and $\mathcal{D}_\ell$ the base changes of $\mathcal{D}$ along $\mathrm{Spec}(k) \to S$ and the induced map $\mathrm{Spec}(\ell) \to S$, respectively. Then for an S-perfect $E \in \mathrm{D}(X)$, we claim that $E_k \in \mathcal{D}_k$ if and only if $E_\ell \in \mathcal{D}_\ell$. Indeed, by the same argument as in the proof of Lemma 9.3, this boils down to the fact that the functor $\mathrm{D}^{\mathrm{b}}(X_k) \to \mathrm{D}^{\mathrm{b}}(X_\ell)$ is conservative. It follows that the condition $E \in \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)(T)$ can be checked fppf locally on T, and therefore that $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is stack.

It remains to show that $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S) \to \mathcal{M}_{\mathrm{pug}}(X/S)$ is representable by open immersions. So let $T \in (\mathrm{Sch}\,/S)$ with a morphism $T \to \mathcal{M}_{\mathrm{pug}}(X/S)$. Since $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is a stack, checking that second projection $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S) \times_{\mathcal{M}_{\mathrm{pug}}(X/S)} T \to T$ is an open immersion reduces to the case where T is affine. Let $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$ be the object corresponding to $T \to \mathcal{M}_{\mathrm{pug}}(X/S)$. Then the fiber product $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S) \times_{\mathcal{M}_{\mathrm{pug}}(X/S)} T$ with its second projection is represented by the set

$$U = \{t \in T \,|\, E_t \in \mathcal{D}_t\}$$

with its inclusion into T. This is an open immersion by Lemma 9.3, so we are done. □

9.2. **Boundedness.** One of our ultimate goals in this paper is to construct well-behaved substacks of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ in the presence of a "stability condition on $\mathcal{D}$ over S", a notion that will be studied in Part IV. In particular, we will be interested in moduli functors which are bounded, in the following sense.

Definition 9.4. A subfunctor $\mathcal{M} \subset \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is *bounded* if there exists a pair $(B, \mathcal{E})$ where B is a scheme of finite type over S and $\mathcal{E} \in \mathcal{M}(B)$ is an object such that for every geometric point $\bar{s}$ over S and $E \in \mathcal{M}(\kappa(\bar{s}))$, there exists a $\kappa(\bar{s})$-rational point b of $B \times_S \operatorname{Spec}(\kappa(\bar{s}))$ such that $\mathcal{E}_b \cong E$.

If $\mathcal{M} \subset \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is an open substack, then following [Tod08], boundedness can be phrased in terms of the intrinsic geometry of $\mathcal{M}$ as follows.

Lemma 9.5. *Let $\mathcal{M}$ be an open and bounded substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$. Then $\mathcal{M}$ is an algebraic stack of finite type over S.*

Proof. Since $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is an algebraic stack locally of finite presentation over S by Proposition 9.2, the same holds for $\mathcal{M}$.

Now consider B as in Definition 9.4. Then $B \to \mathcal{M}$ is surjective by [Sta21, Tag 04ZR], and so $\mathcal{M} \to S$ is quasi-compact by [Sta21, Tag 050X]. Hence it is of finite type. □

The following observation will be needed later.

Lemma 9.6. *Assume S is noetherian. Let $\mathcal{M} \subset \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ be an open substack. Assume there exist bounded subfunctors $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ such that for every geometric point $\bar{s}$ over S and every $E \in \mathcal{M}(\kappa(\bar{s}))$, there is an exact triangle*

$$E_1 \to E \to E_2$$

with $E_1 \in \mathcal{M}_1(\kappa(\bar{s}))$ and $E_2 \in \mathcal{M}_2(\kappa(\bar{s}))$. Then $\mathcal{M}$ is bounded.

Proof. For $i = 1, 2$, let $(B_i, \mathcal{E}_i)$ be a pair as in Definition 9.4 witnessing the boundedness of $\mathcal{M}_i$. Let $B_{12} = B_1 \times_S B_2$, and let $(\mathcal{E}_i)_{B_{12}} \in \mathrm{D}_{\mathrm{pug}}(X_{B_{12}}/B_{12})$ denote the pullback of $\mathcal{E}_i \in \mathrm{D}_{\mathrm{pug}}(X_{B_i}/B_i)$. By Lemma 8.9, the functor

$$\underline{\mathrm{Hom}}_{B_{12}}((\mathcal{E}_2)_{B_{12}}, (\mathcal{E}_1)_{B_{12}}[1]) \colon (\mathrm{Sch}\,/B_{12})^{\mathrm{op}} \to \mathrm{Sets}$$

is covered by a scheme Z which is affine and of finite presentation over B_{12}. By composition with the morphism $B_{12} \to S$, we think of Z as a scheme over S. Note that since each $B_i \to S$ is of finite type, so is $B_{12} \to S$, and hence so is $Z \to S$.

By definition, there is a universal morphism

$$\alpha \in \mathrm{Hom}_{\mathrm{D}(X_Z)}((\mathcal{E}_2)_Z, (\mathcal{E}_1)_Z[1]),$$

where $(\mathcal{E}_i)_Z$ denotes the pullback of $\mathcal{E}_i$ along the base change $Z \to B_{12} \to B_i$. Define $\mathcal{E}$ as the $[-1]$-shifted cone of α, so that there is an exact triangle

$$(\mathcal{E}_1)_Z \to \mathcal{E} \to (\mathcal{E}_2)_Z.$$

Note that since $(\mathcal{E}_1)_Z$ and $(\mathcal{E}_2)_Z$ are Z-perfect objects of $\mathrm{D}(X_Z)$, so is $\mathcal{E}$ by [Sta21, Tag 0DI3]. Hence by [Sta21, Tag 0DLC] the locus in Z where $\mathcal{E}$ is universally gluable is an open subscheme $Z^\circ \subset Z$, i.e., there is an open subscheme $Z^\circ \subset Z$ characterized by the property that $z \in |Z^\circ|$ if and only if $\mathrm{Ext}^i(\mathcal{E}_z, \mathcal{E}_z) = 0$ for $i < 0$. Then $\mathcal{E}_{Z^\circ} \in \mathrm{D}_{\mathrm{pug}}(X_{Z^\circ}/Z^\circ)$.

Moreover, for all $z \in Z$ we have $\mathcal{E}_z \in \mathcal{D}_z$ because the analogous statement is true for $(\mathcal{E}_1)_Z$ and $(\mathcal{E}_2)_Z$ by Lemma 9.3. Thus $\mathcal{E}_{Z^\circ}$ corresponds to a morphism

$$Z^\circ \to \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$$

over S. Let $B \subset Z^\circ$ be the open subscheme given by the preimage of the open substack $\mathcal{M} \subset \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$. Then the restriction $\mathcal{E}_B$ lies in $\mathcal{M}(B)$.

We claim that the pair $(B, \mathcal{E}_B)$ witnesses the boundedness of $\mathcal{M}$. Indeed, as observed above $Z \to S$ is of finite type. Since S is noetherian, so is Z. Therefore, the open immersions $B \subset Z^\circ \subset Z$ are of finite type, and the same holds for the composition $B \to S$. Further, by the assumption of the lemma and the construction of B, for every geometric point $\bar{s}$ over S and $E \in \mathcal{M}(\kappa(\bar{s}))$, there is a $\kappa(\bar{s})$-rational point b of $B \times_S \operatorname{Spec}(\kappa(\bar{s}))$ such that $\mathcal{E}_b \cong E$. □

9.3. **Simple objects.** Finally, we discuss substacks of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ parameterizing objects which are simple in the following sense.

Definition 9.7. An object $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$ is called *simple* if $\operatorname{Hom}(E_{\bar{t}}, E_{\bar{t}}) = \kappa(\bar{t})$ holds for every geometric point $\bar{t}$ of T.

The above definition is equivalent to Lieblich's [Lie06a, Definition 4.3.1], which requires that the automorphism stack of E is given by the multiplicative group over T. Given a subfunctor $\mathcal{M} \subset \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$, we write $s\mathcal{M} \subset \mathcal{M}$ for the subfunctor parameterizing simple objects of $\mathcal{M}$.

Lemma 9.8. *Let $\mathcal{M}$ be an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$. Then $s\mathcal{M}$ is an algebraic stack locally of finite presentation over S and admits the structure of a $\mathbb{G}_m$-gerbe over an algebraic space sM locally of finite presentation over S. If, moreover, $s\mathcal{M}$ is bounded, then $s\mathcal{M}$ and sM are of finite type over S.*

Proof. The simple objects $s\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ form an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ by [Lie06a, Lemma 4.3.2] combined with Proposition 9.2. So $s\mathcal{M}$, being the intersection of $\mathcal{M}$ and $s\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$, is an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ and therefore locally of finite presentation over S. Now as in [Lie06a, Corollary 4.3.3], we obtain sM as the $\mathbb{G}_m$-rigidification of $s\mathcal{M}$. If $s\mathcal{M}$ is bounded, then it is of finite type over S by Lemma 9.5, hence sM is also of finite type. □

10. Fiberwise t-structures

In this section, we consider the following situation:

- $g\colon X \to S$ is a flat, finitely presented morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension.
- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an S-linear strong semiorthogonal component whose projection is of finite cohomological amplitude.

In Section 4 we studied the notion of an S-local t-structure on $\mathcal{D}$, which is a compatible specification of t-structures over every quasi-compact open $U \subset S$. In this section, we consider the notion of a fiberwise collection of t-structures on $\mathcal{D}$, which is just the specification of a t-structure over every point $s \in S$; to get reasonable behavior, we focus on collections where "openness of flatness" holds. If S is regular of finite Krull dimension, then by base change, i.e., Theorem 5.7, a bounded S-local t-structure on $\mathcal{D}$ which is titled-noetherian locally on S

induces a fiberwise collection of t-structures on $\mathcal{D}$. We study the latter, weaker notion since it is sufficient and well-suited for formulating moduli problems.

10.1. **Fiberwise collections of t-structures.** Let us start with the following.

Definition 10.1. A *fiberwise collection of t-structures on $\mathcal{D}$ over S* is a collection $\underline{\tau} = (\tau_s)_{s \in S}$ of t-structures on $\mathcal{D}_s$ for every (closed or non-closed) point $s \in S$.

In this setting, we have enough structure to define flat objects, analogously to Definition 6.1.

Definition 10.2. Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S, and let $T \to S$ be a morphism. Let $t \in T$ and let $s \in S$ be its image. We denote by $\hat{\tau}_t$ the t-structure on $(\mathcal{D}_{\mathrm{qc}})_t$ obtained via Theorem 5.3 by base change of τ_s along $\mathrm{Spec}(\kappa(t)) \to \mathrm{Spec}(\kappa(s))$, and write $(\mathcal{A}_{\mathrm{qc}})_t$ for the heart of $\hat{\tau}_t$. We say an object $E \in \mathrm{D}_{\mathrm{qc}}(X_T)$ is *T-flat with respect to $\underline{\tau}$* if $E_t \in (\mathcal{A}_{\mathrm{qc}})_t$ for every point $t \in T$.

Note that the pullback of any T-flat object along a morphism $T' \to T$ is T'-flat, by t-exactness of the base change functor along field extensions (Theorem 5.3.(4c)).

Remark 10.3. When they both apply, the notions of T-flatness from Definitions 6.1 and 10.2 agree, and hence we may unambiguously use the term "T-flat".

Without any extra conditions, a fiberwise collection of t-structures is not well-behaved, because there is no compatibility imposed between the t-structures on different fibers. To remedy this, we consider the following additional conditions.

Definition 10.4. Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S. Then $\underline{\tau}$ *satisfies openness of flatness* if for every S-perfect object $E \in \mathrm{D}(X)$, the set

$$\{s \in S \,|\, E_s \in (\mathcal{A}_{\mathrm{qc}})_s\}$$

is open. Similarly, $\underline{\tau}$ *universally satisfies openness of flatness* if for every $T \to S$ and every T-perfect object $E \in \mathrm{D}(X_T)$, the set

$$\{t \in T \,|\, E_t \in (\mathcal{A}_{\mathrm{qc}})_t\}$$

is open.

Remark 10.5. In Definition 10.4 we only consider relatively perfect objects, since these are the only type of objects we shall need to consider in our discussion of moduli problems.

In the next result, we show that in Definition 10.4 it suffices to check openness for affine schemes T finitely presented over S.

Lemma 10.6. *Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S. Assume that for every morphism of finite presentation $T \to S$ from an affine scheme T and every T-perfect object $E \in \mathrm{D}(X_T)$, the set*

$$\{t \in T \,|\, E_t \in (\mathcal{A}_{\mathrm{qc}})_t\}$$

is open. Then $\underline{\tau}$ universally satisfies openness of flatness.

Proof. For $T \to S$ an arbitrary morphism and $E \in \mathrm{D}(X_T)$ a T-perfect object we must show the set $\{t \in T \,|\, E_t \in (\mathcal{A}_{\mathrm{qc}})_t\}$ is open, assuming this holds for $T \to S$ of finite presentation with T affine. We may immediately reduce to the case where T is affine. In this case, by Lemma 10.7 below we can write $T = \lim T_i$ as a cofiltered limit of affine schemes $T_i, i \in I$,

which are of finite presentation over S. Assume $t \in T$ is a point such that $E_t \in (\mathcal{A}_{\mathrm{qc}})_t$. Then we must show there is an open $U \subset T$ containing t such that for every $u \in U$ we have $E_u \in (\mathcal{A}_{\mathrm{qc}})_u$.

By Lemma 9.3 we may assume $E \in \mathcal{D}_T$. The object E descends to a T_0-perfect object $E_0 \in \mathrm{D}(X_{T_0})$ for some index $0 \in I$, see [Sta21, Tag 0DI8]. We have $E_0 \in \mathrm{D}^{\mathrm{b}}(X_{T_0})$ by Lemma 8.3.(1), hence we may assume $E_0 \in \mathcal{D}_{T_0}$ by replacing E_0 with its projection into $\mathcal{D}_{T_0}$. If t_0 denotes the image of t under the projection $T \to T_0$, then the assumption $E_t \in (\mathcal{A}_{\mathrm{qc}})_t$ together with Theorem 5.3.(2) shows that $(E_0)_{t_0} \in (\mathcal{A}_{\mathrm{qc}})_{t_0}$. Since $T_0 \to S$ is a finitely presented morphism from an affine scheme, by assumption this means there is an open subset $U_0 \subset T_0$ containing t_0 such that for every $u_0 \in U_0$ we have $(E_0)_{u_0} \in \mathcal{A}_{u_0}$. Then the preimage $U \subset T$ of U_0 under the projection $T \to T_0$ is the sought-for neighborhood of $t \in T$. Indeed, $E_u \in (\mathcal{A}_{\mathrm{qc}})_u$ holds for every $u \in U$ by Theorem 5.3.(4c). □

The following lemma, invoked above, says there are "enough" affine schemes of finite presentation over a quasi-separated scheme.

Lemma 10.7. *Let $T \to S$ be a morphism of schemes with T affine. Then $T \cong \lim T_i$ is a cofiltered limit of affine schemes T_i which are of finite presentation over S.*

Proof. By [Sta21, Tag 09MV], we can write $T = \lim T_i$ as a cofiltered limit of schemes T_i of finite presentation over S with affine transition morphisms. We show that T_i is affine for all i large enough, which gives the result by omitting the first few terms. The scheme T_i is quasi-separated, being of finite presentation over the quasi-separated scheme S. Moreover, as T is affine and thus quasi-compact, it follows from [Sta21, Tag 0CUF] and [Sta21, Tag 01YX] that after shrinking T_i we can assume T_i is quasi-compact for all i. But then T_i is affine for all i large enough by [Sta21, Tag 01Z6]. □

The following result will be useful later in our construction of Quot spaces.

Lemma 10.8. *Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S which universally satisfies openness of flatness. Let $T \to S$ be a morphism from a quasi-compact scheme with affine diagonal. Let $E \to F$ be a morphism of T-perfect and T-flat objects in $\mathrm{D}(X_T)$. Then the set*

$$\{t \in T \mid E_t \to F_t \text{ is surjective in } (\mathcal{A}_{\mathrm{qc}})_t\}$$

is open.

Proof. Let $G \in \mathrm{D}(X_T)$ be the object determined by the exact triangle

$$G \to E \to F.$$

Note that G is T-perfect because E and F are [Sta21, Tag 0DI3], and moreover $G \in \mathcal{D}_{T,\mathrm{qc}}$ by considering its decomposition with respect to the semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X_T) = \langle \mathcal{D}_{T,\mathrm{qc}}, ({}^{\perp}\mathcal{D})_{T,\mathrm{qc}} \rangle.$$

For $t \in T$ we consider the exact triangle

$$G_t \to E_t \to F_t$$

obtained by restriction. By T-flatness we have $E_t, F_t \in (\mathcal{A}_{\mathrm{qc}})_t$, so the long exact cohomology sequence shows that $G_t \in \mathcal{D}_t^{[0,1]}$, and $E_t \to F_t$ is surjective if and only if $G_t \in (\mathcal{A}_{\mathrm{qc}})_t$. Thus the result follows from the assumption that $\underline{\tau}$ universally satisfies openness of flatness. □

Remark 10.9. When τ_s is a tilted-noetherian t-structure for every $s \in S$, then the t-structure on $(\mathcal{D}_{\mathrm{qc}})_t$ descends to one on $\mathcal{D}_t$ by Proposition 5.9; therefore, for a T-perfect object E we can replace the condition $E_t \in (\mathcal{A}_{\mathrm{qc}})_t$ by $E_t \in \mathcal{A}_t$ everywhere.

10.2. **Integrable collections of t-structures.** In this subsection, we compare the notion of a fiberwise collection of t-structures to the notion of a local t-structure from Section 4.

Definition 10.10. Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S, and let $T \to S$ be a morphism from a scheme T which is quasi-compact with affine diagonal. We say $\underline{\tau}$ is *integrable over* T if there exists a t-structure τ_T on $\mathcal{D}_T$ such that for every $t \in T$ the t-structure on $(\mathcal{D}_{\mathrm{qc}})_t$ induced via Theorem 5.3 by base change along the composition $\operatorname{Spec} \kappa(t) \to T$ agrees with the t-structure $\hat{\tau}_t$ from Definition 10.2. In this situation, we say $\underline{\tau}$ *integrates over* T to the t-structure τ_T .

The question of whether a given fiberwise collection of t-structures integrates over a scheme $T \to S$ is subtle. We will be particularly interested in cases where this holds for T a Dedekind scheme. The following two results in this context will be needed later.

Lemma 10.11. *Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S. Let $C \to S$ be a morphism from a Dedekind scheme C. Assume $\underline{\tau}$ integrates over C to a C-local t-structure on $\mathcal{D}_C$ with heart $\mathcal{A}_C$. Then for an object $E \in \mathrm{D}(X_C)$, the following conditions are equivalent:*

(1) *$E \in \mathcal{A}_C$ and E is C-flat.*
(2) *E is C-perfect and C-flat.*

Proof. If $E \in \mathcal{A}_C$ then E is C-perfect by Lemma 8.3.(2), so (1) implies (2). Conversely, assume E is C-perfect and C-flat. Then in particular $E_c \in \mathcal{D}_c$ for every $c \in C$, so by Lemma 9.3.(1) we find that $E \in \mathcal{D}_C$. (Strictly speaking in Section 9, and thus tacitly in Lemma 9.3, the morphism $X \to S$ is assumed to be proper, but this assumption is not used in the proof.) Thus $E \in \mathcal{A}_C$ by Lemma 6.13. □

Lemma 10.12. *Let $\underline{\tau}$ be a fiberwise collection of t-structures on $\mathcal{D}$ over S. Assume $\underline{\tau}$ integrates over C to a C-local t-structure on $\mathcal{D}_C$ whose heart $\mathcal{A}_C$ has a C-torsion theory. Then for a morphism $E \to F$ in $\mathcal{A}_C$, the following conditions are equivalent:*

(1) *$E \to F$ is surjective as a morphism in $\mathcal{A}_C$.*
(2) *For every point $c \in C$ the induced morphism $\mathrm{H}^0_{(\mathcal{A}_{\mathrm{qc}})_c}(E_c) \to \mathrm{H}^0_{(\mathcal{A}_{\mathrm{qc}})_c}(F_c)$ is surjective in $(\mathcal{A}_{\mathrm{qc}})_c$, where $(\mathcal{A}_{\mathrm{qc}})_c \subset (\mathcal{D}_{\mathrm{qc}})_c$ is the heart of the t-structure $\hat{\tau}_c$ from Definition 10.2.*

Proof. Let $G \in \mathcal{A}_C$ be defined by the right exact sequence

$$E \to F \to G \to 0$$

in $\mathcal{A}_C$, so that $E \to F$ is surjective if and only if $G = 0$. But by Proposition 6.19.(1) we have $G = 0$ if and only if for every point $c \in C$ we have $\mathrm{H}^0_{\mathcal{A}_c}(G_c) = 0$, which by right t-exactness of the restriction functor $\mathcal{D} \to \mathcal{D}_c$ is equivalent to condition (2). □

11. Quot spaces

In this section, we work in the following setup:

- $g\colon X \to S$ is a flat, proper, finitely presented morphism of schemes which are quasi-compact with affine diagonal, where X is noetherian of finite Krull dimension.

- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an S-linear strong semiorthogonal component whose projection functor is of finite cohomological amplitude.
- $\underline{\tau}$ is a fiberwise collection of t-structures on $\mathcal{D}$ over S which universally satisfies openness of flatness.

Our goal is to define Quot functors in this setting, and to show that under good conditions they are algebraic spaces that satisfy valuative criteria. We begin by introducing the moduli stack of flat objects, which is used in our proof of representability.

11.1. **Moduli of flat objects.** We have seen in Section 9.1 that there is an algebraic stack $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ locally of finite presentation over S which parametrizes relatively perfect, universally gluable objects of $\mathcal{D}$. We can use $\underline{\tau}$ to cut out a substack of flat objects.

Definition 11.1. We denote by

$$\mathcal{M}_{\underline{\tau}} \colon (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Gpds}$$

the functor whose value on $T \in (\mathrm{Sch}\,/S)$ consists of all objects $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$ which are T-flat with respect to $\underline{\tau}$.

Note that above it would be equivalent to require $E \in \mathrm{D}(X_T)$ is T-perfect and T-flat, since such an object is automatically universally gluable.

Lemma 11.2. *The functor $\mathcal{M}_{\underline{\tau}}$ is an algebraic stack locally of finite presentation over S, and the canonical morphism $\mathcal{M}_{\underline{\tau}} \to \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is an open immersion.*

Proof. The morphism $\mathcal{M}_{\underline{\tau}} \to \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is representable by open immersions since $\underline{\tau}$ universally satisfies openness of flatness. Hence the result follows from Proposition 9.2. □

11.2. **Quot spaces.** Let us start with following definition.

Definition 11.3. Let $E \in \mathcal{D}$ be an S-perfect object. We denote by

$$\mathrm{Quot}_S(E) \colon (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Sets}$$

the functor whose value on $T \in (\mathrm{Sch}\,/S)$ is the set of all morphisms $E_T \to Q$ in $\mathrm{D}(X)$, where:

(1) $Q \in \mathrm{D}(X_T)$ is T-perfect and T-flat with respect to $\underline{\tau}$.
(2) The morphism $\mathrm{H}^0_{(\mathcal{A}_{\mathrm{qc}})_t}(E_t) \to \mathrm{H}^0_{(\mathcal{A}_{\mathrm{qc}})_t}(Q_t) = Q_t$ in $(\mathcal{A}_{\mathrm{qc}})_t$ is surjective for all $t \in T$.

Given $T' \to T$ in $(\mathrm{Sch}\,/S)$, the corresponding map

$$\mathrm{Quot}_S(E)(T) \to \mathrm{Quot}_S(E)(T')$$

takes $E_T \to Q$ to its pullback $E_{T'} \cong (E_T)_{T'} \to Q_{T'}$ along $T' \to T$.

Remark 11.4. It is straightforward to verify that the morphism $E_{T'} \to Q_{T'}$ obtained by pullback in the above definition is indeed in $\mathrm{Quot}_S(E)(T')$.

Remark 11.5. Below we focus on the case where E is S-flat. Then $\mathrm{Quot}_S(E)(T)$ has a slightly simpler description, since $\mathrm{H}^0_{(\mathcal{A}_{\mathrm{qc}})_t}(E_t) = E_t$ for any $t \in T$.

Proposition 11.6. *Let $E \in \mathcal{D}$ be an S-perfect and S-flat object. Then $\mathrm{Quot}_S(E)$ is an algebraic space locally of finite presentation over S.*

The algebraicity of the usual Quot functor of a coherent sheaf can be shown using algebraicity of the stack of coherent sheaves, see [Lie06b], [Sta21, Tag 09TQ]. We follow a similar strategy to prove Proposition 11.6, where the role of the stack of coherent sheaves is replaced by the stack $\mathcal{M}_{\underline{\tau}}$ of flat objects. Towards this, we first prove the following.

Lemma 11.7. *Let $E \in \mathcal{D}$ be an S-perfect and S-flat object. Then $\mathrm{Quot}_S(E)$ is a sheaf in the fppf topology on $(\mathrm{Sch}\,/S)$.*

Proof. Let $T_i \to T$ be an fppf cover in $(\mathrm{Sch}\,/S)$, and let $q_i \colon E_{T_i} \to Q_i$ be an object of $\mathrm{Quot}_S(E)(T_i)$ such that for every i, j the restrictions of q_i and q_j to $T_i \times_T T_j$ agree. By Theorem 8.7 and Lemma 8.9, these morphisms glue uniquely to a morphism $q \colon E_T \to Q$ in $\mathrm{D}_{\mathrm{pug}}(X_T/T)$. To finish we must show $q \colon E_T \to Q$ is in $\mathrm{Quot}_S(E)(T)$, i.e., for every $t \in T$ we have $Q_t \in (\mathcal{A}_{\mathrm{qc}})_t$ and the morphism $E_t \to Q_t$ in $(\mathcal{A}_{\mathrm{qc}})_t$ is surjective. But base change along an extension of fields is t-exact by Theorem 5.3.(4c) and conservative, so these conditions can be checked fppf locally, see Remark 4.9. □

Proof of Proposition 11.6. There is a canonical morphism

$$\mathrm{Quot}_S(E) \to \mathcal{M}_{\underline{\tau}}$$

which for $T \in (\mathrm{Sch}\,/S)$ sends $(E_T \to Q) \in \mathrm{Quot}_S(E)(T)$ to $Q \in \mathcal{M}_{\underline{\tau}}(T)$. It suffices to show $\mathrm{Quot}_S(E) \to \mathcal{M}_{\underline{\tau}}$ is representable by algebraic spaces and locally of finite presentation. Indeed, then since $\mathcal{M}_{\underline{\tau}}$ is algebraic stack locally of finite presentation over S by Lemma 11.2, we conclude that $\mathrm{Quot}_S(E)$ is an algebraic space locally of finite presentation over S by [Sta21, Tag 05UM] and [Sta21, Tag 04SZ].

So let $T \in (\mathrm{Sch}\,/S)$, let $T \to \mathcal{M}_{\underline{\tau}}$ be a morphism, and let

$$Z = \mathrm{Quot}_S(E) \times_{\mathcal{M}_{\underline{\tau}}} T \colon (\mathrm{Sch}\,/T)^{\mathrm{op}} \to \mathrm{Sets}$$

be the fiber product; we must show $Z \to T$ is a locally finitely presented morphism of algebraic spaces. Let $Q \in \mathcal{M}_{\underline{\tau}}(T)$ be the object corresponding to $T \to \mathcal{M}_{\underline{\tau}}$. For $T' \in (\mathrm{Sch}\,/T)$ we have

$$Z(T') = \left\{ \begin{array}{c} \text{morphisms } E_{T'} \to Q_{T'} \text{ in } \mathrm{D}(X_{T'}) \text{ such that} \\ E_{t'} \to Q_{t'} \text{ is surjective in } (\mathcal{A}_{\mathrm{qc}})_{t'} \text{ for all } t' \in T' \end{array} \right\}.$$

Hence $Z \to T$ factors through the forgetful morphism $Z \to \underline{\mathrm{Hom}}_T(E_T, Q)$. Note that $\underline{\mathrm{Hom}}_T(E_T, Q)$ is an algebraic space of finite presentation by Lemma 8.9, whose hypotheses are satisfied since for every $t \in T$ we have $E_t, Q_t \in (\mathcal{A}_{\mathrm{qc}})_t$. Hence the following claim will finish the proof: $Z \to \underline{\mathrm{Hom}}_T(E_T, Q)$ is an open immersion.

To prove the claim, let $T' \in (\mathrm{Sch}\,/T)$, let $T' \to \underline{\mathrm{Hom}}_T(E_T, Q)$ be a morphism, and let

$$Y = Z \times_{\underline{\mathrm{Hom}}_T(E_T, Q)} T' \to T'$$

be the fiber product; we must show $Y \to T'$ is an open immersion. Since $\mathrm{Quot}_S(E)$ is a sheaf in the fppf topology by Lemma 11.7, it follows that Y is too. Hence we may reduce to the case where T' is affine. Let $f \colon E_{T'} \to Q_{T'}$ be the morphism corresponding to $T' \to \underline{\mathrm{Hom}}_T(E_T, Q)$. By Lemma 10.8 the subset

$$U = \left\{ t' \in T' \,\middle|\, f_{t'} \colon E_{t'} \to Q_{t'} \text{ is surjective} \right\}$$

is open in T'. The morphism $U \to T'$ represents $Y \to T'$, so we are done. □

11.3. **Valuative criteria for Quot spaces.** In this subsection, we investigate the valuative criterion for Quot spaces. First we formulate what we mean by valuative criteria; for later use in the paper, we consider the setting of algebraic stacks.

Definition 11.8. Let $f\colon X \to Y$ be a morphism of algebraic stacks. Let R be a valuation ring with field of fractions K, and let $\mathrm{Spec}(R) \to Y$ be a morphism. Then we say f *satisfies the strong existence part of the valuative criterion with respect to* $\mathrm{Spec}(R) \to Y$ if given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{11.1}$$

there exists a dotted arrow making the diagram commute. We say f *satisfies the uniqueness part of the valuative criterion with respect to* $\mathrm{Spec}(R) \to Y$ if for any solid diagram (11.1), the category of dotted arrows (see [Sta21, Tag 0CLA]) is either empty or a setoid with exactly one isomorphism class.

Remark 11.9. The commutativity of the diagram in Definition 11.8 must be understood in the 2-categorical sense, see [Sta21, Tag 0CL9]. In case X and Y are algebraic spaces, this subtlety disappears. Moreover, in this case the uniqueness part of the valuative criterion just says that there exists at most one dotted arrow in (11.1).

Remark 11.10. We use the adjective "strong" because the standard "existence part of the valuative criterion" for algebraic stacks (or spaces) only requires the existence of a dotted arrow after passing to a field extension of K, see [Sta21, Tag 0CLK].

We will show that under a suitable integrability hypothesis, Quot spaces satisfy the valuative criteria formulated above.

Proposition 11.11. *Let $E \in \mathcal{D}$ be an S-perfect and S-flat object. Let R be a discrete valuation ring and let $\mathrm{Spec}(R) \to S$ be a morphism. Assume that $\underline{\tau}$ integrates over $\mathrm{Spec}(R)$ to a bounded $\mathrm{Spec}(R)$-local t-structure on $\mathcal{D}_R$ whose heart $\mathcal{A}_R$ has a $\mathrm{Spec}(R)$-torsion theory. Then the morphism $\mathrm{Quot}_S(E) \to S$ satisfies the strong existence and the uniqueness parts of the valuative criterion with respect to $\mathrm{Spec}(R) \to S$.*

Before proving the proposition, we give an alternate description of the Quot functor in the integrable case.

Lemma 11.12. *Let $E \in \mathcal{D}$ be an S-perfect and S-flat object. Let $C \to S$ be a morphism from a Dedekind scheme. Assume that $\underline{\tau}$ integrates over C to a bounded C-local t-structure on $\mathcal{D}_C$ whose heart $\mathcal{A}_C$ has a C-torsion theory. Then there is an identification*

$$\mathrm{Quot}_S(E)(C) = \left\{ \begin{array}{c} \text{quotients } E_C \to Q \text{ in } \mathcal{A}_C \\ \text{such that } Q \text{ is } C\text{-flat.} \end{array} \right\}$$

Proof. Note that $E_C \in \mathcal{A}_C$ by Lemma 10.11, since E_C is C-perfect and C-flat. By definition, a C-point of $\mathrm{Quot}_S(E)$ is a morphism $E_C \to Q$ in $\mathrm{D}(X_C)$ such that Q is C-perfect and C-flat and $E_c \to Q_c$ is surjective in $(\mathcal{A}_{\mathrm{qc}})_c$ for all $c \in C$. Equivalently, by Lemmas 10.11 and 10.12, Q is a C-flat object of $\mathcal{A}_C$ and $E_C \to Q$ is a quotient in $\mathcal{A}_C$. □

Proof of Proposition 11.11. Let K be the fraction field of R. For the strong existence part of the valuative criterion, by Lemma 11.12 we must show that given a surjection $f\colon E_K \to Q$ in $\mathcal{A}_K$ with Q a K-flat object, there is a surjection $\tilde{f}\colon E_R \to \tilde{Q}$ in $\mathcal{A}_R$ with $\tilde{Q}$ an R-flat object, such that $\tilde{f}$ restricts to f. To this end, we first use Lemma 4.16.(2) to obtain $\tilde{Q}$ and a surjective map $\tilde{f}'\colon E_R \to \tilde{Q}$ extending f. After replacing $\tilde{Q}$ with $\tilde{Q}_{R\text{-tf}}$, we may further assume $\tilde{Q}$ is R-torsion free and hence R-flat by Lemma 6.12, proving the claim.

To show uniqueness, assume we are given two R-flat extensions $\tilde{Q}_1, \tilde{Q}_2$ of Q and surjections $\tilde{f}_i\colon E_R \to \tilde{Q}_i$. By Lemma 6.8, there exists an injective map $i\colon \tilde{Q}_1 \hookrightarrow \tilde{Q}_2 \otimes g^*\mathcal{O}_R(\pi^k)$ for some $k \geqslant 0$ such that after pulling back to K, we have $i \circ f_1 = \pi^k f_2$. Hence the image of $i \circ f_1 - \pi^k f_2$ is R-torsion; since $\tilde{Q}_2$ is R-torsion free by Lemma 6.12, the same holds true of $\tilde{Q}_2 \otimes g^*\mathcal{O}_R(\pi^k)$ so that this image must zero. It follows that the two morphisms are equal already over R. Since π^k acts injectively on R-torsion free objects, we obtain a chain of isomorphisms $\tilde{Q}_1 \cong \operatorname{Im} i = \operatorname{Im} \pi^k \cong \tilde{Q}_2$ of quotients of E_R. □

11.4. **Nagata valuative criteria.** Sometimes, we will only know (via Theorem 5.7) that $\underline{\tau}$ integrates to a bounded $\operatorname{Spec}(R)$-local t-structure when $\operatorname{Spec}(R) \to S$ is essentially of finite type; therefore, we will only assume as much in Definition 20.5. In this section, we prove that when S is Nagata, it is enough to consider such morphisms in the valuative criteria for universal closedness, see Lemma 11.21. This will allow us for example to show universal closedness of moduli spaces of semistable objects in the proof of Theorem 21.24. Otherwise, this section is completely independent of the rest of the paper.

For the definition of the Nagata property see [Sta21, Tag 032E] (for rings), [Sta21, Tag 033R] (for schemes), and [Sta21, Tag 0BAT] (for algebraic spaces). This property holds in essentially all naturally occurring examples. For instance, any quasi-excellent ring is Nagata [Sta21, Tag 07QV].

Recall that a domain A with field of fractions K is called Japanese if for any finite extension of fields $K \subset L$, the integral closure of A in L is finite over A. A ring A is then called universally Japanese if for any finite type ring map $A \to B$ with B a domain, the ring B is Japanese. The key properties we need about Nagata rings are the following.

Lemma 11.13 ([Sta21, Tag 0334])**.** *A ring is Nagata if and only if it is universally Japanese and noetherian.*

Recall that a ring map $A \to B$ is *essentially of finite type* if B is the localization of an A-algebra of finite type. We recall the following result:

Lemma 11.14 ([Sta21, Tag 0334] and [Sta21, Tag 032U])**.** *If A is a Nagata ring and $A \to B$ is a ring map essentially of finite type, then B is Nagata.*

Lemma 11.15. *Let A be a Nagata local domain with fraction field K. Assume that A is not a field. Let $K \subset L$ be a finitely generated field extension. Then there exists a DVR R with fraction field L which dominates A, such that $A \to R$ is essentially of finite type.*

Proof. The following argument is modeled on the proof of [Sta21, Tag 00PH].

If L is not finite over K, choose a transcendence basis $x_1, \dots, x_r$ of L over K. Let $\mathfrak{m} \subset A[x_1, \dots, x_r]$ be the maximal ideal generated by the maximal ideal of A and $x_1, \dots, x_r$, and let $A' = A[x_1, \dots, x_r]_{\mathfrak{m}}$. Then A' is a local domain which dominates A, the map $A \to A'$ is essentially of finite type, and L is finite over the fraction field of A'. Moreover, A' is Nagata by Lemma 11.14. Replacing A with A' we may thus assume L is finite over K.

By [Sta21, Tag 00P8] we may find a noetherian local domain A' of dimension 1 with fraction field K which dominates A and such that $A \to A'$ is essentially of finite type. Again by Lemma 11.14, A' is Nagata. Replacing A with A' we may thus further assume $\dim(A) = 1$.

Let $B \subset L$ be the integral closure of A in L. By Lemma 11.13 the ring A is Japanese, so B is finite over A. Thus we may choose a prime $\mathfrak{q} \subset B$ lying over the maximal ideal of A. Set $R = B_{\mathfrak{q}}$. Then R is a local normal domain of dimension 1 which has fraction field L and dominates A, and the map $A \to R$ is essentially of finite type. By Lemma 11.13 we see that R is also noetherian, and hence a DVR. □

The following notion will be used below.

Definition 11.16. Let Y be an algebraic space. For a scheme Z, we say a morphism $f\colon Z \to Y$ is *essentially (locally) of finite type* if it is either (locally) of finite type or if Z is affine and f factors as

$$Z = \operatorname{Spec}(R[M^{-1}]) \to \operatorname{Spec}(R) \to Y$$

where $\operatorname{Spec}(R) \to Y$ is a morphism (locally) of finite type and $M \subset R$ is a multiplicative system.

Remark 11.17. If Y is a locally noetherian scheme, then for any morphism $\operatorname{Spec}(R) \to Y$ locally of finite type, $\operatorname{Spec}(R)$ is noetherian and hence $\operatorname{Spec}(R) \to Y$ is of finite type. Thus if Y is a locally noetherian scheme, a morphism $Z \to Y$ from an affine scheme Z is essentially locally of finite type if and only if it is essentially of finite type. Since a Nagata scheme Y is locally noetherian by [Sta21, Tag 033Z], we will apply this observation in the results below whenever Y is a Nagata scheme.

Remark 11.18. We have chosen definitions that make our results as general as possible. Note, however, that Definition 11.16 is in a slightly different spirit than our definition of an essentially perfect morphism in Definition 5.4. One can show that for a noetherian scheme Y, any essentially locally of finite type morphism $Z \to Y$ that is also of finite Tor-dimension is essentially perfect, but the converse need not hold.

Lemma 11.19. *Let $f\colon X \to Y$ be a locally of finite type morphism of schemes with Y Nagata. Let $y \rightsquigarrow y'$, $y \neq y'$, be a specialization of points in Y, and let x be a point of X such that $f(x) = y$. Then there exists a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] & Y \end{tikzcd}$$

where R is a DVR with field of fractions K (which may be taken to be $\kappa(x)$), the bottom arrow is essentially of finite type and takes the generic point of $\operatorname{Spec}(R)$ to y and the special point to y', and the image point of the top arrow is x.

Proof. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(\kappa(x)) \arrow[rr] \arrow[d] & & X \arrow[d] \\ \operatorname{Spec}(\kappa(y)) \arrow[r] & \operatorname{Spec}(\mathcal{O}_{Y,y'}) \arrow[r] & Y \end{tikzcd}$$

Let A be image of the ring map $\mathcal{O}_{Y,y'} \to \kappa(y)$. Then A is a local domain with fraction field $\kappa(y)$, and A is not a field since $y \neq y'$. By Lemma 11.14 the ring A is Nagata because it is the quotient of the Nagata ring $\mathcal{O}_{Y,y'}$. The field extension $\kappa(y) \subset \kappa(x)$ is finitely generated since $X \to Y$ is locally of finite type. Thus applying Lemma 11.15 we obtain a DVR R with fraction field $\kappa(x)$ which dominates A, such that $A \to R$ is essentially of finite type. Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(x)) \arrow[rrr] \arrow[d] & & & X \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(\mathcal{O}_{Y,y'}) \arrow[r] & Y
\end{tikzcd}$$

with all of the desired properties. □

We will need the following lemma in our proof of the Nagata valuative criterion for universal closedness below.

Lemma 11.20. *Let $f\colon X \to Y$ be a quasi-compact morphism of algebraic spaces. Assume that for every locally of finite presentation morphism $Z \to Y$ with Z affine, the base change $X_Z \to Z$ is closed. Then f is universally closed.*

Proof. For any algebraic space X, we denote by $|X|$ the underlying topological space of X, see [Sta21, Tag 03BY]. A morphism of algebraic spaces is called closed if the induced map on topological spaces is closed.

By [Sta21, Tag 0CM9], to prove the lemma it suffices to show that for every locally of finite presentation morphism $Z \to Y$ of algebraic spaces, the base change $X_Z \to Z$ is closed. Choose a surjective étale morphism $p\colon Z' \to Z$ where Z' is a disjoint union of affine schemes Z'_i étale over Z. By assumption, the morphism $X_{Z'} \to Z'$ is closed after restriction to each affine Z'_i, and hence closed. Consider the diagram

$$\begin{tikzcd}
|X_{Z'}| \arrow[r, "|p'|"] \arrow[d, "|f_{Z'}|"'] & |X_Z| \arrow[d, "|f_Z|"] \\
|Z'| \arrow[r, "|p|"] & |Z|
\end{tikzcd}$$

of continuous maps of topological spaces. We must show that for any closed subset $T \subset |X_Z|$, the set $|f_Z|(T) \subset |Z|$ is closed. Since $p\colon Z' \to Z$ is a surjective étale morphism, the topology on $|Z|$ is the quotient topology for the surjection $|p|\colon |Z'| \to |Z|$, see [Sta21, Tag 03BX]. Hence we must show $|p|^{-1}|f_Z|(T) \subset |Z'|$ is closed. But since $|X_{Z'}| \to |Z'| \times_{|Z|} |X_Z|$ is surjective it follows that $|p|^{-1}|f_Z|(T) = |f_{Z'}||p'|^{-1}(T)$, which is closed by the observation above that $X_{Z'} \to Z'$ is closed. □

Lemma 11.21. *Let $f\colon X \to Y$ be a morphism of algebraic spaces such that:*

(1) *Y is Nagata.*
(2) *f is of finite type.*
(3) *f satisfies the strong existence part of the valuative criterion with respect to any essentially locally of finite type morphism $\mathrm{Spec}(R) \to Y$ with R a DVR.*

Then f is universally closed.

Proof. By Lemma 11.20 it suffices to show that for every locally of finite presentation morphism $Z \to Y$ with Z affine, the morphism $X_Z \to Z$ is closed. We claim that $X_Z \to Z$ inherits properties (1)-(3). Indeed, (1) holds by [Sta21, Tag 035A], (2) is clear, and (3) follows formally from the fact that $Z \to Y$ is locally of finite presentation.

Thus to prove the lemma, it suffices to assume that Y is affine and show $f\colon X \to Y$ is closed. In this situation, X is a quasi-compact algebraic space since f is of finite type and Y is quasi-compact. Thus by [Sta21, Tag 03H6] we may choose an affine scheme U and a surjective étale morphism $p\colon U \to X$. We must show that if $T \subset |X|$ is a closed subset of the underlying topological space of X, then $|f|(T)$ is closed in Y. The preimage $|p|^{-1}(T)$ is closed in U and hence the set of points of an affine closed subscheme $V \subset U$. Hence $|f|(T)$ coincides with the image of the morphism of affine schemes $V \to Y$, which by [Sta21, Tag 00HY] is closed if and only if it is stable under specialization. In other words, we must show that specializations of points in Y lift along f; but this follows from Lemma 11.19 and property (3). □

Part III. Harder–Narasimhan structures over a curve

In this part of the paper, we develop a theory of Harder–Narasimhan structures over a curve; this will be the basis of our definition of a stability condition over a general base scheme in Part IV. With the exception of Sections 12 and 14 where we review the absolute setting, we work in the following setup.

Setup III.1. Assume:

- $g\colon \mathcal{X} \to C$ is a flat morphism as in the Main Setup and in addition it is projective, where C is a Dedekind scheme;
- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ is a C-linear strong semiorthogonal component of finite cohomological amplitude.

12. Stability conditions and base change

We briefly recall the definition of stability conditions [Bri07, KS08], including the support property, and Bridgeland's deformation result. Then, in Section 12.3, we prove a base change result for numerical stability conditions under arbitrary (not necessarily finitely generated) field extensions, extending the results of [Sos12].

12.1. **Definitions.** Let $\mathcal{D}$ be a triangulated category.

Definition 12.1. A *slicing* $\mathcal{P}$ of $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying:

(1) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$;
(2) if $\phi_1 > \phi_2$ and $E_j \in \mathcal{P}(\phi_j)$, then $\mathrm{Hom}_{\mathcal{D}}(E_1, E_2) = 0$;
(3) (*HN filtrations*) for every nonzero $E \in \mathcal{D}$ there exists a finite sequence of morphisms

$$0 = E_0 \xrightarrow{s_1} E_1 \to \cdots \xrightarrow{s_m} E_m = E$$

such that the cone of s_i is in $\mathcal{P}(\phi_i)$ for some sequence $\phi_1 > \phi_2 > \cdots > \phi_m$ of real numbers.

We write $\phi^+(E) := \phi_1$ and $\phi^-(E) := \phi_m$; moreover, for an interval $I \subset \mathbb{R}$, we write

$$\mathcal{P}(I) := \big\{E\colon \phi^+(E), \phi^-(E) \in I\big\} = \langle \mathcal{P}(\phi)\rangle_{\phi\in I} \subset \mathcal{D}.$$

We also write $\mathcal{P}(\leqslant a) := \mathcal{P}((-\infty, a])$ or $\mathcal{P}(> b) := \mathcal{P}((b, +\infty))$.

Definition 12.2. Let Λ be a finite rank free abelian group with a group homomorphism $v\colon K(\mathcal{D}) \to \Lambda$.

(i) A *pre-stability condition on $\mathcal{D}$ with respect to* Λ is a pair $\sigma = (Z, \mathcal{P})$ where $\mathcal{P}$ is a slicing of $\mathcal{D}$ and $Z\colon \Lambda \to \mathbb{C}$ is a group homomorphism, that satisfy the following condition:

$$\text{for all } 0 \neq E \in \mathcal{P}(\phi), \text{ we have } Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}.$$

We will often abuse notation and write $Z(E)$ for $Z(v(E))$. The nonzero objects of $\mathcal{P}(\phi)$ are called σ-semistable of *phase* ϕ.

(ii) A pre-stability condition $\sigma = (Z, \mathcal{P})$ with respect to Λ satisfies the *support property* if there exists a quadratic form Q on the vector space $\Lambda_{\mathbb{R}}$ such that
- the kernel of Z is negative definite with respect to Q, and

- for any σ-semistable object $E \in \mathcal{D}$, we have
$$Q(v(E)) \geqslant 0.$$

(iii) A *stability condition* (with respect to Λ) is a pre-stability condition with respect to Λ satisfying the support property.

We will briefly review the relevance of the support property for deformations of stability conditions in Section 12.2. For now we note that an easy linear algebra statement implies the following.

Remark 12.3. Assume that Q is negative definite on the kernel of Z. Then for any $C > 0$, there can only be finitely many classes with $|Z(v)| < C$ and $Q(v) \geqslant 0$.

It follows that if $(Z, \mathcal{P})$ satisfies the support property, then $\mathcal{P}(\phi)$ is a finite length category for every $\phi \in \mathbb{R}$, as the image of objects in $\mathcal{P}(\phi)$ in $\mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}$ is a discrete set. In other words, the support property guarantees the existence of a Jordan–Hölder filtration of a semistable object E, which has *stable* filtration quotients; we call them the Jordan–Hölder factors of E.

Definition 12.4. A *stability function Z on an abelian category $\mathcal{A}$* is a morphism of abelian groups $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ is in the semi-closed upper half plane $\mathbb{H} \sqcup \mathbb{R}_{<0} := \{z \in \mathbb{C} \mid \Im z > 0, \text{ or } \Im z = 0 \text{ and } \Re z < 0\}$.

For $0 \neq E \in \mathcal{A}$ we define its *phase* by $\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1]$. An object $E \in \mathcal{A}$ is called *Z-semistable* if for all subobjects $0 \neq A \hookrightarrow E$, we have $\phi(A) \leqslant \phi(E)$.

Definition 12.5. We say that a stability function Z on an abelian category $\mathcal{A}$ *satisfies the HN property* if every object $E \in \mathcal{A}$ admits a Harder–Narasimhan (HN) filtration: a sequence
$$0 = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow E_m = E$$
such that E_i/E_{i-1} is Z-semistable for $i = 1, \ldots, m$, with
$$\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_m/E_{m-1}).$$

If $\mathcal{P}$ is a slicing, then $\mathcal{P}((\phi, \phi + 1])$ is the heart of a bounded t-structure for all $\phi \in \mathbb{R}$; moreover, the slicing induces HN filtrations on $\mathcal{A} := \mathcal{P}((0, 1])$ with respect to Z. The converse also holds:

Lemma 12.6 ([Bri07, Proposition 5.3])**.** *To give a pre-stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure, and a stability function Z on $\mathcal{A}$ with the HN property.*

The support property can be checked at the level of the abelian category, with the same definition. Thus the above lemma can be rephrased in terms of stability conditions as well.

Let X be a projective variety over a field k, and let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ be a k-linear strong semiorthogonal component. Then the Euler characteristic
$$\chi_k(E, F) = \sum_i (-1)^i \dim_k \operatorname{Hom}(E, F[i])$$
induces a pairing
$$\chi_k\colon K(\mathcal{D}_{\mathrm{perf}}) \times K(\mathcal{D}) \to \mathbb{Z}.$$
When the base field is clear from the context, we will always omit it from the notation. We write $K_{\mathrm{num}}(\mathcal{D})$ for the quotient of $K(\mathcal{D})$ by the null space of χ on the right; similarly we write, by some abuse of notation, $K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}})$ for the corresponding quotient on the left.

Lemma 12.7. *Both $K_{\mathrm{num}}(\mathcal{D})$ and $K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}})$ are free and finitely generated abelian groups.*

Proof. By definition $K_{\mathrm{num}}(\mathcal{D})$ and $K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}})$ are torsion free; we need to show they are finitely generated. The pairing χ_k induces an inclusion $K_{\mathrm{num}}(\mathcal{D}) \hookrightarrow \mathrm{Hom}(K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}}), \mathbb{Z})$, so it is enough to prove the statement for $K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}})$. We may also reduce to the case where $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$, as $K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}})$ is a quotient of the image of $K(\mathcal{D}_{\mathrm{perf}})$ in $K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(X))$.

Let $\pi\colon \widetilde{X} \to X$ be an alteration with $\widetilde{X}$ regular and projective (see [dJ96, Theorem 4.1]). If $\mathrm{CH}^*_{\mathrm{num}}(\widetilde{X})$ denotes the Chow ring modulo numerical equivalence, then Riemann–Roch shows that for some integer N, $K_{\mathrm{num}}(\mathrm{D}_{\mathrm{perf}}(\widetilde{X}))$ embeds into $\frac{1}{N}\mathrm{CH}^*_{\mathrm{num}}(\widetilde{X}) \subset \mathrm{CH}^*_{\mathrm{num}}(\widetilde{X}) \otimes \mathbb{Q}$; in particular, since $\mathrm{CH}^*_{\mathrm{num}}(\widetilde{X})$ is finitely generated (in finite characteristic, see e.g. [Ful98, Example 19.1.4]), so is $K_{\mathrm{num}}(\mathrm{D}_{\mathrm{perf}}(\widetilde{X}))$. On the other hand, using that

$$\chi_k(\pi^*(E), F) = \chi_k(E, \pi_*(F)) \quad \text{for} \quad E \in \mathrm{D}_{\mathrm{perf}}(X) \text{ and } F \in \mathrm{D}^{\mathrm{b}}(\widetilde{X}), \tag{12.1}$$

we see that pullback induces a map $K_{\mathrm{num}}(\mathrm{D}_{\mathrm{perf}}(X)) \to K_{\mathrm{num}}(\mathrm{D}_{\mathrm{perf}}(\widetilde{X}))$. To finish the proof, we show this map is injective. Using again the relation (12.1), we reduce to proving the following claim: if $\alpha \in K(\mathrm{D}^{\mathrm{b}}(X))$, then there exists a class $\widetilde{\alpha} \in K(\mathrm{D}^{\mathrm{b}}(\widetilde{X}))$ and an integer d such that $\pi_*(\widetilde{\alpha}) = d\alpha$. This holds by dévissage. Namely, by noetherian induction it suffices to show that if F is a coherent sheaf on X supported on a closed subset $Z \subset X$, then some multiple of $[F] \in K(X)$ lifts to $K(\widetilde{X})$, modulo classes of sheaves supported on proper subsets of Z. But we can find such a lift by considering the class $[\mathcal{O}_{\widetilde{Z}}] \in K(\mathrm{D}^{\mathrm{b}}(\widetilde{X}))$ where $\widetilde{Z} \subset \widetilde{X}$ is a multisection of $\widetilde{X} \to X$ over Z. □

Definition 12.8. We say that a stability condition on $\mathcal{D}$ is *numerical* if the central charge factors via $Z\colon K(\mathcal{D}) \to K_{\mathrm{num}}(\mathcal{D}) \to \mathbb{C}$.

