SOME REMARKS ABOUT THE FM-PARTNERS OF K3 SURFACES WITH PICARD NUMBERS 1 AND 2

PAOLO STELLARI

ABSTRACT. In this paper we prove some results about K3 surfaces with Picard number 1 and 2. In particular, we give a new simple proof of a theorem due to Oguiso which shows that, given an integer N, there is a K3 surface with Picard number 2 and at least N non-isomorphic FM-partners. We describe also the Mukai vectors of the moduli spaces associated to the Fourier-Mukai partners of K3 surfaces with Picard number 1.

1. INTRODUCTION

In some recent papers Hosono, Lian, Oguiso and Yau (see [5] and [13]) gave a formula that counts the number of non-isomorphic Fourier-Mukai partners of a K3 surface. In this paper we are interested in the case of K3 surfaces with Picard number 1 and 2.

In the second paragraph, we recall the formula for the number of the isomorphism classes of Fourier-Mukai partners of a given K3 surface (given in [4]), which allows to count the isomorphism classes of Fourier-Mukai partners of a K3 surface with Picard number 1 (this is also given in [13]). As a first result, we will describe the Mukai vectors of the moduli spaces associated to the Fourier-Mukai partners of such K3 surfaces[1]. This gives some information about the geometry of the Fourier-Mukai partners of the given K3 surface.

In the third paragraph we prove that, given N and d positive integers, there is an elliptic K3 surface with a polarization of degree d and with at least N non-isomorphic elliptic Fourier-Mukai partners (Theorem 3.3). The most interesting consequence of this result is a new simple proof of Theorem 1.7 in [13] (Corollary 3.4 and Remark 3.5).

We start with recalling some essential facts about lattices and K3 surfaces.

1.1. **Lattices and discriminant groups.** A *lattice* $L := (L, b)$ is a free abelian group of finite rank with a non-degenerate symmetric bilinear form $b : L \times L \to \mathbb{Z}$. Two lattices (L_1, b_1) and (L_2, b_2) are *isometric* if there is an isomorphism of abelian groups $f : L_1 \to L_2$ such that $b_1(x, y) = b_2(f(x), f(y))$. We write $\mathrm{O}(L)$ for the group of all autoisometries of the lattice L. A lattice (L, b) is *even* if, for all $x \in L$, $x^2 := b(x, x) \in 2\mathbb{Z}$, it is *odd* if there is $x \in L$ such that $b(x, x) \notin 2\mathbb{Z}$. Given an integral basis for L, we can associate to the bilinear form a symmetric matrix S_L of dimension $\mathrm{rk}L$, uniquely determined up to the action of $\mathrm{GL}(\mathrm{rk}L, \mathbb{Z})$. The integer $\det L := \det S_L$ is called *discriminant* and it is an invariant of the lattice. A lattice is *unimodular* if $\det L = \pm 1$. Given (L, b) and $k \in \mathbb{Z}$, $L(k)$ is the lattice (L, kb).

Given a sublattice V of L with $V \hookrightarrow L$, the embedding is *primitive* if L/V is free. In particular, a sublattice is primitive if its embedding is primitive. Two primitive embeddings $V \hookrightarrow L$ and $V \hookrightarrow L'$ are *isomorphic* if there is an isometry between L and L' which induces the identity on V. For a sublattice V of L we define the *orthogonal* lattice $V^{\perp} := \{x \in L : b(x, y) = 0, \forall y \in V\}$. Given two lattices (L_1, b_1) and (L_2, b_2), their *orthogonal direct sum* is the lattice (L, b), where $L = L_1 \oplus L_2$ and $b(x_1 + y_1, x_2 + y_2) = b_1(x_1, x_2) + b_2(y_1, y_2)$, for $x_1, x_2 \in L_1$ and $y_1, y_2 \in L_2$.

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[1]This result was independently proved by Hosono, Lian, Oguiso and Yau (Theorem 2.1 in [5]).

The *dual lattice* of a lattice (L,b) is $L^\vee := \mathrm{Hom}(L,\mathbb{Z}) \cong \{x \in L\otimes_{\mathbb{Z}}\mathbb{Q} : b(x,y)\in\mathbb{Z}, \forall y \in L\}$. Given the natural inclusion $L \hookrightarrow L^\vee$, $x \mapsto b(-,x)$, we define the *discriminant group* $A_L{:=}L^\vee/L$. The order of A_L is $|\mathrm{det}L\,|$ (see [1], Lemma 2.1, page 12). Moreover, b induces a symmetric bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and a corresponding quadratic form $q_L : A_L \to \mathbb{Q}/\mathbb{Z}$ such that, when L is even, $q_L(\overline{x}) = q(x)$ modulo $2\mathbb{Z}$, where $\overline{x}$ is the image of $x \in L^\vee$ in A_L. The elements of the triple $(t_{(+)}, t_{(-)}, q_L)$, where $t_{(\pm)}$ is the multiplicity of positive/negative eigenvalues of the quadratic form on $L\otimes\mathbb{R}$, are invariants of the lattice L.

If L is unimodular, $L^\vee \cong \{b(-,x) : x \in L\}$. If V is a primitive sublattice of a unimodular lattice L such that $b\,|_V$ is non-degenerate, then there is a natural isometry of groups $\gamma : V^\vee/V \to (V^\perp)^\vee/V^\perp$.