Remark 12.9. Concretely, a stability condition is numerical if Z can be written as

$$Z(_) = \sum_i z_i \chi(F_i, _),$$

for some $z_i \in \mathbb{C}$ and $F_i \in \mathcal{D}_{\mathrm{perf}}$.

12.2. **The space of stability conditions.** We continue to fix a free abelian group Λ of finite rank with a group homomorphism $v\colon K(\mathcal{D}) \to \Lambda$, and write $\mathrm{Stab}_\Lambda(\mathcal{D})$ for the set of stability conditions satisfying the support property with respect to Λ. We will review the deformation result of [Bri07], made somewhat more effective in [BMS16, Appendix A], which shows that $\mathrm{Stab}_\Lambda(\mathcal{D})$ is a complex manifold, locally homeomorphic to $\mathrm{Hom}(\Lambda, \mathbb{C})$.

We first recall the metric topology on the set of slicings:

Proposition and Definition 12.10 ([Bri07, Section 6])**.** *Consider two slicings $\mathcal{P}, \mathcal{Q}$ of $\mathcal{D}$, and write $\phi^+_{\mathcal{P}}(E)$ and $\phi^+_{\mathcal{Q}}(E)$ for the maximal and minimal phase occurring in the HN filtration of the object E with respect to $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then*

$$\begin{aligned} d(\mathcal{P}, \mathcal{Q}) &:= \sup_{E \in \mathcal{D}} \left\{ \left|\phi^+_{\mathcal{Q}}(E) - \phi^+_{\mathcal{P}}(E)\right|, \left|\phi^-_{\mathcal{Q}}(E) - \phi^-_{\mathcal{P}}(E)\right| \right\} \in \mathbb{R}_{\geqslant 0} \cup \{+\infty\} \\ &= \sup_{\phi \in \mathbb{R}, E \in \mathcal{P}(\phi)} \left\{ \left|\phi^+_{\mathcal{Q}}(E) - \phi\right|, \left|\phi^-_{\mathcal{Q}}(E) - \phi\right| \right\} \\ &= \inf_{\phi \in \mathbb{R}} \left\{ \epsilon \in \mathbb{R}_{\geqslant 0} \,\middle|\, \mathcal{P}(\leqslant \phi) \subset \mathcal{Q}(\leqslant \phi + \epsilon) \text{ and } \mathcal{P}(> \phi) \subset \mathcal{Q}(> \phi - \epsilon) \right\}, \end{aligned}$$

and this quantity defines a generalized metric on the set of slicings, i.e., it satisfies the triangle inequality and $d(\mathcal{P}, \mathcal{Q}) = 0$ if and only if $\mathcal{P} = \mathcal{Q}$.

This induces a topology on $\mathrm{Stab}_\Lambda(\mathcal{D})$ as the coarsest topology such that the two forgetful maps to $\mathrm{Hom}(\Lambda, \mathbb{C})$ and to the set of slicings on $\mathcal{D}$ are continuous.

Theorem 12.11 ([Bri07], [BMS16, Proposition A.5])**.** *The space $\mathrm{Stab}_\Lambda(\mathcal{D})$ of stability conditions on $\mathcal{D}$ is a complex manifold, and the canonical map*

$$\mathcal{Z}\colon \mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z$$

is a local isomorphism.

More precisely, assume that $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to the quadratic form Q. Consider the open subset of $\mathrm{Hom}(\Lambda, \mathbb{C})$ consisting of central charges whose kernel is negative definite with respect to Q, and write P_Z for the connected component of this open subset that contains Z. If $U \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ is the connected component of $\mathcal{Z}^{-1}(P_Z)$ containing σ, then $\mathcal{Z}|_U \colon U \to P_Z$ is a covering map.

In other words, a path in P_Z starting at Z lifts uniquely to a path in $\mathrm{Stab}_\Lambda(\mathcal{D})$ starting at σ.

We also note the following standard facts about the hearts of stability conditions, see the example at the end of [Pol07, Section 1] and [AP06, Proposition 5.0.1]:

Lemma 12.12. *Let $\sigma = (\mathcal{A}, Z)$ be a stability condition with respect to Λ.*

(1) *The heart $\mathcal{A}$ is tilted-noetherian.*
(2) *If Z is defined over $\mathbb{Q}[\mathrm{i}]$, i.e., if its image is contained in $\mathbb{Q} \oplus i\mathbb{Q}$, then $\mathcal{A}$ is noetherian.*

The support property combined with some simple linear algebra arguments imply local finiteness of wall-crossing:

Lemma 12.13 ([Bri08, Section 9], [Tod08, Proposition 2.8 and Lemma 2.9], [BM11, Proposition 3.3])**.** *Fix a class $\mathbf{v} \in \Lambda$. Then there exists a locally finite set of walls, i.e., real codimension one submanifolds, $\mathcal{W}_\mathbf{v} \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ with the following properties.*

(1) *Let W be an intersection of finitely many walls in $\mathcal{W}_\mathbf{v}$ (the empty intersection, i.e., $W = \mathrm{Stab}_\Lambda(\mathcal{D})$, being allowed). Let $\mathcal{C} \subset W$ be a* chamber*, namely a connected component of the complement of the union of intersections of W with any other wall. Then the set of σ-semistable objects of class $\mathbf{v}$ is constant for $\sigma \in \mathcal{C}$; the same holds for the set of σ-stable objects.*
(2) *Up to the normalisation*[2] *$Z(\mathbf{v}) \in \mathbb{R}_{<0}$, every such wall in $\mathcal{W}_\mathbf{v}$ is locally described by a linear equation of the form $\Im Z(\mathbf{u}) = 0$ for some $\mathbf{u} \neq 0$, defined over $\mathbb{Q}$. Every chamber $\mathcal{C}$ contains stability conditions with Z defined over $\mathbb{Q}[i]$.*

Proof. The case of σ-semistable objects is treated in [Tod08]. From the proof, one can see that an object that is σ-semistable in a chamber $\mathcal{C}$ is either σ-stable for all $\sigma \in \mathcal{C}$ or strictly σ-semistable for all $\sigma \in \mathcal{C}$. □

12.3. **Stability conditions under base change.** Now consider an algebraic variety X defined over a base field k and a field extension ℓ of k. The pullback $\mathcal{D} \to \mathcal{D}_\ell$, $E \mapsto E_\ell$ induces maps

$$K(\mathcal{D}) \to K(\mathcal{D}_\ell) \quad \text{and} \quad K(\mathcal{D}_{\mathrm{perf}}) \to K(\mathcal{D}_{\ell,\mathrm{perf}})$$

that preserve the Euler characteristic pairing.

[2]This can be achieved via the action of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$, the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$; this group has an action on $\mathrm{Stab}_\Lambda(\mathbb{C})$ covering the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathrm{Hom}(\Lambda, \mathbb{C}) \cong \mathrm{Hom}(\Lambda, \mathbb{R}^2)$.

Lemma 12.14. *Pullback also induces a map* $\eta_{\ell/k}\colon K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}}) \to K_{\mathrm{num}}(\mathcal{D}_{\ell,\mathrm{perf}})$ *on numerical* K*-groups.*

Proof. We have to show that if $E \in \mathcal{D}_{\mathrm{perf}}$ satisfies $\chi_k(E,F) = 0$ for all $F \in \mathcal{D}$, then we also have $\chi_\ell(E_\ell, G) = 0$ for all $G \in \mathcal{D}_\ell$. Since G is defined over some intermediate field extension $k \subset \ell' \subset \ell$ with ℓ'/k finitely generated (as in the proof of Proposition 5.9) and since pullback along ℓ/ℓ' preserves the Euler characteristic, it is sufficient to consider the case where ℓ/k is itself finitely generated. When ℓ/k is finite, the claim follows from adjunction via

$$\chi_\ell(E_\ell, G) = \chi_\ell(E, G^{(k)}) = \dim_k(\ell)\chi_k(E, G^{(k)}) = 0,$$

where $G^{(k)} \in \mathcal{D}$ denotes the pushforward of $G \in \mathcal{D}_\ell$.

By induction, it suffices to prove the claim for $\ell = k(x)$. Let $R = k[x]$. In that case, we can lift $G \in \mathcal{D}_{k(x)}$ to an object $\widetilde{G} \in \mathcal{D}_R$. Then base change implies

$$\chi_\ell(E_\ell, G) = \mathrm{rk}\,\mathcal{H}om_R(E_R, \widetilde{G}) = \chi_k(E, i_0^*\widetilde{G}) = 0$$

where i_0 is the inclusion of the fiber over $0 \in \mathbf{A}^1_k$. □

Dualizing induces a pushforward map

(12.2)

$$\eta^\vee_{\ell/k}\colon K_{\mathrm{num}}(\mathcal{D}_\ell) \hookrightarrow \mathrm{Hom}(K_{\mathrm{num}}(\mathcal{D}_{\ell,\mathrm{perf}}), \mathbb{Z}) \to \mathrm{Hom}(K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}}), \mathbb{Z}) \to K_{\mathrm{num}}(\mathcal{D}) \otimes \mathbb{Q}.$$

Proposition and Definition 12.15. *The image* $K_{\mathrm{num}}(\mathcal{D})_\ell := \mathrm{Im}\,\eta^\vee_{\ell/k}$ *contains* $K_{\mathrm{num}}(\mathcal{D})$ *as a subgroup of finite index; conversely, it is contained in* $K_{\mathrm{num}}(\mathcal{D})_{\overline{k}}$.

Proof. Since $\eta_{\ell/k}$ preserves the Euler characteristic pairing, we have $\eta^\vee_{\ell/k}([F_\ell]) = [F]$ for $F \in \mathcal{D}$, so $K_{\mathrm{num}}(\mathcal{D})_\ell$ contains $K_{\mathrm{num}}(\mathcal{D})$. That it is a finite index subgroup immediately follows from the definitions and Lemma 12.7. For the second claim, it is sufficient to show that if k is algebraically closed, and ℓ/k is finitely generated, then $\mathrm{Im}\,\eta^\vee_{\ell/k}$ is contained in $K_{\mathrm{num}}(\mathcal{D})$. Let R/k be a finitely generated ring with fraction field ℓ; then the same argument as in the end of the proof of Lemma 12.14, with $0 \in \mathbf{A}^1_k$ replaced by any k-rational point of $\mathrm{Spec}\,R$, proves this containment. □

If Z is the central charge of a numerical stability condition on $\mathcal{D}$, we write Z_ℓ for the composition of $Z \circ \eta^\vee_{\ell/k}$. Concretely, if $Z(_) = \sum_i z_i\chi(F_i, _)$ is a central charge as in Remark 12.9, then

$$Z_\ell(_) = \sum_i z_i\chi((F_i)_\ell, _).$$

Now fix a Mukai homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}) \to \Lambda$ as in Definition 12.2. We let

$$\Lambda_\ell \subset \Lambda \otimes \mathbb{Q}$$

be the subgroup generated by Λ and the image of v extended to $K_{\mathrm{num}}(\mathcal{D})_\ell$; again, it contains Λ as a subgroup of finite index. We write

$$v_\ell\colon K_{\mathrm{num}}(\mathcal{D}_\ell) \to \Lambda_\ell$$

for the induced map. If Z factors via a group homomorphism $K_{\mathrm{num}}(\mathcal{D}) \to \Lambda$, then Z_ℓ factors via the composition $K_{\mathrm{num}}(\mathcal{D}_\ell) \to K_{\mathrm{num}}(\mathcal{D})_\ell \to \Lambda_\ell$:

$$\begin{tikzcd}
K(\mathcal{D}_\ell) \arrow[d] \arrow[dr, "v_\ell"] \arrow[drrr, bend left, "Z_\ell"] & K(\mathcal{D}) \arrow[l, "(_)_\ell"'] \arrow[d] \arrow[dr, "v"] \arrow[drrr, bend left, "Z"] & & \\
K_{\mathrm{num}}(\mathcal{D}_\ell) \arrow[r, "\eta^\vee_{\ell/k}"] & K_{\mathrm{num}}(\mathcal{D})_\ell \arrow[r] & \Lambda_\ell \arrow[r] & \mathbb{C}
\end{tikzcd}$$

For an object $E \in \mathcal{D}$ we have the equality

$$v[E] = v_\ell[E_\ell] \in \Lambda, \text{ and so } Z_\ell(E_\ell) = Z(E);$$

on the other hand, if $k \subset \ell$ is finite, $F \in \mathcal{D}_\ell$, and $F^{(k)} \in \mathcal{D}$ denotes its pushforward, then

$$v[F^{(k)}] = \dim_k(\ell) \cdot v_\ell[F].$$

Remark 12.16. As $\Lambda \otimes \mathbb{Q} = \Lambda_\ell \otimes \mathbb{Q}$, the difference between the two lattices is irrelevant when considering central charges. However, we will also use Λ to keep track of classes of semistable objects; then the integral structure will become relevant.

Base change for central charges as above, combined with base change for t-structures as in Proposition 5.9, induces base change for stability conditions for arbitrary field extensions:

Theorem 12.17. *Let $\sigma := (\mathcal{A}, Z)$ be a numerical stability condition with respect to Λ on the strong semiorthogonal component $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$, where X is a variety defined over a field k. Let $k \subset \ell$ be a (not necessarily finitely generated) field extension. Then:*

(1) *The pair $\sigma_\ell := (\mathcal{A}_\ell, Z_\ell)$ defines a numerical stability condition on $\mathcal{D}_\ell$.*
(2) *If there exists a σ_ℓ-semistable object in $\mathcal{D}_\ell$ of class $v \in \Lambda$, then there exists a σ-semistable object in $\mathcal{D}$ of class $n \cdot v$ for some $n \in \mathbb{Z}_{>0}$. (In particular, σ_ℓ satisfies the support property with respect to the same quadratic form as σ.)*
(3) *An object $E \in \mathcal{D}$ is σ-semistable if and only if the pullback $E_\ell \in \mathcal{D}_\ell$ is σ_ℓ-semistable.*
(4) *Let E be an object in $\mathcal{D}$. If $k \subset \overline{k}$ and $\ell \subset \overline{\ell}$ denote the algebraic closures, then $E_{\overline{k}}$ is $\sigma_{\overline{k}}$-stable if and only if $E_{\overline{\ell}}$ is $\sigma_{\overline{\ell}}$-stable.*

The resulting map $\mathrm{Stab}(\mathcal{D}) \to \mathrm{Stab}(\mathcal{D}_\ell)$ is a homeomorphism onto a union of connected components of $\mathrm{Stab}(\mathcal{D}_\ell)$; however, we omit the proof of this fact. Similar results were proved in a slightly different setup for finite separable or Galois field extensions in [Sos12].

Proof. We start with some general observations. For $\phi \in \mathbb{R}$, we write $\mathcal{A}^\phi := \mathcal{P}(\phi - 1, \phi] \subset \mathcal{D}$ (with $\mathcal{A} = \mathcal{A}^1$). By base change via Proposition 5.9 (which applies by Lemma 12.12), this induces a heart $\mathcal{A}^\phi_\ell \subset \mathcal{D}_\ell$ (and we consider $\mathcal{A}_\ell := \mathcal{A}^1_\ell$). We use these hearts to define

$$\mathcal{P}_\ell(\phi) := \bigcap_{\phi' - 1 < \phi \leqslant \phi'} \mathcal{A}^{\phi'}_\ell. \tag{12.3}$$

To prove that $\mathcal{P}_\ell$ is a slicing, we only need to establish existence of HN filtrations, as the first two properties in Definition 12.1 are evident.

It follows from the corresponding statement in Proposition 5.9 that pullback $f^* \colon \mathcal{D} \to \mathcal{D}_\ell$ sends $\mathcal{P}(\phi)$ to $\mathcal{P}_\ell(\phi)$; in particular, statement (3) will be automatic.

Step 1. Claims (1) and (2) hold for finite field extensions $k \subset \ell$.

In this case, the hearts $\mathcal{A}_\ell^\phi$ are already constructed in Theorem 5.7, and by Theorem 5.7.(4c) the pushforward $f_*\colon \mathcal{D}_\ell \to \mathcal{D}$ satisfies

$$f_*\left(\mathcal{A}_\ell^\phi\right) \subset \mathcal{A}^\phi \quad \text{and therefore} \quad f_*\left(\mathcal{P}_\ell(\phi)\right) \in \bigcap_{\phi'-1<\phi\leqslant\phi'} \mathcal{A}^{\phi'} = \mathcal{P}(\phi). \tag{12.4}$$

Since $Z_\ell(E) = \dim_k(\ell)\cdot Z(f_*E)$ this immediately implies the compatibility of Z_ℓ both with $\mathcal{A}_\ell$, and with $\mathcal{P}_\ell$. It also implies part (2), with $n = \dim_k(\ell)$.

It remains to prove the existence of HN filtrations for $E \in \mathcal{A}_\ell$. For each $0 < \phi \leqslant 1$, it follows from [Pol07, Lemma 1.1.2] that $\mathcal{A}_\ell^\phi$ is obtained by tilting $\mathcal{A}_\ell$ at a torsion pair $\left(\mathcal{T}_\ell^{>\phi}, \mathcal{F}_\ell^{\leqslant\phi}\right)$, analogous to the torsion pair $\left(\mathcal{T}^{>\phi}, \mathcal{F}^{\leqslant\phi}\right)$ in $\mathcal{A}$. These torsion pairs induce an a priori infinite filtration

$$0 = T_\ell^{>1} \subset \cdots \subset T_\ell^{>\phi} \subset \cdots \subset T_\ell^{>0} = E \tag{12.5}$$

of E in $\mathcal{A}_\ell$. Since

$$f_*\mathcal{T}_\ell^{>\phi} = f_*\left(\mathcal{A}_\ell \cap \mathcal{A}_\ell^\phi\right) \subset \mathcal{T}^{>\phi} \quad \text{and} \quad f_*\mathcal{F}_\ell^{\leqslant\phi} \subset \mathcal{F}^{\leqslant\phi},$$

the pushforward of this filtration is induced by the HN filtration of $f_*E \in \mathcal{A}$, and thus finite; since f_* is conservative, the original filtration in $\mathcal{A}_\ell$ also has to be finite. The filtration quotients have to lie in $\mathcal{P}_\ell(\phi) = \mathcal{F}_\ell^{\leqslant\phi} \cap \bigcap_{\phi'<\phi} \mathcal{T}_\ell^{>\phi'}$ for appropriate ϕ, and thus this is the desired HN filtration.

Step 2. Claims (1) *and* (2) *hold for* $\ell = k(x)$ *and* Z *defined over* $\mathbb{Q}[\mathrm{i}]$.

Let $R = k[x]$; we write $\mathcal{D}_R$ and $\mathcal{A}_R, \mathcal{A}_R^\phi$ for the triangulated category and hearts obtained from $\mathcal{D}$ and $\mathcal{A}$ via Theorem 3.17 and Theorem 5.7, respectively. Any object in $\mathcal{A}_\ell$ is of the form E_ℓ for some $E \in \mathcal{A}_R$ (see Lemma 3.18). By Lemma 12.12 and Theorem 5.7.(2), $\mathcal{A}_R$ is noetherian; by Remark 6.16 we may assume that E is R-torsion free, and thus R-flat by Lemma 6.12. It follows that $E_p \in \mathcal{A}_p$ for all closed points $p \in \operatorname{Spec} R$; since $[i_{p*}E_p]$ and $[E_\ell]$ define the same class in Λ (via $\mathcal{D}_R \to \mathcal{D}_\ell$ and v_ℓ), this shows that Z_ℓ is compatible with $\mathcal{A}_\ell$. Applying the same argument to $\mathcal{A}^\phi$ instead of $\mathcal{A}$, we obtain by the compatibility of Z_ℓ with all $\mathcal{A}_\ell^\phi$, and thus with $\mathcal{P}_\ell$.

Now consider again the filtration in (12.5). Since Z_ℓ is compatible with $\mathcal{A}_\ell$ and $\mathcal{A}_\ell^\phi$, and since for all $0 < \phi' < \phi < 1$, $T_\ell^{>\phi'}/T_\ell^{>\phi} \in \mathcal{F}_\ell^{\leqslant\phi}$, we have $\Im Z_\ell(T_\ell^{>\phi'}/T_\ell^{>\phi}) > 0$ whenever $T_\ell^{>\phi'}/T_\ell^{>\phi} \neq 0$. Since $\Im Z_\ell$ is discrete, the filtration has to be finite.

Finally, if $E_\ell \in \mathcal{P}_\ell(\phi)$, we may choose a representative $E \in \mathcal{A}_R^\phi$ that is R-torsion free as an object in $\mathcal{A}_R^\phi$. Then $E_p \in \mathcal{A}^\phi$ satisfies $Z(E_p) \in \mathbb{R}_{>0}\cdot e^{\mathrm{i}\pi\phi}$, and thus it is σ-semistable. The equality $v_\ell[E_\ell] = v_\ell[i_{p*}E_p]$ in Λ proves claim (2).

Step 3. Claims (1) *and* (2) *hold when* Z *is defined over* $\mathbb{Q}[\mathrm{i}]$, *and* ℓ *arbitrary.*

By Steps 1 and 2, the claim holds when ℓ is finitely generated over k; the general case follows with the same type of arguments as those used in Proposition 5.9. To prove the existence of HN filtrations for $E \in \mathcal{D}_\ell$, we may assume by Proposition 5.9.(3) that E is obtained by base change from $E_{k'}$ for some finitely generated $k \subset k' \subset \ell$; by construction of $\mathcal{P}_\ell(\phi)$ and $\mathcal{P}_{k'}(\phi)$, the pullback of the HN filtration in $\mathcal{D}_{k'}$ induces one in $\mathcal{D}_\ell$. The compatibility of the central charge Z_ℓ with $\mathcal{P}_\ell$, and claim (2) of the Theorem follow similarly from their analogues for $\mathcal{D}_{k'}$.

Step 4. Claims (1) *and* (2) *hold in full generality.*

Consider a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ that satisfies the support property with respect to a given quadratic form Q. By Theorem 12.11, for any $\epsilon > 0$ there exists a neighborhood $Z \in U \subset \mathrm{Hom}(\Lambda, \mathbb{C})$, depending only on Q, such that U embeds into $\mathrm{Stab}(\mathcal{D})$ with image containing σ, and such that any two stability conditions $\sigma_1 = (Z_1, \mathcal{P}_1)$ and $\sigma_2 = (Z_2, \mathcal{P}_2)$ in the image satisfy $d(\mathcal{P}_1, \mathcal{P}_2) < \epsilon$. Let $U_{\mathbb{Q}} \subset U$ be the dense set of central charges defined over $\mathbb{Q}[\mathrm{i}]$. For any $Z' \in U_{\mathbb{Q}}$ and the corresponding stability condition $\sigma' = (Z', \mathcal{P}')$ on $\mathcal{D}$, we have an induced a stability condition σ'_ℓ on $\mathcal{D}_\ell$ via base change by the previous step.

We first claim that the resulting map $U_{\mathbb{Q}} \to \mathrm{Stab}(\mathcal{D}_\ell)$ is continuous. Indeed, if $\sigma' = (Z', \mathcal{P}')$ and $\sigma'' = (Z'', \mathcal{P}'')$ are two stability conditions with central charges in $U_{\mathbb{Q}}$, and with $d(\mathcal{P}', \mathcal{P}'') < \epsilon$, we recall from Proposition/Definition 12.10 that this is equivalent to $\mathcal{P}'(\leqslant \phi) \subset \mathcal{P}''(\leqslant \phi + \epsilon)$ and $\mathcal{P}'(> \phi) \subset \mathcal{P}''(> \phi - \epsilon)$. Since the construction of the base change t-structure evidently preserves inclusions, this implies $\mathcal{P}'_\ell(\leqslant \phi) \subset \mathcal{P}''_\ell(\leqslant \phi + \epsilon)$ and $\mathcal{P}'_\ell(> \phi) \subset \mathcal{P}''_\ell(> \phi - \epsilon)$, and thus $d(\mathcal{P}'_\ell, \mathcal{P}''_\ell) < \epsilon$.

Since the stability conditions in the image of $U_{\mathbb{Q}}$ satisfy the support property with respect to Q, we can use Theorem 12.11 to extend this map to $U \to \mathrm{Stab}(\mathcal{D}_\ell)$. It remains to show that for $\sigma \in U \setminus U_{\mathbb{Q}}$, the stability condition $\sigma_\ell = (Z_\ell, \overline{\mathcal{P}_\ell})$ obtained via this map satisfies the description in the Theorem; as observed above, it is enough to show that $\overline{\mathcal{P}_\ell} = \mathcal{P}_\ell$, where the latter is defined by equation (12.3). Let $\sigma' = (Z', \mathcal{P}')$ be a stability condition with $Z' \in U_{\mathbb{Q}}$ and $d(\mathcal{P}, \mathcal{P}') < \epsilon$. Then

$$\mathcal{P}_\ell(\leqslant \phi) \subset \mathcal{P}'_\ell(\leqslant \phi + \epsilon) \subset \overline{\mathcal{P}_\ell}(\leqslant \phi + 2\epsilon)$$

for all $\phi \in \mathbb{R}$ by the same argument as above; since this holds for all $\epsilon > 0$, and since $\overline{\mathcal{P}_\ell}$ is a slicing, this shows $\mathcal{P}_\ell(\leqslant \phi) \subset \overline{\mathcal{P}_\ell}(\leqslant \phi)$. The dual argument shows $\mathcal{P}_\ell(> \phi) \subset \overline{\mathcal{P}_\ell}(> \phi)$. Since both $(\mathcal{P}_\ell(> \phi), \mathcal{P}_\ell(\leqslant \phi))$ and $\big(\overline{\mathcal{P}_\ell}(> \phi), \overline{\mathcal{P}_\ell}(\leqslant \phi)\big)$ define t-structures, this gives $\mathcal{P}_\ell = \overline{\mathcal{P}_\ell}$ as desired.

Finally, to extend claim (2) from $U_{\mathbb{Q}}$ to U, let $\sigma \in U \setminus U_{\mathbb{Q}}$, and let $\sigma' \in \mathrm{Stab}(\mathcal{D}_\ell)$ be its image. Then σ' is contained in a chamber $\mathcal{C}$ in the sense of Lemma 12.13.(2). Let E be a σ'-semistable object of class v. Then there is a dense set of stability conditions $\sigma'' \in \mathcal{C}$ in the image of $U_{\mathbb{Q}}$; in particular E is also σ''-semistable, and the proof of (2) in the previous cases gives an object $F \in \mathcal{D}_k$ of class $n \cdot \mathbf{v}$ that is semistable at the corresponding point of $U_{\mathbb{Q}}$. Since the set where an object is semistable is closed, F is also semistable with respect to σ, which proves the claim.

Step 5. Claim (4) *holds.*

First we observe that one direction is clear. Indeed, a non-trivial Jordan–Hölder filtration of $E_{\overline{k}}$ would pullback to a non-trivial Jordan–Hölder filtration of $E_{\overline{\ell}}$. For the reverse direction, suppose that $E \in \mathcal{D}$ is such that $E_{\overline{k}}$ is $\sigma_{\overline{k}}$-stable (and hence $E_{k'}$ is $\sigma_{k'}$ for any finite field extension $k \subset k'$. Then, following the same overall logic as in steps 1–4, we need to show the following: if Z is defined over $\mathbb{Q}[\mathrm{i}]$, and if $k \subset k'$ and $k'(x) \subset \ell$ are finite field extensions, then E_ℓ is σ_ℓ-stable. This is shown similarly to Step 2. Under the assumptions, we can use the $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$-action on σ to achieve that $E \in \mathcal{P}_{k'}(1)$ with Z still defined over $\mathbb{Q}[\mathrm{i}]$; in particular, $\mathcal{A}_{k'}$ is noetherian by Lemma 12.12. We first choose a smooth curve C defined over k' with fraction field ℓ and infinitely many closed points. If E_ℓ is strictly σ_ℓ-semistable, we use Lemma 4.16.(3) to lift its Jordan–Hölder filtration to a filtration of E_C in $\mathcal{A}_C$. Let $U \subset C$ be the open subset where every factor of this filtration is torsion free (which exists

as $\mathcal{A}_{k'}$ is noetherian, so that $\mathcal{A}_C$ is noetherian by Theorem 5.7.(2), and thus has a C-torsion theory by Remark 6.16). Then this induces a non-trivial filtration of $E_c \in \mathcal{A}_c$ for every $c \in U$; since $\Im Z_c(E_c) = 0$, the same is true for each filtration factor. Thus this is in fact a non-trivial filtration in $\mathcal{P}_c(1)$. Since $k(c)/k$ is a finite field extension, pullback via an embedding $k(c) \hookrightarrow \overline{k}$ contradicts the stability of $E_{\overline{k}}$. □

We note that the property of being *σ-stable* is not necessarily preserved by pullback, e.g. for a stable object whose endorphism ring is given by a field extension of k. Consequently, while base change preserves HN filtrations, it does not preserve Jordan–Hölder filtrations. Therefore, we make the following definition:

Definition 12.18. In the setting of Theorem 12.17, we say that an object E is *geometrically σ-stable* if it is stable after base change to the algebraic closure $\overline{k}$ of k.

By Theorem 12.17.(4), this property is preserved by field extensions.

13. Harder–Narasimhan structures over a curve

The aim of this section is to introduce a notion of Harder–Narasimhan (HN) structures over a one-dimensional base. It will include stability conditions on the fibers of $\mathcal{D}$, but additionally provide HN filtrations for every object $E \in \mathcal{D}$; this strengthens the classical notion of relative HN filtrations. As in the case of stability conditions, we will often need the auxiliary notion of *weak Harder–Narasimhan structure*, which will be done in Section 15. In this section, we omit any proofs, as they are essentially the same as for the case of weak HN structures; instead, we introduce the definitions and state their basic properties along with some discussion.

13.1. **Definitions.** We work in Setup III.1. We remind the reader of the notation K, $p \in C$, $W \subset C$, $\mathcal{D}_K$, $\mathcal{D}_p$, and $\mathcal{D}_W$ introduced in Section 2. Given a heart $\mathcal{A}_C \subset \mathcal{D}$ of a bounded t-structure local over C, we similarly write $\mathcal{A}_K$, $\mathcal{A}_p$, and $\mathcal{A}_W$ for the corresponding hearts, given by Theorem 5.6 and Corollary 5.8, respectively. We also refer to Section 6.2 for the categories of C-torsion objects $\mathcal{D}_{C\text{-tor}} \subset \mathcal{D}$ and $\mathcal{A}_{C\text{-tor}} = \mathcal{A} \cap \mathcal{D}_{C\text{-tor}}$. We also introduced there the notion of C-torsion free objects $\mathcal{A}_{C\text{-tf}} \subset \mathcal{A}$, which are objects with no C-torsion subobjects.

The following gives a notion of central charge that is "constant in families".

Definition 13.1. A *central charge on $\mathcal{D}$ over C* is a pair $(Z_K, Z_{C\text{-tor}})$ where

$$Z_K \colon K(\mathcal{D}_K) \to \mathbb{C} \quad \text{and} \quad Z_{C\text{-tor}} \colon K(\mathcal{D}_{C\text{-tor}}) \to \mathbb{C}$$

are group homomorphisms with the following property: for all $E \in \mathcal{D}$, and all proper closed subschemes $W \subset C$, we have

$$Z_K(E_K) = \frac{1}{\operatorname{length} W} Z_{C\text{-tor}}\left(i_{W*}E_W\right). \tag{13.1}$$

Since $\mathcal{D} \to \mathcal{D}_K$ is essentially surjective, Z_K is determined by $Z_{C\text{-tor}}$, and thus (13.1) becomes a consistency condition for $Z_{C\text{-tor}}$: the right-hand-side should be independent of W. We think of this as requiring that $Z_{C\text{-tor}}$ is constant in families of objects over C.

Example 13.2. Equation (13.1) is satisfied whenever $Z_{C\text{-tor}}$ can be written as a linear combination of functions of the form χ_F, for $F \in \mathrm{D}_{\mathrm{perf}}(\mathcal{X})$, defined by

$$\chi_F(E) := \operatorname{length}_{\mathcal{O}_C} g_*(E \otimes F) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{length}_{\mathcal{O}_C} \mathrm{H}^i(g_*(E \otimes F)).$$

Note that this sum is well-defined: for $E \in \mathcal{D}_{C\text{-tor}}$, each cohomology sheaf of $g_*(E \otimes F)$ is a sheaf with zero-dimensional support. Moreover, since E is bounded, F is perfect, and g is projective, we have $g_*(E \otimes F) \in \mathrm{D}^{\mathrm{b}}(C)$. Then, the right-hand-side in (13.1) is independent of W: indeed, by base change it is equal to the rank of the complex $g_*(E \otimes F)$ over C.

Definition 13.3. A *Harder–Narasimhan structure on $\mathcal{D}$ over C* consists of a triple $\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$ where

- $\mathcal{P}$ is a slicing of $\mathcal{D}$, and
- $(Z_K, Z_{C\text{-tor}})$ is a central charge on $\mathcal{D}$ over C,

satisfying the following two properties:

C-linearity: The slicing $\mathcal{P}$ is *local over C*, i.e., for every open $U \subset C$ there exists a slicing $\mathcal{P}_U$ of $\mathcal{D}_U$ such that the pullback sends $\mathcal{P}(\phi)$ to $\mathcal{P}_U(\phi)$.

Compatibility: For all $\phi \in \mathbb{R}$ and all $0 \neq E \in \mathcal{P}(\phi)$, we have either

$$\begin{aligned} E_K \neq 0 \quad &\text{and} \quad Z_K(E_K) \in \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}, \quad \text{or} \\ E \in \mathcal{D}_{C\text{-tor}} \quad &\text{and} \quad Z_{C\text{-tor}}(E) \in \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}. \end{aligned}$$

The nonzero objects of $\mathcal{P}(\phi)$ are said to be *σ_C-semistable* of *phase ϕ*, and the simple objects of $\mathcal{P}(\phi)$ are said to be *σ_C-stable*.

Remark 13.4. Just as in Theorem 4.13 for the case of t-structures, $\mathcal{P}$ is local over C if and only if $\mathcal{P}(\phi)$ is invariant under tensoring with g^*L for all ϕ and all line bundles L on C.

Before giving a concrete example of a HN structure over C, we show how to construct one from an appropriate t-structure; in other words, we prove an analogue of [Bri07, Proposition 5.3] (see also Lemma 12.6).

Definition 13.5. Let $\mathcal{A}_C \subset \mathcal{D}$ be the heart of a bounded C-local t-structure. A *stability function on $\mathcal{A}_C$ over C* is a central charge $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{D}$ over C such that Z_K is a stability function on $\mathcal{A}_K$, and $Z_{C\text{-tor}}$ is a stability function on $\mathcal{A}_{C\text{-tor}}$.

Example 13.6. Let $\mathcal{X} \to C$ be a family of curves, and let $\mathcal{O}_{\mathcal{X}}(1)$ be a relative polarization. For $E \in (\operatorname{Coh} \mathcal{X})_{C\text{-tor}}$, let $p_1(E)$ and $p_0(E)$ be the coefficients of the linear and constant terms, respectively, of the Hilbert polynomial of E, defined as in Example 13.2 via the length of $g_*E(m)$ as $\mathcal{O}_C$-modules. Then

$$Z_{C\text{-tor}}(E) := \mathrm{i}p_1(E) - p_0(E)$$

defines a stability function on $\operatorname{Coh} \mathcal{X}$ over C.

Definition 13.7. Given a stability function on $\mathcal{A}_C$ over C, we define the following *central charge for objects $E \in \mathcal{A}_C$*:

$$Z_C(E) := \begin{cases} Z_K(E_K) & \text{if } Z_K(E_K) \neq 0, \\ Z_{C\text{-tor}}(E) & \text{otherwise.} \end{cases}$$

We then assign to $E \neq 0$ the *slope* $\mu_C(E) \in \mathbb{R} \cup \{+\infty\}$ by

$$\mu_C(E) := \begin{cases} +\infty & \text{if } \Im Z_C(E) = 0, \\ -\frac{\Re Z_C(E)}{\Im Z_C(E)} & \text{otherwise.} \end{cases}$$

Often we will use the phase $\phi(E) := \frac{1}{\pi} \arg Z_C(E) \in (0, 1]$ instead of the slope $\mu_C(E)$.

This slope function satisfies the weak see-saw property (see Lemma 15.5). We can easily construct examples where the strong see-saw property is not satisfied.

Example 13.8. Let $E \in \mathcal{A}_C$ with $E_K \neq 0$ and $p \in C$. Assume that there exists a quotient $i_{p*}E_p \twoheadrightarrow Q \in \mathcal{A}_{C\text{-tor}}$ such that $\mu_C(i_{p*}E_p) \neq \mu_C(Q)$. Then the composition $f\colon E \twoheadrightarrow Q$ satisfies $\mu_C(\ker f) = \mu_C(E) \neq \mu_C(Q)$.

Definition 13.9. An object $E \in \mathcal{A}_C$ is called *Z_C-semistable* if for all proper subobjects $0 \neq A \hookrightarrow E$, we have $\phi(A) \leqslant \phi(E/A)$ (or equivalently, $\mu_C(A) \leqslant \mu_C(E/A)$).

Definition 13.10. We say that a stability function $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{A}_C$ over C *satisfies the HN property* if every object $E \in \mathcal{A}_C$ admits a Harder–Narasimhan (HN) filtration: a sequence

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow E_m = E$$

such that E_i/E_{i-1} is Z_C-semistable for $i = 1, \ldots, m$, with

$$\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_m/E_{m-1}).$$

The objects E_i/E_{i-1} are called the *HN factors* of E.

Since $i_{p*}\colon \mathcal{A}_p \to \mathcal{A}_C$ is exact, fully faithful, and since the image is closed under subobjects and quotients by Corollary 6.10, $Z_{C\text{-tor}}$ automatically induces a compatible notion of semistability on $\mathcal{A}_p$, and if $(Z_K, Z_{C\text{-tor}})$ satisfies the HN property in $\mathcal{A}_C$, then so does $Z_p := Z_{C\text{-tor}} \circ i_{p*}$ on $\mathcal{A}_p$. Combined with Lemma 12.6, this yields:

Lemma 13.11. *A HN structure σ_C on $\mathcal{D}$ over C induces a pre-stability condition $\sigma_p = (\mathcal{A}_p, Z_p)$ on $\mathcal{D}_p$ for every $p \in C$.*

By definition, a HN structure also gives a pre-stability condition $\sigma_K = (\mathcal{A}_K, Z_K)$ on $\mathcal{D}_K$; therefore, a HN structure σ_C induces a pre-stability condition σ_c on $\mathcal{D}_c$ for every point $c \in C$.

The usual notion of relative slope-stability for a family of torsion free sheaves asks that the *generic* fiber is slope-stable; in contrast, Z_C-stability requires stability for *all* fibers (see also Remark 15.8):

Lemma 13.12. *Let $E \in \mathcal{A}_C$ be a C-torsion free object. Then E is Z_C-semistable if and only if E_K is Z_K-semistable and $E_p \in \mathcal{A}_p$ is Z_p-semistable for all closed points $p \in C$.*

Proof. See Lemma 15.7. □

Example 13.13. Consider slope-stability on a family of curves $g\colon \mathcal{X} \to C$ as in Example 13.6. Let $x \in \mathcal{X}$ be a closed point, $p := g(x)$ and let $F \subset \mathcal{X}$ be the fiber containing x. By Lemma 13.12 the ideal sheaf I_x is not semistable: indeed, $i_{p*}(I_x)_p = \mathcal{O}_F(-x) \oplus \mathcal{O}_x$ is not semistable. The HN filtration of I_x is given by $I_F \hookrightarrow I_x$, since I_F is semistable by Lemma 13.12, and the quotient $I_x/I_F = \mathcal{O}_F(-x)$ is the push-forward of a semistable sheaf from the fiber and therefore easily shown to be semistable with respect to $Z_{C\text{-tor}}$ in $\mathcal{A}_{C\text{-tor}}$.

Note that $I_F \hookrightarrow I_x$ is also the simplest possible example of a semistable reduction of a flat family of sheaves; in particular, the notion of HN filtrations with respect to μ_C will require semistable reduction as a necessary ingredient, see Proposition 13.16.

Proposition 13.14. *To give a HN structure on $\mathcal{D}$ over C is equivalent to giving a heart $\mathcal{A}_C \subset \mathcal{D}$ of a bounded C-local t-structure, together with a stability function $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{A}_C$ over C satisfying the HN property.*

Proof. See the proof of Proposition 15.9.

We only stress that for $E \in \mathcal{A}_C$, either all of its HN factors E_i with respect to $\mathcal{P}$ are in $\mathcal{D}_{C\text{-tor}}$, in which case $E \in \mathcal{D}_{C\text{-tor}}$ and $Z_{C\text{-tor}}(E) = \sum_i Z_{C\text{-tor}}(E_i)$ is in the semiclosed upper half plane $\mathbb{H} \sqcup \mathbb{R}_{<0}$; or, otherwise, some of its HN factors have $Z_K(E_i) \neq 0$, and we have $Z_K(E_K) \in \mathbb{H} \sqcup \mathbb{R}_{<0}$. Thus, $(Z_K, Z_{C\text{-tor}})$ is a stability function on $\mathcal{A}_C$ over C. □

Remark 13.15. Let $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ be the universal cover of the group $\mathrm{GL}_2^+(\mathbb{R})$ of real 2×2-matrices with positive determinant. From Definition 13.3, it is evident that $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ acts on the set of pre-stability conditions on $\mathcal{D}$ over C in the same manner as it acts on the set of stability conditions on a triangulated category. Indeed, let $\mathbb{R} \to S^1$ be the universal cover given by $\phi \mapsto e^{\mathrm{i}\pi\phi}$; then $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ can be described as the set of pairs (G, g) where $G \in \mathrm{GL}_2^+(\mathbb{R})$, and $g\colon \mathbb{R} \to \mathbb{R}$ is a choice of a lift of the induced action of G on S^1. Then

$$(G, g)(Z_K, Z_{C\text{-tor}}, \mathcal{P}) = (g^*\mathcal{P}, G \circ Z_K, G \circ Z_{C\text{-tor}})$$

where $g^*\mathcal{P}(\phi) = \mathcal{P}(g^{-1}\phi)$.

13.2. **Existence of HN filtrations.** Finally, we discuss the existence of HN filtrations.

Proposition 13.16. *Let $(Z_K, Z_{C\text{-tor}})$ be a stability function on $\mathcal{A}_C$ over C. If $(Z_K, Z_{C\text{-tor}})$ satisfies the HN property, then all of the following three conditions are satisfied:*

(1) *The pair $(\mathcal{A}_K, Z_K)$ satisfies the HN property.*
(2) *The pair $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ satisfies the HN property.*
(3) *(*Semistable reduction*) For any C-torsion free object $E \in \mathcal{A}_{C\text{-tf}}$ such that $E_K \in \mathcal{A}_K$ is Z_K-semistable, there is a Z_C-semistable subobject $F \subset E$ with $E/F \in \mathcal{A}_{C\text{-tor}}$.*

Moreover, if $\mathcal{A}_C$ has a C-torsion theory $(\mathcal{A}_{C\text{-tor}}, \mathcal{A}_{C\text{-tf}})$, then the converse also holds true.

We show the necessity of the three conditions in the following remark, and refer to Proposition 15.10 for the proof of the converse.

Remark 13.17. We first comment on each of the conditions given in Proposition 13.16.

(1) If $E \in \mathcal{A}_C$ is Z_C-semistable, then by Lemma 13.12 the object $E_K \in \mathcal{A}_K$ is Z_K-semistable. Moreover, given a HN filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_m = E$, then for any $i < j$ we have $\mu_K((E_i/E_{i-1})_K) > \mu_K((E_j/E_{j-1})_K)$ if both quotients are non-zero. Hence, the HN filtration of E in $\mathcal{A}_C$ induces a HN filtration of $E_K \in \mathcal{A}_K$, i.e., condition (1) is necessary. On the other hand, given the HN filtration of $E_K \in \mathcal{A}_K$, we can attempt to construct a HN filtration of $E \in \mathcal{A}_C$ as a refinement of a lift of the HN filtration of E_K to $\mathcal{A}_C$.
(2) Since $\mathcal{A}_{C\text{-tor}}$, as a subcategory of $\mathcal{A}_C$, is closed under subobjects and quotients, the inclusion $\mathcal{A}_{C\text{-tor}} \subset \mathcal{A}_C$ identifies $Z_{C\text{-tor}}$-semistable objects with objects in $\mathcal{A}_{C\text{-tor}}$ that are Z_C-semistable as objects in $\mathcal{A}_C$. In particular, condition (2) is clearly necessary.

(3) Consider the first step of the HN filtration of such an E. It has to be a Z_C-semistable subobject $F \subset E$ with $\mu_C(F) = \mu_C(E)$.

Also recall that when $\mathcal{A}_C$ is noetherian, then the assumption that $(\mathcal{A}_{C\text{-tor}}, \mathcal{A}_{C\text{-tf}})$ is a torsion pair is automatic, see Remark 6.16. However, it is easy to construct examples of hearts local over C that do not satisfy this assumption: for example, consider the heart $\mathcal{B}_C$ obtained by tilting at the torsion pair $(\mathcal{A}_{C\text{-tor}}, \mathcal{A}_{C\text{-tf}})$.

14. Weak stability conditions, tilting, and base change

In order to construct stability conditions on surfaces or higher-dimensional varieties, one may use the auxiliary notion of *weak stability conditions*.[3] The procedure will be analogous for Harder–Narasimhan structures over a curve. In this section, we recall the definition of weak stability conditions following [BMS16, PT19]. Then we study analogues of operations on stability conditions that become more subtle for weak stability conditions, namely tilting (Section 14.2) and base change (Section 14.3).

14.1. Definitions.

We begin by recalling the definition of weak stability conditions by following [BMS16, PT19], and review some of the analogues of basic properties of stability conditions that become more subtle for weak stability conditions.

Definition 14.1. Let $\mathcal{D}$ a triangulated category. A *weak pre-stability condition on* $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ where $\mathcal{P}$ is a slicing of $\mathcal{D}$, and $Z\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism, that satisfy the following condition:

$$\text{For all } 0 \neq E \in \mathcal{P}(\phi), \text{ we have } Z(E) \in \begin{cases} \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi} & \text{if } \phi \notin \mathbb{Z} \\ \mathbb{R}_{\geqslant 0} \cdot e^{\mathrm{i}\pi\phi} & \text{if } \phi \in \mathbb{Z}. \end{cases}$$

As in Definition 12.2, we say that σ is a *weak pre-stability condition on* $\mathcal{D}$ *with respect to* Λ, if Z factors through a group homomorphism $v : K(\mathcal{D}) \to \Lambda$.

Definition 14.2. A *weak stability function* Z *on an abelian category* $\mathcal{A}$ is a morphism of abelian groups $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ is in $\mathbb{H} \sqcup \mathbb{R}_{\leqslant 0} := \{z \in \mathbb{C} \,|\, \Im z > 0, \text{ or } \Im z = 0 \text{ and } \Re z \leqslant 0\}$.

The function Z allows one to define a *slope* for any $0 \neq E \in \mathcal{A}$ by setting

$$\mu_Z(E) := \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)} & \text{if } \Im Z(E) > 0 \\ +\infty & \text{otherwise} \end{cases}$$

and a notion of stability: An object $0 \neq E \in \mathcal{A}$ is *Z-semistable* if for every proper subobject F, we have $\mu_Z(F) \leqslant \mu_Z(E/F)$.

Definition 14.3. Given a weak stability function Z on an abelian category $\mathcal{A}$, we define $\mathcal{A}^0 \subset \mathcal{A}$ as the subcategory of objects $E \in \mathcal{A}$ with $Z(E) = 0$.

HN filtrations and the HN property for weak stability functions are defined exactly as in Definition 12.5.

Lemma 14.4 ([PT19, Section 2.1])**.** *To give a weak pre-stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure, and a weak stability function Z on $\mathcal{A}$ satisfying the HN property.*

[3] In [BMS16, PT19], this notion is called a very weak stability condition.

Example 14.5. Let X be a polarized variety over a field, and let p_n, p_{n-1} be the leading coefficients of the Hilbert polynomial. Then $(\mathrm{i}p_n - p_{n-1}, \operatorname{Coh} X)$ defines a weak stability condition, and $(\operatorname{Coh} X)^0$ is the category of sheaves supported in codimension at least two.

The analogue of Lemma 12.12 is more involved.

Definition 14.6. Let $\mathcal{B} \subset \mathcal{A}$ be an abelian subcategory of an abelian category $\mathcal{A}$. We say that $\mathcal{B}$ is a *noetherian torsion subcategory* of $\mathcal{A}$ if $\mathcal{B}$ is a noetherian abelian category, and if there exists a torsion pair $(\mathcal{B}, \mathcal{B}^\perp)$ in $\mathcal{A}$.

Remark 14.7. An extension-closed abelian subcategory $\mathcal{B} \subset \mathcal{A}$ is a noetherian torsion subcategory if and only if for every object $E \in \mathcal{A}$ any increasing sequence $B_1 \subset B_2 \subset \cdots \subset E$ of subobjects of E with $B_i \in \mathcal{B}$ terminates.

Lemma 14.8 ([PT19, Lemma 2.17])**.** *Let $(\mathcal{A}, Z)$ be a weak pre-stability condition, such that $\mathcal{A}^0 \subset \mathcal{A}$ is a noetherian torsion subcategory, and Z is defined over $\mathbb{Q}[\mathrm{i}]$. Then $\mathcal{A}$ is noetherian.*

Proof. Assume otherwise, and consider a sequence of non-trivial surjections $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots$. Then $\Im Z(E_i)$ is discrete monotone decreasing non-negative function; hence we may assume it to be constant.

Let $M_i \hookrightarrow E_i$ be the maximal subobject with $\Im Z(M_i) = 0$, which exists by the existence of HN filtrations. If C_i is the kernel of $E_i \twoheadrightarrow E_{i+1}$, then $\Im Z(C_i) = 0$, and so $C_i \subset M_i$. Similarly, it follows that the composition $M_i \hookrightarrow E_i \twoheadrightarrow E_{i+1}$ factors via a surjection $M_i \twoheadrightarrow M_{i+1}$ with kernel C_i. Hence we get a sequence of non-trivial surjections $M_1 \twoheadrightarrow M_2 \twoheadrightarrow \cdots$.

Arguing again by discreteness of the central charge, we may assume that $Z(M_i)$ is constant. Then the kernels D_i of the composition $M_1 \twoheadrightarrow M_i$ form a strictly increasing sequence of subobjects of M_1 in $\mathcal{A}^0$. Since $\mathcal{A}^0$ is assumed to be a noetherian torsion subcategory, this is a contradiction to Remark 14.7. □

Finally, we define the support property for weak pre-stability conditions exactly as in Definition 12.2, and call σ a *weak stability condition* if it satisfies the support property with respect to some v, Λ and Q.

Remark 14.9. If σ is a weak stability condition, then $v(\mathcal{A}^0) = 0$. Indeed, every $E \in \mathcal{A}^0$ is automatically semistable, and thus $v(E)$ is a vector in $\operatorname{Ker} Z$ with $Q(v(E)) \geqslant 0$.

Remark 14.10. If $\sigma = (Z, \mathcal{P})$ is a weak stability condition, and E is semistable of phase $\phi \in \mathbb{R} \setminus \mathbb{Z}$, then E admits a Jordan–Hölder filtration.

Lemma 14.11. *Assume that $\sigma = (\mathcal{A}, Z)$ is a weak stability condition, such that $\mathcal{A}^0$ is a noetherian torsion subcategory. Let $\mu \in \mathbb{R} \cup \{+\infty\}$, and let $E_1 \hookrightarrow E_2 \hookrightarrow E_3 \hookrightarrow \cdots \subset E$ be an increasing sequence of subobjects of a fixed object with $\mu(E_i) \geqslant \mu$ for all i. Then this sequence terminates.*

Proof. We have $\mu^+(E_i) \leqslant \mu^+(E)$ and $\mu(E_i) \geqslant \mu$. From this one can deduce that the central charges of all HN factors of all E_i are contained in a compact region. If $\mu^+(E) < +\infty$ (and thus $\mu < +\infty$), this region is the parallelogram with two horizontal edges and two edges corresponding to slope $\mu^+(E)$, and with opposite vertices given by 0 and the complex number z determined by $\Im z = \Im Z(E)$ and $\mu(z) = \mu$. In the case $\mu^+(E) = +\infty$ let $F \subset E$ be the first step of the HN filtration; then we can replace 0 by $Z(F) \in \mathbb{R}_{>0}$ and $\mu^+(E)$ by $\mu^+(E/F)$ in the previous construction of the parallelogram.

It follows by Remark 12.3 that there are only finitely many classes $a_1, \dots, a_m \in \Lambda$ that can occur as the classes of HN factors of E_i. Since $Z(E_i)$ is contained in the same compact region described in the previous paragraph, and since every $Z(E_i)$ is an integral non-negative linear combination of the same finite set of complex numbers $Z(a_k) \in \mathbb{H} \sqcup \mathbb{R}_{<0}, k = 1, \dots, m$, this leaves only finitely possibilities for $Z(E_i)$. Therefore, $Z(E_i)$ has to become constant for $i \geqslant i_0$. But then E_i/E_{i_0} is a subobject of $E/E_{i_0} \in \mathcal{A}^0$ for all $i \geqslant i_0$, and thus the sequence terminates by the assumption that $\mathcal{A}^0$ is noetherian. □

14.2. **Tilting weak stability conditions.** Due to the special role played by objects with central charge zero, there is in general no analogue for weak stability conditions of the $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$-action on stability conditions explained in Remark 13.15. In this subsection, we explore conditions under which a weak stability condition $(\mathcal{A}, Z)$ can nevertheless be tilted.

Given $\beta \in \mathbb{R}$, we can define a pair $(\mathcal{T}^\beta, \mathcal{F}^\beta)$ of subcategories of $\mathcal{A}$ given by

$$\begin{aligned} \mathcal{T}^\beta &:= \langle E \in \mathcal{A}\ \mu\text{-semistable with } \mu(E) > \beta \rangle = \left\{ E : \ \mu^-(E) > \beta \right\}, \\ \mathcal{F}^\beta &:= \langle E \in \mathcal{A}\ \mu\text{-semistable with } \mu(E) \leqslant \beta \rangle = \left\{ E : \ \mu^+(E) \leq \beta \right\}. \end{aligned} \tag{14.1}$$

Existence of HN filtrations combined with the weak see-saw property ensure that $(\mathcal{T}^\beta, \mathcal{F}^\beta)$ is a torsion pair (see Definition 4.6); we write $\mathcal{A}^{\sharp\beta} = \langle \mathcal{F}^\beta[1], \mathcal{T}^\beta \rangle$ for the corresponding tilted heart.

By [BLMS17, Lemma 2.16] and the proof of [BLMS17, Proposition 2.15], the following property guarantees that a weak stability condition can be tilted, see Proposition 14.16 below; we will sketch a proof for completeness.

Definition 14.12. A weak stability condition $\sigma = (\mathcal{A}, Z)$ has the *tilting property* if

(1) $\mathcal{A}^0 \subset \mathcal{A}$ is a noetherian torsion subcategory, and
(2) for every $F \in \mathcal{A}$ with $\mu^+(F) < +\infty$, there exists a short exact sequence $F \hookrightarrow \widetilde{F} \twoheadrightarrow F^0$ with $F^0 \in \mathcal{A}^0$ and $\mathrm{Hom}(\mathcal{A}^0, \widetilde{F}[1]) = 0$.

Example 14.13. Slope stability of sheaves as in Example 14.5 has the tilting property, with $F \hookrightarrow \tilde{F}$ given by the embedding of a torsion free sheaf into its double dual.

Remark 14.14. The tilting property implies that for F as in part (2) of Definition 14.12, any sequence of inclusions $F = F_0 \hookrightarrow F_1 \hookrightarrow F_2 \dots$ with $F_i/F \in \mathcal{A}^0$ terminates: since $\mathrm{Ext}^1(F_i/F, \widetilde{F}) = 0$, the inclusion $F \hookrightarrow \widetilde{F}$ factors via F_i, and thus F_i/F is an increasing sequence of subobjects of $\widetilde{F}/F \in \mathcal{A}^0$.

Remark 14.15. We will see in the proof of Proposition 14.16 that part (2) is equivalent to the condition that $\mathcal{A}^0$ is a torsion subcategory of the heart obtained by tilting $\mathcal{A}$ at the torsion pair $(\mathcal{T}^\beta, \mathcal{F}^\beta)$, for every $\beta \in \mathbb{R}$.

Proposition 14.16. *Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition with the tilting property. Then $\sigma^\beta := \left(\mathcal{A}^{\sharp\beta}, \frac{Z}{\mathrm{i}-\beta}\right)$ is again a weak stability condition, and $(\mathcal{A}^{\sharp\beta})^0 \subset \mathcal{A}^{\sharp\beta}$ is a noetherian torsion subcategory.*

We first notice that $Z^{\sharp\beta} := \frac{Z}{\mathrm{i}-\beta}$ is a weak stability function on $\mathcal{A}^{\sharp\beta}$. Semistable objects can then be easily classified: we omit the proof (see, for example, [PT19, Lemma 2.19]).

Lemma 14.17. *Let $E \in \mathcal{A}^{\sharp\beta}$ with $E \notin (\mathcal{A}^{\sharp\beta})^0$. Then E is $Z^{\sharp\beta}$-semistable if and only if*

(1) *either* $\Im Z(E) \geqslant 0$ *and* $E \in \mathcal{A}$ *is a* Z*-semistable object with* $\mathrm{Hom}(\mathcal{A}^0, E) = 0$*,*
(2) *or* $\Im Z(E) < 0$ *and* E *is an extension*

$$U[1] \to E \to V$$

where $U \in \mathcal{A}$ *is a* Z*-semistable object and* $V \in \mathcal{A}_0$*; moreover, if either* $\Im Z^{\sharp\beta}(E) > 0$ *or* E *is* $Z^{\sharp\beta}$*-stable, then* $\mathrm{Hom}(V', E) = 0$*, for all* $V' \in \mathcal{A}^0$*.*

Proof of Proposition 14.16. We only need to check that $(\mathcal{A}^{\sharp\beta})^0$ is a noetherian torsion subcategory: then, using Lemma 14.17, one can use the HN filtration of $\mathrm{H}^0_{\mathcal{A}}(E)$ and modify the HN filtration of $\mathrm{H}^{-1}_{\mathcal{A}}(E)$ via subobjects and quotients in $(\mathcal{A}^{\sharp\beta})^0$ to construct the HN filtration of $E \in \mathcal{A}^{\sharp\beta}$. Moreover, the Lemma, combined with Remark 14.9, also shows that the set of classes of semistable objects in Λ is unchanged, so that σ^β satisfies the support property. (We will spell out a similar argument in more detail in the proof of Proposition 19.5.)

By construction, $(\mathcal{A}^{\sharp\beta})^0 = \mathcal{A}^0$; in particular, it is noetherian by assumption. We need to show it is a torsion subcategory. Let $F \in \mathcal{A}^{\sharp\beta}$. We can write it as an extension

$$0 \to M[1] \to F \to N \to 0$$

in $\mathcal{A}^{\sharp\beta}$ with $M \in \mathcal{F}^\beta$ and $N \in \mathcal{T}^\beta$. By assumption, there exists a short exact sequence in $\mathcal{A}^{\sharp\beta}$

$$0 \to M^0 \to M[1] \to \widetilde{M}[1] \to 0$$

with $M^0 \in \mathcal{A}^0$ and $\mathrm{Hom}((\mathcal{A}^{\sharp\beta})^0, \widetilde{M}[1]) = 0$.

By replacing F with F/M^0, we can assume $M = \widetilde{M}$. Hence, any injective morphisms $F^0 \hookrightarrow F$ in $\mathcal{A}^{\sharp\beta}$, with $F^0 \in (\mathcal{A}^{\sharp\beta})^0$, induces an injective morphism $F^0 \hookrightarrow N$. Since $\mathcal{A}^0 \subset \mathcal{A}$ is a noetherian torsion subcategory, there is a maximal such subobject $F^0 \hookrightarrow F$, which proves what we wanted. □

Remark 14.18. When Z is defined over $\mathbb{Q}[\mathrm{i}]$ and β is also rational, then Lemma 14.8 implies additionally that $\mathcal{A}^\beta$ is noetherian.

14.3. **Base change for weak stability conditions.** We now explain how to extend our base change result, Theorem 12.17, to the case of weak stability conditions. Throughout this subsection we assume that $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ is a strong semiorthogonal component of the derived category of a variety defined over a field k, and let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition such that $\mathcal{A}^0 \subset \mathcal{A}$ is a noetherian torsion subcategory, and such that Z is defined over $\mathbb{Q}[\mathrm{i}]$.

We first recall the construction of the slicing from the proof of Theorem 12.17.

Definition 14.19. Consider a field extension $k \subset \ell$. Let $\mathcal{A}^\phi := \mathcal{P}(\phi - 1, \phi]$, and write $\mathcal{A}^\phi_\ell$ for the heart in $\mathcal{D}_\ell$ obtained from $\mathcal{A}^\phi$ via base change as in Proposition 5.9. (Since $\mathcal{A}$ is noetherian by Lemma 14.8, $\mathcal{A}^\phi$ is tilted-noetherian.) We define

$$\mathcal{P}_\ell(\phi) := \bigcap_{\phi'-1<\phi\leqslant\phi'} \mathcal{A}^{\phi'}_\ell. \tag{14.2}$$

Proposition 14.20. *Let* $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ *be a strong semiorthogonal component of the derived category of a variety defined over a field* k*, and let* ℓ/k *be a field extension. Let* $\sigma = (\mathcal{A}, Z)$ *be a weak numerical stability condition on* $\mathcal{D}$ *and assume the following:*

(1) $\mathcal{A}^0 \subset \mathcal{A}$ *is a noetherian torsion subcategory, and* Z *is defined over* $\mathbb{Q}[\mathrm{i}]$*.*

(2) *Either ℓ/k is algebraic, or if C is an affine Dedekind scheme essentially of finite type over k, and if $E \in \mathcal{D}_C$ satisfies $E_{K(C)} \in \mathcal{P}_{K(C)}(\phi)$, then there exists an open subset $U \subset C$ such that $E_c \in \mathcal{P}_c(\phi)$ for all $c \in U$.*

Then $\sigma_\ell = (\mathcal{A}_\ell, Z_\ell)$ defines a weak stability condition on $\mathcal{D}_\ell$, with slicing given by $\mathcal{P}_\ell$ as in Definition 14.19. The pullback $\mathcal{D}_k \to \mathcal{D}_\ell$ preserves the properties of being semistable, or geometrically stable, respectively.

We will later consider a variant of the openness condition in (2), see Definition 16.3.

Proof. We follow the proof of Theorem 12.17. Note that by Lemma 14.8, assumption (1) is equivalent to the condition that $\mathcal{A}$ is noetherian and Z is defined over $\mathbb{Q}[\mathrm{i}]$, and thus by Proposition 5.9(4), it is preserved by base change along finitely generated field extensions. Also by our definition of essentially of finite type for affine schemes in Definition 11.16, assumption (2) is automatically preserved under finitely generated field extensions.

Step 1 carries over without any change. Now consider Step 2, the case of $\ell = k(x)$, and again set $R = k[x]$. Since $\mathcal{A}$ and thus $\mathcal{A}_R$ is noetherian, any object $E_{k(x)} \in \mathcal{A}_{k(x)}$ lifts to an R-torsion free object $E_R \in \mathcal{A}_R$, see Remark 6.16. By Lemma 6.12, we have $E_c \in \mathcal{A}_c$ for all closed points $c \in \mathbb{A}^1_k$; since $Z_{k(x)}(E_{k(x)}) = Z_c(E_c)$ this proves the compatibility of $Z_{k(x)}$ with $\mathcal{A}_{k(x)}$.

Our next observation is that if $E \in \mathcal{P}_{k(x)}(1)$, then the same argument combined with condition (2) shows $Z_{k(x)}(E) = 0$. Conversely, if E_R is a torsion free lift of an object $E \in \mathcal{A}_{k(x)}$ with $\Im Z_{k(x)}(E) = 0$, then $E_c \in \mathcal{P}_c(1)$ for all closed points $c \in \mathbb{A}^1_k$ by Step 1, i.e., $E_c \in \bigcap_{0 \leqslant \phi < 1} \mathcal{A}^\phi_c$. By Lemma 6.13, this implies $E_R \in \bigcap_{0 \leqslant \phi < 1} \mathcal{A}^\phi_R$; since $\mathcal{D}_R \to \mathcal{D}_{k(x)}$ is t-exact by Theorem 5.7.(4b), this implies $E \in \bigcap_{0 \leqslant \phi < 1} \mathcal{A}^\phi_{k(x)} = \mathcal{P}_{k(x)}(1)$.

The existence of the HN filtration now follows exactly as in the proof of Theorem 12.17, and the compatibility of $Z_{k(x)}$ with $\mathcal{P}_{k(x)}$ follows again from assumption (2); this concludes Step 2.

Step 3 carries over without change. Finally, Step 5 is trivial when ℓ/k is algebraic, and otherwise becomes easier under our assumption (2): given a Dedekind domain C, a non-trivial Jordan–Hölder filtration over $K(C)$ induces a non-trivial filtration in $\mathcal{P}_c(\phi)$ for the open subset where every Jordan–Hölder factor restricts to a semistable object in $\mathcal{P}_c(\phi)$. □

15. Weak Harder–Narasimhan structures over a curve

In this section, we introduce the notion of a *weak Harder–Narasimhan structure over a curve*. From this section until the end of Part III, we work in Setup III.1.

15.1. Definitions.

The following is the weak version of Definition 13.3.

Definition 15.1. A *weak Harder–Narasimhan structure on $\mathcal{D}$ over C* consists of a triple $\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$, where

- $\mathcal{P}$ is a slicing of $\mathcal{D}$, and
- $(Z_K, Z_{C\text{-tor}})$ is a family of central charges over C

satisfying the following properties:

C-linearity: The slicing $\mathcal{P}$ is local over C.

Compatibility: For all $\phi \in \mathbb{R}$ and all $0 \neq E \in \mathcal{P}(\phi)$, we have either

$$E_K \neq 0 \quad \text{and} \quad Z_K(E_K) \in \begin{cases} \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi} & \text{if } \phi \notin \mathbb{Z} \\ \mathbb{R}_{\geqslant 0} \cdot e^{\mathrm{i}\pi\phi} & \text{if } \phi \in \mathbb{Z} \end{cases}, \quad \text{or}$$

$$E \in \mathcal{D}_{C\text{-tor}} \quad \text{and} \quad Z_{C\text{-tor}}(E) \in \begin{cases} \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi} & \text{if } \phi \notin \mathbb{Z} \\ \mathbb{R}_{\geqslant 0} \cdot e^{\mathrm{i}\pi\phi} & \text{if } \phi \in \mathbb{Z}. \end{cases}$$

We have to keep in mind that for $E \in \mathcal{A}_C$, we no longer have $E_K = 0$ if and only if $Z_K(E_K) = 0$. There is also the following notion of a weak stability function on a local heart; we will see in Proposition 15.9 that weak HN structures can be constructed via a local heart with an appropriate weak stability function.

Definition 15.2. Let $\mathcal{A}_C \subset \mathcal{D}$ be the heart of a bounded C-local t-structure. A *weak stability function for $\mathcal{A}_C$ over C* is a pair of central charges $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{D}$ over C such that Z_K is a weak stability function on $\mathcal{A}_K$, and $Z_{C\text{-tor}}$ is a weak stability function on $\mathcal{A}_{C\text{-tor}}$.

Example 15.3. Assume that $g\colon \mathcal{X} \to C$ has relative dimension n, and let $\mathcal{O}_{\mathcal{X}}(1)$ be a relative polarization. For $E \in \mathcal{A}_{C\text{-tor}}$, let p_n, p_{n-1} be the two leading coefficients of the Hilbert polynomial of E, defined as in Example 13.6. Then

$$Z_{C\text{-tor}}(E) := \mathrm{i}p_n(E) - p_{n-1}(E)$$

defines a weak stability function for $\operatorname{Coh}\mathcal{X}$ over C.

Definition 15.4. Given a weak stability function for $\mathcal{A}_C$ over C, we define

$$Z_C(E) := \begin{cases} Z_K(E_K) & \text{if } E_K \neq 0 \\ Z_{C\text{-tor}}(E) & \text{otherwise.} \end{cases}$$

Then the slope $\mu_C(E) \in \mathbb{R} \cup \{+\infty\}$ for $E \neq 0$ is defined as before in Definition 13.7.

This slope function satisfies a weak version of the see-saw property:

Lemma 15.5. *Given a non-trivial short exact sequence $A \hookrightarrow E \twoheadrightarrow B$ in $\mathcal{A}_C$, we have*

$$\mu_C(A) \leqslant \mu_C(E) \leqslant \mu_C(B) \text{ or } \mu_C(A) \geqslant \mu_C(E) \geqslant \mu_C(B). \tag{15.1}$$

If $E \in \mathcal{A}_{C\text{-tor}}$ and $Z_C(A) \neq 0 \neq Z_C(B)$, we moreover have $\mu_C(A) < \mu_C(B) \Leftrightarrow \mu_C(E) < \mu_C(B)$.

Proof. By Corollary 5.8.(2), pullback to $\operatorname{Spec}(K)$ is t-exact. Hence, if $E_K \neq 0$, the weak seesaw property follows from the corresponding property on $(\mathcal{A}_K, Z_K)$. Similarly, if $E_K = 0$, then $E \in \mathcal{A}_{C\text{-tor}}$, and the weak seesaw property follows from the corresponding property on $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$. □

The weak see-saw property still allows us to define Z_C-semistability for objects in $\mathcal{A}_C$ exactly as in Definition 13.9. HN filtrations and the HN property for weak stability functions for $\mathcal{A}_C$ are defined as in Definition 13.10. As in Lemma 13.11, $Z_{C\text{-tor}}$ induces a weak stability function on $\mathcal{A}_p$ for every $p \in C$, compatible with semistability and HN filtrations, and thus:

Lemma 15.6. *A weak HN structure σ_C on $\mathcal{D}$ over C induces a weak pre-stability condition σ_p on $\mathcal{D}_p$ for every closed point $p \in C$. The pushforward i_{p*} preserves semistability, phases and HN filtrations.*

For a stability function on $\mathcal{A}_C$ over C, Z_C-semistability of a C-torsion free object amounts to semistability of E_p on *all* fibers p, see Lemma 13.12. In the weak case, however, we can only detect the existence of destabilizing quotients $E_p \twoheadrightarrow Q$ with $\mu(E_p) > \mu(Q)$; non-existence of such a quotient is not equivalent to E_p being Z_p-semistable, in case E_p has a subobject in $\mathcal{A}_p^0$.

Lemma 15.7. *Let $E \in \mathcal{A}_C$ be a C-torsion free object. Then E is Z_C-semistable if and only if E_K is Z_K-semistable and $i_{W*}E_W \in \mathcal{A}_{C\text{-tor}}$ does not have a destabilizing quotient for all zero-dimensional subschemes $W \subset C$ or, equivalently, if and only if E_K is Z_K-semistable and $E_p \in \mathcal{A}_p$ does not have a destabilizing quotient for all closed points $p \in C$.*

Let $W \subset C$ be a zero-dimensional subscheme, and consider Lemma 6.9 for E above: since E is torsion free, it shows that $i_{W*}E_W = E/I_W \cdot E \in \mathcal{A}_{C\text{-tor}}$.

Proof. Assume first that E is Z_C-semistable. If E_K is not Z_K-semistable, then we can apply Lemma 4.16.(2) to a destabilizing subobject $A_K \hookrightarrow E_K$ and lift it to a destabilizing subobject $A \hookrightarrow E$. We conclude that E_K is Z_K-semistable. Moreover, for any quotient $i_{W*}E_W \twoheadrightarrow Q$, the composite map $E \twoheadrightarrow i_{W*}E_W \twoheadrightarrow Q$ gives $\mu_C(Q) \geqslant \mu_C(E) = \mu_C(i_{W*}E_W)$. Hence $i_{W*}E_W$ does not have a destabilizing quotient.

Conversely, suppose that we have a destabilizing sequence

$$0 \to A \to E \to B \to 0,$$

with $\mu_C(A) > \mu_C(B)$. Since E is C-torsion free, A is also C-torsion free. If $B_K \neq 0$, then $\mu_K(A_K) = \mu_C(A) > \mu_C(B) = \mu_K(B_K)$ shows that E_K is not Z_K-semistable. If $B \in \mathcal{A}_{C\text{-tor}}$, then let $W \subset C$ be such that B is supported on W. By Lemma 6.9, we have $i_{W*}E_W = E/I_W \cdot E$ since E is torsion free, and the surjection $E \twoheadrightarrow B$ factors via $E \twoheadrightarrow i_{W*}E_W = E/I_W \cdot E \twoheadrightarrow B$. Since $\mu_C(i_{W*}E_W) = \mu_C(E) > \mu_C(B)$, this gives a destabilizing quotient of $i_{W*}E_W$.

Finally, assume that $E_p \in \mathcal{A}_p$, and therefore, $i_{p*}E_p \in \mathcal{A}_{C\text{-tor}}$ does not have a destabilizing quotient for any $p \in C$, but $B \in \mathcal{A}_{C\text{-tor}}$ is a destabilizing quotient of E in $\mathcal{A}_C$. We may assume that the subscheme W supporting B contains a single closed point $p \in W$; so we have $I_W = I_p^m$ for some $m > 0$. Then $B/I_p \cdot B$ is a quotient of $i_{p*}E_p$; by the see-saw property and the assumptions, it follows that $\mu_C(I_p \cdot B) < \mu_C(E)$. However, $I_p \cdot B$ is naturally a quotient of $I_p \otimes E/\left(I_p^{m-1} \cdot I_p \otimes E\right) \cong E/I_p^{m-1}E$ supported on a subscheme of smaller length; proceeding by induction, we get a contradiction. □

Remark 15.8. In the context of Lemma 15.7, assume further that $\mathcal{A}_C$ has a C-torsion theory. If $i_{W*}E_W \in \mathcal{A}_{C\text{-tor}}$ does not have a destabilizing quotient for all zero-dimensional subschemes $W \subset C$, then E_K is automatically Z_K-semistable.

Otherwise, we can lift the destabilizing surjection $E_K \twoheadrightarrow B_K$ to a surjection $E \twoheadrightarrow B$. Let $W \subset C$ be any non-trivial closed subset. Since i_W^* is right t-exact, we have a sequence of surjections

$$E \twoheadrightarrow i_{W*}E_W \twoheadrightarrow i_{W*}\mathrm{H}^0_{\mathcal{A}_W}(B_W) \twoheadrightarrow i_{W*}\mathrm{H}^0_{\mathcal{A}_W}\left((B_{C\text{-tf}})_W\right) = i_{W*}\left((B_{C\text{-tf}})_W\right)$$

where the last equality follows from Lemma 6.12. The composition is destabilizing since $Z\left((B_{C\text{-tf}})_W\right) = Z_K(B_K)$.

We now state and prove the equivalent of Proposition 13.14.

Proposition 15.9. *To give a weak HN structure on $\mathcal{D}$ over C is equivalent to giving a heart $\mathcal{A}_C$ of a bounded C-local t-structure, together with a weak stability function $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{A}_C$ over C satisfying the HN property.*

Proof. Suppose we are given a weak HN structure $\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$. Since the slicing $\mathcal{P}$ is C-local, the same follows for the heart $\mathcal{A}_C := \mathcal{P}(0,1]$. Moreover, for $E \in \mathcal{A}_C$, $Z_{C\text{-tor}}(E)$ and $Z_K(E_K)$ are clearly in the semiclosed upper half plane $\mathbb{H} \sqcup \mathbb{R}_{\leqslant 0}$; thus, $(Z_K, Z_{C\text{-tor}})$ is a weak stability function on $\mathcal{A}_C$ over C.

To show that $(Z_K, Z_{C\text{-tor}})$ satisfies the HN property, it suffices to show that $E \in \mathcal{P}(\phi)$ is Z_C-semistable; then the HN filtration with respect to $\mathcal{P}$ gives the HN filtration with respect to $(Z_K, Z_{C\text{-tor}})$. So given $E \in \mathcal{P}(\phi)$, we first show that E has no subobject with $\mu_C(A) > \mu_C(E)$ using the HN filtration of A with respect to $\mathcal{P}$: indeed, if A_1 is the first step of that filtration, then $A_1 \hookrightarrow A \hookrightarrow E$ would be a chain of inclusions (in particular non-zero), and thus $\phi(A_1) \leqslant \phi$. This leads to a contradiction both when $E_K \neq 0$ and when $E \in \mathcal{A}_{C\text{-tor}}$. Then one can easily show that E has no quotients with $\mu_C(Q) < \mu_C(E)$ by an analogous argument; thus E is Z_C-semistable, as required.

Conversely, for all $\phi \in (0,1]$ we define $\mathcal{P}(\phi)$ to be the category of Z_C-semistable objects $E \in \mathcal{A}_C$ with $Z_C(E) \in \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}$ for $\phi \notin \mathbb{Z}$ and $Z_C(E) \in \mathbb{R}_{\geqslant 0} \cdot e^{\mathrm{i}\pi\phi}$ for $\phi \in \mathbb{Z}$. One verifies that $\mathcal{P}$ is a slicing with the exact same arguments as in the case of weak stability conditions (or stability conditions as in [Bri07, Lemma 5.3]). □

15.2. **Existence of HN filtrations.** Our main tool in proving that a given weak stability function on $\mathcal{A}_C$ over C satisfies the HN property is the following result.

Proposition 15.10. *Let $(Z_K, Z_{C\text{-tor}})$ be a weak stability function on $\mathcal{A}_C$ over C. If it satisfies the HN property, then all of the following three conditions are satisfied:*

(1) *The pair $(\mathcal{A}_K, Z_K)$ satisfies the HN property.*
(2) *The pair $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ satisfies the HN property.*
(3) *(*Semistable reduction*) For any C-torsion free object $E \in \mathcal{A}_{C\text{-tf}}$ such that $E_K \in \mathcal{A}_K$ is Z_K-semistable, there is a Z_C-semistable subobject $F \subset E$ with $E/F \in \mathcal{A}_{C\text{-tor}}$.*

Moreover, if $\mathcal{A}_C$ has a C-torsion theory $(\mathcal{A}_{C\text{-tor}}, \mathcal{A}_{C\text{-tf}})$, then the converse also holds true.

The explanations in Remark 13.17 show that these conditions are necessary. In particular, $(\mathcal{A}_K, Z_K)$ and $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ are weak pre-stability conditions on $\mathcal{D}_K$ and $\mathcal{D}_{C\text{-tor}}$, respectively. We start the proof of the converse with the following observation:

Lemma 15.11. *Let $(Z_K, Z_{C\text{-tor}})$ be a weak stability function on $\mathcal{A}_C$ over C, such that $\mathcal{A}_C$ has a C-torsion theory $(\mathcal{A}_{C\text{-tor}}, \mathcal{A}_{C\text{-tf}})$ and condition (3) of Proposition 15.10 holds. Then the same condition holds for all objects E, not necessarily C-torsion free, with $E_K \in \mathcal{A}_K$ being Z_K-semistable.*

Proof. Let $F \hookrightarrow E_{C\text{-tf}}$ be a semistable subobject as given by assumption (3). We then apply Lemma 6.8 to the isomorphism $(E_{C\text{-tf}})_K \cong E_K$ and obtain an inclusion

$$F \otimes g^* L^{-k} \hookrightarrow E_{C\text{-tf}} \otimes g^* L^{-k} \hookrightarrow E. \qquad \square$$

Let $(Z_K, Z_{C\text{-tor}})$ be a (weak) stability function on $\mathcal{A}_C$ over C. We begin with more definitions:

Definition 15.12. We write

$$\begin{aligned}\mu_C^+(E) &:= \sup\{\mu_C(F) \,|\, 0 \neq F \subseteq E\}\,,\\ \mu_C^-(E) &:= \inf\{\mu_C(Q) \,|\, F \twoheadrightarrow Q \neq 0\}\,.\end{aligned}$$

Definition 15.13. Assume $E \in \mathcal{A}_C$ is not Z_C-semistable. A *maximal destabilizing subobject* (mds) of $E \in \mathcal{A}_C$ is a Z_C-semistable subobject $M \hookrightarrow E$ such that for all $F \hookrightarrow E$ we have $\mu_C(F) \leqslant \mu_C(M)$, and $\mu_C\big(F/(F \cap M)\big) < \mu_C(M)$ whenever F is not a subobject of M.

This is equivalent to M being Z_C-semistable with $\mu_C(M) > \mu_C(F')$ for all $F' \hookrightarrow E/M$; in other words, M is the first step of the HN filtration if it exists.

Proof of Proposition 15.10. Assume conditions (1)-(3) hold and $\mathcal{A}_C$ has a C-torsion theory. We want to construct a mds $M \hookrightarrow E$. By assumption (2), we may assume that $E_K \in \mathcal{A}_K$ is non-zero. By assumption (1), it has a mds with respect to Z_K. Arguing as in the proof of Lemma 15.7, we may assume it is of the form $N_K \subset E_K$ for some subobject $N \subset E$ in $\mathcal{A}_C$.

Since $\mathcal{A}_C$ has a C-torsion theory, we may assume that $N \hookrightarrow E$ is saturated, i.e., that E/N is C-torsion free; in particular

$$E_{C\text{-tor}} = N_{C\text{-tor}}.$$

Write $\mu := \mu_C(N) = \mu_K^+(E_K)$. We distinguish two cases:

(A) $E_{C\text{-tor}} \neq 0$, and $\mu_C^+(E_{C\text{-tor}}) > \mu$.

In this case, we apply assumption (2) and let $M \subset E_{C\text{-tor}}$ be the mds in $\mathcal{A}_{C\text{-tor}}$; in particular $\mu_C(M) = \mu_C^+(E_{C\text{-tor}})$. We claim that M is an mds of E in $\mathcal{A}_C$. Indeed, given any subobject $F \hookrightarrow E$, either $\mu_C(F) \leqslant \mu < \mu_C(M)$ or $\mu_C(F) > \mu$. In the second case, since $\mu = \mu_K^+(E_K)$ it follows that F is C-torsion and factors through $E_{C\text{-tor}} \subset E$, and hence $\mu_C(F) \leqslant \mu_C(M)$ by the choice of M. Thus $\mu_C(F) \leqslant \mu_C(M)$ for any subobject $F \hookrightarrow E$. Now assume F is not a subobject of M. If F is C-torsion, then $F \subset E_{C\text{-tor}}$, so $\mu_C(F/(F \cap M)) < \mu_C(M)$ follows by the choice of M. If F is not C-torsion, then $F_K = (F/(F \cap M))_K$ and again we find $\mu_C(F/(F \cap M)) = \mu_K(F_K) \leqslant \mu < \mu_C(M)$.

(B) $E_{C\text{-tor}} = 0$, or $E_{C\text{-tor}} \neq 0$ with $\mu_C^+(E_{C\text{-tor}}) \leqslant \mu$.

Let $F \subset N$ be the Z_C-semistable subobject given by assumption (3) or Lemma 15.11, respectively. By the existence of HN filtrations in $\mathcal{A}_{C\text{-tor}}$, and the resulting torsion pair as in (14.1), there exists a subobject $F \subseteq M \subseteq N$ such that $\mu_C^-(M/F) \geqslant \mu > \mu_C^+(N/M)$. We claim that M is an mds for E.

Note that $M_K = N_K$ is Z_K-semistable. Thus, if M is not semistable, and if $A \subset M$ is a subobject with $\mu_C(A) > \mu_C(M/A)$, then either A is C-torsion with $\mu_C(A) > \mu$, or M/A is C-torsion with $\mu > \mu_C(M/A)$. The former case is impossible, as A would be a subobject of $N_{C\text{-tor}} = E_{C\text{-tor}}$ and contradict the assumption on $\mu_C^+(E_{C\text{-tor}})$. In the latter case, consider the short exact sequence

$$F/(F \cap A) \hookrightarrow M/A \twoheadrightarrow M/(F + A).$$

We have $\mu_C(F/(F \cap A)) \geqslant \mu$ by semistability of F, and $\mu_C(M/(F+A)) \geqslant \mu_C^-(M/F) \geqslant \mu$ by construction; thus we get a contradiction to the see-saw property.

Since E/N is C-torsion free by assumption, any subobject $F \hookrightarrow E/N$ satisfies $\mu_C(F) = \mu_K(F_K) \leqslant \mu_K^+\left((E/N)_K\right) < \mu$; so $\mu_C^+(E/N) < \mu$. Combined with the inequality $\mu_C^+(N/M) < \mu$, this shows $\mu_C^+(E/M) < \mu$.

We have thus produced a mds for E. To conclude, we need to show that if we replace E by E/M and repeat the above procedure, the process terminates. Indeed, in case (A) we preserve E_K and reduce the length of the HN filtration of $E_{C\text{-tor}}$, and in case (B) we reduce the length of the HN filtration of E_K. □

We will often need $\mathcal{A}^0_{C\text{-tor}}$ and $\mathcal{A}^0_K$ to be noetherian torsion subcategories, see Definition 14.6. For example, the following observation extends Lemma 12.12.

Proposition 15.14. *Assume that the heart $\mathcal{A}_C$ has a C-torsion theory, and that $\mathcal{A}^0_{C\text{-tor}} \subset \mathcal{A}_{C\text{-tor}}$ and $\mathcal{A}^0_K \subset \mathcal{A}_K$ are noetherian torsion subcategories. If there exists a weak HN structure $\sigma_C = (\mathcal{A}_C, Z_K, Z_{C\text{-tor}})$ over C with heart $\mathcal{A}_C$ such that $Z_{C\text{-tor}}$ has discrete image, then $\mathcal{A}_C$ is noetherian.*

Proof. Let $E_1 \twoheadrightarrow E_2 \cdots$ be an infinite sequence of surjections in $\mathcal{A}_C$. Applying Lemma 14.8 to $(\mathcal{A}_K, Z_K)$, we see that the induced sequence of surjections $(E_1)_K \twoheadrightarrow (E_2)_K \twoheadrightarrow \dots$ stabilizes, in other words we may assume that the kernel of every surjection $E_i \twoheadrightarrow E_{i+1}$ is C-torsion. However, by assumption and Lemma 14.8 applied to $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ we know that $\mathcal{A}_{C\text{-tor}} \subset \mathcal{A}_C$ is a noetherian torsion subcategory; therefore, see Remark 14.7, this sequence terminates. □

16. Semistable reduction

The aim of this section is to study in detail condition (3) in Proposition 13.16 or 15.10. We first show that Langton and Maruyama's semistable reduction also works in our context. Then condition (3) follows if semistability satisfies generic openness (see Definition 16.3). This gives us the first example of an HN structure, namely coherent sheaves.

16.1. **The Langton-Maruyama Theorem.** We now state the main result of this section. It is the analogue of [HL10, Theorem 2.B.1] (see [Yos99, Lemma 3.4] or [HMS09, Lemma 2.5]); the proof follows the same lines. The main difference is that we use the existence of a locally finite type Quot space to circumvent the use of completed local rings; indeed, our results in Part I do not prove the existence of a heart on base changes to such rings.

Theorem 16.1 (Langton-Maruyama)**.** *Let $(Z_K, Z_{C\text{-tor}})$ be a (weak) stability function on $\mathcal{A}_C$ over C such that:*

(0) *Given a Dedekind scheme D, a dominant map $f\colon D \to C$ essentially of finite type, a closed point $q \in D$ and $F \in \mathcal{D}_D$, we have*
$$\frac{1}{\dim_{\kappa(f(q))} \kappa(q)} Z_{f(q)}(f_* F_q) = \frac{1}{\dim_K K(D)} Z_K\left(f_* F_{K(D)}\right).$$
(1) *$\mathcal{A}_C$ universally satisfies openness of flatness.*
(2) *$(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ defines a (weak) stability condition on $\mathcal{D}_{C\text{-tor}}$.*
(3) *If $(Z_K, Z_{C\text{-tor}})$ is only a weak stability function, we also assume that $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ has the tilting property.*

Let $E \in \mathcal{A}_C$ be a C-flat object whose restriction $E_K \in \mathcal{A}_K$ to the generic fiber is Z_K-semistable. If there is a closed point $p \in C$ and a quotient $i_{p*}E_p \twoheadrightarrow Q$ with $\mu_C(Q) < \mu_C(i_{p*}E_p)$, then there exists a subobject $E' \subset E$ such that $E/E' \in \mathcal{A}_{C\text{-tor}}$ is supported over p, $\mu_C(E') > \mu_C(E/E')$, and $\mu_C(Q') \geqslant \mu_C(i_{p*}E'_p)$ for all quotients $i_{p*}E'_p \twoheadrightarrow Q'$.

Assumption (0) is a weak version of *universal local constancy of central charges*, see Definition 20.1.(1), which we study later in the context of families of stability conditions. Openness of flatness in assumption (1) refers to Definition 10.4 for the collection of fiberwise t-structures induced by $\mathcal{A}_C$. We remind the reader that under this assumption, Quot spaces are locally of finite type over C by Proposition 11.6. We will explore the support property for $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$, part of assumption (2), in more detail in Section 18.1.

We denote the local ring at p by R, and write π for a generator of its maximal ideal.

Proof. If the claim of the theorem were not true, we could define a descending filtration

$$\cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_0 := E, \tag{16.1}$$

with E_n/E_{n+1} a C-torsion object supported over p and $\mu_C(E_{n+1}) > \mu_C(E_n/E_{n+1})$, as follows. Let Q^n be the maximal destabilizing quotient for $i_{p*}(E_n)_p$, which exists by condition (2); this means that we can write the last step of the HN filtration as $F^n \hookrightarrow i_{p*}(E_n)_p$ where $Q^n = i_{p*}(E_n)_p/F$ is the associated semistable quotient, and $\mu_C^-(F^n) > \mu_C(Q^n)$. Then E_{n+1} is defined as the kernel of the composition $E_n \twoheadrightarrow i_{p*}(E_n)_p \twoheadrightarrow Q^n$; namely, $E_n/E_{n+1} = Q^n$.

Applying $i_{p*}i_p^*$ to $E_{n+1} \hookrightarrow E_n \twoheadrightarrow Q^n$, taking cohomology, and using Lemma 6.7 for Q^n gives an exact sequence

$$0 \to Q^n \to i_{p*}(E_{n+1})_p \to F^n \to 0. \tag{16.2}$$

Consider the sequence of maps $\phi^n \colon Q^n \hookrightarrow i_{p*}(E_{n+1})_p \twoheadrightarrow Q^{n+1}$, which fit into a sequence of commutative diagrams with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & Q^n \arrow[r] \arrow[d] \arrow[dr, "\phi^n"] & i_{p*}(E_{n+1})_p \arrow[r] \arrow[d] & F^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Im}(\phi^n) \arrow[r] \arrow[d] & Q^{n+1} \arrow[r] \arrow[d] & \mathrm{Coker}(\phi^n) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Suppose that $\phi^n = 0$. Then $F^n \twoheadrightarrow Q^{n+1}$, but by definition $\mu_C^-(F^n) > \mu_C(Q^{n+1})$, so as a consequence of the see-saw property, $\mathrm{Hom}(F^n, Q^{n+1}) = 0$. Hence $\phi^n \neq 0$.

Then as Q^n and Q^{n+1} are both Z_C-semistable, we must have

$$\mu_C(Q^n) \leqslant \mu_C(\mathrm{Im}(\phi^n)) \leqslant \mu_C(Q^{n+1}).$$

By construction, we have $\mu_C(Q^n) < \mu_C(E_{n+1}) = \mu_C(E)$, so

$$\mu_C(Q^1) \leqslant \mu_C(Q^n) < \mu_C(E)$$

for all $n \geqslant 1$. Similarly, we have $0 < \Im Z_C(Q^n) \leqslant \Im Z_C(E)$ for all $n \geqslant 1$, so the central charges $\{Z_C(Q^n)\}$ lie in a bounded subset of $\mathbb{C}$. As the Q^n are all Z_C-semistable, it follows from Condition (2) (support property on $\mathcal{D}_{C\text{-tor}}$) and Remark 12.3 that there are only finitely many values for $Z_C(Q^n)$. In particular, the increasing chain $\mu_C(Q^n)$ must stabilize, so we may assume that $\mu_C(Q^n) = \mu_C(Q^{n+1})$ from the outset.

We would like to show that the values $Z_C(Q^n)$ stabilize as well. To that end, assume that $\Im Z_{C\text{-tor}}(\mathrm{Im}(\phi^n)) < \Im Z_{C\text{-tor}}(Q^{n+1})$, so that $\Im Z_{C\text{-tor}}(\mathrm{Coker}(\phi^n)) > 0$. Then as

$$\mu_C(Q^n) = \mu_C(\mathrm{Im}(\phi^n)) = \mu_C(Q^{n+1})$$

we must have $\mu_C(\mathrm{Coker}(\phi^n))$ equal to the same number by the see-saw property in $\mathcal{A}_{C\text{-tor}}$, see Lemma 15.5. Again, by definition $\mu_C^-(F^n) > \mu_C(Q^{n+1}) = \mu_C(\mathrm{Coker}(\phi^n))$, so as a consequence of the see-saw property, $\mathrm{Hom}(F^n, \mathrm{Coker}(\phi^n)) = 0$. This contradicts the fact that we have a surjection $F^n \twoheadrightarrow \mathrm{Coker}(\phi^n)$ by construction, where $\mathrm{Coker}(\phi^n) \neq 0$ by our assumption $\Im Z_{C\text{-tor}}(\mathrm{Im}(\phi^n)) < \Im Z_{C\text{-tor}}(Q^{n+1})$. Thus we must have

$$\Im Z_{C\text{-tor}}(Q^n) \geqslant \Im Z_{C\text{-tor}}(\mathrm{Im}(\phi^n)) = \Im Z_{C\text{-tor}}(Q^{n+1}) > 0.$$

But as the $Z_C(Q^n)$ take only finitely many values, we see that for $n \gg 0$, $Z_C(Q^n)$ must indeed stabilize.

It follows that $\mathrm{Ker}(\phi^n)$ must satisfy $Z_C(\mathrm{Ker}(\phi^n)) = 0$ for $n \gg 0$. But then $\mathrm{Ker}(\phi^n) = 0$ for $n \gg 0$ by the Z_C-semistability of Q^n. So we may assume from the beginning that we have an ascending chain of Z_C-semistable C-torsion objects

$$Q^1 \subset Q^2 \subset \cdots \subset Q^n \subset Q^{n+1} \subset \cdots$$

with $Z_C(Q^n) = Z_C(Q^{n+1})$ for all $n \geqslant 1$.

Now we have $Q^{n+1}/Q^n \in \mathcal{A}^0_{C\text{-tor}}$ for all $n \geqslant 1$. Since $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ has the tilting property by Condition (3), this sequence stabilizes, see Remark 14.14.

We can conclude now that ϕ^n is an isomorphism for all $n \gg 0$, and so the exact sequence (16.2) splits, i.e., $i_{p*}(E_{n+1})_p \cong F^n \oplus Q^n$. In particular, the objects F^n are also constant. Let us set $F := F^n$ and $Q := Q^n$, for all $n \gg 0$. Up to replacing E with E_n, for $n \gg 0$, we can assume that our filtration (16.1) has the property that $i_{p*}(E_n)_p \cong F \oplus Q$ for all n.

We now restrict to the local ring $R = \mathcal{O}_{C,p}$. Note that by Theorem 5.7 there is an induced heart on the base change $\mathcal{D}_R$, such that the restriction functor $\mathcal{D} \to \mathcal{D}_R$ is t-exact. We abuse notation by still using E and Q to denote the restrictions of these objects to $\mathcal{D}_R$. We also set $G^n := E/E_n$. As $E_{n-1}/E_n = Q$, there exists an exact sequence

$$0 \to Q \to G^n \to G^{n-1} \to 0. \tag{16.3}$$

Also, by construction, we have $\pi E_{n-1} \subset E_n$, so by induction it follows that $\pi^n E \subset E_n$. Therefore, the quotient $E \twoheadrightarrow G^n$ factors through $E \twoheadrightarrow E/\pi^n E \twoheadrightarrow G^n$ so that π^n acts as zero on G^n. By Lemma 6.9 it follows that G^n is the pushforward of an object in $\mathcal{A}_{R/(\pi^n)}$, which we also denote by G^n.

Moreover, it is not difficult to see from the construction that E_{n-1} is the span of E_n and πE_{n-2} (as subobjects of E). By induction, this shows that E_{n-1} is the span of E_n and $\pi^{n-1}E$; hence $\pi^i G^n/\pi^{i+1}G^n = E_i/E_{i+1} = Q$, with isomorphisms of the filtration steps induced by multiplication by π. By Lemma 16.2 below, this shows that G^n is a flat object in $\mathcal{A}_{R/(\pi^n)}$.

By Condition (1) and Proposition 11.6, the Quot space $\mathrm{Quot}_{\mathrm{Spec}(R)}(E)$ is an algebraic space locally of finite type over $\mathrm{Spec}(R)$. The quotients $E/\pi^n E \twoheadrightarrow G^n$ give a compatible system of $R/(\pi^n)$ points of $\mathrm{Quot}_{\mathrm{Spec}(R)}(E)$. It follows that the Quot space admits a point finite over K that has the point $E_p \twoheadrightarrow Q$ as a specialization. Therefore, there exists the spectrum D of a DVR over R, an element $E_D \twoheadrightarrow F$ of $\mathrm{Quot}_{\mathrm{Spec}\,R}(E)(D)$ that on the special fiber $q \in D$ is given by a base change of $E_p \twoheadrightarrow Q$, and such that the composition $D \to \mathrm{Spec}\,R \to C$ is dominant and essentially of finite type. As $E_q \twoheadrightarrow F_q$ is destabilizing, the same holds by condition (0) for $E_K^{\oplus \dim_K K(D)} \twoheadrightarrow f_* F_K$. This contradicts the Z_K-semistability of E_K. □

In the proof, we needed the following standard flatness criterion for modules in our context:

Lemma 16.2. *Let $p \in C$, let R be its local ring, and write $R_n = R/(\pi^n)$ for the ring defining the $(n-1)$-th infinitesimal neighborhood of p. Then an object $B \in \mathcal{A}_{R_n}$ is flat if and only if multiplication by π induces isomorphisms*

$$B/\pi B \cong \pi B/\pi^2 B \cong \ldots \cong \pi^{n-1} B.$$

Proof. Let $\mathcal{X}_{R_n}$ be the base change of $\mathcal{X}$ to $\mathrm{Spec}(R_n)$, and write j for the inclusion $\mathcal{X}_p \hookrightarrow \mathcal{X}_{R_n}$. Since j_* is exact, it is enough to test whether $j_* j^* B = B \otimes R_n/(\pi) \in \mathcal{A}_{R_n}$. Using the (2-periodic) minimal resolution $R_n/\pi = \ldots R_n \xrightarrow{\cdot \pi^{n-1}} R_n \xrightarrow{\cdot \pi} R_n$ and its naive truncation at $\leqslant -2$, we obtain an exact triangle

$$j_* j^* B[1] \to \big(B \xrightarrow{\pi} B\big) \to j_* j^* B.$$

Since j^* is right-exact, it follows that $\mathrm{H}^{-1}_{\mathcal{A}_{R_n}}(j_* j^* B) = 0$ if and only if the map $B/\pi \xrightarrow{\cdot \pi^{n-1}} \mathrm{Ker}\, \pi$ is surjective, and $\mathrm{H}^{-2}_{\mathcal{A}_{R_n}}(j_* j^* B) = 0$ if and only if it is injective. This is easily seen to be equivalent to the assumption of the lemma. □

In our examples, condition (3) in Theorem 16.1 will follow from a compatibility of the weak stability condition with the duality functor. Then the existence of HN filtrations will be implied by the following generic openness property.

Definition 16.3. A (weak) stability function $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{A}_C$ over C satisfies *generic openness of semistability* if the following condition holds: given $E \in \mathcal{A}_C$ a C-torsion free object such that E_K is Z_K-semistable, there exists a nonempty open subset $U \subset C$ such that for all $p \in U$ and for all quotients $i_{p*} E_p \twoheadrightarrow Q$, we have $\mu_C(Q) \geqslant \mu_C(E)$.

Remark 16.4. If $(Z_K, Z_{C\text{-tor}})$ is a (weak) stability function on $\mathcal{A}_C$ over C which satisfies the HN property, then generic openness of semistability holds. Indeed, if $E \in \mathcal{A}_C$ is C-torsion free such that E_K is Z_K-semistable, then by Proposition 15.10 there is a Z_C-semistable subobject $F \subset E$ with $F_K = E_K$. Then there exists a nonempty open subset $U \subset C$ such that $F_U = E_U$, and the claim follows from Lemma 15.7 applied to F.

Corollary 16.5. *With the same assumptions as in Theorem 16.1, assume further that $\mathcal{A}_C$ has a C-torsion theory, that $(\mathcal{A}_K, Z_K)$ has the HN property, and that generic openness of semistability holds. Then the (weak) stability function $(Z_K, Z_{C\text{-tor}})$ on $\mathcal{A}_C$ over C satisfies the HN property.*

Theorem 17.1 below shows the existence of a C-torsion theory under the above hypotheses.

Proof. We use Proposition 15.10; we only need to verify assumption (3), semistable reduction.

Let $E \in \mathcal{A}_C$ be a C-torsion free object such that E_K is Z_K-semistable. By generic openness of semistability, there exists a non-empty open subset $U \subset C$ such that $i_{p*} E_p$ does not have a destabilizing quotient for all closed points $p \in U$. Hence, we are left with finitely many points in C, to which we apply Theorem 16.1. We thus obtain a subobject $F \hookrightarrow E$ such that $E/F \in \mathcal{A}_{C\text{-tor}}$ and $i_{p*} F_p$ does not have a destabilizing quotient for all closed points $p \in C$. By Lemma 15.7, F is Z_C-semistable, as we wanted. □

16.2. **An example: coherent sheaves.** Let us consider the stability function $(Z_K, Z_{C\text{-tor}})$ on $\operatorname{Coh}\mathcal{X}$ given by slope-stability on the fibers as in Example 15.3. That is, given a flat morphism $g\colon \mathcal{X} \to C$ of relative dimension n and $\mathcal{O}_{\mathcal{X}}(1)$ a relative polarization, we define

$$Z_{C\text{-tor}}(E) := \mathfrak{i}p_n(E) - p_{n-1}(E),$$

where p_n and p_{n-1} are the two leading coefficients of the Hilbert polynomial of E, defined as in Example 13.6. We define Z_K similarly over the generic fiber. As $\mathcal{O}_{\mathcal{X}}(1)$ is globally defined, $(Z_K, Z_{C\text{-tor}})$ defines a weak stability function for $\operatorname{Coh}\mathcal{X}$ over C. Here the subcategory $(\operatorname{Coh}\mathcal{X})_{C\text{-tor}}$ consists of sheaves supported on fibers of g and $(\operatorname{Coh}\mathcal{X})_{C\text{-tf}}$ consists of flat families of sheaves on the fibers of g. Moreover, it is clear that a Z_C-semistable sheaf $E \in (\operatorname{Coh}\mathcal{X})_{C\text{-tor}}$ is just a sheaf supported on fibers of g that is μ-semistable in the classical sense [HL10, Definition 1.2.12]. It follows that $E \in (\operatorname{Coh}\mathcal{X})_{C\text{-tf}}$ is Z_C-semistable if and only if E is a family of μ-semistable sheaves over C in the classical sense.

Proposition 16.6. *If the fibers of $g\colon \mathcal{X} \to C$ are normal, integral, noetherian schemes, then the weak stability function $(Z_K, Z_{C\text{-tor}})$ on $\operatorname{Coh}\mathcal{X}$ satisfies the HN property.*

Proof. We use Corollary 16.5, for which we must first verify the assumptions of Theorem 16.1. Assumptions (0) and (1) are standard. As $\operatorname{Coh}\mathcal{X}$ is noetherian, it admits a C-torsion theory by Remark 6.16, and of course the subcategory $(\operatorname{Coh}\mathcal{X})^0_{C\text{-tor}}$, which consists of sheaves supported in codimension at least 2 in fibers, is also noetherian. The existence of HN filtrations for objects in $(\operatorname{Coh}\mathcal{X})_{C\text{-tor}}$ or $\operatorname{Coh}\mathcal{X}_K$ follows exactly as in the construction of HN filtrations for classical slope-stability, see [HL10, Theorem 1.6.7]. For the support property, we may simply choose $Q = 0$ and Λ as in [PT19, Section 3.2], which verifies assumption (2).

Next we verify assumption (3). Take $E \in (\operatorname{Coh}\mathcal{X})_{C\text{-tor}}$ with $\mu^+_C(E) < +\infty$, and let us assume without loss of generality that $E = i_{p*}E'$ for $E' \in \operatorname{Coh}\mathcal{X}_p$. The condition $\mu^+_C(E) < \infty$ means that E is a torsion free sheaf on $\mathcal{X}_p$, so $E \hookrightarrow E^{\vee\vee}$ by the integrality of $\mathcal{X}_p$, where $E^{\vee\vee} := \mathcal{H}om(\mathcal{H}om(E, \mathcal{O}_{\mathcal{X}_p}), \mathcal{O}_{\mathcal{X}_p})$. By the normality of $\mathcal{X}_p$, we get that $E^{\vee\vee}$ is reflexive and the quotient $E^{\vee\vee}/E$ is supported on $\mathcal{X}_p$ in codimension at least 2. Thus $E^{\vee\vee}/E \in (\operatorname{Coh}\mathcal{X})^0_{C\text{-tor}}$. The reflexivity of $E^{\vee\vee}$ gives the vanishing of $\operatorname{Hom}((\operatorname{Coh}\mathcal{X})^0_{C\text{-tor}}, E^{\vee\vee}[1])$, finishing the verification of assumption (3).

Finally, as classical slope-stability satisfies generic openness of semistability by [HL10, Proposition 2.3.1], the proposition follows from Corollary 16.5. □

17. Torsion theories and Harder–Narasimhan structures

17.1. **A C-torsion theory via semistable reduction.** The goal of this section is to show that the existence of a C-torsion theory is automatic in our setting; our proof is similar to that of semistable reduction, Theorem 16.1. This gives a partial converse to Proposition 6.19, which shows that the existence of a C-torsion theory implies openness of flatness.

Theorem 17.1. *Let $\mathcal{A}_C$ be heart of a C-local t-structure on $\mathcal{D}$, with the following assumptions:*

(1) *$\mathcal{A}_C$ universally satisfies openness of flatness.*
(2) *For every closed point $p \in C$, there exists a (weak) stability condition $\sigma_p = (\mathcal{A}_p, Z_p)$; in the weak case, we also assume that $\mathcal{A}^0_p \subset \mathcal{A}_p$ is a noetherian torsion subcategory.*

Then $\mathcal{A}_C$ admits a C-torsion theory.

The crucial ingredient will be the following claim:

Lemma 17.2. *With the same assumptions as in Theorem 17.1, if $p \in C$ and $E \in \mathcal{A}_C$, then there exists $n > 0$ such that $I_p^n \cdot E$ has no torsion subobject supported over p.*

Proof of Theorem 17.1. Consider $E \in \mathcal{A}_C$. Since $\mathcal{A}_C$ satisfies openness of flatness, and since $E_K \in \mathcal{A}_K$, there exists a finite set of closed points $p_1, \dots, p_m$ such that E is flat on the complement $C \setminus \{p_1, \dots, p_m\}$. For each i, let n_i be such that $I_{p_i}^{n_i} \cdot E$ has no torsion supported over p_i as in Lemma 17.2. Since $I_{p_i}^{n_i} \cdot E$ and E are isomorphic on the complement of p_i, one sees easily by induction that $\left(I_{p_1}^{n_1} \cdot \dots \cdot I_{p_m}^{n_m} \cdot E\right) \otimes I_{p_1}^{-n_1} \otimes \dots \otimes I_{p_m}^{-n_m}$ is a C-torsion free quotient of E whose kernel is C-torsion. □

We now turn to the proof of Lemma 17.2. Let π be a local generator of I_p around p; it acts on any torsion object supported over p. Consider the exact sequence

$$I_p^{i+1} \cdot E \hookrightarrow I_p^i \cdot E \twoheadrightarrow I_p^i \cdot E / I_p^{i+1} \cdot E$$

for $i \geqslant 0$. Applying $i_{p*} i_p^*$, taking cohomology with respect to $\mathcal{A}_C$, and using Lemmas 6.7 and 6.9 gives a four term exact sequence

$$0 \to \mathrm{Ann}_\pi\left(I_p^{i+1} \cdot E\right) \xrightarrow{\alpha_i} \mathrm{Ann}_\pi(I_p^i \cdot E) \xrightarrow{\beta_i} I_p^i \cdot E / I_p^{i+1} \cdot E \xrightarrow{\gamma_i} I_p^{i+1} \cdot E / I_p^{i+2} \cdot E \to 0 \tag{17.1}$$

where we used the abbreviation $\mathrm{Ann}_\pi(F) = \mathrm{Ann}(I_p; F)$. In particular, there is a sequence of surjections

$$E / I_p \cdot E \xrightarrow{\gamma_0} I_p \cdot E / I_p^2 \cdot E \xrightarrow{\gamma_1} I_p^2 \cdot E / I_p^3 \cdot E \xrightarrow{\gamma_2} \cdots \tag{17.2}$$

Our key claim is the following:

Lemma 17.3. *The sequence* (17.2) *terminates.*

Proof. For $i \geqslant 0$ we define

$$\begin{aligned} F_i &= \mathrm{Im}\left(\mathrm{Ann}_\pi(I_p^i \cdot E) \xrightarrow{\beta_i} I_p^i \cdot E / I_p^{i+1} \cdot E\right), \\ Q_i &= \mathrm{Coker}\left(\mathrm{Ann}_\pi(I_p^{i+1} \cdot E) \xrightarrow{\alpha_0 \circ \dots \circ \alpha_i} \mathrm{Ann}_\pi(E)\right), \\ K_i &= \mathrm{Ker}\left(E / I_p \cdot E \xrightarrow{\gamma_i \circ \dots \circ \gamma_0} I_p^{i+1} \cdot E / I_p^{i+2} \cdot E\right). \end{aligned}$$

Note that by construction, all of these objects are scheme-theoretically supported over p, and hence may be regarded as objects of $\mathcal{A}_p$. Since $\mathrm{Ann}_\pi(E) = i_{p*} \mathrm{H}^{-1}_{\mathcal{A}_p}(E_p)$ by Lemma 6.9, the slope of Q_i is bounded below by $\mu_p(Q_i) \geqslant \mu_p^-(\mathrm{H}^{-1}_{\mathcal{A}_p}(E_p))$. On the other hand, we have $Q_0 = F_0 = K_0$ and short exact sequences

$$K_{i-1} \hookrightarrow K_i \twoheadrightarrow F_i \quad \text{and} \quad F_i \hookrightarrow Q_i \twoheadrightarrow Q_{i-1}.$$

By induction it follows that K_i and Q_i have the same class $K(\mathcal{D}_p)$, and therefore the slope $\mu_p(K_i) = \mu_p(Q_i)$ satisfies the same bound. Thus our claim follows from Lemma 14.11. □

Proof of Lemma 17.2. Let $R_n = \mathcal{O}_C / I_p^n$ be the n-th infinitesimal neighborhood of p. By the previous lemma, after replacing E if necessary, we may assume that the sequence (17.2) is a sequence of isomorphisms; therefore, we have $\mathrm{Ann}_\pi(I_p^i \cdot E) = \mathrm{Ann}_\pi(E)$ for all i. By Lemma 16.2 this means that $F_n := \mathrm{Ann}_{\pi^n}(E) \in \mathcal{A}_{R_n}$ is a flat object over R_n, which we assume to be non-zero for contradiction.

Therefore, we get a sequence of compatible morphisms $\mathrm{Spec}(R_n) \to \mathcal{M}_{\underline{\tau}}$, where $\mathcal{M}_{\underline{\tau}}$ is the functor of flat objects with respect to the fiberwise collection of t-structures induced by $\mathcal{A}_C$. By Lemma 11.2, $\mathcal{M}_{\underline{\tau}}$ is an algebraic stack which is locally of finite type over C. By

Artin approximation, there exists a Dedekind domain $h\colon D \to \mathcal{M}_{\underline{\tau}}$ of finite type over C, together with a point $q \in D$, such that $D \to C$ is dominant, maps q to p and is étale at q, and such that h is induced by F_n in the n-th infinitesimal neighborhood of q, which we can identify with $\operatorname{Spec} R_n$. Let F be the object corresponding to h. We consider the object $\mathcal{H}om_D(F, E_D) \in \mathrm{D}^{\mathrm{b}}(D)$ given by the relative derived sheaf Hom. Then we have

$$\mathrm{H}^{-1}\left(K(D) \otimes \mathcal{H}om_D(F, E_D)\right) = \operatorname{Hom}(F_{K(D)}, E_{K(D)}[-1]) = 0.$$

On the other hand, we have

$$\begin{aligned}\mathrm{H}^{-1}\left(R_n \otimes \mathcal{H}om_D(F, E_D)\right) &= \operatorname{Hom}(F_n, E_{R_n}[-1]) \\ &= \operatorname{Hom}(F_n, \mathrm{H}^{-1}_{\mathcal{A}_{R_n}}(E_{R_n})) \\ &= \operatorname{Hom}(F_n, F_n)\end{aligned}$$

where the second two equalities follow from Lemma 6.9; in particular, this cohomology sheaf contains R_n as a subsheaf for all $n > 0$. This contradicts the above vanishing. □

17.2. **Harder–Narasimhan structures and torsion theories.** In this section, we show that the existence of a C-torsion theory on the heart descends to the slices of a Harder–Narasimhan structure, and vice versa.

Definition 17.4. We say that a (weak) HN structure $\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$ has a C-torsion theory if for every $\phi \in \mathbb{R}$ the category $\mathcal{P}(\phi)$ admits a torsion pair $(\mathcal{P}(\phi)_{C\text{-tor}}, \mathcal{P}(\phi)_{C\text{-tf}})$ where $\mathcal{P}(\phi)_{C\text{-tor}} = \mathcal{P}(\phi) \cap \mathcal{D}_{C\text{-tor}}$ and $\mathcal{P}(\phi)_{C\text{-tf}} = \mathcal{P}(\phi)^{\perp}_{C\text{-tor}}$.

Note that while $\mathcal{P}(\phi)$ is only a quasi-abelian category, the notion of a torsion pair makes sense: we ask that every $E \in \mathcal{P}(\phi)$ fits into a *strict* short exact sequence $E_{C\text{-tor}} \hookrightarrow E \twoheadrightarrow E_{C\text{-tf}}$, which is just an exact triangle in $\mathcal{D}$ with $E_{C\text{-tor}} \in \mathcal{P}(\phi)_{C\text{-tor}}$ and $E_{C\text{-tf}} \in \mathcal{P}(\phi)_{C\text{-tf}}$.

Lemma 17.5. *Let $\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$ be a (weak) HN structure with associated heart $\mathcal{A}_C$. Assume that $E \in \mathcal{P}(\phi)$ for $\phi \in (0, 1]$, and let $W \subset C$ be a closed subset. Then the objects $\operatorname{Ann}(I_W; E)$ and $I_W \cdot E$ in $\mathcal{A}_C$ are also semistable of phase ϕ.*

Proof. The claim is automatic for $I_W \cdot E$, as it is both a quotient of $I_W \otimes E$ and a subobject of E, which are semistable objects of the same phase.

Now consider $A := \operatorname{Ann}(I_W; E) = i_{W*}\mathrm{H}^{-1}_{\mathcal{A}_W}(E_W)$; by semistability of $I_W \otimes E$ we know $\phi(A) \leqslant \phi$. Semistability of E implies that $\phi(E/I_W \cdot E) \geqslant \phi$. On the other hand, we have

$$Z_{C\text{-tor}}(E/I_W \cdot E) - Z_{C\text{-tor}}(A) = Z_{C\text{-tor}}(E_W) = \operatorname{length}(W) \cdot Z_C(E) \in \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}.$$

This is only possible if both inequalities are equalities. So $A \subset I_W \otimes E$ is an inclusion of objects of the same phase, with the latter being semistable; therefore, A is also semistable. □

Proposition 17.6. *Let $\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$ be a (weak) HN structure with associated C-local heart $\mathcal{A}_C$. Then σ_C has a C-torsion theory if and only if $\mathcal{A}_C$ has a C-torsion theory.*

Proof. Assume that σ_C has a C-torsion theory. We first claim that any object $E \in \mathcal{P}(\phi)_{C\text{-tf}}$ is also torsion free as an object in $\mathcal{A}_C$. Indeed, if W is the schematic support of a torsion subobject of E, then $\operatorname{Ann}(I_W; E)$ would be a subobject in $\mathcal{P}(\phi)_{C\text{-tor}}$ by Lemma 17.5. Combined with Lemma 6.17 and the existence of HN filtrations, this shows that every object in $\mathcal{A}_C$ has a maximal C-torsion subobject.

Conversely, assume that $\mathcal{A}_C$ has a C-torsion theory. If $E \in \mathcal{P}(\phi)$ for $\phi \in (0,1]$, then $E_{C\text{-tor}} = \operatorname{Ann}(I_W; E)$ for W the schematic support of $E_{C\text{-tor}}$ by Lemma 6.18. Thus $E_{C\text{-tor}}$ and $E_{C\text{-tf}}$ are also objects of $\mathcal{P}(\phi)$ by Lemma 17.5. This verifies the condition defining the existence of a C-torsion theory for σ_C for $\phi \in (0,1]$, which clearly implies the condition holds for all $\phi \in \mathbb{R}$. □

18. Harder–Narasimhan structures via stability conditions on fibers

18.1. **Support property for HN structures.** Condition (2) of Theorem 16.1 included in particular the assumption that $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ satisfies the support property. In this subsection, we briefly explore the appropriate lattices adapted to the support property on $\mathcal{D}_{C\text{-tor}}$.

Definition 18.1. A *Mukai homomorphism* on $\mathcal{D}$ over C with respect to Λ is a pair $(v_K, v_{C\text{-tor}})$ where

$$v_K \colon K(\mathcal{D}_K) \to \Lambda \quad \text{and} \quad v_{C\text{-tor}} \colon K(\mathcal{D}_{C\text{-tor}}) \to \Lambda$$

are group homomorphisms with the following property: for all $E \in \mathcal{D}$, and all proper closed subschemes $W \subset C$, we have

$$v_K(E_K) = \frac{1}{\operatorname{length} W} v_{C\text{-tor}}\left(i_{W*}E_W\right). \tag{18.1}$$

Notice that (18.1) can be equivalently stated as $v_K(E_K) = v_{C\text{-tor}}\left(i_{p*}E_p\right)$, for all closed point $p \in C$.

Remark 18.2. Given a Mukai homomorphism on $\mathcal{D}$ over C with respect to Λ and a group homomorphism $Z \colon \Lambda \to \mathbb{C}$, we obtain a central charge on $\mathcal{D}$ over C by setting $Z_K = Z \circ v_K$ and $Z_{C\text{-tor}} = Z \circ v_{C\text{-tor}}$.

We denote by $v_p := v_{C\text{-tor}} \circ i_{p*} \colon K(\mathcal{D}_p) \to \Lambda$; thus we have defined $v_c \colon K(\mathcal{D}_c) \to \Lambda$ for every (closed or non-closed) point $c \in C$. The following observation is immediate from the fact that $K(\mathcal{D}_{C\text{-tor}})$ is the direct sum of $i_{p*}(K(\mathcal{D}_p))$ over all closed points $p \in C$.

Lemma 18.3. *To give a Mukai homomorphism is equivalent to giving a collection of homomorphisms $v_c \colon K(\mathcal{D}_c) \to \Lambda$ such that for $E \in \mathcal{D}$, the vector $v_c(E_c)$ is independent of $c \in C$.*

Definition 18.4. We say that a Mukai homomorphism is *numerical on fibers* if v_c factors via $K(\mathcal{D}_c) \to K_{\text{num}}(\mathcal{D}_c)$ for all $c \in C$.

We will discuss Mukai homomorphisms more systematically in Section 21.1.

Definition 18.5. Fix a Mukai homomorphism $(v_K, v_{C\text{-tor}})$ with lattice Λ. A (weak) HN structure satisfies the support property with respect to a quadratic form Q on $\Lambda_{\mathbb{R}}$ if $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ satisfies the support property with respect to $v_{C\text{-tor}}$ and Q.

Recall Lemmas 13.11 and 15.6, which associate to a (weak) HN structure σ_C a (weak) pre-stability condition σ_c on $\mathcal{D}_c$ for all $c \in C$. The support property for σ_C is equivalent to the collection $(\sigma_c)_{c \in C}$ satisfying the support property with respect to a uniform quadratic form:

Lemma 18.6. *Let σ_C be a (weak) HN structure with a C-torsion theory. Then σ_C satisfies the support property with respect to a quadratic form Q if and only if for every $c \in C$, or equivalently for every closed point $c \in C$, the induced (weak) pre-stability condition σ_c satisfies the support property with respect to Q and v_c.*

Proof. Assume σ_C satisfies the support property. The claim is automatic for closed points p from the construction, as the inclusion $i_{p*}\colon \mathcal{A}_p \to \mathcal{A}_{C\text{-tor}}$ preserves semistable objects. As for the generic point K, by the existence of a C-torsion theory, every semistable object $E_K \in \mathcal{A}_K$ lifts to a torsion free object $E \in \mathcal{A}_C$. By Proposition 15.10, we may assume E is Z_C-semistable. If σ_C is a HN structure, then $i_{p*}E_p$ is semistable by Lemma 13.12, and thus $Q(v_K(E_K)) = Q(v_p(E_p)) \geqslant 0$. In the weak case, we deduce from Lemma 15.7 that the HN filtration of E_p is of the form $E_p^0 \hookrightarrow E_p \twoheadrightarrow E_p/E_p^0$ with $E_p^0 \in \mathcal{A}_p^0$. By Remark 14.9, it follows that $Q(v_K(E_K)) = Q(v_p(E_p)) = Q(v_p(E_p/E_p^0)) \geqslant 0$.

Conversely, assume that the σ_p uniformly satisfy the support property for all closed points $p \in C$. Every indecomposable semistable object $E \in \mathcal{A}_{C\text{-tor}}$ is set-theoretically supported over a point $p \in C$; we consider the filtration $0 = \pi^{n+1}\cdot E \subset \pi^n \cdot E \subset \cdots \subset \pi \cdot E \subset E$ given by Lemma 6.11. Since all filtration quotients $\pi^i \cdot E/\pi^{i+1}\cdot E \in \mathcal{A}_p$ are also quotients of $E/\pi\cdot E$ of the same the phase, they do not have destabilizing quotients. Hence $Q(v_{C\text{-tor}}(E)) \geqslant 0$ follows from the simple linear algebra argument in [BMS16, Lemma A.6]. The same linear algebra argument gives the result for the general case of a decomposable E. □

18.2. **HN structures via stability conditions on fibers.** To conclude our investigation of HN structures, we show that under appropriate assumptions, they can be constructed via stability conditions on fibers.

Theorem 18.7. *Let $\mathcal{A}_C$ be a C-local heart in $\mathcal{D}$ that universally satisfies openness of flatness. Fix a group homomorphism $Z\colon \Lambda \to \mathbb{C}$, a quadratic form Q on $\Lambda_{\mathbb{R}}$, and a Mukai homomorphism $(v_K, v_{C\text{-tor}})$. Assume that the induced pair $(Z_K, Z_{C\text{-tor}})$ satisfies condition (0) of Theorem 16.1. Then the triple $\sigma_C = (\mathcal{A}_C, Z_K, Z_{C\text{-tor}}) = (\mathcal{A}_C, Z\circ v_K, Z\circ v_{C\text{-tor}})$ is a HN structure with a C-torsion theory and satisfying the support property with respect to Q if and only if*

(1) *$\sigma_c = (\mathcal{A}_c, Z\circ v_c)$ gives a pre-stability condition on $\mathcal{D}_c$ for all (closed or non-closed) $c \in C$;*
(2) *all the σ_c satisfy the support property with respect to Q; and*
(3) *generic openness of semistability holds (Definition 16.3).*

Similarly, if σ_C is a weak HN structure, then (1)-(3) hold. Conversely, if, in addition to (1)-(3), we also have

(4) *σ_c has the tilting property (Definition 14.12),*

then σ_C is a weak HN structure with a C-torsion theory and satisfying the support property with respect to Q. Moreover, σ_C induces the stability conditions σ_c.

Proof. The proof is divided in three steps.

Step 1. (Weak) HN structure with support property and a C-torsion theory $\Rightarrow$ (weak) stability conditions on the fibers satisfying (1)-(3).

We have already seen, in Lemmas 13.11 (resp. 15.6) and 18.6 that a HN structure (resp. a weak HN structure) σ_C induces on the fibers stability conditions (resp. weak stability conditions) σ_c that satisfy the support property with respect to Q. This gives conditions (1) and (2). Condition (3), generic openness of semistability, holds by Remark 16.4.

Step 2. Stability conditions on the fibers satisfying (1)-(3) $\Rightarrow$ HN structure with a C-torsion theory and satisfying the support property

Now assume that the $\sigma_c = (\mathcal{A}_c, Z \circ v_c)$ are stability conditions on the fibers, satisfying the support property with respect to a uniform quadratic form Q, such that generic openness of semistability holds. By Theorem 17.1, the heart $\mathcal{A}_C$ has a C-torsion theory, and we will use Corollary 16.5 to prove that σ_C is a HN structure. Then it follows that σ_C has a C-torsion theory by Proposition 17.6 and that σ_C satisfies the support property with respect to Q by Lemma 18.6, as required.

We first observe that $(Z_K, Z_{C\text{-tor}})$ is a stability function on $\mathcal{A}_C$ over C. Indeed, Z_K is a stability function on $\mathcal{A}_K$ by assumption, and by decomposing $E \in \mathcal{A}_{C\text{-tor}}$ according to its set-theoretic support and then using the filtration of Lemma 6.11, it follows that $Z_{C\text{-tor}}$ is a stability function on $\mathcal{A}_{C\text{-tor}}$ from the fact that Z_p is a stability function on $\mathcal{A}_p$. Moreover, by assumption Z_K has the HN property on $\mathcal{A}_K$.

Our next claim is that $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ has the HN property so that assumption (2) of Theorem 16.1 is satisfied by Lemma 12.6. Given an object $E \in \mathcal{A}_{C\text{-tor}}$, it is not difficult to see that by considering the decomposition of E according to its set-theoretic support, which is a direct sum of objects set-theoretically supported over distinct closed points, it suffices to show that HN filtrations exist for objects set-theoretically supported over a single closed point $p \in C$. Given such an E, we proceed by induction on the length of its schematic support in C. Let π be a local generator of I_p. Since both $(\pi \cdot E) \otimes I_p^{-1}$ and $E/\pi \cdot E$ have a HN filtration by the induction assumption, and since both are quotients of E, one can construct a $Z_{C\text{-tor}}$-semistable quotient $E \twoheadrightarrow Q^0$ of phase $\phi(Q^0) = \min\{\phi^-(E/\pi \cdot E), \phi^-(\pi \cdot E)\}$. It is not difficult to show from the see-saw property, Lemma 15.5, that Q^0 is of minimal phase, i.e., every other quotient $E \twoheadrightarrow Q'$ satisfies $\phi(Q') \geqslant \phi(Q^0)$. Repeating this argument for the kernel E^1, we obtain a $Z_{C\text{-tor}}$-semistable quotient $E^1 \twoheadrightarrow Q^1$ of minimal phase with kernel E^2. As E/E^2 is itself a quotient of E, so that $\phi(E/E_2) \geqslant \phi(Q^0)$, the see-saw property gives $\phi(E^1/E^2) \geqslant \phi(E/E^1)$. Continuing in this way gives a decreasing filtration $E = E^0 \supset E^1 \supset E^2 \supset \dots$ such that the filtration quotients are $Z_{C\text{-tor}}$-semistable with $\phi(E^0/E^1) \leqslant \phi(E^1/E^2) \leqslant \dots$.

We need to show that this process terminates. Every central charge $Z_{C\text{-tor}}(E^i/E^{i+1})$ of any of the semistable filtration quotients is contained in the parallelogram with adjacent edges of angle $\pi\phi^-(E)$ and π, and with 0 and $Z_{C\text{-tor}}(E)$ as opposite vertices. Since each E^i/E^{i+1} satisfies the support property inequality of σ_p (by the second part of the proof of Lemma 18.6), it follows, see Remark 12.3, that there are only finitely many possible classes $v_{C\text{-tor}}(E^i/E^{i+1})$. Therefore there is an i with $\Im Z_{C\text{-tor}}(E^i/E^{i+1}) = 0$ for all i, which is only possible if E^i itself is semistable with $\Im Z_{C\text{-tor}}(E^i) = 0$, so the process terminates. Grouping the consecutive i with filtration quotients of equal phase gives the HN filtration of E. Therefore, $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ has the HN property.

The remaining assumptions of Theorem 16.1 and Corollary 16.5 are part of our hypotheses, so σ_C is indeed a HN structure as claimed.

Step 3. Weak stability conditions on the fibers satisfying (1)-(4) $\Rightarrow$ *weak HN structure with a C-torsion theory and satisfying the support property*

The structure of the arguments carries over exactly, with assumption (4) needed to ensure that all assumptions of Theorems 16.1 and 17.1 are satisfied. The only additional argument needed is to show that if $\sigma_p = (\mathcal{A}_p, Z_p)$ has the tilting property for all closed points $p \in C$, then the same holds for $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$.

For both conditions in Definition 14.12, it suffices to consider objects set-theoretically supported over a single closed point $p \in C$; we let π be a local generator of its maximal ideal. We first show noetherianity of $\mathcal{A}^0_{C\text{-tor}}$. Given $E \in \mathcal{A}^0_{C\text{-tor}}$ set-theoretically supported over p, consider the filtration induced by π, as in Lemma 6.11. Then any surjection $E \twoheadrightarrow Q$ induces surjections $\pi^i \cdot E/\pi^{i+1} \cdot E \twoheadrightarrow \pi^i \cdot Q/\pi^{i+1} \cdot Q$. Since these are surjections in the noetherian category $\mathcal{A}^0_p$, we easily conclude that any sequence of surjections $E \twoheadrightarrow E_1 \twoheadrightarrow E_2 \cdots$ terminates, i.e., that $\mathcal{A}^0_{C\text{-tor}}$ is noetherian. A similar inductive argument, using the same filtration, shows that $\mathcal{A}^0_{C\text{-tor}}$ is a torsion subcategory.

Next we verify condition (2) of Definition 14.12 for $(\mathcal{A}_{C\text{-tor}}, Z_{C\text{-tor}})$ for $F \in \mathcal{A}_{C\text{-tor}}$, $\mu^+(F) < +\infty$, set-theoretically supported over p. As explained in Remark 14.15, we must show $F[1]$ has a maximal subobject in $\mathcal{A}^0_{C\text{-tor}}$ with respect to the heart $\mathcal{A}^{\sharp\beta}_{C\text{-tor}}$ for $\beta \geqslant \mu^+(F)$. It follows that using the short exact sequence $\pi \cdot F \hookrightarrow F \twoheadrightarrow F/\pi \cdot F$ and induction on the length of the support of F, we can reduce to the case where F is scheme-theoretically supported over p. In this case, $F = i_{p*}E$ and by assumption there exists a short exact sequence $E \hookrightarrow \widetilde{E} \twoheadrightarrow E^0$ in $\mathcal{A}_p$ with $E^0 \in \mathcal{A}^0_p$ and $\mathrm{Hom}(\mathcal{A}^0_p, \widetilde{E}[1]) = 0$. We claim that the pushforward by i_{p*} of this sequence gives the desired exact sequence for F. It suffices to show $\mathrm{Hom}(\mathcal{A}^0_{C\text{-tor}}, i_{p*}\widetilde{E}[1]) = 0$. This reduces to showing $\mathrm{Hom}(i_{p*}T, i_{p*}\widetilde{E}[1]) = 0$ for every $T \in \mathcal{A}^0_p$. Using adjunction and Lemma 6.7, this reduces to showing $\mathrm{Hom}(T, \widetilde{E}) = 0 = \mathrm{Hom}(T, \widetilde{E}[1]) = 0$. The first equality holds since $\mu^+(\widetilde{E}) < +\infty$, and the second holds by our choice of $\widetilde{E}$. □

19. Tilting weak Harder–Narasimhan structures over a curve

Recall from Remark 13.15 that the universal cover $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ of $\mathrm{GL}_2^+(\mathbb{R})$ acts on the set of Harder–Narasimhan structures on $\mathcal{D}$ over C. In terms of hearts and a family of stability functions as in Proposition 13.14, this corresponds to tilting the corresponding heart $\mathcal{A}_C$.

As we already saw in the case of weak stability conditions in Section 14.2, this procedure is much more subtle for weak HN structures, and may not exist in general, due to the special role of objects with central charge 0, which are arbitrarily defined to have phase 1 if they are in the heart. In particular, HN filtrations are *not* preserved under tilting. In this section, we give a criterion to ensure that a weak HN structure can be tilted, extending Proposition 14.16.

Consider a weak HN structure $\sigma_C = (\mathcal{A}_C, Z_K, Z_{C\text{-tor}})$ on $\mathcal{D}$ over C. We write $(\mathcal{T}^\beta_C, \mathcal{F}^\beta_C)$ for the torsion pair defined as in (14.1), with μ replaced by μ_C. The tilted heart $\mathcal{A}^{\sharp\beta}_C := \langle \mathcal{F}^\beta_C[1], \mathcal{T}^\beta_C \rangle$ is C-local, since Z_C-semistability is invariant under tensoring with the pullback of a line bundle from C.

We first need a relative analogue of the tilting property defined in Definition 14.12:

Definition 19.1. Given a weak HN structure $\sigma_C = (\mathcal{A}_C, Z_K, Z_{C\text{-tor}})$, we write $\mathcal{A}^0_C \subset \mathcal{A}_C$ for the subcategory objects E with $Z_C(F) = 0$ for every subquotient F of E. We say that σ_C has the *tilting property* if

(1) $\mathcal{A}^0_C \subset \mathcal{A}_C$ is a noetherian torsion subcategory, and
(2) for every $F \in \mathcal{A}_C$ with $\mu^+_C(F) < +\infty$, there exists a short exact sequence $F \hookrightarrow \widetilde{F} \twoheadrightarrow F^0$ with $F^0 \in \mathcal{A}^0_C$ and $\mathrm{Hom}(\mathcal{A}^0_C, \widetilde{F}[1]) = 0$.

Remark 19.2. Let E be an object with $E_K \neq 0$ but $Z_K(E_K) = 0$, and $F \in \mathcal{A}_p$ for some closed point $p \in C$. Then $Z_C(E \oplus i_{p*}F) = 0$, but $E \oplus i_{p*}F \in \mathcal{A}^0_C$ only holds if $Z_p(F) = 0$.

Remark 19.3. Assume that $\mathcal{A}_C^0$ is a noetherian torsion subcategory, and consider a short exact sequence $0 \to A \to E \to B \to 0$ of objects in $\mathcal{A}_C$ with no morphisms from $\mathcal{A}_C^0$. Then if A, B satisfy the condition in (2), the same holds for E. This is seen most easily by considering the tilt of $\mathcal{A}_C$ at the torsion pair $(\mathcal{A}_C^0, (\mathcal{A}_C^0)^\perp)$: the titled heart contains the objects $A[1], E[1], B[1]$, and we have to show that each of them has a maximal subobject in $\mathcal{A}_C^0$.

Example 19.4. Consider relative slope stability σ_C for a smooth family $\mathcal{X} \to C$ of higher-dimensional varieties as a weak HN structure as in Example 15.3. Then $\mathcal{A}_C^0$ consists of sheaves whose support has codimension $\geqslant 2$ in every fiber, and σ_C has the tilting property. Indeed, by Remark 19.3, it is enough to consider the case where $E \in \operatorname{Coh} \mathcal{X}$ is either torsion free, or the pushforward $E = i_{p*}F$ of a torsionfree sheaf $F \in \operatorname{Coh} \mathcal{X}_p$. In the latter case, we use the double dual of F as in Example 14.13. In the former case, let $E^{\vee\vee}$ be the double dual of E in $\operatorname{Coh} \mathcal{X}$, and let $G \subset E^{\vee\vee}/E$ be the maximal subsheaf whose support has codimension $\geqslant 2$ in every fiber. The preimage $\tilde{E} \subset E^{\vee\vee}$ of G has the desired property.

Proposition 19.5. *Let $\mathcal{A}_C$ be the heart of a C-local t-structure on $\mathcal{D}$ that satisfies universal openness of flatness. Let $\sigma_C = (\mathcal{A}_C, Z_K, Z_{C\text{-tor}})$ be a weak HN structure on $\mathcal{D}$ over C. Assume that:*

(1) *σ_C has the tilting property, and*
(2) *the induced weak stability condition σ_c has the tilting property for every $c \in C$.*

Then $(\mathcal{A}_C^{\sharp\beta})^0 \subset \mathcal{A}_C^{\sharp\beta}$ is a noetherian torsion subcategory, and $\sigma_C^{\sharp\beta} = (\mathcal{A}_C^{\sharp\beta}, \frac{Z_K}{\mathrm{i}-\beta}, \frac{Z_{C\text{-tor}}}{\mathrm{i}-\beta})$ is a weak HN structure.

Proof. The first claim follows just as in the proof of Proposition 14.16.

We need to prove that the rotated central charge, which we denote by $Z_C^{\sharp\beta}$, satisfies the HN property on $\mathcal{A}_C^{\sharp\beta}$. We first observe, in analogy with Lemma 14.17, that $Z_C^{\sharp\beta}$-semistable objects in $\mathcal{A}_C^{\sharp\beta}$ are either

(1) Z_C-semistable objects of $\mathcal{T}_C^{\sharp\beta}$, or
(2) objects E such that $\mathrm{H}^{-1}_{\mathcal{A}_C}(E)$ is a Z_C-semistable object of $\mathcal{F}_C^{\sharp\beta}$ and $\mathrm{H}^0_{\mathcal{A}_C}(E) \in \mathcal{A}_C^0$; in addition, we require $\Im Z_C(E) = 0$ or $\operatorname{Hom}(\mathcal{A}_C^0, E) = 0$.

The HN filtrations of objects in $\mathcal{T}^{\sharp\beta}$ remain unchanged, and thus it remains to consider objects in $\mathcal{F}_C^{\sharp\beta}[1]$. To be able to proceed by induction, we consider slightly more generally an object $E \in \mathcal{A}_C^{\sharp\beta}$ with $\mathrm{H}^0_{\mathcal{A}_C}(E) \in \mathcal{A}_C^0$. Let $F = \mathrm{H}^{-1}_{\mathcal{A}_C}(E)$, and let $G \hookrightarrow F \twoheadrightarrow Q$ be the last step of the HN filtration of F in $\mathcal{A}_C$, with Q being Z_C-semistable. Then $E' = E/G[1]$ is an object where $\mathrm{H}^{-1}_{\mathcal{A}_C}(E')$ is Z_C-semistable and $\mathrm{H}^0_{\mathcal{A}_C}(E') = \mathrm{H}^0_{\mathcal{A}_C}(E) \in \mathcal{A}_C^0$. Replacing E' with the quotient Q' by its maximal subobject in $\mathcal{A}_C^0$, we obtain a semistable quotient of E in $\mathcal{A}_C^{\sharp\beta}$ of the same slope as $Q[1]$. Moreover, the kernel G' of $E \twoheadrightarrow Q'$ will again be an object with $\mathrm{H}^0_{\mathcal{A}_C}(G') \in \mathcal{A}_C^0$, and the length of the HN filtration of $\mathrm{H}^{-1}_{\mathcal{A}_C}(G')$ is smaller than that of F. Therefore, this procedure terminates. □

Remark 19.6. We can now explain why the definition of Z_C-semistability for a C-torsion free object $E \in \mathcal{A}_C$ requires *all* fibers E_p to be semistable, rather than just the general fiber, as e.g. in the definition of *relative slope stability*. Let $E \in \mathcal{A}_C$ be a torsion free object with $\mu_C(E) = \beta$, such that E_K is semistable. Let $p \in C$ be a closed point such that $i_{p*}E_p$ admits a destabilizing short exact sequence $A \hookrightarrow i_{p*}E_p \twoheadrightarrow Q$, with A, Q semistable and $\mu_C(A) > \mu_C(i_{p*}E_p) = \mu_C(E) > \mu_C(Q)$. If E was defined to be Z_C-semistable, then

$E[1] \in \mathcal{A}_C^{\sharp\beta}$. On the other hand, $A \in \mathcal{T}_C^{\beta} \subset \mathcal{A}_C^{\sharp\beta}$; if F denotes the kernel of the surjection $E \otimes I_p^{-1} \twoheadrightarrow i_{p*}E_p \twoheadrightarrow Q$ in $\mathcal{A}_C$, then we would obtain a short exact sequence $A \hookrightarrow E[1] \twoheadrightarrow F[1]$ in $\mathcal{A}_C^{\sharp\beta}$. In other words, despite E being semistable and torsion free in $\mathcal{A}_C$, the shift $E[1]$ would not be torsion free in $\mathcal{A}_C^{\sharp\beta}$. Moreover, the existence of a maximal torsion subsheaf of $E[1]$ is essentially equivalent to semistable reduction for E; thus we have instead built semistable reduction into our basic setup of Z_C-stability.

Part IV. Stability conditions over a higher-dimensional base

In this part of the paper, we introduce a notion of stability conditions for a category $\mathcal{D}$ over a higher-dimensional base S; its key property will be that it comes equipped with relative moduli spaces. Throughout, we work in the following setup.

Setup IV.1. Assume:

- $g\colon \mathcal{X} \to S$ is a flat morphism as in the Main Setup and in addition it is projective;
- $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ is an S-linear strong semiorthogonal component of finite cohomological amplitude.

20. Flat families of fiberwise (weak) stability conditions

In this section we introduce the notion of a flat family of fiberwise (weak) stability conditions, and prove some basic results in this context. In Section 21 we will define the notion of a (weak) stability condition over a base by further imposing a suitable support property.

20.1. **Definitions.** Our goal is to provide a notion of a stability condition over S that is strong enough to yield relative moduli spaces of stable objects, and flexible enough to allow for deformation results, and constructions via Bogomolov–Gieseker type inequalities. To do so, we will consider various compatibility conditions on fiberwise collections of stability conditions.

Definition 20.1. Let $\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s\in S}$ be a collection of (weak) numerical stability conditions on $\mathcal{D}_s$ for every (closed or non-closed) point $s \in S$. In the weak case, we assume that σ_s satisfies the assumptions of Proposition 14.20 for all $s \in S$.

(1) $\underline{\sigma}$ *universally has locally constant central charges* if for every morphism $T \to S$ and every T-perfect object $E \in \mathrm{D}(\mathcal{X}_T)$ such that $E_t \in \mathcal{D}_t$ for all $t \in T$, the function $T \to \mathbb{C}$ given by $t \mapsto Z_t(E_t)$ is locally constant. In this situation, if T is connected we often write $Z(E)$, $\phi(E)$, and $\mu(E)$ for the constant values $Z_t(E_t)$, $\phi_t(E_t)$, and $\mu_t(E_t)$.

(2) $\underline{\sigma}$ *universally satisfies openness of geometric stability* if for every morphism $T \to S$ and every T-perfect object $E \in \mathrm{D}(\mathcal{X}_T)$, the set

$$\{t \in T \mid E_t \in \mathcal{D}_t \text{ and is geometrically } \sigma_t\text{-stable}\}$$

is open.

(3) $\underline{\sigma}$ *universally satisfies openness of lying in* $\mathcal{P}(I)$ for an interval $I \subset \mathbb{R}$ if for every morphism $T \to S$ and every T-perfect object $E \in \mathrm{D}(\mathcal{X}_T)$, the set

$$\{t \in T \mid E_t \in \mathcal{P}_t(I)\}$$

is open.

The (weak) stability conditions σ_t appearing in Definition 20.1 are given by the base change results Theorem 12.17 and Proposition 14.20, and geometric stability is meant in the sense of Definition 12.18. Further, we note that if T is quasi-compact with affine diagonal, then by Lemma 9.3 the condition on $E \in \mathrm{D}(X_T)$ in (1) is equivalent to $E \in \mathcal{D}_T$.

Remark 20.2. In the situation of Definition 20.1, for any $\phi \in \mathbb{R}$ we obtain a fiberwise collection of t-structures given by

$$\mathcal{D}_s^{\leqslant 0} = \mathcal{P}_s(> \phi), \quad \mathcal{D}_s^{\geqslant 0} = \mathcal{P}_s(\leqslant \phi + 1).$$

For $\phi = 0$, we call this the *fiberwise collection of t-structures underlying* $\underline{\sigma}$. Note that for $I = (0, 1]$, universal openness of lying in $\mathcal{P}(I)$ is precisely the condition of universal openness of flatness (in the sense of Definition 10.4) of this fiberwise collection of t-structures.

Lemma 20.3. *Let $\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s\in S}$ be a collection of (weak) numerical stability conditions as in Definition 20.1. Then the properties of universal local constancy of central charges, universal openness of geometric stability, and universal openness of lying in $\mathcal{P}(I)$ for an interval $I \subset \mathbb{R}$ can be checked on morphisms $T \to S$ of finite type from an affine scheme.*

Proof. Using parts (3) and (4) of Theorem 12.17 and the analogous statements in Proposition 14.20, this follows by the argument in the proof of Lemma 10.6. □

We next show that for collections of stability conditions, universal openness of geometric stability implies universal generic openness of semistability in a suitable sense:

Lemma 20.4. *Let $\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s\in S}$ be a collection of numerical stability conditions which universally has locally constant central charges and universally satisfies openness of geometric stability. If $T \to S$ is a morphism from an irreducible, quasi-compact scheme with affine diagonal, and if $E \in \mathcal{D}_T$ is a T-perfect object whose generic fiber $E_{K(T)}$ is $\sigma_{K(T)}$-semistable, then there exists a nonempty open subset $U \subset T$ such that E_t is σ_t-semistable for all $t \in U$.*

Proof. Let $K(T) \subset \overline{K}$ be the algebraic closure, and consider the Jordan–Hölder filtration for $E_{\overline{K}}$. By Proposition 5.9.(3), this filtration is defined over a finitely generated field extension $K(T) \subset L$; let E^i_L be the filtration factors, which are by definition geometrically σ_L-stable. Let $f\colon T' \to T$ be a morphism of finite type such that $L = K(T')$ as field extensions of $K(T)$. By Lemma 4.16.(3), we can lift the JH filtration of E_L to a sequence of morphisms

$$0 = F^0_{T'} \xrightarrow{f_1} F^1_{T'} \xrightarrow{f_2} \cdots \to F^m_{T'} = E_{T'}$$

in $\mathcal{D}_{T'}$ such that $E^i_{T'} := \operatorname{cone}(f_i)$ is a lift of E^i_L to $\mathcal{D}_{T'}$. Moreover, in light of [Lie06a, Proposition 2.2.1], by shrinking T' we may assume that all $F^i_{T'}$ are T'-perfect. By our assumption on universal openness of geometric stability, by further shrinking T' we may assume $E^i_{t'}$ is geometrically $\sigma_{t'}$-stable for all $t' \in T'$ and for all i. Since the central charges are universally locally constant, the objects $E^i_{t'}$ also have the same phase. It follows that $E_{t'}$ is $\sigma_{t'}$-semistable, and hence E_t is σ_t-semistable for all $t \in f(T')$ by Theorem 12.17.(3). Since $f(T')$ contains an open set, this concludes the proof of the lemma. □

The following, combined with Definition 21.15 later on, is the main definition of this paper.

Definition 20.5. A *flat family of fiberwise stability conditions on $\mathcal{D}$ over S* is a collection of numerical stability conditions $\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s\in S}$ on $\mathcal{D}_s$ for every (closed or non-closed) point $s \in S$ such that:

(1) $\underline{\sigma}$ universally has locally constant central charges.
(2) $\underline{\sigma}$ universally satisfies openness of geometric stability.
(3) $\underline{\sigma}$ integrates to a HN structure over any Dedekind scheme $C \to S$ essentially of finite type (see Definition 11.16) over S, i.e., the stability conditions σ_c for $c \in C$ are induced, in the sense of Lemma 13.11, by a HN structure σ_C on $\mathcal{D}_C$ over C.

We define a *flat family of fiberwise weak stability conditions* analogously via weak stability conditions on the fibers, but with some additional assumptions:

(0) For each $s \in S$, the central charge Z_s is defined over $\mathbb{Q}[\mathrm{i}]$, and $\mathcal{A}_s^0 \subset \mathcal{A}_s$ (see Definition 14.3) is a noetherian torsion subcategory (see Definition 14.6).

(2') In addition to universal openness of geometric stability, $\underline{\sigma}$ satisfies the following property: For any morphism $T \to S$ essentially of finite type with T irreducible and any T-perfect $E \in \mathcal{D}_T$ whose generic fiber $E_{K(T)}$ is $\sigma_{K(T)}$-semistable, there exists a nonempty open subset $U \subset T$ such that E_t is σ_t-semistable for all $t \in U$.

(3') For any Dedekind scheme $C \to S$ essentially of finite type over S, the weak stability conditions σ_c for $c \in C$ are induced, in the sense of Lemma 15.6, by a weak HN structure σ_C on $\mathcal{D}_C$ over C with the additional property that $\mathcal{A}_C^0$ (see Definition 19.1) is a noetherian torsion subcategory.

Remark 20.6. Let us elaborate on the conditions appearing in Definition 20.5:

(1) Universal openness of geometric stability in (2) is the main consistency condition relating the slicings for different fibers and it is necessary for the existence of moduli spaces of semistable objects, see Theorem 21.24. In Proposition 20.8 below, we show that a flat family of fiberwise (weak) stability conditions automatically satisfies the other compatibility condition introduced above — universal openness of lying in $\mathcal{P}(I)$ for any $I \subset \mathbb{R}$. In the case of weak stability conditions, we initially interpret condition (2') in the sense of Definition 14.19; this in combination with condition (0) means that the assumptions of Proposition 14.20 are satisfied, and thus the notions of universal openness of geometric stability and lying in $\mathcal{P}(I)$ are indeed well-defined.

(2) By Lemma 20.4 condition (2') automatically holds for a flat family of fiberwise stability conditions, but in the weak case we need to include generic openness of semistability as an extra assumption.

(3) If S is of finite type over a field, or more generally a *Jacobson scheme*, since the closed points of S are dense in every closed subset of S, $\underline{\sigma}$ is determined by σ_s for all closed points s: indeed this follows from universal openness of geometric stability and universal generic openness, once we invoke [Lie06a, Proposition 2.2.1] to lift objects to relatively perfect ones.[4] In particular, from (1) it follows that the central charge at a point s is determined by the central charge at any specialization of s. The existence of central charges on non-closed points is a consistency condition for central charges on closed points, generalizing Definition 13.1. The existence of slicings for non-closed points axiomatizes the classical notion of generic stability and generic HN filtrations. This notion is significantly strengthened by condition (3) when the base is a Dedekind scheme.

(4) Clearly, a flat family of fiberwise stability conditions over S induces one over T for a base change $T \to S$ essentially of finite type.

(5) In case $S = \mathrm{Spec}(k)$ is a point, a "flat family of fiberwise stability conditions on $\mathcal{D}$" (and, similarly, "a stability condition on $\mathcal{D}$ over k" in Definition 21.15 below) is stronger than the notion of a stability condition on $\mathcal{D}$ from Section 12. Indeed, Definition 20.5 includes openness of stability as a requirement (while Definition 21.15 will include boundedness).

We single out the following remark for emphasis.

[4]We have learned a related statement for complete DVRs from Fabian Haiden.

Remark 20.7. Consider a Dedekind scheme $C \to S$. The key requirement in condition (3) of Definition 20.5 is the existence of a t-structure on $\mathcal{D}_C$ integrating the t-structures on the fibers $\mathcal{D}_c$ induced by the stability conditions σ_c and universally satisfying openness of flatness. Indeed, we will see that, given $\underline{\sigma}$, the heart $\mathcal{A}_C$ universally satisfies openness of flatness (Proposition 20.8) and has a C-torsion theory (Corollary 20.10). Conversely, if a heart $\mathcal{A}_C$ exists and universally satisfies openness of flatness, then by Theorem 17.1 it does admit a C-torsion theory; once we impose a support property which is uniform across all fibers, see Definition 21.15, the existence of a HN structure then follows from Theorem 18.7 and Lemma 20.4.

In the case of weak stability conditions, the same logic holds if in addition, every weak stability condition σ_s has the tilting property, this is preserved under base change, and $\mathcal{A}^0_C$ is a noetherian torsion subcategory.

Conversely, a (weak) HN structure on $S = C$ satisfying support property on fibers induces a flat family of fiberwise (weak) stability conditions only if we additionally require various base change compatibilities; for example, universal openness of geometric stability (and universal generic openness, in the weak stability case) rather than just generic openness for objects that are semistable over $K(C)$ (see Definition 16.3).

20.2. **Universal openness of flatness.** In order to take advantage of the results of Part II, in particular the existence of Quot spaces shown in Section 11, we have to show that the fiberwise collection of t-structures underlying a flat family of (weak) stability conditions universally satisfies openness of flatness. More generally (see Remark 20.2), we show the following.

Proposition 20.8. *Let $\underline{\sigma}$ be a flat family of fiberwise (weak) stability conditions on $\mathcal{D}$ over S. Then $\underline{\sigma}$ universally satisfies openness of lying in $\mathcal{P}(I)$ for any interval $I \subset \mathbb{R}$.*

Given a base change $T \to S$ of finite type and a T-perfect object $E \in \mathcal{D}_T$, we obtain two functions

$$\phi^+_E \colon T \to \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad \phi^-_E \colon T \to \mathbb{R} \cup \{+\infty\} \tag{20.1}$$

that assign to $t \in T$ the maximal and minimal phase $\phi^\pm(E_t)$ of the HN filtration of E_t with respect to the slicing $\mathcal{P}_t$; here we set $\phi^\pm$ of the zero object to be $\mp\infty$ for convenience. The key observation underlying Proposition 20.8 is the semicontinuity of these functions.

Lemma 20.9. *The functions ϕ^+_E and ϕ^-_E are, respectively, upper and lower semicontinuous constructible functions on T.*

Proof of Proposition 20.8. By Lemma 20.3, it is enough to prove openness of lying in $\mathcal{P}(I)$ for a finite type base change $T \to S$ and $E \in \mathcal{D}_T$ as above. Suppose for concreteness that the interval I is of the form $I = (a, b]$; the argument is the same for other types of intervals. Then the set

$$\{t \in T \mid E_t \in \mathcal{P}_t(I)\} = \left(\phi^-_E\right)^{-1}(a, +\infty) \cap \left(\phi^+_E\right)^{-1}(-\infty, b]$$

is open by Lemma 20.9. □

Proof of Lemma 20.9. We first note that $\phi^\pm_E$ are preserved by arbitrary base change, as the pull-back under base change for field extensions in Theorem 12.17 or Proposition 14.20 preserves the HN filtration.

Step 1. ϕ^+_E and ϕ^-_E are constructible functions.

Due to the compatibility with base change, and since T is noetherian, it is enough to prove that if T is irreducible, then there exists an open set $U \subset T$ on which $\phi_E^{\pm}$ is constant. To prove this claim, let $0 \to E_1 \to E_2 \to \cdots \to E_m = E$ be an arbitrary lift to $\mathcal{D}_T$ of the HN filtration of E_η, where $\eta \in T$ is the generic point, given by Lemma 4.16.(1).

Let F_i be the cone of the map $E_{i-1} \to E_i$; by construction, the objects $(F_i)_\eta$ are the HN factors of E_η, and thus semistable. By the generic openness of semistability (see Lemma 20.4), there exists a nonempty open set $U \subset T$ such that every $(F_i)_t$ is semistable for all points $t \in U$. It follows that the $(E_i)_t$ induce the HN filtration of E_t for all $t \in U$, and in particular that $\phi_E^{\pm}$ are constant on U.

Step 2. ϕ_E^+ and ϕ_E^- are, respectively, monotone increasing and decreasing under specialization.

As $\phi_E^{\pm}$ are preserved by base change, and since we assume S (and thus T) to be Nagata, it suffices by Lemma 11.19 to consider a DVR R essentially of finite type over T: if k, K are the special and generic point of $\operatorname{Spec} R$, respectively, we have to show $\phi^+(E_k) \geqslant \phi^+(E_K)$ and $\phi^-(E_k) \leqslant \phi^-(E_K)$. We may assume $E_K \neq 0$, otherwise the claim is trivial.

By Definition 20.5 assumption (3), the (weak) stability conditions σ_k and σ_K are induced by a (weak) Harder–Narasimhan structure σ_R over $\operatorname{Spec} R$. Let

$$0 = E_0 \to E_1 \to \cdots \to E_m = E_R$$

be the HN filtration of E_R with respect to σ_R. Its base change to the fraction field induces the HN filtration of E_K (in the sense that some filtration quotients might be R-torsion, inducing isomorphisms $E_i \to E_{i+1}$ after base change to K); therefore $\phi^+(E_K) \leqslant \phi(E_1) = \phi^+(E_R)$, and likewise $\phi^-(E_K) \geqslant \phi^-(E_R)$.

It remains to prove $\phi^-(E_R) \geqslant \phi^-(E_k)$ and $\phi^+(E_R) \leqslant \phi^+(E_k)$. Since i_{k*} is t-exact with respect to the t-structures obtained from $\mathcal{P}(> \phi)$ for any $\phi \in \mathbb{R}$, see Lemma 15.6, it follows that i_k^* is right t-exact, and $i_k^! = i_k^*[-1]$ is left t-exact; in other words,

$$E_k \in \mathcal{P}_k[\phi^-(E_R), \phi^+(E_R) + 1]. \tag{20.2}$$

This immediately implies the desired claim for ϕ^+. Indeed, consider the triangle $E_1 \to E_R \to G$ obtained from the HN filtration, and let A be the first HN factor of $(E_1)_k$. Then

$$\phi(A) \geqslant \phi(E_1) = \phi^+(E_R) > \phi^+(G) \geqslant \phi^+(G_k[-1]).$$

Hence the composition $A \to (E_1)_k \to E_k$ is non-zero, and so $\phi^+(E_k) \geqslant \phi(A) \geqslant \phi^+(E_R)$.

An analogous argument reduces the case of ϕ^- to the following claim: if $E \in \mathcal{A}_R$ is σ_R-semistable, then $\phi(E) = \phi^-(E_k)$. To prove this claim, first suppose $Z_K(E_K) \neq 0$. Then $Z_k(E_k) = Z_K(E_K) \in \mathbb{R}_{>0} \cdot e^{\mathrm{i}\pi\phi}$. Together with $E_k \in \mathcal{P}_k[\phi, \phi + 1]$, this is only possible if $\phi^-(E_k) = \phi$ as claimed.

Now suppose $Z_K(E_K) = 0$. Then $\phi = 1$, so $E/\pi E \in \mathcal{P}_k(1)$ by our analysis above; to conclude, it suffices to prove $E/\pi E \neq 0$. Assume otherwise. It follows immediately that $E_k = \mathrm{H}^{-1}_{\mathcal{A}_k}(E_k)[1]$ with $Z_k\left(\mathrm{H}^{-1}_{\mathcal{A}_k}(E_k)\right) = 0$; the same holds for all quotients of E in $\mathcal{A}_R$.

Recall from Definition 20.5.(3') that $\mathcal{A}_R^0$ is noetherian, so if E were in $\mathcal{A}_R^0$, then it would have a maximal R-torsion subobject which would contradict $E/\pi E = 0$ by Remark 6.20. Thus $E \notin \mathcal{A}_R^0$, so there exist quotients $E \twoheadrightarrow Q_1 \twoheadrightarrow Q_2$ with a short exact sequence $A \hookrightarrow Q_1 \twoheadrightarrow Q_2$ and $Z_R(A) \neq 0$. Since $Z_K((Q_i)_K) = 0$, this is only possible if A is R-torsion.

On the other hand, the surjection $\mathrm{H}^{-1}_{\mathcal{A}_k}((Q_2)_k) \twoheadrightarrow A/\pi A$ in $\mathcal{A}_k$ shows $Z_k(A/\pi A) = 0$; via Lemma 6.11 this gives a contradiction. □

Finally, as consequence of Proposition 20.8 and Theorem 17.1, we have:

Corollary 20.10. *Let $\underline{\sigma}$ be a flat family of fiberwise (weak) stability conditions on $\mathcal{D}$ over S. Let $C \to S$ be essentially of finite type, with C Dedekind, and let $\mathcal{A}_C$ be the heart of the (weak) HN structure on $\mathcal{D}_C$. Then $\mathcal{A}_C$ has a C-torsion theory.*

20.3. **Product stability conditions.** In this subsection, we assume that X_0 and S are of finite type over a field k with X_0 projective, and let $X = X_0 \times_{\operatorname{Spec} k} S$ be the product. Further, assume that $\mathcal{D}_0 \subset \mathrm{D}^{\mathrm{b}}(X_0)$ is a k-linear strong semiorthogonal component of finite cohomological amplitude, and $\mathcal{D} = (\mathcal{D}_0)_S \subset \mathrm{D}^{\mathrm{b}}(X)$ is its base change to S.

By Theorem 12.17, any stability condition σ_0 on $\mathcal{D}_0$ induces a collection $\underline{\sigma}$ of fiberwise stability conditions on $\mathcal{D}$ over S; we call such a $\underline{\sigma}$ a *product stability condition*. If σ_0 universally satisfies openness of geometric stability and the heart of σ_0 universally satisfies openness of flatness, then $\underline{\sigma}$ is a flat family of fiberwise stability conditions by Theorems 18.7 and 5.7. As a partial converse, we have the following result, strengthening [LPZ18, Proposition 2.6].

Proposition 20.11. *If S as above is connected and has a k-rational point, then the only flat families of fiberwise stability conditions on $\mathcal{D}$ over S are product stability conditions.*

Proof. Let $\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s\in S}$ be a flat family of fiberwise stability conditions on $\mathcal{D}_S$ over S. By Remark 20.6.(3), $\underline{\sigma}$ is determined by σ_s for all closed points $s \in S$.

Step 1. Assume that S is irreducible, and that $s, t \in S$ are two k-rational points. Then $\sigma_s = \sigma_t$.

Let $F \in \mathcal{P}_s(\phi) \subset \mathcal{D}_s$. We claim that $F \in \mathcal{P}_t(\phi)$ and $Z_s(F) = Z_t(F)$.

Indeed, if we consider $E := p^*F$, where $p\colon X \to X_0$ is the natural projection, then E is S-perfect and $E_t = E_s = F$, so by Definition 20.5.(1) $Z_s(F) = Z_t(F)$. Now, suppose that $E_t = F$ is not σ_t-semistable. Then, if we consider the mds F_1 of F with respect to σ_t, we have $\phi(F_1) > \phi(F)$. By Definition 20.5.(2), there exist open sets $U \ni s$ and $V \ni t$, such that F is semistable for all points in U and F_1 is semistable for all points in V. Since $U \cap V \neq \emptyset$, the inequality $\phi(F_1) > \phi(F)$ contradicts the existence of a non-trivial morphism $F_1 \to F$.

Step 2. Assume that S is irreducible, and that it contains a point $s \in S$ such that σ_s is obtained by base change from a stability condition σ_0 on $\mathcal{D}_0$. Then $\underline{\sigma}$ is a product stability condition.

Let $t \in S$; we need to show that σ_t is equal to the stability condition $(\sigma_0)_{k(t)}$ obtained by base change from σ_0 via the field extension $k \subset k(t)$. Let ℓ be a field extension of k containing both $k(t)$ and $k(s)$. Consider the base change $S_\ell := S \times_{\operatorname{Spec} k} \operatorname{Spec} \ell \to S$. Then every irreducible component of S_ℓ contains ℓ-rational points s_ℓ, t_ℓ mapping to s, t, respectively. By Step 1, we have

$$(\sigma_t)_\ell = (\sigma_s)_\ell = (\sigma_0)_\ell = \big((\sigma_0)_{k(t)}\big)_\ell .$$

By Theorem 12.17, two stability conditions that become equal after base change are equal. This proves the claim.

Step 3. The general case.

If S is any connected scheme, then by induction on the number of irreducible components and the previous steps, every irreducible component contains a point t such that σ_t is obtained by base change from a stability condition σ_0 on $\mathcal{D}_0$. By Step 2, the result follows. □

21. Stability conditions over S

In this section we give the definition of a (weak) stability condition over a given base S. To this end we need to define a suitable *support property* for a flat family of fiberwise (weak) stability conditions. This consists of two properties: the existence of a uniform quadratic form controlling the central charges, and boundedness of geometrically stable objects. We show that a stability condition over S has well-behaved moduli spaces of semistable objects.

21.1. **Support property.** So far, we have assumed the support property on each fiber; to make this notion useful, we will have to assume the existence of a uniform quadratic form controlling the central charges and classes of semistable objects on all fibers.

Recall the definition of the numerical K-groups $K_{\mathrm{num}}(\mathcal{D}_s)$ underlying Definition 12.8, the existence of a pushforward map $\eta^{\vee}_{t/s}\colon K_{\mathrm{num}}(\mathcal{D}_t) \twoheadrightarrow K_{\mathrm{num}}(\mathcal{D}_s)_t \subset K_{\mathrm{num}}(\mathcal{D}_s) \otimes \mathbb{Q}$ for every point t over s, see (12.2), and the property $K_{\mathrm{num}}(\mathcal{D}_s)_t \subset K_{\mathrm{num}}(\mathcal{D}_s)_{\overline{s}}$ shown in Proposition and Definition 12.15. Condition (1) on universal local constancy of central charges in Definition 20.5 naturally leads to the following.

Definition 21.1. We define the *relative numerical Grothendieck group* $K_{\mathrm{num}}(\mathcal{D}/S)$ as the quotient of $\bigoplus_{s\in S} K_{\mathrm{num}}(\mathcal{D}_s)_{\overline{s}}$ by the saturation of the subgroup generated by elements of the form

$$\eta^{\vee}_{t_1/f(t_1)}[E_{t_1}] - \eta^{\vee}_{t_2/f(t_2)}[E_{t_2}] \tag{21.1}$$

for all tuples (f, E, t_1, t_2) where $f\colon T \to S$ is a morphism from a connected scheme T, $E \in \mathrm{D}(X_T)$ is a T-perfect object such that $E_t \in \mathcal{D}_t$ for all $t \in T$, and $t_1, t_2 \in T$.

Remark 21.2. Analogous to Lemma 20.3, we would obtain the same group $K_{\mathrm{num}}(\mathcal{D}/S)$ if we only considered morphisms $f\colon T \to S$ of finite type from a connected affine scheme in the definition.

Given $f\colon T \to S$ and E as in Definition 21.1, we write $[E]$ for the element of $K_{\mathrm{num}}(\mathcal{D}/S)$ given by the image of $[E_t] \in K_{\mathrm{num}}(\mathcal{D}_t)$ under the composition

$$K_{\mathrm{num}}(\mathcal{D}_t) \to K_{\mathrm{num}}(\mathcal{D}_{f(t)})_t \hookrightarrow K_{\mathrm{num}}(\mathcal{D}_{f(t)})_{\overline{f(t)}} \to K_{\mathrm{num}}(\mathcal{D}/S)$$

for any $t \in T$, which is independent of $t \in T$ by the definition of $K_{\mathrm{num}}(\mathcal{D}/S)$.

By the evident universal property of $K_{\mathrm{num}}(\mathcal{D}/S)$, for any flat family of fiberwise (weak) stability conditions there exists a central charge $Z\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \mathbb{C}$ such that for all $s \in S$, the central charge Z_s factors as $Z_s\colon K_{\mathrm{num}}(\mathcal{D}_s) \to K_{\mathrm{num}}(\mathcal{D}/S) \xrightarrow{Z} \mathbb{C}$.

Definition 21.3. Let Λ be a finite rank free abelian group. A *relative Mukai homomorphism* for $\mathcal{D}$ over S with respect to Λ is a group homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \Lambda$.

Remark 21.4. By definition, a relative Mukai homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \Lambda$ is *universally locally constant* in the sense that for every morphism $T \to S$ and every T-perfect object $E \in \mathrm{D}(X_T)$ such that $E_t \in \mathcal{D}_t$ for all $t \in T$, the function $T \to \Lambda$ given by $t \mapsto v([E_t])$ is locally constant.

We will always restrict our attention to flat families of fiberwise (weak) stability conditions where $Z\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \mathbb{C}$ factors via a relative Mukai homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \Lambda$. We will frequently use the following choice.

Proposition and Definition 21.5. *There is an Euler characteristic pairing*

$$\chi\colon K(\mathcal{D}_{\mathrm{perf}}) \times K_{\mathrm{num}}(\mathcal{D}/S) \to \mathbb{Z} \quad \textit{given by} \quad \chi([F],[E]) = \chi(F_t, E)$$

for some point t over S and objects $E \in \mathcal{D}_t$ and $F \in \mathcal{D}_{\mathrm{perf}}$. We write $\mathcal{N}(\mathcal{D}/S)$ for the quotient

$$\mathcal{N}(\mathcal{D}/S) := K_{\mathrm{num}}(\mathcal{D}/S)/\operatorname{Ker}\chi,$$

and call it the uniformly numerical relative Grothendieck group *of $\mathcal{D}$ over S. If S is quasi-projective over a field, then $\mathcal{N}(\mathcal{D}/S)$ is a free abelian group of finite rank.*

Proof. We need to show that χ satisfies the relation given in (21.1). This follows since $\chi(F_{t_1}, E_{t_1})$ and $\chi(F_{t_2}, E_{t_2})$ both compute the rank of the object $\mathcal{H}om_T(F_T, E)$ in $\mathrm{D}^{\mathrm{b}}(T)$.

To prove the claim on being a free abelian group of finite rank, we first observe that $K_{\mathrm{num}}(\mathcal{D}/S)$ is generated by objects defined over closed points. Therefore, $\mathcal{N}(\mathcal{D}/S)$ is a subgroup of the numerical Grothendieck group for compactly supported objects in $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ considered in [BCZ17, Section 5.1]. Since $\mathcal{X}$ is quasi-projective over a field, we can compactify it and then use directly the argument in the proof of [BCZ17, Lemma 5.1.1], by using Lemma 12.7 to avoid both the normality and the characteristic zero assumptions in [BCZ17, Lemma 5.1.1]. □

Example 21.6. Let $g\colon \mathcal{X} \to S$ be a smooth family of polarized surfaces with S connected and whose geometric generic fiber has Picard rank one, and let $\mathcal{O}_{\mathcal{X}}(1)$ denote a relatively ample line bundle. Then the coefficients of the relative Hilbert polynomial with respect to $\mathcal{O}_{\mathcal{X}}(1)$ as in Example 15.3 (or, equivalently, the degrees of the Chern character on the fibers with respect to $\mathcal{O}_{\mathcal{X}}(1)$) induce an isomorphism $\mathcal{N}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \cong \mathbb{Z}^3$.

Example 21.7 (Yoshioka's trick)**.** Consider the previous example in the case where the base $S = C$ is a curve. We can modify this construction at a single closed point $c \in C$ whose fiber $\mathcal{X}_c$ has higher Picard rank as follows. Let $L \subset K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_c))$ be the saturated subgroup generated by (the restrictions of) $\mathcal{O}_{\mathcal{X}}$, $\mathcal{O}_{\mathcal{X}}(1)$ and the class of a point. Then the quotient map to $K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_c))/L$ extends to a map $K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \to K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_c))/L$ by setting it identically to zero for objects supported over any point $c' \in C, c \neq c'$. The choice of

$$K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \to \mathcal{N}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \oplus K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_c))/L$$

will allow us to deform the central charge specifically for objects supported over c.

This can be generalized to an arbitrary family over C by letting L be the subgroup of $K_{\mathrm{num}}(\mathcal{D}_c)$ generated by all classes of the form $\eta^{\vee}_{t/c}[E_t]$ for objects E defined over a scheme $f\colon T \to C$ with f dominant, $t \in T$ and $f(t) = c$.

Example 21.8. Generalizing Example 21.6, let $g\colon \mathcal{X} \to S$ be a flat projective morphism of relative dimension n, with normal and integral fibers. The coefficients $p_n, \dots, p_0$ of the Hilbert polynomial with respect to $\mathcal{O}_X(1)$ define, up to normalization, a morphism $\mathcal{N}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \to \mathbb{Z}^{n+1}$.

The assignment $\underline{\sigma} = \big(\sigma_s = (\mathrm{i}p_n - p_{n-1}, \operatorname{Coh}\mathcal{X}_s)\big)_{s\in S}$ is a flat family of weak fiberwise stability conditions on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$: condition (2) follows from [HL10, Proposition 2.3.1], while condition (3') follows from Section 16.2. Moreover, for each s the subcategory $\mathcal{A}^0_s$ is the category of torsion sheaves supported in codimension $\geqslant 2$, which satisfies condition (0) because

$\operatorname{Coh}\mathcal{X}_s$ is noetherian. Similarly, to verify condition (3'), we note that $\mathcal{A}^0_C$ consists of sheaves whose support intersects every fiber in codimension $\geqslant 2$.

This can be generalized to modules $\operatorname{Coh}(\mathcal{X},\mathcal{B})$ over a sheaf of coherent algebras $\mathcal{B}$ on $\mathcal{X}$ (or, more generally, a "sheaf of differential operators" in the sense of [Sim94, Section 2]); the Hilbert polynomial being the one of the underlying sheaf. In the example of cubic fourfolds (see Part VI), we will use the case when $\mathcal{B}$ is a sheaf of the even parts of a Clifford algebra.

From now on, we fix Λ and a relative Mukai homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}/S)\to\Lambda$.

Definition 21.9. Given a flat family of fiberwise (weak) stability conditions $\underline{\sigma}$, for which Z factors via v, we let Λ_0 be the saturated subgroup of Λ generated by $v([E_t])$ for all $E_t\in\mathcal{A}^0_t$ (see Definition 14.3) and all points t over S. Let $\overline{\Lambda}$ be the free abelian group Λ/Λ_0, and $\overline{v}$ the composition of v with the quotient map.

Remark 21.10. If each σ_s is a stability condition, then $\Lambda_0=\{0\}$ and $\overline{\Lambda}=\Lambda$.

In our definition of the support property for a flat family of fiberwise (weak) stability conditions, we will need to impose a boundedness assumption on moduli spaces. The following definition summarizes the relevant moduli spaces we can consider in our context.

Definition 21.11. Let $\underline{\sigma}$ be a flat family of fiberwise (weak) stability conditions on $\mathcal{D}$ over S. Fix a vector $\mathbf{v}\in\Lambda$ and $\phi\in\mathbb{R}$ such that $Z(\mathbf{v})\in\mathbb{R}_{>0}e^{\mathrm{i}\pi\phi}$.

(1) We denote by
$$\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})\colon(\mathrm{Sch}\,/S)^{\mathrm{op}}\to\mathrm{Gpds}$$
the functor whose value on $T\in(\mathrm{Sch}\,/S)$ consists of all T-perfect objects $E\in\mathrm{D}(\mathcal{X}_T)$ such that for all $t\in T$ we have $E_t\in\mathcal{D}_t$, E_t is geometrically σ_t-stable of phase ϕ, and $v([E_t])=\mathbf{v}$. An object $E\in\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})(T)$ is called a *family of geometrically $\underline{\sigma}$-stable objects of class* $\mathbf{v}$ over T.

(2) We denote by
$$\mathcal{M}_{\underline{\sigma}}(\mathbf{v})\colon(\mathrm{Sch}\,/S)^{\mathrm{op}}\to\mathrm{Gpds}$$
the functor whose value on $T\in(\mathrm{Sch}\,/S)$ consists of all T-perfect objects $E\in\mathrm{D}(\mathcal{X}_T)$ such that for all $t\in T$ we have $E_t\in\mathcal{D}_t$, E_t is σ_t-semistable of phase ϕ, and $v([E_t])=\mathbf{v}$. An object $E\in\mathcal{M}_{\underline{\sigma}}(\mathbf{v})(T)$ is called a *family of $\underline{\sigma}$-semistable objects of class* $\mathbf{v}$ over T.

(3) For an interval $I\subset\mathbb{R}$, we denote by
$$\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})\colon(\mathrm{Sch}\,/S)^{\mathrm{op}}\to\mathrm{Gpds}$$
the functor whose value on $T\in(\mathrm{Sch}\,/S)$ consists of all T-perfect objects $E\in\mathrm{D}(\mathcal{X}_T)$ such that for all $t\in T$ we have $E_t\in\mathcal{P}_t(I)\subset\mathcal{D}_t$ and $v([E_t])=\mathbf{v}$.

We will always omit the phase ϕ from the notation. Note that by Theorem 12.17 and Proposition 14.20 the above prescriptions indeed define functors, i.e., the conditions considered on $E\in\mathrm{D}(X_T)$ are stable under base change. Also note that $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})=\mathcal{P}_{\underline{\sigma}}([\phi,\phi];\mathbf{v})$.

Recall from Section 9 the moduli stack $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ of objects in $\mathcal{D}$.

Lemma 21.12. *Let $\underline{\sigma}$ be a flat family of fiberwise (weak) stability conditions on $\mathcal{D}$ over S, and fix a vector $\mathbf{v}\in\Lambda$ and an interval $I\subset\mathbb{R}$. Then $\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$, $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$, and $\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})$ are open substacks of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$.*

Proof. Since the relative Mukai homomorphism v is universally locally constant (by Remark 21.4) and $\underline{\sigma}$ universally satisfies openness of geometric stability, $\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$ is an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$. Similarly, since $\underline{\sigma}$ also universally satisfies openness of lying in $\mathcal{P}(I)$ by Proposition 20.8, $\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})$ (and thus $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ as a special case) is an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$. □

Definition 21.13. Let $\underline{\sigma}$ be a flat family of fiberwise (weak) stability conditions on $\mathcal{D}$ over S. We say that $\underline{\sigma}$ *satisfies boundedness* if $\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$ is bounded in the sense of Definition 9.4 for every $\mathbf{v} \in \Lambda$.

Remark 21.14. In Section 21.2 we show that in the case of stability conditions, we also obtain boundedness of $\mathcal{P}_{\underline{\sigma}}(I,\mathbf{v})$ for I of length less than 1, and of certain Quot spaces.

Now we can finally define the support property, completing our main definition. We use the notation of Definition 21.9, and write the central charge as the composition of $Z\colon \Lambda \to \mathbb{C}$ with the fixed relative Mukai homomorphism v.

Definition 21.15. We say that a flat family of fiberwise (weak) stability conditions $\underline{\sigma}$ satisfies the *support property with respect to* Λ if:

(4) There exists a quadratic form Q on $\overline{\Lambda}_{\mathbb{R}} = \overline{\Lambda} \otimes \mathbb{R}$ such that
  (a) the kernel $(\ker Z)/\Lambda_0 \subset \overline{\Lambda}$ is negative definite with respect to Q, and
  (b) for every $s \in S$ and for every σ_s-semistable object $E \in \mathcal{D}_s$, we have $Q(v(E)) \geqslant 0$.

(5) $\underline{\sigma}$ satisfies boundedness.

In this case, we call $\underline{\sigma}$ a *(weak) stability condition on $\mathcal{D}$ over S with respect to* Λ.

Remark 21.16. If the flat family $\underline{\sigma}$ of fiberwise (weak) stability conditions satisfies the support property, then every σ_s satisfies the support property as a stability condition on $\mathcal{D}_s$ with respect to Q and the composition $K_{\mathrm{num}}(\mathcal{D}_s) \to K_{\mathrm{num}}(\mathcal{D}/S) \to \Lambda$. Similarly, consider a Dedekind scheme C and a morphism $C \to S$ essentially of finite type. Our relative Mukai homomorphism induces a Mukai homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}_{C\text{-tor}}) \to \Lambda$ by Lemma 18.3, and the induced (weak) stability condition on $\mathcal{D}_{C\text{-tor}}$ satisfies the support property by Lemma 18.6.

Remark 21.17. Let us explain the role of assumption (5) in the support property. In the absolute case, the support property is the key behind the deformation result for stability conditions in Theorem 12.11. It implies that under a small deformation of a central charge, there are only finitely many classes of objects that could destabilize a given object E. Now consider an object $E \in \mathcal{D}_C$ defined over a curve C. To ensure that openness of stability is preserved, we need to show that unless the generic fiber of E gets destabilized, each such class can only destabilize E_c for finitely many closed points $c \in C$. This can only be ensured by showing that the set of potentially destabilizing quotients is bounded; see Section 22 for the full proof.

Example 21.18. In the setting of Example 21.8, $\underline{\sigma} = \big(\sigma_s = (Z_s := \mathrm{i}p_n - p_{n-1}, \operatorname{Coh}\mathcal{X}_s)\big)_{s\in S}$ gives a weak stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S. Indeed, $\Lambda_0 = \ker Z$ is the saturated subgroup generated by $(p_{n-2},\dots,p_0)$ and so $\overline{\Lambda} \cong \mathbb{Z}^2$. We can therefore choose any non-negative quadratic form Q on $\overline{\Lambda}_{\mathbb{R}}$ to satisfy (4a) and (4b). Boundedness is [Lan04, Theorem 4.2].

As before, this generalizes to the case $\operatorname{Coh}(\mathcal{X},\mathcal{B})$; as remarked in [Lan04, Section 4], boundedness can be proved over an arbitrary base S with the same argument as in [Sim94, Proposition 3.5].

Remark 21.19. The action of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ on stability conditions on the fibers, and on HN structures over curves (see Remark 13.15), preserves all properties in Definitions 20.5 and 21.15, and thus acts on the set of stability conditions over S.

21.2. **Boundedness results.** Now we show that the boundedness of a flat family of fiberwise stability conditions leads to boundedness of other moduli functors.

Lemma 21.20. *Let $\underline{\sigma}$ be a stability condition on $\mathcal{D}$ over S. Let $I \subset \mathbb{R}$ be an interval of length less than 1, and $\mathbf{v} \in \Lambda$. Then $\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})$ is a bounded (in the sense of Definition 9.4) algebraic stack of finite type over S; in particular, so is $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$.*

Proof. Let ϕ_0, ϕ_1 be the endpoints of (the closure of) I, and consider an object $E \in \mathcal{P}_t(I)$ of class $\mathbf{v}$ for a point t over S; let $\bar{t}$ denote the algebraic closure of t. Then every *stable factor* (i.e., a Jordan–Hölder factor of one of its HN factors) of $E_{\bar{t}}$ with respect to $\sigma_{\bar{t}}$ has central charge in the parallelogram with angles $\pi\phi_0$ and $\pi\phi_1$ and with 0 and $Z(\mathbf{v})$ as opposite vertices. By Remark 12.3, this means that as t ranges over all points over S, there are only finitely many possible classes occurring as stable factors of $E_{\bar{t}}$. Combining this with the boundedness of $\underline{\sigma}$ for stable objects—condition (5) in Definition 21.15—, and the fact that $\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v}) \subset \mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$ is open by Lemma 21.12, we conclude by Lemma 9.6 that $\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})$ is bounded. Thus by Lemma 9.5, $\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})$ is an algebraic stack of finite type over S. □

Recall from Section 11 that given a fiberwise collection of t-structures $\underline{\tau}$ universally satisfying openness of flatness, we have defined a moduli stack $\mathcal{M}_{\underline{\tau}}$ of flat objects in $\mathcal{D}$, as well as a Quot space $\mathrm{Quot}_S(E)$ for any object $E \in \mathcal{M}_{\underline{\tau}}(S)$. Note that if $\underline{\sigma}$ is a flat family of fiberwise (weak) stability conditions on $\mathcal{D}$ over S, then by Proposition 20.8 the fiberwise collection of t-structures underlying $\underline{\sigma}$ universally satisfies openness of flatness.

Lemma 21.21. *Let $\underline{\sigma}$ be a stability condition on $\mathcal{D}$ over S. Let $E \in \mathcal{M}_{\underline{\tau}}(S)$ where $\underline{\tau}$ is the fiberwise collection of t-structures underlying $\underline{\sigma}$. For $\phi \in (0,1)$, let $\mathrm{Quot}_S^{\leqslant\phi}(E)$ be the subfunctor of $\mathrm{Quot}_S(E)$ which assigns to $T \in (\mathrm{Sch}/S)$ the set of $(E_T \to Q) \in \mathrm{Quot}_S(E)(T)$ satisfying $\phi(Q_t) \leqslant \phi$ for all $t \in T$. Then $\mathrm{Quot}_S^{\leqslant\phi}(E)$ is an algebraic space of finite type over S, and the morphism $\mathrm{Quot}_S^{\leqslant\phi}(E) \to S$ is universally closed.*

Proof. The canonical morphism $\mathrm{Quot}_S^{\leqslant\phi}(E) \to \mathrm{Quot}_S(E)$ is representable by open immersions because $\underline{\sigma}$ universally has locally constant central charges. Since $\mathrm{Quot}_S(E)$ is an algebraic space locally of finite type over S by Proposition 11.6, it follows that $\mathrm{Quot}_S^{\leqslant\phi}(E)$ is too.

Next we prove that $\mathrm{Quot}_S^{\leqslant\phi}(E)$ is in fact of finite type over S. Recall the function ϕ_E^- defined in (20.1). By Lemma 20.9, ϕ_E^- is a constructible function on the noetherian topological space S, and hence has a minimum $\phi_0 > 0$. In particular, given $(E_T \to Q) \in \mathrm{Quot}_S^{\leqslant\phi}(E)(T)$ and a point $t \in T$, every stable factor of the base change $Q_{\bar{t}}$ to the algebraic closure has phase $\geqslant \phi_0$. Since we also have $\Im Z(Q) \leqslant \Im Z(E)$ and $\phi(Q) \leqslant \phi$, it follows that every stable factor of $Q_{\bar{t}}$ has central charge in the parallelogram with angles $\pi\phi_0$ and π and with 0 and z as opposite vertices, where $z \in \mathbb{C}$ is the complex number of phase ϕ such that $\Im z = \Im Z(E)$. By Remark 12.3 it follows that there is a finite set $\Gamma \subset \Lambda$ of classes occurring as stable factors of $Q_{\bar{t}}$; moreover, this set Γ depends only on E, ϕ, and Z, and hence works uniformly for any $(E_T \to Q) \in \mathrm{Quot}_S^{\leqslant\phi}(E)(T)$ and $t \in T$. As the central charge of Q lies in the triangle with vertices 0, z, and w, where w is complex number of phase ϕ_0 such that $\Im w = \Im Z(E)$, the set

of sums of classes in Γ whose central charge lie in this triangle is another uniform finite set Γ', which contains all of the possible classes of Q. Therefore, the morphism $\mathrm{Quot}_S^{\leqslant\phi}(E) \to \mathcal{M}_{\underline{\tau}}$ sending $E_T \to Q$ to Q factors through the canonical morphism $\coprod_{\mathbf{v}\in\Gamma'} \mathcal{P}_{\underline{\sigma}}([\phi_0, 1]; \mathbf{v}) \to \mathcal{M}_{\underline{\tau}}$. The proof of Proposition 11.6 shows that the morphism $\mathrm{Quot}_S^{\leqslant\phi}(E) \to \coprod_{\mathbf{v}\in\Gamma'} \mathcal{P}_{\underline{\sigma}}([\phi_0, 1]; \mathbf{v})$ is of finite type. The target of this morphism is of finite type over S by Lemma 21.20, hence so is $\mathrm{Quot}_S^{\leqslant\phi}(E)$.

By combining condition (3) of Definition 20.5, Corollary 20.10, and Proposition 11.11, we find that the morphism $\mathrm{Quot}_S(E) \to S$ satisfies the strong existence part of the valuative criterion with respect to any essentially of finite type morphism $\mathrm{Spec}(R) \to S$ with R a DVR. By universal local constancy of the central charges of $\underline{\sigma}$, the same holds for $\mathrm{Quot}_S^{\leqslant\phi}(E) \to S$. Thus Lemma 11.21 shows the morphism $\mathrm{Quot}_S^{\leqslant\phi}(E) \to S$ is universally closed. (Note that $\mathrm{Spec}(R) \to S$ is essentially of finite type if and only if it is essentially locally of finite type, see Remark 11.17.) □

21.3. **Relative moduli spaces.** In this section, we show that in our setting, relative moduli spaces of semistable objects are well-behaved. First we prove that if $\underline{\sigma}$ is a stability condition on $\mathcal{D}$ over S, then the moduli stack $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ is *quasi-proper* over S, that is, it satisfies the strong existence part of the valuative criterion. The proof is essentially the same as [AP06, Theorem 4.1.1], and is the reason we require the existence of HN structures after base change to Dedekind schemes. In characteristic zero, we further use [AHLH18] to show that it admits a good moduli space $M_{\underline{\sigma}}(\mathbf{v})$ (in the sense of Alper) which is proper over S.

Lemma 21.22. *Let $\underline{\sigma}$ be a stability condition on $\mathcal{D}$ over S. Then for every $\mathbf{v} \in \Lambda$, the morphism $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) \to S$ satisfies the strong existence part of the valuative criterion for any DVR essentially of finite type over S.*

Proof. Let $\mathrm{Spec}(R) \to S$ be a morphism from a DVR that is essentially of finite type, let K be its field of fractions, k its residue field, and assume we are given a lift $\mathrm{Spec}(K) \to \mathcal{M}_{\underline{\sigma}}(\mathbf{v})$, corresponding to a σ_K-semistable object in $\mathcal{D}_K$ of class $\mathbf{v}$. We may assume that it is the base change E_K of $E_R \in \mathcal{D}_R$.

By assumption, we have a HN structure σ_R on $\mathcal{D}_R$ over $\mathrm{Spec}\, R$ satisfying the support property. Since the HN filtration of E_R induces the one of E_K, we may assume that E_R is σ_R-semistable. By Corollary 20.10 and Proposition 17.6 we can assume that it is R-torsion free. By Lemma 13.12, its special fiber E_k is σ_k-semistable. □

We recall the notion of a good moduli space from [Alp13].

Definition 21.23. Let $\mathcal{Y}$ be an algebraic stack over S. We say that $\mathcal{Y}$ admits a *good moduli space* if there exist an algebraic space Y over S and an S-morphism $\pi\colon \mathcal{Y} \to Y$ such that:

(1) π is quasi-compact and the functor $\pi_*\colon \mathrm{QCoh}\mathcal{Y} \to \mathrm{QCoh}Y$ is exact; and
(2) the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_{\mathcal{Y}}$ is an isomorphism.

Theorem 21.24. *Let $\underline{\sigma}$ be a (weak) stability condition on $\mathcal{D}$ over S, and let $\mathbf{v} \in \Lambda$.*

(1) *$\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$ is an algebraic stack of finite type over S.*
(2) *If $\underline{\sigma}$ is a stability condition, then $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ is an algebraic stack of finite type over S. Moreover, if $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) = \mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$, then it is a $\mathbb{G}_m$-gerbe over its coarse moduli space $M_{\underline{\sigma}}(\mathbf{v})$, which is an algebraic space proper over S.*

(3) *Suppose further that S has characteristic 0. If $\underline{\sigma}$ is a stability condition, then $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ admits a good moduli space $M_{\underline{\sigma}}(\mathbf{v})$ which is an algebraic space proper over S.*

Proof. Claim (1) follows immediately from Lemma 9.5 and the definitions. Now let us assume that $\underline{\sigma}$ is a stability condition. We have already seen in Lemma 21.20 and Proposition 20.8 that the functor $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ is bounded and an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$, and hence an algebraic stack of finite type over S again by Lemma 9.5.

If $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) = \mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$, then every object $E \in \mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ is simple (see Definition 9.7), as every stable object over an algebraically closed field has only trivial endomorphisms. The $\mathbb{G}_m$-structure over its coarse moduli space follows from Lemma 9.8.

Lemma 11.21 and Lemma 21.22 show that $M_{\underline{\sigma}}(\mathbf{v})$ is universally closed over S.[5]

Now consider an étale covering $\widetilde{M} \to M_{\underline{\sigma}}(\mathbf{v})$ from a scheme $\widetilde{M}$ that admits a universal family. The pullback of the diagonal in $M_{\underline{\sigma}}(\mathbf{v}) \times_S M_{\underline{\sigma}}(\mathbf{v})$ to $\widetilde{M} \times_S \widetilde{M}$ is the locus where the two pullbacks $\mathcal{E}_1, \mathcal{E}_2$ of the universal family have isomorphic fibers. Since the fibers are stable of the same phase, this is also the locus where there exists a non-trivial morphism between the fibers of $\mathcal{E}_1, \mathcal{E}_2$, which is represented by a closed immersion. Since the property of being a closed immersion is étale local on the base, this shows that $M_{\underline{\sigma}}(\mathbf{v})$ is separated over S.

Finally, to prove (3) we use the recent groundbreaking result [AHLH18]. By [Alp13, Proposition 7.9], we can reduce to the case where S is affine. In this case, we can argue exactly as in the proof of [AHLH18, Theorem 7.25] to obtain that $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ admits a separated good moduli space $M_{\underline{\sigma}}(\mathbf{v})$. To show properness, we can directly use [AHLH18, Proposition 3.43(3)] together with Lemma 21.22. □

We finish this section by extending the *Positivity Lemma*, [BM14b, Theorem 4.1], which gives a numerically positive divisor class on every fiber of $M_{\underline{\sigma}}(\mathbf{v})$, in the case where semistability and stability coincide. We show that it is induced by a divisor class on $M_{\underline{\sigma}}(\mathbf{v})$ when the central charge factors via the uniformly numerical relative Grothendieck group $\mathcal{N}(\mathcal{D}/S)$ of $\mathcal{D}$ over S given by Proposition and Definition 21.5, and that it descends to good moduli spaces.

The group $N^1(\mathcal{M}_{\underline{\sigma}}(\mathbf{v})/S)$ of relative real numerical Cartier divisors on $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ over S is the quotient of the group of real Cartier divisors on $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ modulo those that have degree zero on every curve $C \to \mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ contracted to a point in S. A class in $N^1(\mathcal{M}_{\underline{\sigma}}(\mathbf{v})/S)$ has a well-defined degree on every such curve; hence we can talk about *relatively nef* or *relatively strictly nef* classes that pair non-negatively, or positively, with every such contracted curve, respectively.

Theorem 21.25. *In the setting of Theorem 21.24, assume that the Mukai morphism to Λ factors via $\mathcal{N}(\mathcal{D}/S)$. Let $M_{\underline{\sigma}}(\mathbf{v})$ be either the coarse moduli space (when semistability and stability coincide) or the good moduli space (in characteristic 0). Then there is a relative real numerical Cartier divisor class $\ell_{\underline{\sigma}} \in N^1(\mathcal{M}_{\underline{\sigma}(\mathbf{v})}/S)$, naturally associated to $\underline{\sigma}$, that is relatively nef. Moreover, we have $\ell_{\underline{\sigma}}.C = 0$ if and only if C is a curve of S-equivalent objects (i.e., if C is contracted in the morphism $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) \to M_{\underline{\sigma}}(\mathbf{v})$). It descends to a relative numerical Cartier divisor class $l_{\underline{\sigma}} \in N^1(M_{\underline{\sigma}}(\mathbf{v})/S)$ that is relatively strictly nef.*

Proof. In order to simplify notation, we first apply the action of $\mathbb{C} \subset \widetilde{\mathrm{GL}_2^+(\mathbb{R})}$, the universal cover of $\mathbb{C}^* \subset \mathrm{GL}_2^+(\mathbb{R})$, so that we may assume that $Z(\mathbf{v}) = -1$. We assume for simplicity

[5]Recall that we assume in the Main Setup that S is Nagata, so checking the valuative criterion on DVRs essentially of finite type over S is sufficient.

that Z is defined over $\mathbb{Q}[\mathrm{i}]$; once we prove Theorem 22.2, the general case can be reduced to that one as every wall of $\mathrm{Stab}(\mathcal{D}/S)$ is defined over $\mathbb{Q}$. From the definition of $\mathcal{N}(\mathcal{D}/S)$ and our assumption on the Mukai morphism, there exists $F \in \mathcal{D}_{\mathrm{perf}}$ and $a \in \mathbb{Q}$ such that $\Im Z(_) = a\chi([F], _)$; by linearity it is enough to consider the case $a = 1$.

Now recall that the numerical Cartier divisor class ℓ_{σ_s} is determined by

$$\ell_{\sigma_s}.C = \Im Z_s(v_s(p_{\mathcal{X}_s *}\mathcal{E}_C)),$$

where $C \to \mathcal{M}_{\sigma_s}(\mathbf{v})$ is a curve in the moduli stack, $\mathcal{E}_C \in \mathcal{D}_C$ is the associated family of σ_s-semistable objects, and $p_{\mathcal{X}_s *}\colon \mathcal{D}_C \to \mathcal{D}_s$ is the pushforward. In the notation above, we have $\ell_{\sigma_s}.C = \chi(F_s, p_{\mathcal{X}_s *}\mathcal{E}_C)$.

Noting that $\chi(F_s, E) = \Im Z_s(E) = 0$ for any object $E \in \mathcal{D}_s$ with $v_s(E) = \mathbf{v}$, it follows that the (dual of the) determinant line bundle construction can be applied to F: namely, if $\widetilde{F}$ denotes the pullback of F to $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) \times_S \mathcal{X}$, and $\mathcal{E}$ is the universal family, then $\mathcal{H}om_{\mathcal{M}_{\underline{\sigma}}(\mathbf{v})}(\widetilde{F}, \mathcal{E})$ is a complex of rank zero, and its determinant

$$\mathcal{L}_F := \det(\mathcal{H}om_{\mathcal{M}_{\underline{\sigma}}(\mathbf{v})}(\widetilde{F}, \mathcal{E}))$$

is a line bundle whose degree on a curve C as above agrees with $\ell_{\sigma_s}.C$ by adjunction (see also [BM14b, Proposition 4.4] for an analogous argument). We can thus define $\ell_{\underline{\sigma}}$ by

$$\ell_{\underline{\sigma}} := [\mathcal{L}_F] \in N^1(\mathcal{M}_{\underline{\sigma}}(\mathbf{v})/S).$$

As $\ell_{\underline{\sigma}}|_{\mathcal{M}_{\sigma_s}} = \ell_{\sigma_s}$ by construction, its claimed positivity properties are purely a statement on fibers, given by [BM14b, Positivity Lemma 3.3].

It remains to show that it descends to a class on the coarse or good moduli space; we will prove the latter case. By [Alp13, Theorem 10.3], it is enough to show that stabilizer groups of geometric points of $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ with closed image act trivially on $\mathcal{L}_F$. Such a point corresponds to a polystable object $E = \oplus_i E_i^{\oplus m_i}$ where $E_i \in \mathcal{D}_{\overline{s}}$ are distinct stable objects of the same phase, for some geometric point $\overline{s}$ over S; in particular, $\Im Z_{\overline{s}}(E_i) = 0$. Its stabilizer group is $\prod_i \mathrm{GL}(m_i, \kappa(\overline{s}))$; each factor acts on $\mathcal{L}_F|_E$ via $\det^{\chi(F_{\overline{s}}, E_i)} = \det^{\Im Z_{\overline{s}}(E_i)} = 1$. □

22. Deforming stability conditions over S

The goal of this section is an analogue of Bridgeland's deformation result for stability conditions, Theorem 12.11, for the case of stability conditions over S.

We continue to fix a group homomorphism $v\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \Lambda$, and let $\mathrm{Stab}_\Lambda(\mathcal{D}/S)$ denote the set of stability conditions on $\mathcal{D}$ over S with respect to Λ.

Definition 22.1. We define the topology on $\mathrm{Stab}_\Lambda(\mathcal{D}/S)$ as the coarsest topology such that the canonical map

$$\mathrm{Stab}_\Lambda(\mathcal{D}/S) \to \mathrm{Stab}_\Lambda(\mathcal{D}_s), \quad \underline{\sigma} \mapsto \sigma_s$$

is continuous for every $s \in S$.

Theorem 22.2. *The space $\mathrm{Stab}_\Lambda(\mathcal{D}/S)$ of stability conditions on $\mathcal{D}$ over S is a complex manifold, and the forgetful map*

$$\mathcal{Z}\colon \mathrm{Stab}_\Lambda(\mathcal{D}/S) \to \mathrm{Hom}(\Lambda, \mathbb{C}),$$

is a local isomorphism.

More precisely, assume that $\underline{\sigma}$ satisfies the support property with respect to the quadratic form Q, and write $P_Z \subset \mathrm{Hom}(\Lambda, \mathbb{C})$ for the connected component containing Z of the set

of central charges whose kernel is negative definite with respect to Q. Then there is an open neighborhood $\underline{\sigma} \in U \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ such that $\mathcal{Z}|_U \colon U \to P_Z$ is a covering.

We follow the proof strategy in [Bay19]. As pointed out in Remark 21.19, $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ acts on $\mathrm{Stab}_\Lambda(\mathcal{D}/S)$, lifting the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathrm{Hom}(\Lambda, \mathbb{C}) \cong \mathrm{Hom}(\Lambda, \mathbb{R}^2)$. Therefore, we can use the same simplification as in [Bay19] and only treat the case of a purely *real* variation of the central charge. This implies that the hearts $\mathcal{A}_s$ on the fibers, as well as the local hearts $\mathcal{A}_C$ for Dedekind schemes $C \to S$ in condition (3), remain unchanged.

More precisely, we can assume by [Bay19, Section 7] that Q has signature $(2, \mathrm{rk}\,\Lambda - 2)$, that Z and Q satisfy the normalization of [Bay19, Lemma 4.2], and consider a deformation of the form $W = Z + u \circ p$ where p is the orthogonal projection $\Lambda_\mathbb{R} \to \mathrm{Ker}\, Z$ (with respect to Q), and $u \colon \mathrm{Ker}\, Z \to \mathbb{R}$ is a linear map with operator norm (with respect to the standard norm on $\mathbb{R}$, and the norm $\|\cdot\|$ induced by $-Q$ on $\mathrm{Ker}\, Z$) satisfying $\|u\| < 1$; since it is sufficient to prove the theorem for small deformations of Z, we may later choose a smaller bound $\|u\| < \delta$. We also note that by the normalization in [Bay19, Lemma 4.2] $|W(v) - Z(v)| \leqslant \|u\|\,|Z(v)|$ for all v with $Q(v) \geqslant 0$.

Let $\underline{\varsigma}$ denote the collection of stability conditions on the fibers with central charge W, where each $\varsigma_s = (W, \mathcal{Q}_s)$ is induced from $\sigma_s = (Z, \mathcal{P}_s)$ via Theorem 12.11. To show that $\underline{\varsigma}$ is a stability condition on $\mathcal{D}$, we only need to show that properties (2)–(3) of Definition 20.5 and (4)–(5) Definition 21.15 are also satisfied for $\underline{\varsigma}$.

We begin with a standard argument comparing the slicings at each point. For $\phi \in (0, 1)$, let $\epsilon \in [0, \frac{1}{2}]$ be such that $\frac{\sin \pi\epsilon}{\sin \pi\phi} = \|u\|$; then $0 < \phi - \epsilon < \phi + \epsilon < 1$.

Lemma 22.3. *For all $s \in S$ we have $\mathcal{Q}_s(\phi) \subset \mathcal{P}_s[\phi - \epsilon, \phi + \epsilon]$.*

Proof. Let $E \in \mathcal{Q}_s(\phi)$ and let $A \subset E$ be the first step of its HN filtration with respect to Z. Then as $Q(A) \geqslant 0$,

$$\phi \geqslant \phi(W(A)) \geqslant \phi\big(Z(A) + \|u\|\,|Z(A)|\big).$$

The last term depends only on $\phi(Z(A))$, and the law of sines in the triangle $0, Z(A), Z(A) + \|u\|\,|Z(A)|$ shows that it equals ϕ for $\phi(Z(A)) = \phi + \epsilon$. Therefore, $\phi^+(E) = \phi(Z(A)) \leqslant \phi + \epsilon$. An analogous argument with the maximal destabilizing quotient of E shows $\phi^-(E) \geqslant \phi - \epsilon$. □

Proof of Theorem 22.2. Since each ς_s satisfies the support property with respect to the same quadratic form as σ_s, property (4) will be automatic. Also, (1) holds by construction of $K_{\mathrm{num}}(\mathcal{D}/S)$.

By Proposition 20.8, $\mathcal{P}[\phi - \epsilon, \phi + \epsilon]$ is an open substack of $\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$; to prove that $\underline{\varsigma}$ satisfies (2), it thus remains to show that $\mathcal{M}^{\mathrm{st}}_{\underline{\varsigma}}(\mathbf{v}) \subset \mathcal{P}[\phi - \epsilon, \phi + \epsilon]$ is an open substack. So consider an object $E \in \mathcal{D}_T$ with $E_t \in \mathcal{P}_t[\phi - \epsilon, \phi + \epsilon]$ for all $t \in T$. Applying Lemma 22.3 to a quotient $E_{\bar{t}} \twoheadrightarrow Q$ after base change to the algebraic closure $\mathcal{A}_{\bar{t}}$, we see that any such quotient that is W-semistable with $\phi(W(Q)) \leqslant \phi$ satisfies $\phi(Z(Q)) \leqslant \phi + \epsilon$. In particular, any such quotient occurs in the Quot scheme $\mathrm{Quot}_T^{\leqslant \phi+\epsilon}(E)$ featured in our Grothendieck Lemma 21.21.

Since the class of objects in $K_{\mathrm{num}}(\mathcal{D}/S)$ is locally constant in families, the condition $\phi(W(Q)) \leqslant \phi$ picks out a union of finitely many connected components of $\mathrm{Quot}_T^{\leqslant \phi+\epsilon}(E)$. The union of their images in T is the locus where E is not geometrically W-stable. The image of each component is closed by universal closedness of the Quot scheme in Lemma 21.21. Therefore, openness of geometric stability holds for $\underline{\varsigma}$, verifying (2).

Now fix $\mathbf{v} \in \Lambda$. By Lemma 22.3, we have $\mathcal{M}_{\underline{\varsigma}}(\mathbf{v}) \subset \mathcal{P}([\phi - \epsilon, \phi + \epsilon]; \mathbf{v})$. The latter is bounded by Lemma 21.20, hence also the former, establishing condition (5).

Finally, given a Dedekind scheme $C \to S$ essentially of finite type over S, the fiberwise collection of t-structures on $\mathcal{D}_c$ for $c \in C$ induced by $\underline{\sigma}$ integrates to a C-local heart $\mathcal{A}_C$, in the sense of Definition 10.10; it universally satisfies openness of flatness by Proposition 20.8. Since the hearts $\mathcal{A}_c$ on the fibers are the same for σ_c and ς_c, the same holds for $\underline{\varsigma}$. We conclude with Theorem 18.7. □

23. Inducing stability conditions over S

Let $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ be an S-linear strong semiorthogonal component of finite cohomological amplitude, with a relative exceptional collection $E_1, \dots, E_m$ (see Definition 3.19). We write $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ for the S-linear semiorthogonal decomposition given by Lemma 3.25 with $\mathcal{D}_2 = \langle \alpha_{E_1}(\mathrm{D}^{\mathrm{b}}(S)), \dots, \alpha_{E_m}(\mathrm{D}^{\mathrm{b}}(S)) \rangle$. The goal of this section is a criterion to induce a stability condition on $\mathcal{D}_1$ over S from a weak stability condition on $\mathcal{D}$ over S; this generalizes the absolute case treated in [BLMS17, Proposition 5.1]. Recall that we already studied how to induce local t-structures on $\mathcal{D}_1$ in Section 7.

We consider the saturated subgroup generated by the image of v

$$\Lambda_1 := \langle v(K_{\mathrm{num}}(\mathcal{D}_1/S)) \rangle \subset \Lambda. \tag{23.1}$$

Also recall the subgroup $\Lambda_0 \subset \Lambda$ introduced in Definition 21.9, generated by classes of semistable objects with vanishing central charge.

Theorem 23.1. *In Setup IV.1 we further assume that g is smooth. Let $\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{A}_s))_{s \in S}$ be a weak stability condition on $\mathcal{D}$ over S. Assume the following conditions hold:*

(1) $(E_i)_s \in \mathcal{A}_s$ *for all i and $s \in S$.*
(2) $\mathrm{S}_{\mathcal{D}_s}((E_i)_s) \in \mathcal{A}_s[1]$ *for all i and $s \in S$, where $\mathrm{S}_{\mathcal{D}_s}$ denote the Serre functor of $\mathcal{D}_s$.*
(3) $v(E_i) \notin \Lambda_0$ *for all i.*
(4) $\Lambda_0 \cap \Lambda_1 = 0$.
(5) *For all $\mathbf{v} \in \Lambda$, the set*

$$\{F \in M_{\underline{\sigma}}(\mathbf{v}') \colon \mathbf{v}' \in \mathbf{v} + \Lambda_0, \chi(E_i, F) \geqslant 0 \text{ for all } i = 1, \dots, m\}$$

is bounded.

For each $s \in S$, let $\mathcal{A}_{s,1}$ be the heart in $(\mathcal{D}_1)_s$ given by Corollary 7.6, and let $Z_{s,1}$ be the central charge given by the restriction of Z_s along $K((\mathcal{D}_1)_s) \to K(\mathcal{D}_s)$. Then the collection

$$\underline{\sigma}_1 = ((\sigma_s)_1 = (Z_{s,1}, \mathcal{A}_{s,1}))_{s \in S}$$

is a stability condition on $\mathcal{D}_1$ over S with respect to Λ_1.

Proof. By [BLMS17, Proposition 5.1], $\underline{\sigma}_1 = ((\sigma_s)_1 = (Z_{s,1}, \mathcal{A}_{s,1}))_{s \in S}$ is a collection of numerical stability conditions. We need to check that $\underline{\sigma}_1$ satisfies the conditions (1)-(5) in Definitions 20.5 and 21.15. To simplify notation, we assume throughout the proof that there is only one exceptional object E; the general case holds by similar arguments.

Step 1. $\underline{\sigma}_1$ universally has locally constant central charges, i.e., condition (1) holds.

This is automatic because the central charges of $\underline{\sigma}_1$ are the restrictions of those of $\underline{\sigma}$.

Step 2. The fiberwise collection of t-structures $\underline{\tau}_1$ underlying $\underline{\sigma}_1$ universally satisfies openness of flatness. Moreover, for any Dedekind scheme $C \to S$ essentially of finite type over S, $\underline{\tau}_1$ integrates over C to a bounded C-local t-structure whose heart has a C-torsion theory.

By Proposition 20.8 the fiberwise collection of t-structures underlying $\underline{\sigma}$ universally satisfies openness of flatness, so by construction the same follows for $\underline{\sigma}_1$. Let $\mathcal{A}_C$ be the heart of the HN structure on $\mathcal{D}_C$ that integrates $\underline{\sigma}$ over C. Since by assumption $E_c \in \mathcal{A}_c$ for all $c \in C$, Lemma 6.13 shows $E_C \in \mathcal{A}_C$. Now Corollary 7.6 applies to show $(\mathcal{A}_C)_1$ is the heart of a bounded C-local t-structure on $\mathcal{D}_1$, which by Remark 7.8 integrates $\underline{\tau}_1$ over C. Finally, $(\mathcal{A}_C)_1$ has a C-torsion theory by Theorem 17.1.

Step 3. $\underline{\sigma}_1$ satisfies condition (4) in the support property.

The existence of the uniform quadratic form Q on $(\Lambda_1)_{\mathbb{R}}$ with properties (4a) and (4b) follows directly from the proof of [BLMS17, Proposition 5.1], as the construction of the quadratic form on each fiber given there depends only on the slopes of the exceptional objects.

Step 4. $\underline{\sigma}_1$ satisfies the following boundedness property: given $\mathbf{v} \in \Lambda_1$ and $0 < \phi_0 < \phi_1 \leqslant 1$, there exists a scheme B of finite type over S and a $\underline{\tau}_1$-flat object $\widetilde{F} \in (\mathcal{D}_1)_B$, such that every geometric point of $\mathcal{P}_{\underline{\sigma}_1}([\phi_0, \phi_1], \mathbf{v})$ is of the form $\widetilde{F}_{\bar{b}}$ for some geometric point $\bar{b}$ of B.

Indeed, under the simplifying assumption that we have only one exceptional object E, let

$$\phi^- := \min\left(\phi_0, \phi^-(E)\right),$$

let $\mathbf{v} \in \Lambda_1$, let $\bar{s}$ be a geometric point of S, and let $F \in \mathcal{P}_{\underline{\sigma}_1}([\phi_0, \phi_1], \mathbf{v})(\bar{s})$. By [BLMS17, Remark 5.12], we have $F \in \mathcal{P}_{\underline{\sigma}}([\phi^-, 1])$. Now consider any short exact sequence $A \hookrightarrow F \twoheadrightarrow A'$ in $\mathcal{A}$ corresponding to a Harder-Narasimhan filtration step of F with respect to $\sigma_{\bar{s}}$. The class of A' in Λ/Λ_0 is bounded by the support property. Moreover, as in the proof of Corollary 7.6 we see that $\mathrm{Ext}^k(E, A') = 0$ for $k \geqslant 2$, and similarly $\mathrm{Ext}^1(E, A') = \mathrm{Ext}^2(E, A) = 0$. Therefore, $\chi(E, A') \geqslant 0$, and the set of such A' is bounded by assumption (5). This in turn implies that the set of classes $v(A) \in \Lambda$ of A, and hence also the set of A itself is bounded. Using Lemma 9.6 one concludes the proof of Step 4.

Step 5. $\underline{\sigma}_1$ universally satisfies openness of geometric stability and satisfies boundedness, i.e., conditions (2) and (5) hold.

By Lemma 20.3, to show (2) it suffices to show that if $T \to S$ is finite type morphism from a connected affine scheme and $F \in \mathrm{D}(\mathcal{X}_T)$ is a T-perfect object, then the set

$$U = \{t \in T \mid F_t \in (\mathcal{D}_1)_t \text{ and is geometrically } \sigma_{1,t}\text{-stable}\}$$

is open. By Step 2 the fiberwise collection of t-structures $\underline{\tau}_1$ underlying $\underline{\sigma}_1$ universally satisfies openness of flatness, so we may assume $F_t \in \mathcal{A}_{t,1}$ for all $t \in T$, i.e., $F \in \mathcal{M}_{\underline{\tau}_1}(T)$. By [BLMS17, Remark 5.12], if F_t is geometrically $\sigma_{1,t}$-stable, then we have

$$\phi^-_{\sigma_t}(F_t) \geqslant \phi_0 := \min(\phi_{\sigma_{1,t}}(F_t), \phi^-(E_T)),$$

and thus $F_t \in \mathcal{P}_{\sigma_t}(\phi_0, 1]$. By Proposition 20.8, $\underline{\sigma}$ universally satisfies openness of lying in $\mathcal{P}(\phi_0, 1]$, so we may therefore assume $F_t \in \mathcal{P}_{\sigma_t}(\phi_0, 1]$ for all $t \in T$.

Let $\mathrm{Quot}_{T,\underline{\tau}_1}(F) \to T$ denote the Quot space of F with respect to the fiberwise collection of t-structures $\underline{\tau}_1$. As in Lemma 21.21, for $\phi \in (0, 1)$ we let $\mathrm{Quot}^{\leq\phi}_{T,\underline{\tau}_1}(F)$ be the subfunctor of $\mathrm{Quot}_{T,\underline{\tau}_1}(F)$ which assigns to $T' \in (\mathrm{Sch}\,/T)$ the set of $(F_{T'} \to Q) \in \mathrm{Quot}_{T,\underline{\tau}_1}(F)(T')$

satisfying $\phi(Q_{t'}) \leqslant \phi$ for all $t' \in T'$. We claim that $\mathrm{Quot}^{\leq\phi}_{T,\underline{\tau}_1}(F)$ is an algebraic space of finite type over T, and the morphism $\mathrm{Quot}^{\leq\phi}_{T,\underline{\tau}_1}(F) \to T$ is universally closed.

Indeed, using the same arguments as in the proof of Proposition 11.6 it is enough to know that the quotients occurring in $\mathrm{Quot}^{\leq\phi}_{T,\underline{\tau}_1}(F)$ belong to a bounded $\underline{\tau}_1$-flat family of objects; since each such quotient lies in $\mathcal{P}_{\underline{\sigma}_1}(\phi_0, 1](\mathbf{v})$ for some $\mathbf{v} \in \Lambda_1$, and since the set of possible $\mathbf{v}$ is finite by the same arguments as in Lemma 21.21, this follows from the previous steps.

The non-geometrically stable locus of F is the union of the images of all connected components of $\mathrm{Quot}^{\leq\phi}(F)$ except those where the quotient Q satisfies $v(Q) = v(F)$; hence it is closed by Proposition 11.11, and thus being geometrically stable is open in T.

To conclude Step 5, we note that universal openness of geometric stability, combined with the weaker boundedness statement in Step 4, immediately implies boundedness of $\underline{\sigma}_1$.

Step 6. $\underline{\sigma}_1$ integrates to a HN structure along any Dedekind scheme $C \to S$ essentially of finite type over S, i.e., condition (3) *holds.*

This follows from the previous steps, as explained in Remark 20.7. □

Part V. Construction

24. Main construction statements

The main goal of Part V is to construct stability conditions for families of surfaces, or of threefolds that individually admit stability conditions. The first step is to show that we can *tilt* the weak stability condition given by slope-stability on the fibers in order to obtain a relative version of *tilt-stability* (see Section 25); for threefolds, we have to tilt a second time in order to produce a stability condition over the base (see Section 26).

In more detail, we formulate slope stability as a weak stability condition with heart $\operatorname{Coh} X$, and central charge depending on $\mathrm{ch}_0, \mathrm{ch}_1$ only. We use this to obtain a new heart $\operatorname{Coh}^\beta X$ via tilting. Using the classical Bogomolov–Gieseker inequality[6], one then constructs a weak stability condition, called *tilt-stability*, with heart $\operatorname{Coh}^\beta X$ and with central charge depending on $\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2$. Tilting again produces a heart $\mathcal{A}^{\alpha,\beta}$. Assuming a conjectural Bogomolov–Gieseker type inequality for tilt-stable objects in $\operatorname{Coh}^\beta X$, proposed in [BMT14], one can then produce an actual stability condition with heart $\mathcal{A}^{\alpha,\beta}$.

Generalizing each of these steps to families of threefolds will lead to the following result; the existence of moduli spaces generalizes [PT19, Theorem 1.5] to the relative setting.

Theorem 24.1. *Let $g\colon \mathcal{X} \to S$ be a polarized flat family of smooth projective varieties.*

(1) *If the fibers of g are either one-dimensional, or two-dimensional satisfying the classical Bogomolov–Gieseker inequality, then the standard construction of stability conditions on curves or surfaces produces a stability condition $\underline{\sigma}$ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S.*
(2) *If the fibers of g are three-dimensional and additionally satisfy the conjectural Bogomolov–Gieseker inequality of* [BMT14, BMS16]*, then the construction of stability conditions proposed in [ibid.] produces a stability condition $\underline{\sigma}$ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S.*

In each of these situations, given a vector $\mathbf{v} \in \Lambda$ for the corresponding choice of Λ, the relative moduli space $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ exists as an algebraic stack of finite type over S, and the map $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) \to S$ satisfies the strong valuative criterion of universal closedness. If $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) = \mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$, then $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ has a coarse moduli space $M_{\underline{\sigma}}(\mathbf{v})$, proper over S. Finally, in characteristic 0, $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ always admits a good moduli space $M_{\underline{\sigma}}(\mathbf{v}) \to S$ which is proper over S.

We in fact prove a slightly stronger version of (1) in Proposition 25.3, and a weak version of (2) but for arbitrary dimension, and without assuming the conjectural inequality, in Proposition 26.1. The latter will be crucial for us in the case of cubic fourfolds in Part VI.

Remark 24.2. The construction of stability conditions via tilting depends on two globally defined $\mathbb{Q}$-divisors ω and B on $\mathcal{X}$, with ω relatively ample. As in [BMS16], we assume that ω and B are parallel, namely

$$\omega = \alpha H \qquad \text{and} \qquad B = \beta H,$$

where $H = c_1(\mathcal{O}_{\mathcal{X}}(1))$ is the polarization. Using [PT19, Section 3] one can extend our arguments to B and ω not necessarily parallel.

[6] The classical Bogomolov–Gieseker inequality [Rei78, Bog78, Gie79] holds in characteristic zero. In positive characteristic it holds e.g. for varieties that can be lifted to characteristic zero, see [Lan15, Theorem 1]; the weaker version of [Lan04] is not enough to define tilt-stability.

25. Tilting slope-stability in families

In this section we show that the notion of tilt-stability from [BMT14] works in families. We will divide the construction into two steps. We first rotate slope-stability and show that this gives a family of weak stability conditions as well. Then we use deformation of families of weak stability conditions to prove the analogue statement for tilt-stability. When specialized to families of curves or surfaces, this will prove part (1) of Theorem 24.1.

We continue to work in Setup IV.1, with the additional assumptions that $g\colon \mathcal{X} \to S$ is a smooth projective morphism of relative dimension $n \leqslant 3$. We fix a relatively ample divisor $\mathcal{O}_{\mathcal{X}}(1)$.

Since $g\colon \mathcal{X} \to S$ is smooth, instead of working with the relative Hilbert polynomial as in Example 21.8 we can work with Chern characters. To make this precise, observe that the Chern characters on the fibers, when paired with the appropriate power of the relative ample class $H = c_1(\mathcal{O}_{\mathcal{X}}(1))$, yield maps, for all $s \in S$,

$$\mathrm{ch}_{\mathcal{X}_s}\colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_s)) \to \mathbb{Q}^{n+1}.$$

These maps are locally constant in families, and thus they factor through a map

$$\mathrm{ch}_{\mathcal{X}/S} = \oplus_{i=0}^{n} \mathrm{ch}_{\mathcal{X}/S,i}\colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \to \mathbb{Q}^{n+1}.$$

By the Hirzebruch-Riemann-Roch theorem, the image of $\mathrm{ch}_{\mathcal{X}/S}$ coincides with the image of the Hilbert polynomial; we denote it by Λ.

Consider the weak stability condition

$$\underline{\sigma} := \left(\sigma_s = \left(\mathrm{i}\,\mathrm{ch}_{\mathcal{X}/S,0} - \mathrm{ch}_{\mathcal{X}/S,1}, \mathrm{Coh}\,\mathcal{X}_s\right)\right)$$

on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by slope-stability on each fiber; its properties can be verified as in Example 21.18. The weak stability conditions σ_s have the tilting property, and, for every Dedekind domain C, the HN structure σ_C has the tilting property as well, see Examples 14.13 and 19.4.

If $n \geqslant 2$, we let $\Lambda_0 \subset \Lambda$ be the subgroup generated by the image of $\mathrm{ch}_{\mathcal{X}/S,2} \oplus \mathrm{ch}_{\mathcal{X}/S,3}$ and, if $n = 3$, we let $\Lambda_0^\sharp \subset \Lambda_0$ be the one generated by the image of $\mathrm{ch}_{\mathcal{X}/S,3}$. We write $\boldsymbol{\eta}$ for the class of a point, which generates $\Lambda_0^\sharp$. Finally, we denote by $\overline{\Lambda} := \Lambda/\Lambda_0$ and by $\overline{\Lambda}^\sharp := \Lambda/\Lambda_0^\sharp$.

25.1. **Rotating slope-stability in families.** Let $\beta \in \mathbb{Q}$. To simplify the notation, we write

$$\mathrm{ch}^{\beta}_{\mathcal{X}/S}(_) = \mathrm{ch}_{\mathcal{X}/S}\left(e^{-\beta H} \cdot _\right) \in \Lambda_{\mathbb{Q}}.$$

For every $s \in S$ we define

$$\sigma_s^{\sharp\beta} := \left(Z_s = \mathrm{i}\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,1} + \mathrm{ch}^{\beta}_{\mathcal{X}_s,0}, \mathrm{Coh}^{\beta}\,\mathcal{X}_s\right),$$

where $\mathrm{Coh}^{\beta}\,\mathcal{X}_s$ is the tilt of $\mathrm{Coh}\,\mathcal{X}_s$ at the slope β, defined as in Section 14.2.

Proposition 25.1. *The collection $\underline{\sigma}^{\sharp\beta} := (\sigma_s^{\sharp\beta})$ is a (weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S with respect to $\overline{\Lambda}$. Moreover, if $n = 3$ and we fix $\mathbf{v} \in \Lambda$, then $\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}^{\sharp\beta}}(\mathbf{v} + b\boldsymbol{\eta}) = \emptyset$ for $b \gg 0$.*

Remark 25.2. In the case of slope-stable torsion free sheaves, the analogue of the last claim follows from boundedness: for every slope-stable sheaf E of class $\mathbf{v} + b\boldsymbol{\eta}$, the kernel of any surjection $E \twoheadrightarrow T$ for T a sheaf of length b will be stable of class $\mathbf{v}$; thus the dimension of $M^{\mathrm{st}}_{\sigma}(\mathbf{v})$ is at least $3b$.

Proof. For $n = 1$ this was already observed in Remark 21.19, so we assume $n \geqslant 2$.

As shown in Proposition 14.16, $\sigma_s^{\sharp\beta}$ is a weak stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_s)$ for all $s \in S$. There is nothing to prove for Definition 21.15.(4). Conditions (0) and (3') in Definition 20.5 follow from Propositions 14.16 and 19.5, respectively, and condition (1) is immediate.

Recall from Lemma 14.17 the classification of $\sigma_s^{\sharp\beta}$-semistable objects $E \in \mathrm{Coh}^\beta \mathcal{X}_s$:

(1) if $\mathrm{ch}_{\mathcal{X}_s,0}(E) \geqslant 0$, then E is a torsion free slope-semistable sheaf, or a torsion sheaf with support either pure of codimension one, or of codimension $\geqslant 2$;
(2) if $\mathrm{ch}_{\mathcal{X}_s,0}(E) < 0$, then E is an extension

$$U[1] \to E \to V \tag{25.1}$$

where U is a torsion free slope-semistable sheaf and V is a torsion sheaf supported in codimension $\geqslant 2$. Moreover, if either $\mathrm{ch}^\beta_{\mathcal{X}_s,1}(E) > 0$ or E is $\sigma_s^{\sharp\beta}$-stable, then $\mathrm{Hom}(V', E) = 0$, for all sheaves $V' \in \mathrm{Coh}\,\mathcal{X}_s$ supported in codimension $\geqslant 2$; in particular, U is reflexive.

We use this to show openness in the sense of conditions (2) and (2') for geometrically stable/semistable objects of the form (25.1). Standard arguments show openness of the condition that H^{-1} is torsion-free, and that H^0 is supported codimension two. Using a flattening stratification for $\mathrm{H}^i(E)$ shows that the semistable locus is constructible, since being reflexive or semistable is open in flat families of sheaves. It remains to show that the unstable locus is closed under specialization. This is easy to show for both the Hom-vanishing from sheaves supported in codimension two and the slope-semistability of H^{-1}: in both cases, we first lift the destabilizing sheaf from the generic point to the appropriate DVR, and then extend the morphism to one that specializes to a non-zero morphism.

To finish the proof, we will simultaneously prove the second claim of the proposition and boundedness in the sense of Definition 21.15.(5) by proving that $\coprod_{b\geqslant 0} \mathcal{M}^{\mathrm{st}}_{\underline{\sigma}^{\sharp\beta}}(\mathbf{v} + b\boldsymbol{\eta})$ is bounded. First we consider the case in which $\mathrm{ch}_{\mathcal{X}_s,0}(E) \geqslant 0$. Torsion sheaves can never be strictly $\sigma^{\sharp\beta}$-stable; thus the claim follows directly from boundedness of slope-stable sheaves, [Lan04, Theorem 4.2], and Remark 25.2.

Thus we are left to consider the case $\mathrm{ch}_{\mathcal{X}_s,0}(E) < 0$. After modifying β slightly if necessary (which will not affect stability of objects of class $\mathbf{v} + b\boldsymbol{\eta}$), we may assume $\mathrm{ch}^\beta_{\mathcal{X}_s,1}(E) > 0$. Let $\mathbb{D}_s = \mathcal{H}om(-, \mathcal{O}_{\mathcal{X}_s})[1]$. By [BLMS17, Lemma 2.19], for such E there is an exact triangle

$$E^\sharp \to \mathbb{D}_s(E) \to W[-1]$$

where $E^\sharp$ is a torsion free slope-stable sheaf and W is supported in codimension 3. Moreover, for all $i = 0, \dots, 3$, $\mathrm{ch}_{\mathcal{X}_s,i}(\mathbb{D}_s(E)) = (-1)^{i+1}\,\mathrm{ch}_{\mathcal{X}_s,i}(E)$. Hence

$$\begin{aligned} &\mathrm{ch}_{\mathcal{X}_s,i}(E^\sharp) = (-1)^{i+1}\,\mathrm{ch}_{\mathcal{X}_s,i}(E), \text{ for all } i = 0, 1, 2, \text{ and} \\ &\mathrm{ch}_{\mathcal{X}_s,3}(E^\sharp) = \mathrm{ch}_{\mathcal{X}_s,3}(E) + \mathrm{ch}_{\mathcal{X}_s,3}(W) \geqslant \mathrm{ch}_{\mathcal{X}_s,3}(E). \end{aligned}$$

Since $E^\sharp$ is a torsion free slope-stable sheaf, the first case in [Lan04, Theorem 4.4] shows that it belongs to a bounded family. This also implies that $\mathrm{ch}_{\mathcal{X}_s,3}(W)$ can only take finitely many values, and so W also belongs to a bounded family. By Lemma 9.6, the same holds for $\mathbb{D}_s(E)$. Since the morphism g is smooth, the duality functor exists in families and commutes with base change; therefore, E also belongs to a bounded family, as we wanted. □

25.2. **Tilt-stability in families.** Let $\alpha, \beta \in \mathbb{Q}$, $\alpha > 0$. We now deform the weak stability condition $\underline{\sigma}^{\sharp\beta}$ of Section 25.1 with respect to α.

For every $s \in S$ we define

$$\sigma_s^{\alpha,\beta} := \left(Z_s^{\alpha,\beta} = \mathrm{i}\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,1} + \frac{\alpha^2}{2}\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,0} - \mathrm{ch}^{\beta}_{\mathcal{X}_s,2}, \mathrm{Coh}^{\beta}\,\mathcal{X}_s \right).$$

We can now prove the following.

Proposition 25.3. *Assume that the Bogomolov–Gieseker inequality holds for slope-stable sheaves on the fibers of $g\colon \mathcal{X} \to S$, namely for all $s \in S$ and for all σ_s-stable sheaves E*

$$\Delta_s(E) := \mathrm{ch}_{\mathcal{X}_s,1}(E)^2 - 2\,\mathrm{ch}_{\mathcal{X}_s,0}(E)\,\mathrm{ch}_{\mathcal{X}_s,2}(E) \geqslant 0.$$

Then the collection $\underline{\sigma}^{\alpha,\beta} := (\sigma_s^{\alpha,\beta})$ is a (weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S with respect to $\overline{\Lambda}^{\sharp}$. Moreover $\sigma_s^{\alpha,\beta}$ has the tilting property, for all $s \in S$, and for any base change $C \to S$ from a Dedekind scheme C, the HN structure $\sigma_C^{\alpha,\beta}$ has the tilting property as well. Finally, if $n = 3$ and we fix $\mathbf{v} \in \Lambda$, then $\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}^{\alpha,\beta}}(\mathbf{v} + b\boldsymbol{\eta}) = \emptyset$ for $b \gg 0$.

Combined with Theorem 22.2 in case $n = 1, 2$, this in particular gives Theorem 24.1.(1).

Proof. The proof is a relative version of [PT19, Section 4.5]. The extension can be done analogously as in the proof of Theorem 22.2; the difference is that at $\sigma^{\infty,\beta} := \sigma^{\sharp\beta}$ we start with the weak stability condition given in Proposition 25.1. The key observation is the following:

Claim 1. Given $\overline{\mathbf{v}} \in \overline{\Lambda}^{\sharp}$, then for $\sigma^{\alpha,\beta}$-semistable objects $E \in \mathrm{Coh}^{\beta}\,\mathcal{X}_s$ of class $\overline{\mathbf{v}}$, there are only finitely many possible classes in $\overline{\Lambda}^{\sharp}$ of HN factors of E with respect to $\sigma^{\infty,\beta}$.

This claim is purely a statement on fibers, and is shown for example in the proof [BMT14, Theorem 7.3.1] or [BMS16, Theorem 3.5]; see also [PT19, Lemma 2.26]. The implied finiteness of wall-crossing as $\alpha \to \infty$ also shows that the Bogomolov–Gieseker inequality is preserved: for each wall-crossing, it follows by simple linear algebra, see [BMS16, Lemma A.6].

Applying the strong boundedness statement of Proposition 25.1 to each HN factor of E, in combination with the previous claim then immediately gives the following refinement.

Claim 2. Given $\mathbf{v} \in \Lambda$, then for $\sigma^{\alpha,\beta}$-semistable objects $E \in \mathrm{Coh}^{\beta}\,\mathcal{X}_s$ of class $\mathbf{v}$, there are only finitely many possible classes in Λ of HN factors of E with respect to $\sigma^{\infty,\beta}$.

Boundedness as in Definition 21.15.(5), as well as the stronger boundedness claimed in the last statement of the Proposition, follow immediately; see also [PT19, Corollary 4.18].

The condition on (0) and the second part of (3') in Definition 20.5 follow trivially from the corresponding properties of $\underline{\sigma}^{\sharp\beta}$, as $Z^{\alpha,\beta}(_) = 0$ for objects of $\mathrm{Coh}^{\beta}\,\mathcal{X}_s$ or $\mathrm{Coh}^{\beta}\,\mathcal{X}_C$ is a stronger condition than $Z^{\sharp\beta}(_) = 0$, and as the property of being a noetherian torsion subcategory is clearly preserved by passing to a smaller subcategory.

We now want to show universal openness of geometric stability; we can restrict to phases ϕ with $0 < \phi < 1$ (as $\sigma^{\alpha,\beta}$-semistable objects of phase 1 are the same as $\sigma^{\sharp\beta}$-semistable objects). Lemma 22.3 applies literally in our situation, which means we can restrict to the situation where $E \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_T)$ is contained in $\mathcal{P}^{\sharp\beta}[\phi - \epsilon, \phi + \epsilon]$. After possibly replacing T be an open subset, we can further assume $E_t \in \mathcal{P}_t^{\sharp\beta}[\phi - \epsilon, \phi + \epsilon]$, for all $t \in T$.

To show openness of geometric stability, note that we have already verified assumption (1) of Proposition 14.20 above; therefore, we can base change to the algebraic closure $\overline{t}$ and argue as in the proof of Theorem 22.2 to deduce it from boundedness of the Quot space

$\mathrm{Quot}_T^{\leqslant\phi+\epsilon}(E)$. This is defined analogously as in Grothendieck Lemma 21.21 as the subfunctor of the Quot space such that $\mathrm{Quot}_T^{\leqslant\phi+\epsilon}(E)(T')$ parametrizes quotients $E_{T'} \to Q$ that satisfy $\phi(Q_t) \leqslant \phi + \epsilon$ for all $t \in T'$, where the phase is calculated with respect to the weak stability condition $\underline{\sigma}^{\sharp\beta}$. By Claim 2, the classes of possible quotients are finite in Λ. Hence, the Quot space is bounded, and we can conclude the proof of (2) in Definition 20.5.

Property (1) in Definition 20.5 is immediate. The proof of universal generic openness, namely property (2') in Definition 20.5, now follows as in the proof of Lemma 20.4 when the phase is in $(0, 1)$ (and so JH filtrations exist), since as we observed $\sigma_s^{\alpha,\beta}$ can be base changed over any field extension and, for phase 1, it follows from the corresponding property of $\sigma_s^{\sharp\beta}$.

To prove part (3), or the remaining part of (3') in Definition 20.5, we use semistable reduction and Theorem 18.7. Let C be a Dedekind scheme. Then, by Proposition 25.1 and Proposition 20.8, the tilted category $\mathrm{Coh}^\beta \mathcal{X}_C$ universally satisfies openness of flatness. As we observed before, $\sigma_c^{\alpha,\beta}$ has the tilting property and so the assumptions of Theorem 18.7 are met, thus giving a HN structure $\sigma_C^{\alpha,\beta}$. Finally, as in the sheaf case discussed in Example 19.4, by using the dual functor $\mathbb{D}_C$ on $\mathrm{D^b}(\mathcal{X}_C)$ and Remark 19.3, it is not hard to check that $\sigma_C^{\alpha,\beta}$ also has the tilting property. □

26. Tilting tilt-stability in families of threefolds

In this section we consider the case of families of threefolds and we show that the double-tilt construction from [BMT14] works in families. As before, we first rotate tilt-stability, and show that this provides a family of weak stability conditions. Then we deform to complete the proof of part (2) of Theorem 24.1.

We keep the notation and setup of Section 25, to which we add the following assumptions: the morphism $g\colon \mathcal{X} \to S$ has relative dimension $n = 3$ and the Bogomolov–Gieseker inequality holds for slope-stable sheaves on its fibers.

26.1. **Rotating tilt-stability in families of threefolds.** Let $\alpha, \beta, \gamma \in \mathbb{Q}$ with $\alpha > 0$. Let $u_\gamma \in \mathbb{C}$ be the unit vector in the upper half plane such that $\gamma = -\frac{\Re u_\gamma}{\Im u_\gamma}$.

For every $s \in S$ we define

$$\sigma_s^{\alpha,\beta\sharp\gamma} := \left(Z_s^{\alpha,\beta\sharp\gamma} = \frac{1}{u_\gamma} \cdot Z_s^{\alpha,\beta}, \mathcal{A}_{\alpha,\beta,s}^\gamma \right),$$

where $\mathcal{A}_{\alpha,\beta,s}^\gamma$ is the tilted category of $\mathrm{Coh}^\beta \mathcal{X}_s$ at tilt-slope γ.

Proposition 26.1. *The collection $\underline{\sigma}^{\alpha,\beta\sharp\gamma} := (\sigma_s^{\alpha,\beta\sharp\gamma})$ is a weak stability condition on $\mathrm{D^b}(\mathcal{X})$ over S with respect to $\overline{\Lambda}^\sharp$.*

Proof. The argument is very similar to Proposition 25.1.

By [BLMS17, Proposition 2.15], for all $s \in S$, $\sigma_s^{\alpha,\beta\sharp\gamma}$ is a weak stability condition on $\mathrm{D^b}(\mathcal{X}_s)$. The classification of $\sigma_s^{\alpha,\beta\sharp\gamma}$-stable objects is identical to the one in the proof of Proposition 25.1, where the role of $\mathrm{ch}_{\mathcal{X}_s,0}^\beta$ and $\mathrm{ch}_{\mathcal{X}_s,1}^\beta$ is replaced by the real and imaginary parts of $Z_s^{\alpha,\beta\sharp\gamma}$, respectively. Moreover, the objects V_s in case (2) (in the proof of Proposition 25.1) are torsion sheaves supported on points.

Now, the proof works exactly in the same way as in Proposition 25.1. Here we use the derived dual functor $\mathbb{D}_{2,s} := \mathcal{H}om(-, \mathcal{O}_{\mathcal{X}_s})[2]$ and the fact that $\mathbb{D}_{2,s}(E_s)$ is directly a $\sigma_s^{\alpha,\beta\sharp\gamma}$-stable object in $\mathcal{A}_{\alpha,\beta,s}^\gamma$ in the proof of property (5) in Definition 21.15. □

Let X be a Fano threefold of Picard rank 1 over an algebraically closed field k. Let us denote by $\mathcal{K}u(X)$ its Kuznetsov component as defined in [Kuz09] and [BLMS17, Section 6]. By Lemma 3.25, the definition behaves nicely for smooth families.

By [BLMS17, Theorem 1.1], if X is not a complete intersection of a quadric and a cubic in $\mathbb{P}^5$, then Bridgeland stability conditions exist on $\mathcal{K}u(X)$. The construction is done by rotating tilt-stability and by inducing stability. By Proposition 26.1 and Theorem 23.1, these two steps work in families as well; see also Section 30.2 where a similar (more involved) argument is used in the cubic fourfold case. Hence, we obtain the following result.

Corollary 26.2. *Let $\mathcal{X} \to S$ be a smooth family of Fano threefolds of Picard rank 1 which are not complete intersections of a quadric and a cubic in $\mathbb{P}^5$. Let $\mathcal{K}u(\mathcal{X}/S)$ denote the relative Kuznetsov component. Then the space of numerical stability conditions on $\mathcal{K}u(\mathcal{X}/S)$ over S is non-empty.*

26.2. **Bridgeland stability in families of threefolds.** Let $\alpha, \beta, a, b \in \mathbb{Q}$ such that $\alpha > 0$ and

$$a > \frac{1}{6}\alpha^2 + \frac{1}{2}|b|\alpha.$$

We keep the notation in the previous section and fix $\gamma = 0$. For $s \in S$, we set

$$\mathcal{A}_{\alpha,\beta,s} := \mathcal{A}^{\gamma=0}_{\alpha,\beta,s},$$

$$Z^{a,b}_{\alpha,\beta,s} := \mathrm{i}\left(\mathrm{ch}^{\beta}_{\mathcal{X}_s,2} - \frac{\alpha^2}{2}\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,0}\right) + a\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,1} + b\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,2} - \mathrm{ch}^{\beta}_{\mathcal{X}_s,3},$$

and

$$\underline{\sigma}^{a,b}_{\alpha,\beta} := \left(\sigma^{a,b}_{\alpha,\beta,s} = \left(Z^{a,b}_{\alpha,\beta,s}, \mathcal{A}_{\alpha,\beta,s}\right)\right).$$

We can now prove the following which is the family version of [BMS16, Theorem 8.2].

Proposition 26.3. *Assume that the generalized Bogomolov–Gieseker inequality holds for tilt-stable objects on the fibers of g, namely for all $s \in S$ and for all $\sigma_{\alpha,\beta,s}$-stable objects E*

$$\nabla_{\beta,s}(E) := 4\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,2}(E)^2 - 6\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,1}(E)\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,3}(E) - \alpha^2\Delta_s(E) \geqslant 0. \tag{26.1}$$

Then the collection $\underline{\sigma}^{a,b}_{\alpha,\beta}$ is a stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S with respect to Λ.

Proof. The proof is the relative version of [PT19, Section 4.6]. The extension can be done analogously as in the proof of Proposition 25.3, the limit case being Proposition 26.1. □

By Theorem 22.2 this completes the proof of Theorem 24.1.(2); in particular, we can take $\alpha, \beta, a, b \in \mathbb{R}$.

Remark 26.4. The generalized Bogomolov–Gieseker inequality (26.1) was first proven for $\mathbb{P}^3$ in [Mac14] and, soon after, for the smooth quadric hypersurface in $\mathbb{P}^4$ in [Sch14]. The case of Fano threefolds of Picard rank one was treated in [Li19b]. The case of abelian threefolds is covered independently by [MP16] and [BMS16] (the full support property is now also known, see [OPT18]). Recently, the case of the quintic threefolds has finally been settled in [Li19a]. For other cases with higher Picard rank, we refer to [BMSZ17, Piy17, Kos18, Kos20]. Unfortunately, the inequality does not hold in general, at least for higher Picard rank, see [Sch17], as well as [Kos18, Appendix A] and [MS19b].

27. Stability conditions from degeneration

As an application of Proposition 25.3, one can use degeneration to prove the generalized Bogomolov–Gieseker inequality. The following is a variation of [Kos20, Proposition 3.2].

Proposition 27.1. *Let $g\colon \mathcal{X} \to C$ be a smooth family of polarized threefolds over a Dedekind scheme C of characteristic zero, and fix a point $0 \in C$. Consider an arbitrary $\mathbb{Q}$-divisor B on $\mathcal{X}$. Let B_0 (resp. B_η) be the restriction of B to the special fiber $\mathcal{X}_0 := g^{-1}(0)$ (resp. the general fiber $\mathcal{X}_\eta$). If the generalized Bogomolov–Gieseker inequality holds for tilt-stable objects on $\mathcal{X}_0$ with respect to B_0, i.e., for all σ_{α,B_0}-stable objects E*

$$\nabla_{B_0}(E) := 4\,\mathrm{ch}^{B_0}_{\mathcal{X}_0,2}(E)^2 - 6\,\mathrm{ch}^{B_0}_{\mathcal{X}_0,1}(E)\,\mathrm{ch}^{B_0}_{\mathcal{X}_0,3}(E) - \alpha^2\Delta_0(E) \geqslant 0,$$

then it also holds for tilt-stable objects on $\mathcal{X}_\eta$ with respect to H_η, B_η.

Proof. Assume that there exists a tilt-stable object E_η on $\mathcal{X}_\eta$ violating the inequality. We consider the relative tilt stability condition $\underline{\sigma}^{\alpha,B}$ coming from the analogue of Proposition 25.3 for non parallel classes αH and B (see Remark 24.2) and the relative moduli space $M_{\underline{\sigma}^{\alpha,B}}(\mathbf{v})$ over C with Chern character $\mathbf{v}$ as the Chern character of a C-flat lift E of E_η to $\mathcal{X}$.

Since $M_{\underline{\sigma}^{\alpha,B}}(\mathbf{v})$ is proper over C (by Theorem 24.1, based on Theorem 21.24.(2)) and non-empty at the generic point, it is non-empty on the special fiber $\mathcal{X}_0$, which is a contradiction. □

Remark 27.2. The proof of [Kos20, Proposition 3.2] gives the following variants. If $C = \mathbf{A}^1_k$, and if all fibers $\mathcal{X}_b$ for $b \neq 0$ are isomorphic (as in the case of a toric degeneration), then the generalized Bogomolov-Gieseker inequality for $\mathcal{X}_0$ implies the same inequality for *all* $\mathcal{X}_b$. Without this assumption, we obtain the result for *very general* $b \in C$, as any counterexample lives in a moduli space $M_{\underline{\sigma}}(\mathbf{v})$ proper over C, and there are countably many choices for $\mathbf{v}$.

28. Donaldson–Thomas invariants

As pointed out in [PT19], an immediate application of properness of the relative moduli space is that counting invariants of Donaldson–Thomas type arising from moduli spaces of stable objects in the derived category are actually deformation-invariant.

Let X be a smooth projective Calabi–Yau threefold with $H^1(X, \mathcal{O}_X) = 0$ over the complex numbers. We assume that the generalized Bogomolov–Gieseker inequality holds for tilt-stable objects in $\mathrm{D^b}(X)$; for example, by [Li19a], this holds for the quintic threefold. We consider the open subset $\mathrm{Stab}^\dagger(\mathrm{D^b}(X))$ of the space of stability conditions on $\mathrm{D^b}(X)$ constructed in [BMT14, BMS16] via the generalized Bogomolov–Gieseker inequality (or a variant of it).

For a stability condition $\sigma \in \mathrm{Stab}^\dagger(\mathrm{D^b}(X))$ and a numerical class $\mathbf{v} \in K_{\mathrm{num}}(\mathrm{D^b}(X))$, we consider the moduli stack $\mathcal{M}_\sigma(\mathbf{v})$. In the case $\mathcal{M}_\sigma(\mathbf{v}) = \mathcal{M}^{\mathrm{st}}_\sigma(\mathbf{v})$ the results of [HT10] show that the coarse moduli space $M_\sigma(\mathbf{v})$ has a symmetric perfect obstruction theory, and so a zero-dimensional virtual class $[M_\sigma(\mathbf{v})]^{\mathrm{vir}}$ and a *Donaldson–Thomas invariant*

$$\mathrm{DT}_\sigma(\mathbf{v}) := \int_{[M_\sigma(\mathbf{v})]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

We can use Theorem 21.24.(2) and Theorem 24.1 together with [PT19, Remark 5.4] (which is based on [BF97] and [HT10, Corollary 4.3]) to deduce the invariance of $\mathrm{DT}_\sigma(\mathbf{v})$ under complex deformations of X. In particular, by [Li19a], we get the following result:

Theorem 28.1. *The Donaldson–Thomas invariants $\mathrm{DT}_\sigma(\mathbf{v})$ counting stable objects on a smooth quintic threefold X, with respect to $\sigma \in \mathrm{Stab}^\dagger(X)$, are deformation-invariant.*

Part VI. Moduli spaces for Kuznetsov components of cubic fourfolds

29. Main applications to cubic fourfolds

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. Its *Kuznetsov component* is the admissible subcategory defined by

$$\mathcal{K}u(X) := \mathcal{O}_X^{\perp} \cap \mathcal{O}_X(H)^{\perp} \cap \mathcal{O}_X(2H)^{\perp} \subset \mathrm{D}^{\mathrm{b}}(X),$$

where H denotes the hyperplane class. The goal of this final part of the paper is to describe the structure of moduli spaces of stable objects in $\mathcal{K}u(X)$. We will work over the complex numbers throughout. Many of our arguments can be adapted to positive characteristic (and have interesting applications in that setting that will be discussed elsewhere), but the strongest results can be proved over $\mathbb{C}$.

The category $\mathcal{K}u(X)$ shares many properties with the derived category of K3 surfaces. Its foundations were developed in [Kuz10, AT14, Huy17]; see [Huy19, MS19a] for surveys of those results. In particular, we recall:

(1) $\mathcal{K}u(X)$ is a 2-Calabi–Yau category: $\mathrm{Hom}(E,F) = \mathrm{Hom}(F,E[2])^{\vee}$.
(2) The topological K-theory of $\mathcal{K}u(X)$, along with the faithful functor $\mathcal{K}u(X) \to \mathrm{D}^{\mathrm{b}}(X)$ and the Hodge structure on $H^4(X,\mathbb{Z})$ equips $\mathcal{K}u(X)$ with an extended *Mukai lattice*, which we denote by $\widetilde{H}(\mathcal{K}u(X),\mathbb{Z})$: as a lattice, it is isomorphic to $H^*(S,\mathbb{Z})$, for any K3 surface S; it carries a weight two Hodge structure with $h^{2,0} = 1$; and it admits a Mukai vector $v\colon K(\mathcal{K}u(X)) \to \widetilde{H}(\mathcal{K}u(X),\mathbb{Z})$ satisfying $(v(E),v(F)) = -\chi(E,F)$.

The Mukai lattice embeds into $K_{\mathrm{top}}(X) \subset H^*(X,\mathbb{Q})$ as the right orthogonal complement of the classes of $\mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H)$. We denote the sublattice of integral $(1,1)$-classes by $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z})$ and the image of the Mukai vector by $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z})$; the latter is isomorphic to $K_{\mathrm{num}}(\mathcal{K}u(X))$. The rational Hodge conjecture for cubic fourfolds (proved in [Zuc77]; see also [CM78] for a short proof) shows that $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Q})$ is isomorphic to $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Q})$. By the integral Hodge conjecture [Voi07], the two groups are actually isomorphic over $\mathbb{Z}$. While the rational Hodge conjecture will be used later in the proof of Lemma 30.4, the integral version is not needed in our argument, and in fact it will also follow from our results (see Corollary 29.8).

By [BLMS17, Theorem 1.2 and Remark 9.11], one can explicitly describe a non-empty connected open subset $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ in the space of numerical Bridgeland stability conditions on $\mathcal{K}u(X)$ (with respect to the lattice $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z})$ and the Mukai vector); it is the covering of a certain period domain, which is defined analogously to the case of K3 surfaces, treated in [Bri08]. We can then extend [Bri08, Theorem 1.1] as follows.

Theorem 29.1. *The open subset* $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ *is a connected component in* $\mathrm{Stab}(\mathcal{K}u(X))$.

The connected component $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ is realized as a covering $\eta\colon \mathrm{Stab}^{\dagger}(\mathcal{K}u(X)) \to \mathfrak{P}_0^+$, where $\mathfrak{P}_0^+$ is a period domain defined as follows. We take $\mathfrak{P} \subset \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{C})$ as the open subset consisting of those vectors whose real and imaginary parts span positive-definite two-planes. With $\Delta := \{\delta \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z})\colon (\delta,\delta) = -2\}$, we set

$$\mathfrak{P}_0 := \mathfrak{P} \setminus \bigcup_{\delta\in\Delta} \delta^{\perp},$$

which has two connected components; we let $\mathfrak{P}_0^+$ be the one containing the image under η of the examples of stability conditions constructed in [BLMS17, Theorem 1.2].

Let $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ be a non-zero primitive class, and let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$. Theorem 29.1 relies on our second main result, which concerns the existence and non-emptiness of the moduli space $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ of σ-stable objects in $\mathcal{K}u(X)$ with Mukai vector $\mathbf{v}$. It is the analogue of a long series of results [Bea83, Muk84, Muk87, O'G97, Huy97, Yos01] on moduli spaces of sheaves on K3 surfaces, the last one being [Yos01, Theorems 0.1 and 8.1].

Theorem 29.2. *Let X be a cubic fourfold. Then*

$$\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z}) = \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}).$$

Moreover, assume that $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ is a non-zero primitive vector and let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ be a stability condition on $\mathcal{K}u(X)$ that is generic with respect to $\mathbf{v}$. Then

(1) *$M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is nonempty if and only if $\mathbf{v}^2 \geqslant -2$. Moreover, in this case, it is a smooth projective irreducible holomorphic symplectic variety of dimension $\mathbf{v}^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.*

(2) *If $\mathbf{v}^2 \geqslant 0$, then there exists a natural Hodge isometry*

$$\theta\colon H^2(M_\sigma(\mathcal{K}u(X), \mathbf{v}), \mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbf{v}^\perp & \text{if } \mathbf{v}^2 > 0 \\ \mathbf{v}^\perp/\mathbb{Z}\mathbf{v} & \text{if } \mathbf{v}^2 = 0, \end{cases}$$

where the orthogonal is taken in $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

Here *generic* means that σ is not on a wall: since $\mathbf{v}$ is primitive, this means that stability and semistability coincide for objects of Mukai vector $\mathbf{v}$.

The embedding $\mathbf{v}^\perp \hookrightarrow \widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$ identifies the latter with the Markman–Mukai lattice of $M_\sigma(\mathcal{K}u(X), \mathbf{v})$, which determines the birational class of $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ by Markman's global Torelli theorem [Mar11, Corollary 9.9]; moreover, the Hodge classes of $\widetilde{H}(\mathcal{K}u(X, \mathbb{Z})$ control the Mori cone by [BHT15, Mon15].

Remark 29.3. Assume now that $\mathbf{v}$ is not primitive, i.e., $\mathbf{v} = m\mathbf{v}_0$, for some $m > 1$, and σ a $\mathbf{v}$-generic stability condition. Then the previous theorem implies immediately that the moduli space $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is non-empty if $\mathbf{v}_0^2 \geqslant -2$; conversely, if $\mathbf{v}_0^2 < -2$, one can show easily by induction on m that $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is empty. If the good moduli space $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is normal, one can prove further that $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is an irreducible proper algebraic space (by using a similar argument as in [KLS06, Theorem 4.4]). Moreover, either $\dim M_\sigma(\mathcal{K}u(X), \mathbf{v}) = \mathbf{v}^2 + 2$ and $M_\sigma^{\mathrm{st}}(\mathcal{K}u(X), \mathbf{v}) \neq \emptyset$, or $m > 1$ and $\mathbf{v}^2 \leqslant 0$.

Theorem 29.2 is proved by deformation to the case where $\mathcal{K}u(X)$ is known to be equivalent to the derived category of a K3 surface. Such deformation arguments rely on relative moduli spaces of Bridgeland stable objects, given by the following result.

Theorem 29.4. *Let $\mathcal{X} \to S$ be a family of cubic fourfolds, where S is a connected quasi-projective variety over $\mathbb{C}$. Let $\mathbf{v}$ be a primitive section of the local system of the Mukai lattices $\widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$ of the fibers, such that $\mathbf{v}$ is algebraic on all fibers. Assume that, for a closed point $s_0 \in S$ not contained in any Hodge locus, there exists a stability condition $\tau_{s_0} \in \mathrm{Stab}^\dagger(\mathcal{K}u(\mathcal{X}_{s_0}))$ that is generic with respect to $\mathbf{v}$, and whose central charge $Z_{s_0}\colon \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_{s_0}), \mathbb{Z}) \to \mathbb{C}$ is invariant under the monodromy action induced by the inclusion $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_{s_0}), \mathbb{Z}) \subset H^*(X, \mathbb{Q})$.*

(1) *If $S = C$ is a curve, then there exists an algebraic space $\widetilde{M}(\mathbf{v})$, and a smooth proper morphism $\widetilde{M}(\mathbf{v}) \to C$ that makes $\widetilde{M}(\mathbf{v})$ a relative moduli space over C: the fiber over any point $c \in C$ is a coarse moduli space $M_{\sigma_c}(\mathcal{K}u(\mathcal{X}_c), \mathbf{v})$ of stable objects in the Kuznetsov component of the corresponding cubic fourfold for some stability condition σ_c.*
(2) *There exist a non-empty open subset $S^0 \subset S$, a quasi-projective variety $M^0(\mathbf{v})$, and a smooth* projective *morphism $M^0(\mathbf{v}) \to S^0$ that makes $M^0(\mathbf{v})$ a relative moduli space over S^0.*
(3) *There exists an algebraic space $M(\mathbf{v})$ and a proper morphism $M(\mathbf{v}) \to S$ such that every fiber is a good moduli space $M_{\sigma_s}(\mathcal{K}u(\mathcal{X}_s), \mathbf{v})$ of semistable objects.*

In all cases, we can choose $\underline{\sigma}$ such that $M_{\sigma_{s_0}}(\mathcal{K}u(\mathcal{X}_{s_0}), \mathbf{v}) = M_{\tau_{s_0}}(\mathcal{K}u(\mathcal{X}_{s_0}), \mathbf{v})$.

Note that every fiber of the morphism $\widetilde{M}(\mathbf{v}) \to C$ is projective, but the morphism itself might not be. In contrast, we expect that the morphism $M(\mathbf{v}) \to S$ is always projective.

For a very general cubic fourfold, $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ is isomorphic to the lattice A_2 generated by two roots $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ with $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = -1$ and $\boldsymbol{\lambda}_1^2 = \boldsymbol{\lambda}_2^2 = 2$; for example, we can set

$$\boldsymbol{\lambda}_1 = v(p(\mathcal{O}_L(1))) \quad \text{and} \quad \boldsymbol{\lambda}_2 = v(p(\mathcal{O}_L(2))), \tag{29.1}$$

where L is a line in X and p is the left adjoint of $\mathcal{K}u(X) \hookrightarrow \mathrm{D}^{\mathrm{b}}(X)$. Applying the theorem above to classes in A_2 yields the following result.

Corollary 29.5. *For any pair (a, b) of coprime integers, there is a unirational locally complete 20-dimensional family, over an open subset of the moduli space of cubic fourfolds, of smooth polarized irreducible holomorphic symplectic manifolds of dimension $2n + 2$, where $n = a^2 - ab + b^2$. The polarization has either degree $6n$ and divisibility $2n$ if 3 does not divide n, or degree and divisibility $\frac{2}{3}n$ otherwise.*

It may be worth pointing out that, as observed for example in [Add16], an even integer $2n$ has the form $2n = 2(a^2 - ab + b^2)$ if and only if it satisfies Hassett's condition:

$(**)$ $2n$ is not divisible by 4, 9 or any odd prime $p \equiv 2 \pmod 3$.

By [Has00, Theorem 1.0.2], integers satisfying $(**)$ are related to cubic fourfolds with Hodge-theoretically associated K3 surfaces discussed in Corollary 29.7.

Example 29.6. Let S be the moduli space of cubic fourfolds. If we choose $\mathbf{v} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$ in Theorem 29.4, then by [LPZ18] $S^0 = S$, and $M(\mathbf{v})$ is the relative Fano variety of lines over S. For $\mathbf{v} = 2\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$, still by [LPZ18] (see also [LLMS18]), we have that $S^0 \subset S$ is the complement of cubics containing a plane, and $M^0(\mathbf{v})$ is the family of irreducible holomorphic symplectic eightfolds constructed by Lehn, Lehn, Sorger and van Straten [LLSvS17]. Finally, for $\mathbf{v} = 2\boldsymbol{\lambda}_1 + 2\boldsymbol{\lambda}_2$, in [LPZ20] the authors obtain an algebraic construction of a 20-dimensional family of 10-dimensional O'Grady spaces compactifying the twisted intermediate Jacobian fibration of the cubic hyperplane sections, birational to the one constructed in [Voi18].

Recall from Hassett's work on cubic fourfolds, [Has00], that there is a countable union of divisors of special cubics with a Hodge-theoretically associated K3 surface. In our notation, a cubic is contained in one of Hassett's special divisors if and only if $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ contains a hyperbolic plane.

Corollary 29.7. *Let X be a cubic fourfold. Then X has a Hodge-theoretically associated K3 if and only if there exists a smooth projective K3 surface S and an equivalence $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$.*

This (literally) completes a result by Addington and Thomas, [AT14, Theorem 1.1], who proved that every divisor described by Hassett contains an open subset of cubics admitting a derived equivalence as above. A version of the corollary also holds for K3 surfaces with a Brauer twist, completing a result by Huybrechts [Huy17, Theorem 1.4]; the corresponding Hodge-theoretic condition is the existence of a square-zero class in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ (see Proposition 33.1). Partial results were also obtained in [Kuz10, Mos18].

As pointed out to us by Voisin, the non-emptiness of moduli spaces also produces enough algebraic cohomology classes to reprove her result on the integral Hodge conjecture for cubic fourfolds; we also refer to [MO20, Corollary 0.3] for a different recent proof:

Corollary 29.8 ([Voi07, Theorem 18])**.** *The integral Hodge conjecture holds for X.*

Remark 29.9. In [Per20] our argument for proving the integral Hodge conjecture for cubic fourfolds has been further developed and generalized. In particular, a version of the integral Hodge conjecture is proved for suitable CY2 categories, without the use of stability conditions.

Our results also provide the full machinery of [BM14a], describing the birational geometry of $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ in terms of wall-crossing, but we will not discuss the details here.

Let us discuss the line of argument in the proofs of Theorems 29.1, 29.2 and 29.4. The key point is to generalize a deformation argument by Mukai, and show that the deformation of simple objects in the Kuznetsov components along a deformation of cubic fourfolds is unobstructed as long as their Mukai vector remains algebraic; combined with openness of stability this shows smoothness of the relative moduli space. The existence of the family of stability conditions is explained in Section 30, while the above deformation argument is in Section 31. The proofs of all results are then in Section 33.

30. Stability conditions on families of Kuznetsov components

In this section we recast Kuznetsov's work in [Kuz08] in the relative setting along the lines of [BLMS17]. The aim is to construct stability conditions on families of Kuznetsov components of cubic fourfolds.

30.1. **The Kuznetsov component in families.** In this section we study Kuznetsov components for families of smooth cubic fourfolds. We start by reconsidering the results in [Kuz08] and [BLMS17, Section 7] for families of cubic fourfolds. The proofs discussed here are very close to those presented in [BLMS17], hence we will be concise.

Let $g\colon \mathcal{X} \to S$ be a family of cubic fourfolds, where S is a quasi-projective variety over $\mathbb{C}$. We let $\mathcal{O}_{\mathcal{X}}(1)$ denote the relative very ample line bundle.

Lemma 30.1. *There exist an admissible subcategory $\mathcal{K}u(\mathcal{X}) \hookrightarrow \mathrm{D}^{\mathrm{b}}(\mathcal{X})$ and a strong S-linear semiorthogonal decomposition of finite cohomological amplitude*

$$\mathrm{D}^{\mathrm{b}}(\mathcal{X}) = \langle \mathcal{K}u(\mathcal{X}), \alpha_{\mathcal{O}_{\mathcal{X}}}(\mathrm{D}^{\mathrm{b}}(S)), \alpha_{\mathcal{O}_{\mathcal{X}}(1)}(\mathrm{D}^{\mathrm{b}}(S)), \alpha_{\mathcal{O}_{\mathcal{X}}(2)}(\mathrm{D}^{\mathrm{b}}(S))\rangle,$$

where $\alpha_{\mathcal{O}_{\mathcal{X}}(n)}(\mathrm{D}^{\mathrm{b}}(S)) := g^* \mathrm{D}^{\mathrm{b}}(S) \otimes \mathcal{O}_{\mathcal{X}}(n)$.

Proof. This follows immediately from Lemma 3.25, since $\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}(1), \mathcal{O}_{\mathcal{X}}(2)$ is a relative exceptional collection in $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$. □

Let us assume further that the family $g\colon \mathcal{X} \to S$ comes with a family $\mathcal{L} \subset \mathcal{X}$ over S of lines contained in the fibers of g

$$\begin{tikzcd} \mathbb{P}_S(g_*\mathcal{O}_{\mathcal{L}}(1)) = \mathcal{L} \arrow[r, hook] \arrow[dr] & \mathcal{X} \arrow[r, hook] \arrow[d] & \mathbb{P}_S(g_*\mathcal{O}_{\mathcal{X}}(1)) \arrow[dl] \\ & S & \end{tikzcd} \tag{30.1}$$

such that, for all $s \in S$, the line $\mathcal{L}_s$ is not contained in a plane in $\mathcal{X}_s$.

Example 30.2. Given any family $g\colon \mathcal{X} \to S$ of cubic fourfolds, we can always find a base change $g'\colon \mathcal{X}' \to S'$ satisfying this existence of an appropriate family of lines. For example, we can take S' to be the open subset $F^0(\mathcal{X}/S)$ of the relative Fano variety of lines $F(\mathcal{X}/S)$ in $\mathcal{X}$ consisting of all lines which are not contained in planes inside the fibers of g. The base-change $S' \to S$ can also be taken to be finite.

Let $\widetilde{\mathbb{P}}_S \to \mathbb{P}_S(g_*\mathcal{O}_{\mathcal{X}}(1))$ be the blow-up along $\mathcal{L}$. We denote by $\varepsilon\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ the strict transform of $\mathcal{X}$ via this blow-up (or equivalently the blow-up of $\mathcal{X}$ along $\mathcal{L}$). Consider the projective bundle $\mathbb{P}_S(g_*\mathscr{I}_{\mathcal{L}}(1))$ which, for simplicity, we denote by $\mathbb{P}^3_S$, even though it is not trivial. We let $\widetilde{\mathbb{P}}_S \to \mathbb{P}^3_S$ be the $\mathbb{P}^2$-bundle induced by the projection from $\mathcal{L}$, whose restriction to $\widetilde{\mathcal{X}}$ induces a conic fibration $q\colon \widetilde{\mathcal{X}} \to \mathbb{P}^3_S$.

As in [Kuz08, Section 3], we denote by $\mathcal{B}^S_0$ (resp. $\mathcal{B}^S_1$) the sheaf on $\mathbb{P}^3_S$ of even (resp. odd) parts of Clifford algebras corresponding to this fibration and, for all $m \in \mathbb{Z}$,

$$\mathcal{B}^S_{2m+1} = \mathcal{B}^S_1 \otimes \mathcal{O}_{\mathbb{P}^3_S}(m) \quad \text{and} \quad \mathcal{B}^S_{2m} = \mathcal{B}^S_0 \otimes \mathcal{O}_{\mathbb{P}^3_S}(m).$$

According to [Kuz08, Theorem 4.2] there is a strong S-linear semiorthogonal decomposition of finite cohomological amplitude

$$\mathrm{D}^{\mathrm{b}}(\widetilde{\mathcal{X}}) = \langle \Phi(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0)), q^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S)\rangle.$$

Here $\Phi\colon \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0) \to \mathrm{D}^{\mathrm{b}}(\widetilde{\mathcal{X}})$ is the fully faithful Fourier–Mukai functor whose kernel is explicitly described in [Kuz08, Section 4] (it corresponds to $\Phi_{-1,0}$ in [Kuz08, Proposition 4.9]), but such an explicit description is not needed in this paper. Denote by Ψ its left adjoint.

Proposition 30.3. *The functor $\Psi \circ \varepsilon^*\colon \mathcal{K}u(\mathcal{X}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0)$ is fully faithful and it induces a strong S-linear semiorthogonal decomposition of finite cohomological amplitude*

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0) = \left\langle \Psi(\varepsilon^*\mathcal{K}u(\mathcal{X})), \alpha_{\mathcal{B}^S_1}(\mathrm{D}^{\mathrm{b}}(S)), \alpha_{\mathcal{B}^S_2}(\mathrm{D}^{\mathrm{b}}(S)), \alpha_{\mathcal{B}^S_3}(\mathrm{D}^{\mathrm{b}}(S)) \right\rangle.$$

Proof. The result follows by repeating line by line the same proof as in [BLMS17, Proposition 7.7] in the relative setting above. □

30.2. **Existence of stability conditions in families.** The next step consists in constructing stability conditions on $\mathcal{K}u(\mathcal{X})$ over S in the sense of Definitions 20.5 and 21.15.

Let $g\colon \mathcal{X} \to S$ be a family of cubic fourfolds over a connected quasi-projective variety S over $\mathbb{C}$. Assume that $g\colon \mathcal{X} \to S$ is equipped with a family $\mathcal{L} \to S$ of lines which are not contained in planes in the fibers of g. Let $G := \mathrm{Mon}(g)$ be the monodromy group of g. Its action on $H^*(\mathcal{X}_s, \mathbb{Q})$ preserves, and thus acts on, $\widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$, for all $s \in S$; for very general $s \in S$, it also preserves Hodge classes on $\mathcal{X}_s$ (see [Voi13, Theorem 4.1]), and hence acts on $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$. The sublattice $\mathrm{Fix}(G) := \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})^G$, for $s \in S$ a very

general point, is then naturally identified with a saturated sublattice of $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s),\mathbb{Z})$, for all closed $s \in S$, by parallel transport.

Let M be a saturated sublattice of $\mathrm{Fix}(G)$ containing A_2 generated as in (29.1), and denote by $M^\vee$ the dual of M. By assumption, we have a sequence of natural homomorphisms

$$v_s\colon K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X}_s)) \hookrightarrow \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s),\mathbb{Z}) \hookrightarrow \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s),\mathbb{Z})^\vee \to M^\vee,$$

for all closed points $s \in S$. These homomorphisms determine a relative Mukai homomorphism in the sense of Definition 21.3.

Lemma 30.4. *There is a relative Mukai homomorphism $v\colon K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X})/S) \to M^\vee$, such that for any closed point $s \in S$ the composition*

$$K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X}_s)) \to K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X})/S) \xrightarrow{v} M^\vee$$

is equal to v_s. If S is smooth, v factors via a homomorphism $\mathcal{N}(\mathcal{K}u(\mathcal{X})/S) \to M^\vee$ out of the uniformly numerical relative Grothendieckgroup of $\mathcal{K}u(\mathcal{X})$.

Proof. For any non-closed point $s \in S$ we define a homomorphism

$$v_s\colon K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X}_s))_{\overline{s}} \to M^\vee$$

as follows:

(1) For $[F] \in K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X}_s))_{\overline{s}}$, choose a finite type morphism $f\colon T \to S$ from a connected $\mathbb{C}$-variety T with a point $t \in T$ such that $f(t) = s$, and a T-perfect object $E \in \mathrm{D}(\mathcal{X}_T)$ such that $E_u \in \mathcal{K}u(\mathcal{X}_u)$ for all $u \in T$ and $\eta^\vee_{t/s}[E_t] = [F]$ (recall Proposition and Definition 12.15).

(2) For $[F]$, T, and E as in (1), choose a closed point $q \in T$, let $p = f(q)$, and set

$$v_s([F]) = v_p(\eta^\vee_{q/p}[E_q]).$$

We first explain why the choice in (1) is always possible. Write $[F] = \eta^\vee_{K/\kappa(s)}[E]$ where $K/\kappa(s)$ is a finite field extension and $E_K \in \mathcal{K}u(\mathcal{X}_K)$; choose a finite type morphism $f\colon T \to S$ from a smooth irreducible $\mathbb{C}$-variety with fraction field K realising the given field extension $K/\kappa(s)$; finally, extend $E_K \in \mathcal{K}u(\mathcal{X}_t)$ to an object $E \in \mathcal{K}u(\mathcal{X}_T)$ (Lemma 3.18), which by the smoothness of T and Lemma 8.3(2) is automatically T-perfect.

To prove independence of all choices, and that the v_s induce a homomorphism from the relative numerical K-group $K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X})/S)$, we instead prove the stronger statement that they induce a homomorphism out of the uniformly numerical relative Grothendieck group $\mathcal{N}(\mathcal{K}u(\mathcal{X})/S)$ when S is smooth; the statement in the general case follows by replacing S with a resolution of singularities.

Thus it is enough to show that for S smooth, and $s \in S$ a very general closed point, the composition

$$K(\mathcal{K}u(\mathcal{X})_{\mathrm{perf}})\otimes\mathbb{Q} = K(\mathcal{K}u(\mathcal{X}))\otimes\mathbb{Q} \to K(\mathcal{K}u(\mathcal{X}_s))\otimes\mathbb{Q} \xrightarrow{v_s} \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s),\mathbb{Q}) \to M\otimes\mathbb{Q}$$

is surjective, where the last map is given by orthogonal projection. This in turn is follows from the surjectivity of the composition

$$K(\mathrm{D}^{\mathrm{b}}(\mathcal{X})) \to K(D^b(\mathcal{X}_s)) \to H^*_{\mathrm{Hdg}}(\mathcal{X}_s,\mathbb{Q}),$$

where we use the morphism $H^*_{\mathrm{Hdg}}(\mathcal{X}_s,\mathbb{Q}) \to M \otimes \mathbb{Q}$ that is compatible with the projection $\mathrm{D}^{\mathrm{b}}(\mathcal{X}) \to \mathcal{K}u(\mathcal{X})$. Finally, the surjectivity of the last map is a consequence of the rational

Hodge conjecture [Zuc77] for the very general fiber of $\mathcal{X} \to S$. Indeed, the only non-trivial statement is the surjectivity onto $H^{2,2}(\mathcal{X}_s, \mathbb{Q})$. The rational Hodge conjecture implies that classes of the form $\mathrm{ch}_2(I)$, for I an ideal sheaf of a surface in $\mathcal{X}_s$, where I is in a component of the relative Hilbert scheme that dominates S, generate $H^{2,2}(\mathcal{X}_s, \mathbb{Q})$. Taking a finite base change $T \to S$ admitting a rational section to the component containing I, extending the pullback of the universal ideal sheaf to $\mathcal{X}_T$ in an arbitrary manner, and pushing forward along $\mathcal{X}_T \to \mathcal{X}$ produces an object in $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ whose restriction to $\mathcal{X}_s$ has class proportional to $[I] \in K(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_s))$. This proves the claim. □

Proposition 30.5. *In the above setup, there exists a stability condition $\underline{\sigma}$ on $\mathcal{K}u(\mathcal{X})$ over S with respect to $M^\vee$.*

Proof. We first consider the case $M = A_2$. By Proposition 30.3, we can realize $\mathcal{K}u(\mathcal{X})$ as an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0)$. For any $s \in S$, as in [BLMS17, Definition 9.1], we can define a twisted Chern character $\mathrm{ch}_{\mathcal{B}^s_0} \colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_s, \mathcal{B}^s_0)) \to \mathbb{Q}^4$; this induces a global morphism

$$\mathrm{ch}_{\mathcal{B}^S_0} := \prod \mathrm{ch}_{\mathcal{B}^s_0} \colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0)/S) \to \mathbb{Q}^4.$$

In analogy to the notation in Section 25 we set:

$$\Lambda := \mathrm{im}\left(\mathrm{ch}_{\mathcal{B}^S_0}\right) \qquad \Lambda_0 := \mathrm{im}\left(\mathrm{ch}_{\mathcal{B}^S_0,2} \oplus \mathrm{ch}_{\mathcal{B}^S_0,3}\right) \qquad \Lambda^\sharp_0 := \mathrm{im}\left(\mathrm{ch}_{\mathcal{B}^S_0,3}\right)$$
$$\overline{\Lambda} := \Lambda/\Lambda_0 \qquad \overline{\Lambda}^\sharp := \Lambda/\Lambda^\sharp_0.$$

By Example 21.8 we have a weak stability condition $\underline{\sigma}^1$ on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0)$ with respect to the lattice $\overline{\Lambda}$. Propositions 25.1 and 25.3 also apply in the twisted setting that we are considering here; the required Bogomolov inequality is given by [BLMS17, Theorem 8.3]. Thus we get a weak stability condition $\underline{\sigma}^{\alpha,\beta}$ on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3_S, \mathcal{B}^S_0)$ over S with respect to $\overline{\Lambda}^\sharp$, which coincides fiberwise with the weak stability conditions constructed in [BLMS17, Proposition 9.3]. We can rotate one more time by Proposition 26.1 getting the weak stability condition $\underline{\sigma}^{\alpha,\beta\sharp\gamma}$.

Finally, we can apply Theorem 23.1 and get a stability condition on $\mathcal{K}u(\mathcal{X})$ over S with respect to the lattice $\Lambda_1 := \mathrm{ch}_{\mathcal{B}^S_0}(K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X})/S))$; the boundedness assumption 23.1.(5) follows from the analogue of the boundedness statement in Proposition 25.3. By [BLMS17, Proposition 9.10], we have $\Lambda_1 = A_2$, thus completing the proof in the case $M = A_2$.

Now consider the case of a lattice M containing A_2. Let $\mathfrak{P}(M) \subset \mathrm{Hom}(M^\vee, \mathbb{C})$ be the set of central charges such that $\Im Z$ and $\Re Z$ span a positive definite two-plane in $M \otimes \mathbb{R}$. Let $\Delta \subset \widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$, for some arbitrary $s \in S$, be the set of (-2)-classes δ that are not orthogonal to A_2. (This excludes classes that would become algebraic only over the Hassett divisor $\mathcal{C}_2$, see [Has00, Definition 3.1.3 and Section 4.4].) Let $\mathfrak{P}_0(M) \subset \mathcal{P}(M)$ be the open subset where $\Im Z, \Re Z$ are not orthogonal to any $\delta \in \Delta$; by standard arguments, e.g. as in [Bri07, Section 8], this is an open subset, and it contains the central charge of the stability condition we constructed above for $M = A_2$. Now consider the Mukai vector $\mathbf{v}' \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_{s'}), \mathbb{Z})$ of any stable object in any fiber $\mathcal{K}u(\mathcal{X}_{s'})$; either parallel transport (which does not change its class in $M^\vee$) identifies it with a class in Δ, or it satisfies $\mathbf{v}'^2 \geqslant 0$. In both cases, it is not contained in the kernel of any central charge in $\mathfrak{P}_0(M)$. The same arguments as in [Bri07, Section 8] therefore imply the support property with respect to M for any stability condition with central charge in $\mathfrak{P}_0(M)$. □

Remark 30.6. One can also show, just as in [Bri07, Section 8], that Theorem 22.2 implies the existence of an open set in $\mathrm{Stab}_M(\mathcal{K}u(\mathcal{X})/S)$ that covers a connected component of $\mathfrak{P}_0(M)$.

Remark 30.7. It will be crucial for us to be able to modify slightly the lattice M at a closed subset, similar to the case of a one-dimensional base considered in Example 21.7. Let $S' \subseteq S$ be a closed subvariety of S which is an irreducible component of the Hodge locus of the family g (see [CDK95]). Consider the base changed family $g'\colon \mathcal{X}' \to S'$. Let $G' := \mathrm{Mon}(g')$ and consider a saturated sublattice $M' \subseteq \mathrm{Fix}(G')$, such that M' is orthogonal to all classes that remain algebraic along all of S. We can then modify the morphism v to a map

$$v'\colon K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X})/S) \to M^\vee \oplus M'^\vee,$$

by setting the second component to be zero for $s \notin S'$. Since the full support property holds on each fiber of g, the statement in Proposition 30.5 holds true with respect to the image of v' inside $(M \oplus M')^\vee$.

For example if S is a curve and S' is a finite set of closed points in the Hodge locus of S, we can choose $M' := \bigoplus_{s\in S'} \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$.

Finally, we specialize to families over a one-dimensional base. Let $g\colon \mathcal{X} \to C$ be a family of cubic fourfolds, where C is a quasi-projective irreducible curve. Assume that g comes with a family $\mathcal{L} \to C$ of lines not contained in a plane and that the fixed locus M of the monodromy group G of g is $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbb{Z})$, for c a very general point of C. Let $\mathbf{v}$ be a primitive section of the local system of the Mukai lattices $\widetilde{H}(\mathcal{K}u(\mathcal{X}_c), \mathbb{Z})$ of the fibers, such that $\mathbf{v}$ is of Hodge type on all fibers $\mathcal{X}_c$. In particular, in our case, this means that A_2 and $\mathbf{v}$ are contained in M.

Proposition 30.8. *In the assumptions above, let $c_0 \in C$ be a closed point and let $\tau_{c_0} \in \mathrm{Stab}^\dagger(\mathcal{K}u(\mathcal{X}_{c_0}))$ be a stability condition which is $\mathbf{v}$-generic. Then there exist a lattice $M' \supseteq M$ and a stability condition $\underline{\sigma}$ on $\mathcal{K}u(\mathcal{X})$ over C with respect to $(M')^\vee$ such that:*

(1) *σ_c is $\mathbf{v}$-generic for all $c \in C$;*
(2) *σ_{c_0} is a small deformation of τ_{c_0} satisfying $M_{\sigma_{c_0}}(\mathcal{K}u(\mathcal{X}_{c_0}), \mathbf{v}) = M_{\tau_{c_0}}(\mathcal{K}u(\mathcal{X}_{c_0}), \mathbf{v})$;*
(3) *if $c_0 \in C$ is a smooth point and the central charge Z_{c_0} of τ_{c_0} factors via $M^\vee$, then there is an open neighborhood $U \subset C$ of c_0 such that the central charge of $\underline{\sigma}$ restricted over U factors through $\mathcal{N}(\mathcal{K}u(\mathcal{X}_U)/U)$.*

Note that, if we are in the setting of Theorem 29.4.(1), then up to taking a finite cover $C' \to C$ we can always assume that the assumptions in Proposition 30.8 are satisfied, as well as the one in part (3) (see [Voi13, Theorem 4.1]).

Proof. By Proposition 30.5 there exists a stability condition $\underline{\sigma}'$ on $\mathcal{K}u(\mathcal{X})$ over C with respect to $M^\vee$ such that σ'_c is $\mathbf{v}$-generic for c in a non-empty subset of C. Let C' be the subset of C where σ'_c is not $\mathbf{v}$-generic. By openness of geometric stability and boundedness of the relative moduli space C' is a finite set. For each $c \in C'$ there is (by the support property) a finite set in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbb{Z})$ of Mukai vectors of Jordan–Hölder factors of objects in $\mathcal{M}_{\underline{\sigma}'}(\mathbf{v})$. We now apply Remark 30.7 to enlarge the lattice M to $M' := M \oplus_{c\in C'} M_c$ such that these classes are no longer proportional to $\mathbf{v}$ in $M^\vee \oplus M_c^\vee$; for $c = c_0$ we just take M_{c_0} such that the central charge $Z_{\tau_{c_0}}$ is defined over it. By Remark 30.6, there exists a deformation $\underline{\sigma}$ of $\underline{\sigma}'$ such that (1) and (2) are satisfied.

Under the additional assumption in (3), we do not need to modify M in the fiber over c_0 and thus we can apply directly Lemma 30.4. □

Remark 30.9. Consider the analogous situation over a higher-dimensional base S. Then $S' \subset S$ becomes a closed subset, contained in finite union of Hodge loci S'_i. Assume that the monodromy group for each S'_i acts trivially on the Mukai vectors of Jordan–Hölder factors of objects in $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ occurring along S'_i. Then we can apply exactly the same procedure as above and again produce a stability condition over S that is $\mathbf{v}$-generic on all fibers.

31. Generalized Mukai's theorem

The result of this section is the following generalization to $\mathcal{K}u(X)$ of well-known results by Mukai [Muk84] and Inaba [Ina11] for K3 surfaces.

Let $g\colon \mathcal{X} \to S$ be a smooth family of cubic fourfolds, where S is reduced and of finite type over $\mathbb{C}$. Let $\mathbf{v}$ be a primitive section of the local system given by the Mukai lattices $\widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$ of the fibers over $s \in S(\mathbb{C})$, such that $\mathbf{v}$ is of Hodge type on all fibers. Consider the locally of finite type algebraic stack $s\mathcal{M}_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v}) \to S$ parameterizing simple universally gluable S-perfect objects in $\mathcal{K}u(\mathcal{X})$ with Mukai vector $\mathbf{v}$. This is a $\mathbb{G}_m$-gerbe over a locally of finite type algebraic space $\rho\colon sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v}) \to S$, see Lemma 9.8.

Theorem 31.1 (Mukai)**.** *The morphism $\rho\colon sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v}) \to S$ is smooth.*

Remark 31.2. A generalization of Theorem 31.1 was proved in [Per20], where $\mathcal{K}u(\mathcal{X})$ is replaced by any "family of CY2 categories" over S. In fact, the proof of Theorem 31.1 below incorporates some clarifications and corrections from [Per20] to the argument from the first version of this paper, in particular the use of the T^1-lifting theorem to reduce to split square-zero extensions in the deformation argument.

The proof of Theorem 31.1 involves a combination of Mukai's original argument and the methods of [KM09], which in turn build on ideas of Buchweitz and Flenner. We start by recalling a result on the deformation theory of objects in the derived category; for a more complete treatment of this subject and related arguments, see [HT10, Lie06a, HMS09, AT14].

Let $\mathcal{Y} \to \operatorname{Spec} R$ be a smooth projective morphism, and let $\mathcal{Y}_0 \to \operatorname{Spec} R_0$ be its base change along $R \to R_0 = R/I$ where $I \subset R$ is a square-zero ideal. Note that $\mathcal{Y} \to \operatorname{Spec} R$ is a deformation of $\mathcal{Y}_0 \to \operatorname{Spec} R_0$ over R, and recall that the set of isomorphism classes of such deformations of $\mathcal{Y}_0$ forms a torsor under $\mathrm{H}^1(\mathrm{T}_{\mathcal{Y}_0/\operatorname{Spec} R_0} \otimes I)$. If $R \to R_0$ is a split square-zero extension, i.e. admits a section $R_0 \to R$, then there is a trivial deformation of $\mathcal{Y}_0$ over R given by base change along the section, so the set of deformation classes is canonically identified with $\mathrm{H}^1(\mathrm{T}_{\mathcal{Y}_0/\operatorname{Spec} R_0} \otimes I)$ with the trivial deformation corresponding to 0. In this case, we write

$$\kappa(\mathcal{Y}) \in \mathrm{H}^1(\mathrm{T}_{\mathcal{Y}_0/\operatorname{Spec} R_0} \otimes I) \cong \operatorname{Ext}^1(\Omega_{\mathcal{Y}_0/R_0}, I)$$

for the element corresponding to the deformation $\mathcal{Y} \to \operatorname{Spec} R$, called the *Kodaira–Spencer class*. Further, recall that for any object $E_0 \in \mathrm{D}_{\mathrm{perf}}(\mathcal{Y}_0)$ there is a canonical element

$$A(E_0) \in \operatorname{Ext}^1(E_0, E_0 \otimes \Omega_{\mathcal{Y}_0/R_0})$$

called the *Atiyah class*. The following is then the main result of [HT10], simplified to the case where the structure morphism is smooth and the square-zero extension is split.

Theorem 31.3. *Let $\mathcal{Y} \to \operatorname{Spec} R$ be a smooth projective morphism, let $R \to R_0 = R/I$ be a split square-zero extension, and let $\mathcal{Y}_0 \to \operatorname{Spec} R_0$ be the base change of $\mathcal{Y}$. For any*

$E_0 \in \mathrm{D}_{\mathrm{perf}}(\mathcal{Y}_0)$, there exists an object $E \in \mathrm{D}_{\mathrm{perf}}(\mathcal{Y})$ such that $E_{\mathcal{Y}_0} \cong E_0$ if and only if

$$\kappa(\mathcal{Y}) \cdot A(E_0) = 0 \in \mathrm{Ext}^2(E_0, E_0 \otimes I),$$

in which case the set of isomorphism classes of such E forms a torsor under $\mathrm{Ext}^1(E_0, E_0 \otimes I)$.

Proof of Theorem 31.1. First we claim that it is enough to prove the theorem when the base S is smooth. Indeed, first note that for any morphism $S' \to S$ with based changed family $\mathcal{X}'$, the relative moduli space $\rho'\colon sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X}')/S')(\mathbf{v}) \to S'$ is the base change of $\rho\colon sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v}) \to S$ along $S' \to S$. Applied to closed points of S this shows that the fibers of ρ are smooth. Applied to the spectrum of a DVR over S (which factors via a smooth scheme of finite type over S), this proves the valuative criterion for flatness [Gro66, 11.8.1]. Combined, this shows that ρ is smooth.

Thus from now on we may assume that S is smooth. Assume we have proven that in this case the total space $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ is smooth. Let $0 \in S(\mathbb{C})$ be a closed point, let $\mathcal{X}_0$ be the fiber of $\mathcal{X} \to S$, and let $\mathbf{v}_0 \in \widetilde{H}(\mathcal{K}u(\mathcal{X}_0), \mathbb{Z})$ be the fiber of $\mathbf{v}$. Then the moduli space $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X}_0))(\mathbf{v}_0)$ —i.e., the fiber of ρ over 0— is smooth by our assumption (applied to the case where S is a point), and of dimension $\mathbf{v}_0^2 + 2$ (as this computes $\dim \mathrm{Ext}^1(E_0, E_0)$ for a simple universally gluable object in $\mathcal{K}u(\mathcal{X}_0)$ of class $\mathbf{v}_0$). Therefore $\rho\colon sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v}) \to S$ is a locally finite type morphism between smooth spaces whose closed fibers are smooth of constant dimension. It follows that ρ is smooth.

Thus to prove the theorem it suffices to show that $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ is smooth if S is smooth. Since $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ is locally of finite type over $\mathbb{C}$, we just need to show that it is formally smooth at any $\mathbb{C}$-point. More precisely, let E_0 be a $\mathbb{C}$-point of $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ lying over a closed point $0 \in S$. Let $\mathrm{Art}_{\mathbb{C}}$ be the category of Artinian local $\mathbb{C}$-algebras with residue field $\mathbb{C}$, and let $F\colon \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$ be the deformation functor of the point E_0, i.e. $F(A)$ consists of pairs $(\mathrm{Spec}(A) \to S, E)$ where $\mathrm{Spec}(A) \to S$ takes the closed point $p \in \mathrm{Spec}(A)$ to $0 \in S$, and E is an A-point of $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ whose restriction over $p \in S$ is isomorphic to E_0; for simplicity we often simply write $E \in F(A)$ suppressing the map $\mathrm{Spec}(A) \to S$ from the notation. (Note that in the definition of $F(A)$, instead of requiring E to be an A-point of $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$, it is equivalent to require $E \in \mathrm{D}_{\mathrm{perf}}(\mathcal{X}_A)$; indeed, an A-point E of $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ must lie in $\mathrm{D}_{\mathrm{perf}}(\mathcal{X}_A)$ since it is relatively perfect over $\mathrm{Spec}(A)$ and $\mathcal{X}_A \to \mathrm{Spec}(A)$ is smooth, and conversely, the condition that an object $E \in \mathrm{D}_{\mathrm{perf}}(\mathcal{X}_A)$ restricts over p to E_0 guarantees that E lies in $\mathcal{K}u(\mathcal{X}_A)$, has class $\mathbf{v}$, and is universally gluable, as E_0 has these properties.) To prove formal smoothness of $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X})/S)(\mathbf{v})$ at E_0, we must show that F is a smooth functor, i.e. for any surjection $A' \to A$ in $\mathrm{Art}_{\mathbb{C}}$, the map $F(A') \to F(A)$ is surjective.

It will also be useful to consider the deformation functor $G\colon \mathrm{Art}_{\mathbb{C}} \to \mathrm{Sets}$ of the point $0 \in S$, whose value on $A \in \mathrm{Art}_{\mathbb{C}}$ consists of morphisms $\mathrm{Spec}(A) \to S$ taking the closed point to 0. By definition, there is a morphism of functors $F \to G$.

We start by considering a split square-zero extension $A' \to A$, that is $A' \cong A[\varepsilon]/(\varepsilon^2)$. We will prove that the fiber $\mathrm{Def}_E(A')$ of the map

$$F(A') \to G(A') \times_{G(A)} F(A)$$

over any point $(\mathrm{Spec}(A') \to S, E)$ is a torsor under $\mathrm{Ext}^1(E, E)$. Let $\mathcal{X}_{A'} \to \mathrm{Spec}\, A'$ and $\mathcal{X}_A \to \mathrm{Spec}\, A$ be the base changes of our family $\mathcal{X} \to S$ of cubic fourfolds. By Theorem 31.3,

it suffices to show that

$$\kappa(\mathcal{X}_{A'}) \cdot A(E) = 0 \in \mathrm{Ext}^2(E, E). \tag{31.1}$$

Note that $\mathrm{Hom}(E, E) \cong A$ since E is simple, and relative Serre duality over A gives that $\mathrm{Ext}^2(E, E)$ is a line bundle on $\mathrm{Spec}\, A$. More precisely, a version of the arguments in [KM09, Theorem 4.3] relative to A shows that

$$\mathrm{Ext}^2(E, E) \xrightarrow{\mathrm{Tr}_E(_\circ A(E))} \mathrm{Ext}^3(\mathcal{O}_{\mathcal{X}_A}, \Omega_{\mathcal{X}_A/A}) \tag{31.2}$$

is an isomorphism. Now we claim that

$$0 = \mathrm{Tr}_E\left(\kappa(\mathcal{X}_{A'}) \cdot A(E)^2\right) \in \mathrm{Ext}^3(\mathcal{O}_{\mathcal{X}_A}, \Omega_{\mathcal{X}_A/A}),$$

so that by the isomorphism (31.2) the desired vanishing (31.1) holds. Indeed, consider the formal exponential

$$\exp(A(E))\colon E \to \bigoplus_{i \geqslant 0} E \otimes \Omega^i_{\mathcal{X}_A/A}[i].$$

Due to the vanishing of $H^{i+2,i}$ for $i \neq 1$, we have

$$\mathrm{Tr}_E\left(\kappa(\mathcal{X}_{A'}) \cdot A(E)^2\right) = 2\,\mathrm{Tr}_E(\kappa(\mathcal{X}_{A'}) \cdot \exp(A(E))) = 2\kappa(\mathcal{X}_{A'}) \cdot \mathrm{ch}(E),$$

which vanishes, as it is precisely the obstruction to $\mathbf{v}$, or equivalently $\mathrm{ch}(E)$, remaining of Hodge type along A'.

Now we prove that F is a smooth functor. We use the T^1-lifting theorem [Kaw92, FM99] to reduce to the case of split square-zero extensions handled above. Namely, for any integer $n \geqslant 0$ set $A_n = \mathbb{C}[t]/(t^{n+1})$ and $A'_n = A_n[\varepsilon]/(\varepsilon^2)$. The T^1-lifting theorem says that F is smooth if for every $n \geqslant 0$ the natural map

$$F(A'_{n+1}) \to F(A'_n) \times_{F(A_n)} F(A_{n+1})$$

is surjective. Note that this map fits into a commutative diagram

$$\begin{tikzcd}
F(A'_{n+1}) \arrow[r] \arrow[d] & F(A'_n) \times_{F(A_n)} F(A_{n+1}) \arrow[d] \\
G(A'_{n+1}) \arrow[r] & G(A'_n) \times_{G(A_n)} G(A_{n+1})
\end{tikzcd}$$

where the bottom horizontal arrow is surjective because $0 \in S$ is a smooth point. Therefore, it suffices to prove the map

$$F(A'_{n+1}) \to G(A'_{n+1}) \times_{G(A'_n) \times_{G(A_n)} G(A_{n+1})} (F(A'_n) \times_{F(A_n)} F(A_{n+1})) \tag{31.3}$$

is surjective. Let $(\mathrm{Spec}(A'_{n+1}) \to S, E'_n, E_{n+1})$ be a point of the target of this map, and set $E_n = (E_{n+1})_{A_n} \cong (E'_n)_{A_n}$. By the previous paragraph, the set $\mathrm{Def}_{E_{n+1}}(A'_{n+1})$ of deformations of E_{n+1} over A'_{n+1} is an $\mathrm{Ext}^1(E_{n+1}, E_{n+1})$-torsor, and the set $\mathrm{Def}_{E_n}(A'_n)$ of deformations of E_n over A'_n is an $\mathrm{Ext}^1(E_n, E_n)$-torsor. Moreover, the restriction map $\mathrm{Def}_{E_{n+1}}(A'_{n+1}) \to \mathrm{Def}_{E_n}(A'_n)$ is compatible with the torsor structures under the restriction map

$$\mathrm{Ext}^1(E_{n+1}, E_{n+1}) \to \mathrm{Ext}^1(E_n, E_n). \tag{31.4}$$

We claim the map on Ext^1 groups, and hence also $\mathrm{Def}_{E_{n+1}}(A'_{n+1}) \to \mathrm{Def}_{E_n}(A'_n)$, is surjective. Given this claim, we can choose an element $E'_{n+1} \in \mathrm{Def}_{E_{n+1}}(A'_{n+1})$ restricting to

$E'_n \in \mathrm{Def}_{E_n}(A'_n)$; then E'_{n+1} maps to $(\mathrm{Spec}(A'_{n+1}) \to S, E'_n, E_{n+1})$ under (31.3), proving the required surjectivity.

It remains to prove the map (31.4) is surjective. In fact, we will prove there is an isomorphism

$$\mathrm{Ext}^1(E_{n+1}, E_{n+1}) \otimes_{A_{n+1}} A_n \cong \mathrm{Ext}^1(E_n, E_n) \tag{31.5}$$

which implies the surjectivity of (31.4). The isomorphism (31.5) is a special case of [Per20, Lemma 7.4(2)], but for completeness we include the argument. By base change (Lemma 3.21), we have

$$\mathcal{H}om_{A_{n+1}}(E_{n+1}, E_{n+1}) \otimes_{A_{n+1}} A_n \cong \mathcal{H}om_{A_n}(E_n, E_n).$$

We will prove the cohomology groups $\mathrm{Ext}^i(E_{n+1}, E_{n+1})$ of $\mathcal{H}om_{A_{n+1}}(E_{n+1}, E_{n+1})$ are locally free A_{n+1}-modules for all i, which by the above base change formula implies the isomorphism (31.5). Serre duality shows that $\mathrm{Ext}^i(E_{n+1}, E_{n+1})$ vanishes for $i \notin [0, 2]$, and for $i = 0$ or $i = 2$ we already observed above that it is a line bundle. For $i = 1$, by the local criterion for flatness it suffices to show that $\mathrm{H}^{-1}(j_0^* \mathrm{Ext}^1(E_{n+1}, E_{n+1})) = 0$, where $j_0 \colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A_{n+1})$ is the closed point. Note that we have a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{H}^j(j_0^* \mathrm{Ext}^i(E_{n+1}, E_{n+1})) \implies \mathrm{Ext}^{i+j}(E_0, E_0).$$

But $E_2^{0,j}$ and $E_2^{2,j}$ are 1-dimensional for $j = 0$ and vanish for $j \neq 0$, and $\mathrm{Ext}^0(E_0, E_0)$ is 1-dimensional, so the vanishing $\mathrm{H}^{-1}(j_0^* \mathrm{Ext}^1(E_{n+1}, E_{n+1})) = 0$ follows. □

32. Specializing Kuznetsov components to twisted K3 surfaces

Combining Proposition 30.8, Theorem 21.24, and Theorem 31.1, we obtain the following.

Corollary 32.1. *Let X be a cubic fourfold, let $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ be a primitive vector, and let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ be a $\mathbf{v}$-generic stability condition. Let X' be another cubic fourfold in the Hodge locus where $\mathbf{v}$ stays a Hodge class. Then there exist a family $g \colon \mathcal{X} \to C$ of cubic fourfolds over a smooth connected quasi-projective curve and a stability condition $\underline{\sigma}$ on $\mathcal{K}u(\mathcal{X})$ over C such that:*

(1) *$\mathbf{v}$ is a primitive vector in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbb{Z})$ for all closed points $c \in C$;*
(2) *there exists closed points $c_0, c_1 \in C$ such that $\mathcal{X}_{c_0} = X$ and $\mathcal{X}_{c_1} = X'$;*
(3) *σ_{c_0} is a small deformation of σ satisfying $M_{\sigma_{c_0}}(\mathcal{K}u(\mathcal{X}_{c_0}), \mathbf{v}) = M_\sigma(\mathcal{K}u(\mathcal{X}_{c_0}), \mathbf{v})$;*
(4) *σ_c is $\mathbf{v}$-generic for all $c \in C$;*
(5) *the relative moduli space $M_{\underline{\sigma}}(\mathbf{v})$ is smooth and proper over C.*

Furthermore, if we assume that the central charge Z of σ is defined over $\mathbb{Q}[\mathrm{i}]$ and that X' is also in the Hodge locus where $\ell_\sigma \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Q})$ stays a Hodge class, then we can choose $\underline{\sigma}$ so that there is an open neighborhood $U \subset C$ of c_0 such that the central charge of $\underline{\sigma}$ restricted over U factors through $\mathcal{N}(\mathcal{K}u(\mathcal{X}_U)/U)$.

Proof. Parts (1)–(5) follow directly from Proposition 30.8. To clarify the last statement, we recall that the class $\ell_\sigma \in N^1(M_\sigma(\mathbf{v}))$ of Theorem 21.25 corresponds via the Mukai morphism (see Theorem 29.2.(2)) to a class in $\widetilde{H}(\mathcal{K}u(X), \mathbb{R})$, which is actually rational under our assumptions. Then, up to slightly deforming σ, we can assume that its central charge factors through $M := \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbb{Z})$, for c a very general point in C. The conclusion follows then directly from Proposition 30.8.(3). □

Corollary 32.1 shows that to prove deformation invariant properties about $M_\sigma(\mathcal{K}u(X), \mathbf{v})$, we may specialize the cubic fourfold X within the Hodge locus for $\mathbf{v}$. In the proofs of our results stated in Section 29, we will use this observation to specialize to the case where $\mathcal{K}u(X)$ is equivalent to the derived category of a twisted K3 surface (S, α), for which many results are already known [BM14b]. There is one subtlety: moduli spaces of σ-stable objects in $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ are only well-understood when σ lies in the connected component $\mathrm{Stab}^\dagger(S, \alpha)$ containing geometric stability conditions, i.e., those for which skyscraper sheaves of points are stable of the same phase.

Remark 32.2. In the first version of this paper, the above subtlety was overlooked, but in the meantime it was addressed in [PPZ19, Section 5.2] in the context of Gushel–Mukai varieties. The arguments below closely follow those of [PPZ19, Section 5.2], but are included for completeness.

Definition 32.3. Let X be a cubic fourfold and (S, α) a twisted K3 surface. A †-*equivalence* $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$ is an equivalence under which $\mathrm{Stab}^\dagger(\mathcal{K}u(X))$ maps to $\mathrm{Stab}^\dagger(S, \alpha)$.

The following proposition gives the desired type of specialization, and is the main result of this section.

Proposition 32.4. *Let X be a cubic fourfold and let $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$. Then X is deformation equivalent within the Hodge locus for $\mathbf{v}$ to a cubic fourfold X' such that there is a †-equivalence $\mathcal{K}u(X') \simeq \mathrm{D}^{\mathrm{b}}(S', \alpha')$ for a twisted K3 surface (S', α').*

We will prove the proposition at the end of this section, after some preliminary results.

Lemma 32.5. *Let X be a cubic fourfold, let $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ be primitive vector with $\mathbf{v}^2 = 0$, and let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ be a $\mathbf{v}$-generic stability condition. Assume there exists a σ-stable object in $\mathcal{K}u(X)$ of class $\mathbf{v}$. Then $S = M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is a smooth K3 surface and there is a †-equivalence $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$ for a certain Brauer class $\alpha \in \mathrm{Br}(S)$.*

Proof. The results of Section 30.2 applied in the case where the base is a point show that σ is a stability condition *over* $\mathrm{Spec}(\mathbb{C})$. Thus by Theorem 21.24.(3), S is a proper algebraic space over $\mathrm{Spec}(\mathbb{C})$. Moreover, by Theorem 31.1, S is also smooth over $\mathrm{Spec}(\mathbb{C})$. Since $\mathcal{K}u(X)$ is a 2-Calabi–Yau category, standard arguments show that S has dimension $\mathbf{v}^2 + 2 = 2$ and is equipped with a symplectic form (see [KM09]). Since S is a smooth proper 2-dimensional algebraic space, it is in fact a smooth projective surface. Moreover, by [BLMS17, Proposition A.7], S is also connected.

Now let $\mathcal{E}$ be a quasi-universal family over $S \times X$ and $\alpha \in \mathrm{Br}(S)$ the associated Brauer class. By [Bri99], the exact functor $\Phi_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(S, \alpha) \to \mathcal{K}u(X)$ is fully faithful. Since the 0-th Hochschild cohomology $\mathrm{HH}^0(\mathcal{K}u(X))$ is one-dimensional, $\Phi_{\mathcal{E}}$ is also essentially surjective (Bridgeland's trick; see for example [Kuz19, Proposition 5.1]). By construction, the equivalence $\Phi_{\mathcal{E}}$ is a †-equivalence.

As S has a symplectic form, it is either a K3 or abelian surface. Since $\mathrm{D}^{\mathrm{b}}(S, \alpha) \simeq \mathcal{K}u(X)$ has the same Hochschild homology as a K3 surface, S must in fact be a K3 surface. □

Next we observe that the existence of a †-equivalence deforms along Hodge loci for square-zero classes.

Lemma 32.6. *Let X and X' be cubic fourfolds which are deformation equivalent within the Hodge locus for a vector $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ with $\mathbf{v}^2 = 0$. Then $\mathcal{K}u(X)$ is †-equivalent to the derived category of a twisted K3 surface if and only if $\mathcal{K}u(X')$ is.*

Proof. Assume $\mathcal{K}u(X)$ is †-equivalent to the derived category of a twisted K3 surface. Then if $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ is $\mathbf{v}$-generic, the moduli space $M_\sigma(\mathcal{K}u(X), \mathbf{v})$ is a smooth K3 surface [BM14b]. By Corollary 32.1, it follows that for any $\mathbf{v}$-generic $\sigma' \in \mathrm{Stab}^\dagger(\mathcal{K}u(X'))$, the moduli space $M_{\sigma'}(\mathcal{K}u(X'), \mathbf{v})$ is also a smooth K3 surface. By Lemma 32.5, we conclude $\mathcal{K}u(X')$ is †-equivalent to the derived category of a twisted K3 surface. □

The following is our key technical ingredient. It allows us to replace a given equivalence $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$ with a †-equivalence, provided X admits deformations with suitable Hodge-theoretic properties.

Lemma 32.7. *Let X be a cubic fourfold such that:*

(1) *There is an equivalence $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$ for a twisted K3 surface (S, α).*
(2) *There exists a vector $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbf{Z})$ with $\mathbf{v}^2 = 0$ such that X is deformation equivalent within the Hodge locus for $\mathbf{v}$ to a cubic fourfold X' with the property that $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X'), \mathbf{Z})$ contains no vectors $\mathbf{w}$ with $\mathbf{w}^2 = -2$.*

Then there exists a twisted K3 surface (T, β), possibly different from (S, α), and a †-equivalence $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(T, \beta)$.

Proof. Let $\mathcal{X} \to C$ be a family of cubic fourfolds over a smooth connected quasi-projective curve C, such that $\mathcal{X}_0 = X$ and $\mathcal{X}_1 = X'$ for some points $0, 1 \in C(\mathbb{C})$, and $\mathbf{v}$ remains a Hodge class along C. There exists a simple object $E \in \mathcal{K}u(X)$ of class $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbf{Z})$ with $\mathrm{Ext}^{<0}(E, E) = 0$; indeed, even stronger, there exist σ-stable objects of class $\mathbf{v}$ for a $\mathbf{v}$-generic $\sigma \in \mathrm{Stab}^\dagger(S, \alpha)$ by [BM14b]. Thus by Theorem 31.1 there is a Zariski open $U \subset C$ such that for any $c \in U(\mathbb{C})$ there exists a simple object $E_c \in \mathcal{K}u(\mathcal{X}_c)$ of class $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbf{Z})$ with $\mathrm{Ext}^{<0}(E_c, E_c) = 0$; in particular, it follows that $\mathrm{Ext}^1(E_c, E_c) \cong \mathbb{C}^2$.

The condition that $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbf{Z})$ contains no vectors $\mathbf{w}$ with $\mathbf{w}^2 = -2$ holds for a very general $c \in U(\mathbb{C})$, because it holds for $c = 1$ by assumption. Therefore, after possibly replacing X' by a different fiber of $\mathcal{X} \to C$, we may assume there is an object $E' \in \mathcal{K}u(X')$ such that $\mathrm{Ext}^1(E', E') \cong \mathbb{C}^2$. It then follows from [BLMS17, Lemma A.4] that E' is σ-stable for any $\sigma \in \mathrm{Stab}(\mathcal{K}u(X'))$. Thus, Lemma 32.5 shows $\mathcal{K}u(X')$ is †-equivalent to the derived category of a twisted K3 surface, and by Lemma 32.6 we conclude the same for $\mathcal{K}u(X)$. □

Proof of Proposition 32.4. By [AT14, Theorem 4.1], X is deformation equivalent within the Hodge locus for $\mathbf{v}$ to a cubic fourfold in $\mathcal{C}_8$, where $\mathcal{C}_8$ is the discriminant 8 Hassett divisor. Recall that $\mathcal{C}_8$ can be described either as the irreducible divisor parameterizing cubic fourfolds containing a plane, or as the Hodge locus for a certain square-zero vector in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbf{Z})$, see [Huy17]. Therefore, to finish the proof, by Lemma 32.6 it suffices to show there exists a $X \in \mathcal{C}_8$ such that $\mathcal{K}u(X)$ is †-equivalent to the derived category of a twisted K3 surface.

By [Huy17, Proposition 2.15], there exists a Hassett divisor $\mathcal{C}_d$ such that $\mathcal{C}_d$ is the Hodge locus for a square-zero vector in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbf{Z})$, and the very general point of $\mathcal{C}_d$ parameterizes a cubic fourfold X such that $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbf{Z})$ contains no vectors $\mathbf{w}$ with $\mathbf{w}^2 = -2$. Moreover, again by [AT14, Theorem 4.1], there exists a cubic fourfold $X \in \mathcal{C}_d \cap \mathcal{C}_8$ such that $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$ for a twisted K3 surface (S, α) (and in fact we may take $\alpha = 0$). Thus,

we conclude by Lemma 32.7 that $\mathcal{K}u(X)$ is †-equivalent to the derived category of a twisted K3 surface, as required. □

33. Proofs of the main results

In this section, we apply the previous results to prove Theorems 29.1, 29.2, and 29.4, and their consequences, Corollaries 29.5, 29.7, and 29.8.

Proof of Theorem 29.4.(1). Let $\mathcal{X} \to C$ be a family of cubic fourfolds. Let $\widetilde{C} \to C$ be a finite (not necessarily étale) Galois cover such that $\mathcal{X}_{\widetilde{C}}$ admits a family of lines in the fibers that are not contained in a plane. Let $\underline{\sigma}$ be the stability condition over $\widetilde{C}$ given by Proposition 30.8 that is $\mathbf{v}$-generic in every fiber; in the construction it is easy to ensure that $\underline{\sigma}$ is Galois-invariant, in the naive sense that in each orbit, the stability conditions in the corresponding Kuznetsov component are identical.

Now consider the relative moduli space $M_{\underline{\sigma}}(\mathbf{v})$. By construction, there are no properly semistable objects, and so $M_{\underline{\sigma}}(\mathbf{v})$ is an open subspace of $sM_{\mathrm{pug}}(\mathcal{K}u(\mathcal{X}_{\widetilde{C}})/\widetilde{C})$. By Theorem 21.24.(2), it is an algebraic space, proper over $\widetilde{C}$. By Theorem 31.1, it is also smooth over $\widetilde{C}$.

Finally, by the Galois invariance of $\underline{\sigma}$, the Galois group of $\widetilde{C} \to C$ also acts on $M_{\underline{\sigma}}(\mathbf{v})$; therefore it descends to a smooth and proper morphism $\widetilde{M}(\mathbf{v}) \to C$ with the properties described in the statement of the theorem. □

Proof of Theorem 29.4.(3). In the case where S admits a family of lines over S, none of which are contained in a plane, this is just Theorem 21.24 in our context. The general case can be reduced to that situation using a cover of S, just as in the previous proof. □

Next we prove Corollary 29.7, in the following general form.

Proposition 33.1. *Let X be a cubic fourfold. Then there exist a smooth projective K3 surface S and a Brauer class $\alpha \in \mathrm{Br}(S)$ such that $\mathcal{K}u(X) \simeq \mathrm{D}^{\mathrm{b}}(S,\alpha)$ if and only if there exists a non-zero primitive Mukai vector $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z})$ such that $\mathbf{v}^2 = 0$. Moreover, the class α can be chosen to be trivial if and only if there exists another Mukai vector $\mathbf{v}' \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z})$ such that $(\mathbf{v},\mathbf{v}') = 1$.*

Note the following generalization of [Muk87], which can also be regarded as an elaboration on Lemma 32.5 (with slightly different hypotheses).

Lemma 33.2. *Let $\mathcal{D}$ be a 2-Calabi–Yau category, i.e. an admissible subcategory of the derived category of a smooth projective variety, with Serre functor $\mathrm{S}_{\mathcal{D}} = [2]$. Assume $\mathrm{HH}^0(\mathcal{D}) = \mathbb{C}$.*

(1) *If there exists a K3 surface S and a Brauer class $\alpha \in \mathrm{Br}(S)$ such that $\mathcal{D} \simeq \mathrm{D}^{\mathrm{b}}(S,\alpha)$, then there exists a primitive Mukai vector $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{D},\mathbb{Z})$ with $\mathbf{v}^2 = 0$.*
(2) *If there exists a primitive Mukai vector $\mathbf{v}$ with $\mathbf{v}^2 = 0$ and a Bridgeland stability condition σ which is $\mathbf{v}$-generic and such that $M_\sigma(\mathbf{v})$ exists and it is isomorphic to a K3 surface S, then $\mathcal{D} \simeq \mathrm{D}^{\mathrm{b}}(S,\alpha)$, for a certain $\alpha \in \mathrm{Br}(S)$.*

Moreover, the Brauer class α can be chosen to be trivial if and only if there exists another Mukai vector $\mathbf{v}' \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{D},\mathbb{Z})$ with $(\mathbf{v}',\mathbf{v}) = 1$.

Remark 33.3. The group of Hodge classes $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{D}, \mathbf{Z})$ appearing in the lemma can be defined as in [Per20]; however, in this paper we will only need the case where $\mathcal{D} = \mathcal{K}u(X)$ for a cubic fourfold X.

Proof. Part (1) is clear, by taking $\mathbf{v}$ to be the Mukai vector of a skyscraper sheaf (and, if the Brauer class is trivial, $\mathbf{v}'$ to be that of the structure sheaf). Part (2) holds as in the proof of Lemma 32.5. Finally, if there exists another Mukai vector $\mathbf{v}' \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{D}, \mathbb{Z})$ with $(\mathbf{v}', \mathbf{v}) = 1$, then the quasi-universal family is universal, and so α can be chosen to be trivial. □

Proof of Proposition 33.1. By Lemma 33.2, we only need to show that, given a non-zero primitive Mukai vector $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ such that $\mathbf{v}^2 = 0$, there exists a $\mathbf{v}$-generic stability condition σ for which the moduli space $M_\sigma(\mathbf{v})$ is a K3 surface. This follows from Proposition 32.4, Corollary 32.1, and the analogous statement for moduli spaces for stability conditions in the distinguished component of the stability manifold of a twisted K3 surface [BM14b]. □

The above proposition is enough to extend [Huy17, Theorem 1.5.ii)], by using the Derived Torelli Theorem for twisted K3 surfaces [Orl97, HS06].

Corollary 33.4. *Let d be a positive integer such that $d \equiv 0, 2 \pmod 6$ and $n_i \equiv 0 \pmod 2$ for all primes $p_i \equiv 2 \pmod 3$ occuring in the factorization $2d = \prod p_i^{n_i}$. Then for cubic fourfolds $X, X' \in \mathcal{C}_d$, there exists an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ if and only if there exists a Hodge isometry $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z}) \cong \widetilde{H}(\mathcal{K}u(X'), \mathbb{Z})$.*

Proof of Theorem 29.2. We divide the proof in a few steps. For the moment, we think of $\mathrm{Stab}^\dagger(\mathcal{K}u(X))$ as a non-empty connected open subset. We postpone the proof that it is a whole connected component of the space of stability conditions to later on in the section.

Non-emptiness. We first deal with the non-emptiness statement in part (1) of the theorem. One implication follows immediately from the properties of the Mukai pairing: since $\mathbf{v}$ is primitive and σ is $\mathbf{v}$-generic, then $M_\sigma(\mathbf{v}) = M_\sigma^{\mathrm{st}}(\mathbf{v})$ and any object E in $M_\sigma(\mathbf{v})$ satisfies $\mathbf{v}^2 = -\chi(E, E) \geqslant -2$.

For the converse, let $\mathbf{v} \in \widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ be a primitive Mukai vector with $\mathbf{v}^2 \geqslant -2$ and $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ be a $\mathbf{v}$-generic stability condition. We apply one more time Corollary 32.1 as in the proof of Proposition 33.1: the Hodge locus where $\mathbf{v}$ stays a Hodge class will intersect the divisor $\mathcal{C}_8$. Since, by [BM14b, Corollary 6.9], the moduli space is non-empty there, it is non-empty on all fibers, in particular over X, as we wanted.

Hodge classes are algebraic. The equality $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z}) = \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ is now immediate: every Hodge class can be written as sum of Hodge classes having positive square. The non-emptiness statement above guarantees that these are algebraic, as we wanted.

Projectivity. The moduli space $M_\sigma(\mathbf{v})$ is smooth and proper, by Theorem 31.1 and Theorem 21.24.(3). Moreover, by [KM09, Theorem 2.2], since $M_\sigma(\mathbf{v})$ parameterizes stable objects in the K3 category $\mathcal{K}u(X)$, Serre duality gives a non-degenerate closed symplectic 2-form on $M_\sigma(\mathbf{v})$, and so it has trivial canonical bundle. We want to prove that $M_\sigma(\mathbf{v})$ is projective.[7]

[7] The following argument was suggested to us by Giulia Saccà; it greatly simplifies the one in an earlier version of this paper.

We claim, in a slightly more precise version of Corollary 32.1, that there exists a family $g\colon \mathcal{X} \to C$ of cubic fourfolds, parameterized by a smooth connected quasi-projective curve C, $c_0, c_1 \in C$, and a stability condition $\underline{\sigma}$ on $\mathcal{K}u(\mathcal{X})$ over C whose central charge factors via $\mathcal{N}(\mathcal{K}u(\mathcal{X})/C)$ such that:

(1) $\mathbf{v}$ is a primitive vector in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_c), \mathbb{Z})$ for all closed points $c \in C$;
(2) $\mathcal{X}_{c_0} = X$ and $M_{\sigma_{c_0}}(\mathcal{K}u(\mathcal{X}_{c_0}), \mathbf{v}) = M_\sigma(\mathcal{K}u(\mathcal{X}_{c_0}), \mathbf{v})$;
(3) $\mathcal{K}u(\mathcal{X}_{c_1}) \cong \mathrm{D}^{\mathrm{b}}(S, \alpha)$, for some twisted K3 surface (S, α);
(4) σ_c is $\mathbf{v}$-generic for all $c \in C$;
(5) the relative moduli space $M_{\underline{\sigma}}(\mathbf{v})$ is smooth and proper over C.

If we assume the claim, we can conclude the proof as follows. Let us consider the relatively nef real numerical Cartier divisor class $\ell_{\underline{\sigma}} \in N^1(M_{\underline{\sigma}}(\mathbf{v})/C)$ of Theorem 21.25. By slightly deforming $\underline{\sigma}$ and taking multiples if necessary, we can assume it is actually an integral class. By Proposition 32.4 and [BM14b, Corollary 7.5], we have that the divisor class $\ell_{\sigma_{c_1}}$ is ample. Since ampleness is an open property, ℓ_{σ_c} is ample for all c in a Zariski open subset U of C. By relative Serre vanishing, we can further assume it has no higher cohomology, for all $c \in U$. Hence, $h^0(M_{\sigma_c}(\mathbf{v}), \ell_{\sigma_c}^{\otimes m}) = \chi(M_{\sigma_c}(\mathbf{v}), \ell_{\sigma_c}^{\otimes m})$, for all $c \in U$ and $m > 0$, and thus it is independent on c.

By semicontinuity, this shows that $h^0(M_\sigma(\mathbf{v}), \ell_\sigma^{\otimes m})$ has maximal growth, for $m \gg 0$, and therefore it is big on $M_\sigma(\mathbf{v})$. Since $M_\sigma(\mathbf{v})$ has trivial canonical bundle and ℓ_σ is nef as well, the Base Point Free Theorem [KM98, Theorem 3.3] (for algebraic spaces see also [Anc87]) implies that a multiple of ℓ_σ is globally generated on $M_\sigma(\mathbf{v})$. But ℓ_σ has the stronger positivity property of intersecting any curve strictly positively; this shows that ℓ_σ is actually ample, and thus $M_\sigma(\mathbf{v})$ is projective, as we wanted.

Finally, we prove the claim above. By Corollary 32.1, there exists a family $g\colon \mathcal{X} \to C$ and a stability condition $\underline{\sigma}$ with all the properties except that the central charge might not factor via $\mathcal{N}(\mathcal{K}u(\mathcal{X})/C)$. This works if we replace C with an open subset $C' \subseteq C$ containing c_0 but c_1 might not be in C'.

The claim follows if we show that we can take C with infinitely many points satisfying (3). The argument is rather elementary and we will briefly discuss it. Let N be the smallest saturated sublattice of $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$ generated by the natural sublattice A_2, $\mathbf{v}$ and ℓ_σ. By [AT14, Proposition 2.3], the saturated sublattice $N^\perp$ of $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$ identifies with a saturated sublattice N' in $H^4(X, \mathbb{Z})$ (up to sign). Thus X is contained in the Hodge locus $\mathcal{C}_M$ of the moduli space of cubic fourfolds $\mathcal{C}$ determined by the lattice $M := (N')^\perp$. By Proposition 33.1, the set of cubic fourfolds satisfying (3) is a countable collection of divisors in $\mathcal{C}$ and each of them is a Hodge locus determined by a rank-2 sublattice of the fourth integral cohomology lattice of the cubic fourfold with non-trivial intersection with M, due to the signature of the lattice orthogonal to the saturated sublattice generated by N and the additional class of square zero. Thus the intersection of $\mathcal{C}_M$ with any such divisor $\mathcal{C}'$ has codimension at most 1 in $\mathcal{C}_M$. In order to show that the intersection $\mathcal{C}_M \cap \mathcal{C}'$ is nonempty for infinitely many $\mathcal{C}'$ as above, by the explicit description of the image of the period map for cubic fourfolds [Loo09, Laz10], we just need to show that this intersection is not contained in the two special divisors $\mathcal{C}_2$ and $\mathcal{C}_6$ in the period domain of cubic fourfolds. But if, for infinitely many $\mathcal{C}'$ the intersection is contained in either of the two divisors then, by the density of such Hodge loci, we would get that $\mathcal{C}_M$ is contained either in $\mathcal{C}_2$ or $\mathcal{C}_6$ which contradicts the fact that X is in $\mathcal{C}_M$. Hence we

can pick the curve C in $\mathcal{C}_M$ so that all the properties above are satisfied and (3) holds true for infinitely many points. The same is then true when passing to an open subset C' containing c_0.

The holomorphic symplectic structure. Since M is deformation equivalent to a Hilbert scheme of points on a K3 surface, it is a irreducible holomorphic symplectic manifold. Moreover, the existence of a quasi-universal family on the relative moduli space guarantees that the morphism θ does behave well in family as well, and thus Theorem 29.2.(2) follows by again reducing to the case of (twisted) K3 surfaces and using [BM14b, Theorem 6.10] therein. Theorem 29.2 is proven. □

We can finally complete the proof of Theorem 29.4.

Proof of Theorem 29.4.(2). First of all, as in the first part of the proof of Proposition 30.8, we can base change with a finite cover $u\colon \widetilde{S} \to S$ to have a family of lines over $\widetilde{S}$ and so that the fixed locus M of the monodromy group is $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(\mathcal{X}_{s_0}), \mathbb{Z})$. This gives a stability condition $\underline{\sigma}$ on $\mathcal{K}u(\mathcal{X})$ over $\widetilde{S}$ with respect to $M^\vee$ and with the property that $\underline{\sigma}_{s_0} = \sigma_{s_0}$. We can shrink S and assume further that the cover $\widetilde{S} \to S$ is étale.

Stability is open; in particular, the set of points $s \in \widetilde{S}$ such that $\underline{\sigma}_s$ is $\mathbf{v}$-generic is open. We set $\widetilde{S}^0$ this open subset. We consider the relative moduli space $\widetilde{M}^0(\mathbf{v}) := M_{\underline{\sigma}}(\mathbf{v})$ over $\widetilde{S}^0$. This is proper and smooth over $\widetilde{S}^0$, and since now the lattice is fixed, comes also with a relatively ample divisor class $l_{\underline{\sigma}}$, as defined in the proof of Theorem 29.2.

By [Sta21, Tag 0D30], $\widetilde{M}^0(\mathbf{v})$ is a smooth projective integral scheme over $\widetilde{S}^0$. Let $S^0 := u(\widetilde{S}^0) \subset S$; it is open. By Proposition 20.11, $\widetilde{M}^0(\mathbf{v})$ gives a descent datum for $\widetilde{M}^0(\mathbf{v})/\widetilde{S}^0/S^0$ (see [Sta21, Tag 023U]). Hence, $\widetilde{M}^0(\mathbf{v})$ does descend to an algebraic space $M^0(\mathbf{v}) \to S^0$. Since the central charge is monodromy-invariant, the relatively ample divisor class does descend as well. Therefore $M^0(\mathbf{v})$ is a scheme which is smooth projective over S and a relative moduli space, which is what we wanted. □

Proof of Corollary 29.5. Let S be an open subset of the 20-dimensional moduli space of smooth cubic fourfolds admitting a universal family $g\colon \mathcal{X} \to S$; for example, we can choose the open set of cubics that have no automorphism. For a very general point $s_0 \in S$, we have $K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X}_{s_0})) = A_2$, which is monodromy-invariant by construction.

For a pair of integers (a, b) as in the statement, consider the vector $\mathbf{v} := a\boldsymbol{\lambda}_1 + b\boldsymbol{\lambda}_2 \in A_2$. Take $\sigma_{s_0} \in \mathrm{Stab}^\dagger(\mathcal{K}u(\mathcal{X}_{s_0}))$ a $\mathbf{v}$-generic stability condition. Since $\eta(\sigma_{s_0}) \in A_2 \otimes \mathbb{C}$, the central charge of σ_{s_0} is monodromy invariant.

By Theorem 29.4.(2), there is a non-empty open subset $S^0 \subseteq S$ and a relative moduli space $g\colon M^0(\mathbf{v}) \to S^0$. By Theorem 29.2, the fibers of g are irreducible holomorphic symplectic manifolds of dimension $\mathbf{v}^2 + 2 = 2(a^2 - ab + b^2) + 2$. This proves the first part of the statement.

As explained in the proof of Theorem 29.2, $M^0(\mathbf{v})$ is endowed with a relative line bundle l such that $l_s = l_{\sigma_s}$. By Theorem 29.2.(2), l_s is orthogonal to $\mathbf{v}$ for all $s \in S^0$ and must be a vector in A_2, since for a very general point s_0 we have $K_{\mathrm{num}}(\mathcal{K}u(\mathcal{X}_{s_0})) = A_2$. Hence l_s is proportional to $\mathbf{w} = -(2b - a)\boldsymbol{\lambda}_1 + (2a - b)\boldsymbol{\lambda}_2$. Given a, b coprime, the class $\mathbf{w}$ is primitive if $a^2 - ab + b^2$ is not divisible by 3, and divisible by 3 otherwise. A simple computation yields the degree. Another application of Theorem 29.2.(2) shows that the divisibility of the

polarisation is equal to the divisibility of $\mathbf{w}$ or $\frac{\mathbf{w}}{3}$ in $\mathbf{v}^\perp$. The claim then follows from the fact that $\mathbf{v}^\perp$ has discriminant $2n$, whereas $\mathbf{v}^\perp \cap \mathbf{w}^\perp = A_2^\perp$ has discriminant 3.

To get a unirational family, we observe that S^0 is open in the moduli space of cubic fourfolds which is unirational by construction. □

Remark 33.5. Using a descent argument as in the proof of Theorem 29.4.(2), one can extend this family to a larger subset of the DM stack of cubic fourfolds. The precise locus $S \setminus S^0$ we are forced to omit can be determined as in [DM19] as the locus where a stability condition σ with $\eta(\sigma) \in A_2 \otimes \mathbb{C}$ cannot be $\mathbf{v}$-generic, or, equivalently, where l cannot extend to a polarisation.

Example 33.6. Let S be the moduli space of smooth cubic fourfolds, and $\mathbf{v} = 2\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$ as in Example 29.6. By [LPZ18, Theorem 1.2], we can take S^0 to be the complement of the divisor $\mathcal{C}_8$ of cubics containing a plane. (Alternatively, this consequence could also be deduced from the extension of [BM14a, Theorem 5.7] to our context; this would also show that along $\mathcal{C}_8$, stability conditions with central charge in A_2 lie on a wall, induced by the additional Hodge classes; in other words, S^0 is the maximal possible subset in which Theorem 29.4.(2) holds.) It follows from the first statement of [Has00, Proposition 5.2.1] that the monodromy on $\mathcal{C}_8$ acts trivially on these additional Hodge classes. Therefore, we are in the setting of Remark 30.9, and can deform the stability conditions with central charge in A_2 to some that are generic on all fibers; the associated relative coarse moduli space extends $M^0(\mathbf{v}) \to S^0$ to a proper morphism $M(\mathbf{v}) \to S$ of algebraic spaces, with all fibers being smooth and projective. Over $\mathcal{C}_8$, it agrees with the moduli spaces of stable objects constructed by Ouchi in [Ouc17].

Proof of Theorem 29.1. Let σ be a stability condition in the boundary of the open subset $\mathrm{Stab}^\dagger(\mathcal{K}u(X)) \subset \mathrm{Stab}(\mathcal{K}u(X))$; by the covering map property, its central charge is on the boundary of $\mathfrak{P}_0^+(\mathcal{K}u(X))$. This means that in the kernel of Z there is either a root $\delta \in \widetilde{H}_{\mathrm{Hdg}}(X,\mathbb{Z}), \delta^2 = -2$, or there is a *real* class $\mathbf{w} \in \widetilde{H}_{\mathrm{Hdg}}(X,\mathbb{Z}) \otimes \mathbb{R}$ with $\mathbf{w}^2 \geqslant 0$.

By definition of the topology on $\mathrm{Stab}(\mathcal{K}u(X))$, the set where a given object is semistable is closed; therefore, the non-emptiness of Theorem 29.2 still holds for σ. In the former case, this is a direct contradiction to $Z(\delta) = 0$. In the latter case, let Q be the quadratic form giving the support property for σ, and consider a sequence $\mathbf{w}_i \in \widetilde{H}_{\mathrm{Hdg}}(X,\mathbb{Z}) \otimes \mathbb{Q}$ with $\mathbf{w}_i^2 \geqslant 0$ and $\mathbf{w}_i \to \mathbf{w}$. As an integral multiple of $\mathbf{w}_i$ is the Mukai vector of a σ-semistable object, we have $Q(\mathbf{w}_i) \geqslant 0$, a contradiction to $Q(\mathbf{w}) < 0$. □

Proof of Corollary 29.8. Take a class $\mathbf{v} \in H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$. By [AH61, Section 2.5] (see also [AT14, Theorem 2.1(3)]), there is $\mathbf{w} \in K_{\mathrm{top}}(X)$ such that $v(\mathbf{w}) = \mathbf{v} + \widetilde{\mathbf{v}}$, where $\widetilde{\mathbf{v}} \in H^6(X,\mathbb{Q}) \oplus H^8(X,\mathbb{Q})$.

Consider the projection $\mathbf{w}'$ of $\mathbf{w}$ to $\widetilde{H}(\mathcal{K}u(X),\mathbb{Z})$ which is induced by the projection functor. It is clear that the identity $\mathbf{w}' = \mathbf{w} + a_0[\mathcal{O}_W] + a_1[\mathcal{O}_W(1)] + a_2[\mathcal{O}_W(2)]$ holds in $K_{\mathrm{top}}(X)$, where for $a_0, a_1, a_2 \in \mathbb{Z}$ (see, for example, [AT14, Section 2.4]). Note that $c_2(\mathbf{w}')$ and $c_2(\mathbf{w}) = c_2(\mathbf{v})$ differ by a multiple of h^2, so one is algebraic if and only if the other one is. The vector $\mathbf{w}'$ is in $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X),\mathbb{Z}) = \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z})$ since the projection preserves the Hodge structure (for the latter equality we used Theorem 29.2). Hence, there exists $E \in \mathcal{K}u(X)$ such that $v(E) = \mathbf{w}'$ and $c_2(\mathbf{w}')$ is algebraic. □

REFERENCES

[Add16] Nicolas Addington. On two rationality conjectures for cubic fourfolds. *Math. Res. Lett.*, 23(1):1–13, 2016. arXiv:1405.4902. 120

[AE21] Benjamin Antieau and Elden Elmanto. Descent for semiorthogonal decompositions. *Adv. Math.*, 380:107600, 2021. 12

[AH61] Michel F. Atiyah and Fredric Hirzebruch. *Vector bundles and homogeneous spaces*. Proc. Sympos. Pure Math., Vol. III. American Mathematical Society, Providence, R.I., 1961. 136

[AHLH18] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. Existence of moduli spaces for algebraic stacks, 2018. arXiv:1812.01128. 4, 104, 105

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013. arXiv:0804.2242. 4, 104, 105, 106

[Anc87] Vincenzo Ancona. Vanishing and nonvanishing theorems for numerically effective line bundles on complex spaces. *Ann. Mat. Pura Appl. (4)*, 149:153–164, 1987. 134

[AP06] Dan Abramovich and Alexander Polishchuk. Sheaves of t-structures and valuative criteria for stable complexes. *J. Reine Angew. Math.*, 590:89–130, 2006. arXiv:math/0309435. 3, 6, 22, 24, 26, 32, 34, 36, 39, 42, 62, 104

[AT14] Nicolas Addington and Richard Thomas. Hodge theory and derived categories of cubic fourfolds. *Duke Math. J.*, 163(10):1885–1927, 2014. arXiv:1211.3758. 5, 6, 118, 121, 126, 131, 134, 136

[Bay19] Arend Bayer. A short proof of the deformation property of Bridgeland stability conditions. *Math. Ann.*, 375(3-4):1597–1613, 2019. arXiv:1606.02169. 107

[BBD82] Alexander A. Beĭlinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982. 22

[BCZ17] Arend Bayer, Alastair Craw, and Ziyu Zhang. Nef divisors for moduli spaces of complexes with compact support. *Selecta Math. (N.S.)*, 23(2):1507–1561, 2017. arXiv:1602.00863. 100

[BD85] Arnaud Beauville and Ron Donagi. La variété des droites d'une hypersurface cubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(14):703–706, 1985. 6

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983. 5, 119

[Beh09] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. *Ann. of Math. (2)*, 170(3):1307–1338, 2009. arXiv:math/0507523. 5

[BF97] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997. arXiv:alg-geom/9601010. 117

[BHT15] Arend Bayer, Brendan Hassett, and Yuri Tschinkel. Mori cones of holomorphic symplectic varieties of $K3$ type. *Annales Scientifiques de l'ENS*, 48(5):941–950, 2015. arXiv:1307.2291. 119

[BLMS17] Arend Bayer, Martí Lahoz, Emanuele Macrì, and Paolo Stellari. Stability conditions on Kuznetsov components, 2017. Appendix about the Torelli theorem for cubic fourfolds by A. Bayer, M. Lahoz, E. Macrì, P. Stellari, and X. Zhao. arXiv:1703.10839. 5, 39, 40, 41, 73, 108, 109, 113, 115, 116, 118, 121, 122, 124, 130, 131

[BM11] Arend Bayer and Emanuele Macrì. The space of stability conditions on the local projective plane. *Duke Math. J.*, 160(2):263–322, 2011. arXiv:0912.0043. 62

[BM14a] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014. arXiv:1301.6968. 121, 136

[BM14b] Arend Bayer and Emanuele Macrì. Projectivity and birational geometry of Bridgeland moduli spaces. *J. Amer. Math. Soc.*, 27(3):707–752, 2014. arXiv:1203.4613. 5, 105, 106, 130, 131, 133, 134, 135

[BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. *Invent. Math.*, 206(3):869–933, 2016. arXiv:1410.1585. 4, 5, 61, 62, 71, 88, 111, 114, 116, 117

[BMSZ17] Marcello Bernardara, Emanuele Macrì, Benjamin Schmidt, and Xiaolei Zhao. Bridgeland stability conditions on Fano threefolds. *Épijournal Geom. Algébrique*, 1:Art. 2, 24, 2017. arXiv:1607.08199. 116

[BMT14] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014. arXiv:1103.5010. 4, 5, 111, 112, 114, 115, 117

[Bog78] Fedor A. Bogomolov. Holomorphic tensors and vector bundles on projective manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(6):1227–1287, 1439, 1978. 111

[BOR20] Pieter Belmans, Shinnosuke Okawa, and Andrea T. Ricolfi. Moduli spaces of semiorthogonal decompositions in families, 2020. arXiv:2002.03303. 12

[Bri99] Tom Bridgeland. Equivalences of triangulated categories and Fourier-Mukai transforms. *Bull. London Math. Soc.*, 31(1):25–34, 1999. arXiv:math/9809114. 130

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007. arXiv:math/0212237. 2, 4, 22, 59, 60, 61, 62, 68, 78, 124, 125

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008. arXiv:math/0307164. 22, 62, 118

[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and Drinfeld centers in derived algebraic geometry. *J. Amer. Math. Soc.*, 23(4):909–966, 2010. arXiv:0805.0157. 13, 16

[CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995. 125

[CM78] Albert Conte and Jacob P. Murre. The Hodge conjecture for fourfolds admitting a covering by rational curves. *Math. Ann.*, 238(1):79–88, 1978. 118

[DIM15] Olivier Debarre, Atanas Iliev, and Laurent Manivel. Special prime Fano fourfolds of degree 10 and index 2. In *Recent advances in algebraic geometry*, volume 417 of *London Math. Soc. Lecture Note Ser.*, pages 123–155. Cambridge Univ. Press, Cambridge, 2015. arXiv:1302.1398. 7

[dJ96] Aise J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, 83:51–93, 1996. 61

[DK18] Olivier Debarre and Alexander Kuznetsov. Gushel-Mukai varieties: classification and birationalities. *Algebr. Geom.*, 5(1):15–76, 2018. arXiv:1510.05448. 7

[DK19] Olivier Debarre and Alexander Kuznetsov. Gushel-Mukai varieties: linear spaces and periods. *Kyoto J. Math.*, 59(4):897–953, 2019. arXiv:1605.05648. 7

[DM19] Olivier Debarre and Emanuele Macrì. On the period map for polarized hyperkähler fourfolds. *Int. Math. Res. Not. IMRN*, 22:6887–6923, 2019. arXiv:1704.01439. 136

[Dou02] Michael R. Douglas. Dirichlet branes, homological mirror symmetry, and stability. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 395–408, Beijing, 2002. Higher Ed. Press. arXiv:math/0207021. 3

[DV10] Olivier Debarre and Claire Voisin. Hyper-Kähler fourfolds and Grassmann geometry. *J. Reine Angew. Math.*, 649:63–87, 2010. arXiv:0904.3974. 7

[FM99] Barbara Fantechi and Marco Manetti. On the T^1-lifting theorem. *J. Algebraic Geom.*, 8(1):31–39, 1999. 128

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 1998. 61

[Gie79] David Gieseker. On a theorem of Bogomolov on Chern classes of stable bundles. *Amer. J. Math.*, 101(1):77–85, 1979. 111

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. 30

[Gro66] Alexander Grothendieck. Éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, 28:255, 1966. 127

[Has00] Brendan Hassett. Special cubic fourfolds. *Compositio Math.*, 120(1):1–23, 2000. 120, 124, 136

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. 80, 84, 100

[HL14] Daniel Halpern-Leistner. On the structure of instability in moduli theory. *arXiv:1411.0627*, 2014. 27

[HMS09] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Derived equivalences of $K3$ surfaces and orientation. *Duke Math. J.*, 149(3):461–507, 2009. arXiv:0710.1645. 80, 126

[HRS96] Dieter Happel, Idun Reiten, and Sverre O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.*, 120(575):viii+ 88, 1996. 22

[HS06] Daniel Huybrechts and Paolo Stellari. Proof of Căldăraru's conjecture. Appendix to: "Moduli spaces of twisted sheaves on a projective variety" by K. Yoshioka. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 31–42. Math. Soc. Japan, Tokyo, 2006. arXiv:math/0411541. 133

[HT10] Daniel Huybrechts and Richard P. Thomas. Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes. *Math. Ann.*, 346(3):545–569, 2010. arXiv:0805.3527. 117, 126

[Huy97] Daniel Huybrechts. Birational symplectic manifolds and their deformations. *J. Differential Geom.*, 45(3):488–513, 1997. arXiv:alg-geom/9601015. 5, 119

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006. 35

[Huy17] Daniel Huybrechts. The K3 category of a cubic fourfold. *Compos. Math.*, 153(3):586–620, 2017. arXiv:1505.01775. 5, 6, 118, 121, 131, 133

[Huy19] Daniel Huybrechts. Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories. In *Birational Geometry of Hypersurfaces*, pages 165–198. Springer, 2019. arXiv:1811.02876. 118

[IM11] Atanas Iliev and Laurent Manivel. Fano manifolds of degree ten and EPW sextics. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(3):393–426, 2011. arXiv:0907.2781. 7

[Ina11] Michi-aki Inaba. Smoothness of the moduli space of complexes of coherent sheaves on an abelian or a projective $K3$ surface. *Adv. Math.*, 227(4):1399–1412, 2011. arXiv:1002.0379. 126

[JS12] Dominic Joyce and Yinan Song. A theory of generalized Donaldson-Thomas invariants. *Mem. Amer. Math. Soc.*, 217(1020):iv+199, 2012. arXiv:0810.5645. 4

[Kaw92] Yujiro Kawamata. Unobstructed deformations. A remark on a paper of Z. Ran: "Deformations of manifolds with torsion or negative canonical bundle" [J. Algebraic Geom. **1** (1992), no. 2, 279–291; MR1144440 (93e:14015)]. *J. Algebraic Geom.*, 1(2):183–190, 1992. 128

[KLS06] Dmitry Kaledin, Manfred Lehn, and Christoph Sorger. Singular symplectic moduli spaces. *Invent. Math.*, 164(3):591–614, 2006. arXiv:math/0504202. 119

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. 134

[KM09] Alexander Kuznetsov and Dimitri Markushevich. Symplectic structures on moduli spaces of sheaves via the Atiyah class. *J. Geom. Phys.*, 59(7):843–860, 2009. arXiv:math/0703264. 126, 128, 130, 133

[Kos18] Naoki Koseki. Stability conditions on product threefolds of projective spaces and Abelian varieties. *Bull. Lond. Math. Soc.*, 50(2):229–244, 2018. arXiv:1703.07042. 116

[Kos20] Naoki Koseki. Stability conditions on threefolds with nef tangent bundles. *Adv. Math.*, 372:28, 2020. Id/No 107316. arXiv:1811.03267. 5, 116, 117

[KP18] Alexander Kuznetsov and Alexander Perry. Derived categories of Gushel-Mukai varieties. *Compos. Math.*, 154(7):1362–1406, 2018. arXiv:1605.06568. 7

[KP21] Alexander Kuznetsov and Alexander Perry. Categorical joins. *J. Amer. Math. Soc.*, 34(2):505–564, 2021. arXiv:1804.00144. 7

[KS08] Maxim Kontsevich and Yan Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, 2008. arXiv:0811.2435. 2, 4, 59

[Kuz08] Alexander Kuznetsov. Derived categories of quadric fibrations and intersections of quadrics. *Adv. Math.*, 218(5):1340–1369, 2008. arXiv:math/0510670. 121, 122

[Kuz09] Alexander Kuznetsov. Derived categories of Fano threefolds. *Proc. Steklov Inst. Math.*, 264(1):110–122, 2009. arXiv:0809.0225. 116

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 219–243. Birkhäuser Boston Inc., Boston, MA, 2010. arXiv:0808.3351. 5, 118, 121

[Kuz11] Alexander Kuznetsov. Base change for semiorthogonal decompositions. *Compos. Math.*, 147(3):852–876, 2011. arXiv:0711.1734. 6, 11, 12, 13, 14, 15, 16

[Kuz19] Alexander Kuznetsov. Calabi-Yau and fractional Calabi-Yau categories. *J. Reine Angew. Math.*, 753:239–267, 2019. arXiv:1509.07657. 130

[Lan04] Adrian Langer. Semistable sheaves in positive characteristic. *Ann. of Math. (2)*, 159(1):251–276, 2004. 102, 111, 113

[Lan15] Adrian Langer. Bogomolov's inequality for Higgs sheaves in positive characteristic. *Invent. Math.*, 199(3):889–920, 2015. 111

[Laz10] Radu Laza. The moduli space of cubic fourfolds via the period map. *Ann. of Math. (2)*, 172(1):673–711, 2010. 134

[Li19a] Chunyi Li. On stability conditions for the quintic threefold. *Invent. Math.*, 218(1):301–340, 2019. arXiv:1810.03434. 5, 116, 117

[Li19b] Chunyi Li. Stability conditions on Fano threefolds of Picard number 1. *J. Eur. Math. Soc. (JEMS)*, 21(3):709–726, 2019. arXiv:1510.04089. 116

[Lie06a] Max Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.*, 15(1):175–206, 2006. arXiv:math/0502198. 3, 4, 43, 44, 48, 94, 95, 126

[Lie06b] Max Lieblich. Remarks on the stack of coherent algebras. *Int. Math. Res. Not.*, pages Art. ID 75273, 12, 2006. arXiv:math/0603034. 6, 53

[LLMS18] Martí Lahoz, Manfred Lehn, Emanuele Macrì, and Paolo Stellari. Generalized twisted cubics on a cubic fourfold as a moduli space of stable objects. *J. Math. Pures Appl. (9)*, 114:85–117, 2018. arXiv:1609.04573. 6, 120

[LLSvS17] Christian Lehn, Manfred Lehn, Christoph Sorger, and Duco van Straten. Twisted cubics on cubic fourfolds. *J. Reine Angew. Math.*, 731:87–128, 2017. arXiv:1305.0178. 6, 120

[Loo09] Eduard Looijenga. The period map for cubic fourfolds. *Invent. Math.*, 177(1):213–233, 2009. arXiv:0705.0951. 134

[LPZ18] Chunyi Li, Laura Pertusi, and Xiaolei Zhao. Twisted cubics on cubic fourfolds and stability conditions, 2018. arXiv:1802.01134. 6, 98, 120, 136

[LPZ20] Chunyi Li, Laura Pertusi, and Xiaolei Zhao. Elliptic quintics on cubic fourfolds, O'Grady 10, and Lagrangian fibrations, 2020. arXiv:2007.14108. 7, 120

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. 13

[Lur17] Jacob Lurie. Higher algebra. Available at `http://www.math.harvard.edu/~lurie/`, 2017. 15, 27

[Lur18] Jacob Lurie. Spectral algebraic geometry. Available at `http://www.math.harvard.edu/~lurie/`, 2018. 29

[Mac14] Emanuele Macrì. A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space. *Algebra Number Theory*, 8(1):173–190, 2014. arXiv:1207.4980. 116

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011. arXiv:1101.4606. 119

[MO20] Giovanni Mongardi and John Christian Ottem. Curve classes on irreducible holomorphic symplectic varieties. *Commun. Contemp. Math.*, 22(7):1950078, 15, 2020. arXiv:1805.12019. 121

[Mon15] Giovanni Mongardi. A note on the Kähler and Mori cones of hyperkähler manifolds. *Asian J. Math.*, 19(4):583–591, 2015. arXiv:1307.0393. 119

[Mos18] Riccardo Moschetti. The derived category of a non generic cubic fourfold containing a plane. *Math. Res. Lett.*, 25(5):1525–1545, 2018. arXiv:1607.06392. 121

[MP16] Antony Maciocia and Dulip Piyaratne. Fourier-Mukai transforms and Bridgeland stability conditions on abelian threefolds II. *Internat. J. Math.*, 27(1):1650007, 27, 2016. arXiv:1310.0299. 116

[MS19a] Emanuele Macrì and Paolo Stellari. Lectures on non-commutative K3 surfaces, Bridgeland stability, and moduli spaces. In *Birational Geometry of Hypersurfaces*, pages 199–265. Springer, 2019. arXiv:1807.06169. 5, 118

[MS19b] Cristian Martinez and Benjamin Schmidt. Bridgeland stability on blow ups and counterexamples. *Math. Z.*, 292(3-4):1495–1510, 2019. arXiv:1708.08567. 116

[Muk84] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface. *Invent. Math.*, 77(1):101–116, 1984. 5, 119, 126

[Muk87] Shigeru Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987. 5, 119, 132

[Nee96] Amnon Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996. arXiv:alg-geom/9412022. 13, 19

[O'G97] Kieran G. O'Grady. The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface. *J. Algebraic Geom.*, 6(4):599–644, 1997. arXiv:alg-geom/9510001. 5, 119

[OPT18] Georg Oberdieck, Dulip Piyaratne, and Yukinobu Toda. Donaldson-Thomas invariants of abelian threefolds and Bridgeland stability conditions, 2018. arXiv:1808.02735. 116

[Orl97] Dmitri Orlov. Equivalences of derived categories and $K3$ surfaces. *J. Math. Sci. (New York)*, 84(5):1361–1381, 1997. Algebraic geometry, 7. arXiv:alg-geom/9606006. 133

[Orl06] Dmitri Orlov. Triangulated categories of singularities, and equivalences between Landau-Ginzburg models. *Mat. Sb.*, 197(12):117–132, 2006. arXiv:math/0503630. 13

[Ouc17] Genki Ouchi. Lagrangian embeddings of cubic fourfolds containing a plane. *Compos. Math.*, 153(5):947–972, 2017. arXiv:1407.7265. 136

[Per19] Alexander Perry. Noncommutative homological projective duality. *Adv. Math.*, 350:877–972, 2019. arXiv:1804.00132. 14, 16, 21, 29

[Per20] Alexander Perry. The integral hodge conjecture for two-dimensional Calabi-Yau categories, 2020. arXiv:2004.03163. 121, 126, 129, 133

[Piy17] Dulip Piyaratne. Stability conditions, Bogomolov-Gieseker type inequalities and Fano 3-folds, 2017. arXiv:1705.04011. 116

[Pol07] Alexander Polishchuk. Constant families of t-structures on derived categories of coherent sheaves. *Mosc. Math. J.*, 7(1):109–134, 167, 2007. arXiv:math/0606013. 3, 6, 18, 22, 23, 24, 26, 27, 31, 32, 33, 62, 65

[PPZ19] Alexander Perry, Laura Pertusi, and Xiaolei Zhao. Stability conditions and moduli spaces for Kuznetsov components of Gushel-Mukai varieties, 2019. arXiv:1912.06935. 7, 130

[PT19] Dulip Piyaratne and Yukinobu Toda. Moduli of Bridgeland semistable objects on 3-folds and Donaldson–Thomas invariants. *J. Reine Angew. Math.*, 747:175–219, 2019. arXiv:1504.01177. 4, 71, 72, 73, 84, 111, 114, 116, 117

[Rei78] Miles Reid. Bogomolov's theorem $c_1^2 \leq 4c_2$. In *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, pages 623–642, Tokyo, 1978. Kinokuniya Book Store. 111

[Sch14] Benjamin Schmidt. A generalized Bogomolov-Gieseker inequality for the smooth quadric threefold. *Bull. Lond. Math. Soc.*, 46(5):915–923, 2014. arXiv:1309.4265. 116

[Sch17] Benjamin Schmidt. Counterexample to the generalized Bogomolov-Gieseker inequality for threefolds. *Int. Math. Res. Not. IMRN*, 8:2562–2566, 2017. arXiv:1602.05055. 116

[Sim94] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, 79:47–129, 1994. 101, 102

[Sos12] Pawel Sosna. Stability conditions under change of base field. *Math. Nachr.*, 285(2-3):364–376, 2012. arXiv:1004.4835. 59, 64

[Sta21] The Stacks Project Authors. Stacks project, 2021. Available at `http://stacks.math.columbia.edu`. 8, 9, 14, 15, 18, 19, 30, 32, 33, 34, 37, 43, 44, 45, 46, 47, 50, 53, 54, 55, 56, 57, 58, 135

[Tho00] Richard Thomas. A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations. *J. Differential Geom.*, 54(2):367–438, 2000. arXiv:math/9806111. 5

[Tod08] Yukinobu Toda. Moduli stacks and invariants of semistable objects on $K3$ surfaces. *Adv. Math.*, 217(6):2736–2781, 2008. arXiv:math.AG/0703590. 4, 5, 47, 62

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990. 13

[Voi07] Claire Voisin. Some aspects of the Hodge conjecture. *Jpn. J. Math.*, 2(2):261–296, 2007. 118, 121

[Voi13] Claire Voisin. Hodge loci. In *Handbook of moduli. Vol. III*, volume 26 of *Adv. Lect. Math. (ALM)*, pages 507–546. Int. Press, Somerville, MA, 2013. 122, 125

[Voi18] Claire Voisin. Hyper-Kähler compactification of the intermediate Jacobian fibration of a cubic fourfold: The twisted case. In *Local and global methods in algebraic geometry*, volume 712 of *Contemp. Math.*, pages 341–355. Amer. Math. Soc., Providence, RI, 2018. arXiv:1611.06679. 7, 120

[Yos99] Kōta Yoshioka. Some notes on the moduli of stable sheaves on elliptic surfaces. *Nagoya Math. J.*, 154:73–102, 1999. arXiv:alg-geom/9705007. 80

[Yos01] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.*, 321(4):817–884, 2001. arXiv:math/0009001. 5, 119

[Zuc77] Steven Zucker. The Hodge conjecture for cubic fourfolds. *Compositio Math.*, 34(2):199–209, 1977. 118, 124

Index of Notations

$\mathcal{A}$, heart of a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$, 22
$\mathcal{A}^0 \subset \mathcal{A}$ subcategory of $\mathcal{A}$ where Z vanishes, 71
$\mathcal{A}^{\gamma}_{\alpha,\beta,s}$, tilted heart of $\operatorname{Coh}^{\beta} \mathcal{X}_s$ at tilt-slope γ, 115
$\mathcal{A}_K$, base change of the heart of a bounded tilted-noetherian t-structure to K the function field of S, 33
$\mathcal{A}_\ell$, base change of the heart of a bounded tilted-noetherian t-structure to a field extension $k \subset \ell$, 33
$\alpha_E \colon \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(X)$, pullback functor, tensor with a relative exceptional object E, 20
$\mathcal{A}_{\mathrm{qc}}$, induced heart on the quasi-coherent component $\mathcal{D}_{\mathrm{qc}}$, 23
$\mathcal{A}_s$, base change of the heart of a bounded t-structure to a point $s \in S$
  S regular, 33
$\mathcal{A}^\sharp = \langle \mathcal{F}[1], \mathcal{T} \rangle$, tilted heart at a torsion pair, 22
$\mathcal{A}^{\sharp\beta} = \langle \mathcal{F}^\beta[1], \mathcal{T}^\beta \rangle$, tilted heart at β, 73
$\mathcal{A}_{S\text{-tf}}$, subcategory of S-torsion free objects in $\mathcal{A}_S$, 35
$\mathcal{A}_{S\text{-tor}}$, subcategory of S-torsion objects in $\mathcal{A}_S$, 35
$\mathcal{A}_T$, base change of the heart of a bounded t-structure via $T \to S$
  T affine, 27
  $T \to S$ essentially of finite type, 31

$\operatorname{ch}_{\mathcal{X}/S} \colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})/S) \to \mathbb{Q}^{n+1}$, 112
$\operatorname{ch}_{\mathcal{X}_s} \colon K_{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X}_s)) \to \mathbb{Q}^{n+1}$, Chern characters on the fibers, 112
$\operatorname{ch}^{\beta}_{\mathcal{X}/S}$, twisted Chern characters, 112
$\operatorname{Coh}^{\beta} X_s$, tilted heart of $\operatorname{Coh} X_s$ at slope β, 73, 111
$\operatorname{Coh}^{\beta} \mathcal{X}_s$, tilted heart of $\operatorname{Coh} \mathcal{X}_s$ at slope β, 112

$(\mathcal{D}^{\sharp,\leqslant 0}, \mathcal{D}^{\sharp,\geqslant 0})$, tilted t-structure at a torsion pair, 22
$(\mathcal{D}^{\leqslant 0}_T, \mathcal{D}^{\geqslant 0}_T)$, base change of a t-structure via $T \to S$
  T affine, 27
  $T \to S$ essentially of finite type, 31
$\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$, semiorthogonal decomposition, 11
  S-linear, 12
  strong, 12
†-equivalence preserving geometric components, 130
$\Delta_s := \operatorname{ch}^2_{\mathcal{X}_s,1} - 2 \operatorname{ch}_{\mathcal{X}_s,0} \operatorname{ch}_{\mathcal{X}_s,2}$, discriminant, 114
$\mathcal{D}_\ell$, base change of $\mathcal{D}$ to $\operatorname{Spec}(\ell)$, 33
$\mathrm{D}_{\mathrm{pug}}(X/S)$, category of universally gluable S-perfect objects, 44
$\mathcal{D}_{\mathrm{qc}}$, quasi-coherent component, 13
$\mathcal{D}_{S\text{-tor}}$, subcategory of S-torsion objects in $\mathcal{D}$, 35
$\mathcal{D}_T$, base change of an S-linear strong semiorthogonal component $\mathcal{D} \subset \mathrm{D}^{\mathrm{b}}(X)$ via $T \to S$, 17
$\mathrm{DT}_\sigma(\mathbf{v})$, Donaldson–Thomas invariant, 117

$E_1, \dots, E_m$, relative exceptional objects in $\mathcal{D}$, 19
$\eta^{\vee}_{\ell/k} \colon K_{\mathrm{num}}(\mathcal{D}_\ell) \to K_{\mathrm{num}}(\mathcal{D}) \otimes \mathbb{Q}$, pushforward map, 63

$\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$, extended Mukai lattice, 118
$\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \cong K_{\mathrm{num}}(\mathcal{K}u(X))$, sublattice of algebraic classes, 118, 119
$\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$, sublattice of integral $(1,1)$-classes, 118, 119

integrable t-structure over a scheme, 51

$K_{\mathrm{num}}(\mathcal{D})$, numerical K-group, 60
$K_{\mathrm{num}}(\mathcal{D}_{\mathrm{perf}})$, numerical K-group, 60
$K_{\mathrm{num}}(\mathcal{D})_\ell$, field extension $k \subset \ell$, base changed numerical K-group, 63
$K_{\mathrm{num}}(\mathcal{D}/S)$, relative numerical K-group, 99
$\mathcal{K}u(X)$, Kuznetsov component of cubic fourfolds, 118

Λ_0, saturated subgroup of Λ generated by Mukai vectors of elements in $\mathcal{A}^0_s$, 101
$\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$, roots of the A_2-lattice inside $\widetilde{H}_{\mathrm{Hdg}}(\mathcal{K}u(X), \mathbb{Z})$, 120
$\overline{\Lambda}$, free abelian group Λ/Λ_0, 101
Λ_ℓ, field extension $k \subset \ell$, base changed lattice, 63
$\mathrm{L}_{\mathcal{C}}$, left mutation functor through $\mathcal{C}$, 11

$\mathcal{M}_{\mathrm{pug}}(X/S)$, functor of universally gluable T-perfect objects in $\mathrm{D}_{\mathrm{pug}}(X_T/T)$, 44
$\mathcal{M}_{\mathrm{pug}}(\mathcal{D}/S)$, functor of universally gluable T-perfect objects in $\mathrm{D}_{\mathrm{pug}}(X_T/T)$ such that $E_t \in \mathcal{D}_t$ for all $t \in T$, 45
$\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$, stack parameterizing geometrically $\underline{\sigma}$-stable objects of class $\mathbf{v}$ and phase ϕ, 101
$M_{\underline{\sigma}}(\mathbf{v})$, coarse moduli space of $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$, 105
$M_{\underline{\sigma}}(\mathbf{v})$, coarse moduli space of $\mathcal{M}_{\underline{\sigma}}(\mathbf{v}) = \mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v})$, 111
$\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$, stack parameterizing $\underline{\sigma}$-semistable objects of class $\mathbf{v}$ and phase ϕ, 101
$\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ $(\mathcal{M}^{\mathrm{st}}_{\underline{\sigma}}(\mathbf{v}))$, substacks of $\mathcal{M}_{\underline{\tau}}$, parameterizing $\underline{\sigma}$-semistable (geometrically $\underline{\sigma}$-stable) objects of class $\mathbf{v}$ and phase ϕ, 105, 111
$\mathcal{M}_{\underline{\tau}}$, functor of of all objects $E \in \mathrm{D}_{\mathrm{pug}}(X_T/T)$ which are T-flat with respect to $\underline{\tau}$, 52

μ_C, slope for objects in $\mathcal{A}_C$, 69, 76
$\mu_C^+(E)$ ($\mu_C^-(E)$), maximal (minimal) slope for objects in $\mathcal{A}_C$, 79
μ_Z, slope function, 71

$\mathcal{N}(\mathcal{D}/S)$, uniformly numerical relative Grothendieck group, 100
$\nabla_{\beta,s} := 4(\mathrm{ch}^\beta_{\mathcal{X}_s,2})^2 - 6\,\mathrm{ch}^\beta_{\mathcal{X}_s,1}\,\mathrm{ch}^\beta_{\mathcal{X}_s,3} - \alpha^2\Delta_s$, 116

$\mathcal{P}$, $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$, slicing, 59
$\mathfrak{P}$, generalized period domain, 118
$\mathcal{P}(I)$, slicing for an interval I, 59
$\mathcal{P}_{\underline{\sigma}}(I;\mathbf{v})$, stack parameterizing objects in $\mathcal{P}(I)$ of class $\mathbf{v}$, 101
$\phi(E)$, phase of an object, 60, 69
ϕ_E^+ (ϕ_E^-), maximal (minimal) phase functions, 96
$\phi^+(E)$ ($\phi^-(E)$), maximal (minimal) phase of an object, 59
$\mathcal{P}_\ell(\phi)$, field extension $k \subset \ell$, base changed slicing, 64, 74

$\mathrm{Quot}_S(E)$, Quot functor, 52
$\mathrm{Quot}_S^{\leqslant\phi}(E)$, subfunctor parametrizing quotients with phase $\leqslant \phi$, 103

$\mathrm{R}_{\mathcal{C}}$, right mutation functor through $\mathcal{C}$, 11

$\sigma = (Z, \mathcal{P})$
(pre-)stability condition, 59
weak (pre-)stability condition, 71, 72
having the tilting property, 73
$\sigma = (\mathcal{A}, Z)$,
(pre-)stability condition, 60
weak (pre-)stability condition, 71
$\sigma^\beta = (\mathcal{A}^{\sharp\beta}, Z^{\sharp\beta})$,
tilted weak stability condition at $\beta \in \mathbb{R}$, 73
$\sigma_C = (Z_K, Z_{C\text{-tor}}, \mathcal{P})$, C a Dedekind scheme
Harder–Narasimhan structure on $\mathcal{D}$ over C, 68
satisfying the support property, 87
weak HN structure on $\mathcal{D}$ over C, 75
having the tilting property, 90
satisfying the support property, 87
$\sigma_C = (\mathcal{A}_C, Z_K, Z_{C\text{-tor}})$, C a Dedekind scheme
Harder–Narasimhan structure on $\mathcal{D}$ over C, 70
weak HN structure on $\mathcal{D}$ over C, 78
$\sigma_c = (\mathcal{A}_c, Z_c)$, $c \in C$ Dedekind scheme,
induced stability condition on $\mathcal{D}_c$, 88
when $c = p$ is a closed point, 69, 76
$\sigma_C^{\sharp\beta}$, tilted weak HN structure at slope β, 91
$\sigma_\ell = (\mathcal{A}_\ell, Z_\ell)$, field extension $k \subset \ell$,
base changed stability condition, 64
base changed weak stability condition, 75
$\underline{\sigma} = (\sigma_s = (Z_s, \mathcal{P}_s))_{s\in S}$,
(weak) stability condition on $\mathcal{D}$ over S with respect to Λ, 102
flat family of fiberwise (weak) stability conditions on $\mathcal{D}$, 94
$\underline{\sigma} := (\sigma_s = (\mathrm{i}\,\mathrm{ch}_{X/S,0} - \mathrm{ch}_{X/S,1}, \mathrm{Coh}\,X_s))$
(weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by slope-stability on each fiber, 100
$\underline{\sigma} := (\sigma_s = (\mathrm{i}\,\mathrm{ch}_{\mathcal{X}/S,0} - \mathrm{ch}_{\mathcal{X}/S,1}, \mathrm{Coh}\,X_s))$
(weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by slope-stability on each fiber, 102
$\underline{\sigma} := (\sigma_s = (\mathrm{i}\,\mathrm{ch}_{\mathcal{X}/S,0} - \mathrm{ch}_{\mathcal{X}/S,1}, \mathrm{Coh}\,\mathcal{X}_s))$
(weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by slope-stability on each fiber, 112
$\underline{\sigma}^{\sharp\beta} := (\sigma_s^{\sharp\beta} = (\mathrm{i}\,\mathrm{ch}^\beta_{\mathcal{X}_s,1} + \mathrm{ch}^\beta_{\mathcal{X}_s,0}, \mathrm{Coh}^\beta\,\mathcal{X}_s))$
(weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by rotating slope-stability on each fiber, 112
$\underline{\sigma}^{\alpha,\beta} := (\sigma_s^{\alpha,\beta} = (Z_s^{\alpha,\beta}, \mathrm{Coh}^\beta\,\mathcal{X}_s))$
(weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by tilt-stability on each fiber, 114
$\underline{\sigma}^{\alpha,\beta\sharp\gamma} := (\sigma_s^{\alpha,\beta\sharp\gamma} = (\frac{1}{u_\gamma} \cdot Z_s^{\alpha,\beta}, \mathcal{A}^\gamma_{\alpha,\beta,s}))$,
(weak) stability condition on $\mathrm{D}^{\mathrm{b}}(\mathcal{X})$ over S given by rotating tilt-stability on each fiber, 115
$s\mathcal{M} \subset \mathcal{M}$, subfunctor parameterizing simple objects of $\mathcal{M}$, 48
sM, algebraic moduli space, 48
$\mathrm{Stab}^\dagger(\mathcal{K}u(X))$, connected component of $\mathrm{Stab}(\mathcal{K}u(X))$ containing geometric stability conditions, 118
$\mathrm{Stab}^\dagger(S,\alpha)$, connected component of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(S,\alpha))$ containing geometric stability conditions, 130
$\mathrm{Stab}_\Lambda(\mathcal{D})$, space of stability conditions on $\mathcal{D}$, 62
$\mathrm{Stab}_\Lambda(\mathcal{D}/S)$, space of stability conditions on $\mathcal{D}$ over S, 106

$(\mathcal{T}, \mathcal{F})$, torsion pair of an abelian category $\mathcal{A}$, 22
$(\mathcal{T}^\beta, \mathcal{F}^\beta)$, torsion pair at $\beta \in \mathbb{R}$, 73
$\tau = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$, t-structure on $\mathcal{D}$, 21
S-local, 23
bounded, 22
noetherian, 23
tilted-noetherian, 23
$\underline{\tau} = (\tau_s)_{s\in S}$, fiberwise collection of t-structures on $\mathcal{D}$ over S, 49

$v\colon K_{\mathrm{num}}(\mathcal{D}/S) \to \Lambda$, relative Mukai homomorphism, 99
$(v_K, v_{C\text{-tor}})$,
Mukai homomorphism on $\mathcal{D}$ over C, 87
numerical on fibers, 87
v_ℓ, field extension $k \subset \ell$,

base changed Mukai homomorphism, 64
$\mathbf{v} \in \Lambda$, Mukai vector, 62, 101

$Z\colon K(\mathcal{A}) \to \mathbb{C}$,
stability function on $\mathcal{A}$, 60
weak stability function on $\mathcal{A}$, 71
Z_C, central charge for objects in $\mathcal{A}_C$, 68, 76
$(Z_K, Z_{C\text{-tor}})$, central charge on $\mathcal{D}$ over C, 67
$Z_s^{\alpha,\beta} := \mathrm{i}\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,1} + \frac{\alpha^2}{2}\,\mathrm{ch}^{\beta}_{\mathcal{X}_s,0} - \mathrm{ch}^{\beta}_{\mathcal{X}_s,2}$,
central charge for tilt-stability, 114

A.B.: School of Mathematics and Maxwell Institute, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom
E-mail address: arend.bayer@ed.ac.uk
URL: http://www.maths.ed.ac.uk/~abayer/

M.L.: Université Paris Diderot – Paris 7, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France
E-mail address: marti.lahoz@ub.edu
URL: http://www.ub.edu/geomap/lahoz/
Current address: Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Spain

E.M.: Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115, USA
E-mail address: emanuele.macri@universite-paris-saclay.fr
URL: https://www.imo.universite-paris-saclay.fr/~macri/
Current address: Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, Rue Michel Magat, Bât. 307, 91405 Orsay, France

H.N.: Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115, USA
E-mail address: hnuer@technion.ac.il
URL: https://sites.google.com/site/howardnuermath/home
Current address: Department of Mathematics, Technion, Israel Institute of Technology, Amado 914, Haifa 32000, Israel

A.P.: Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA
E-mail address: arper@umich.edu
URL: http://www-personal.umich.edu/~arper/
Current address: Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA

P.S.: Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy
E-mail address: paolo.stellari@unimi.it
URL: https://sites.unimi.it/stellari