1.2. **K3 surfaces and M-polarizations.** A *K3 surface* is a 2-dimensional complex projective smooth variety with trivial canonical bundle and first Betti number $b_1 = 0$. From now on, X will be a K3 surface. The group $H^2(X,\mathbb{Z})$ with the cup product is an even unimodular lattice and it is isomorphic to the lattice $\Lambda := U^3 \oplus E_8(-1)^2$ (for the meaning of U and E_8 see [1] page 14). The lattice Λ is called *K3 lattice* and it is unimodular and even.

Given the lattice $H^2(X,\mathbb{Z})$, the *Néron-Severi group* $\mathrm{NS}(X)$ is a primitive sublattice. $T_X := \mathrm{NS}(X)^\perp$ is the *transcendental lattice*. The rank of the Néron-Severi group $\rho(X) :=\mathrm{rkNS}(X)$ is called the *Picard number*, and the signature of the Néron-Severi group is $(1,\rho-1)$, while the one of the transcendental lattice is $(2,20-\rho)$. If X and Y are two K3 surfaces, $f : T_X \to T_Y$ is an *Hodge isometry* if it is an isometry of lattices and the complexification of f is such that $f_{\mathbb{C}}(\mathbb{C}\omega_X) = \mathbb{C}\omega_Y$, where $H^{2,0}(X) = \mathbb{C}\omega_X$ and $H^{2,0}(Y) = \mathbb{C}\omega_Y$. We write $(T_X,\mathbb{C}\omega_X) \cong (T_Y,\mathbb{C}\omega_Y)$ to say that there is an Hodge isometry between the two transcendental lattices.

A *marking* for a K3 surface X is an isometry $\varphi : H^2(X,\mathbb{Z}) \to \Lambda$. We write (X,φ) for a K3 surface X with a marking φ. Given $\Lambda_{\mathbb{C}} := \Lambda\otimes\mathbb{C}$ and given $\omega \in \Lambda_{\mathbb{C}}$ we denote by $[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}})$ the corresponding line and we define the set $\Omega := \{[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : \omega\cdot\omega = 0, \omega\cdot\overline{\omega} > 0\}$. The image in $\mathbb{P}(\Lambda_{\mathbb{C}})$ of the line spanned by $\varphi_{\mathbb{C}}(\omega_X)$ belongs to Ω and is called *period point* (or period) of the marked surface (X,φ). From now on, the period point of a marked K3 surface (X,φ) will be indicated either by $\mathbb{C}\varphi_{\mathbb{C}}(\omega_X)$ or by $[\varphi_{\mathbb{C}}(\omega_X)]$.

Given two K3 surfaces X and Y, we say that they are *Fourier-Mukai-partners* (or FM-partners) if there is an equivalence between the bounded derived categories of coherent sheaves $\mathrm{D}^b_{coh}(X)$ and $\mathrm{D}^b_{coh}(Y)$. By results due to Mukai and Orlov, this is equivalent to say that there is an Hodge isometry $(T_X,\mathbb{C}\omega_X) \to (T_Y,\mathbb{C}\omega_Y)$. We define $FM(X)$ to be the set of the isomorphism classes of the FM-partners of X.

Let M be a primitive sublattice of Λ with signature $(1,t)$. A K3 surface X with a marking $\varphi : H^2(X,\mathbb{Z}) \to \Lambda$ is a *marked M-polarized K3 surface* if $\varphi^{-1}(M) \subseteq \mathrm{NS}(X)$. A K3 surface is *$M$-polarizable* if there is a marking φ such that (X,φ) is a marked M-polarized K3 surface. Two marked and M-polarized surfaces (X,φ) and (X',φ') are isomorphic if there is an isomorphism $\psi : X \to X'$ such that $\varphi' = \varphi\circ\psi^*$. Form now on, we will consider the case of lattices $M := \langle h\rangle$, with $h^2 = 2d$ and $d > 0$. The pair (X,h), where X is a K3 surface and $h \in \mathrm{NS}(X)$, with $h^2 = 2d$, means a K3 surface with a polarization of degree $2d$.

2. FM-partners of a K3 surface with $\rho = 1$ and associated Mukai vectors

In this section we want to describe the Mukai vectors of the moduli spaces associated to the M-polarized FM-partners of a K3 surface X with Picard number 1. By Orlov's results ([14]), $q = |FM(X)|$ is the same as the number of non-isomorphic compact 2-dimensional fine moduli spaces of stable sheaves on X. Obviously, on a K3 surface with Picard number 1 and $\mathrm{NS}(X) = \langle h\rangle$ there is only one $\langle h\rangle$-polarization of degree $h^2 = 2d$. So the concept of FM-partner and the concept of M-polarized FM-partner coincide. If $M = \langle h\rangle$ we are sure, by Orlov, that if we find

q non-isomorphic moduli spaces, then these are representatives of all the isomorphism classes of M-polarized FM-partners of X.

We recall briefly the counting formula for the isomorphism classes of FM-partners of a given K3 surface. Given a lattice S, the *genus* of S is the set $\mathcal{G}(S)$ of all the isometry classes of lattices S' such that $A_S \cong A_{S'}$ and the signature of S' is equal to the one of S.

Let T_X be the transcendental lattice of an abelian surface or of a K3 surface X with period $\mathbb{C}\omega_X$. We can define the group

$$G := O_{Hodge}(T_X, \mathbb{C}\omega_X) = \{g \in \mathrm{O}(T_X) : g(\mathbb{C}\omega_X) = \mathbb{C}\omega_X\}.$$

We know (see [2] Theorem 1.1, page 128), that the genus of a lattice, with fixed rank and discriminant, is finite. The map $\mathrm{O}(S) \to \mathrm{O}(A_S)$ defines an action of $\mathrm{O}(S)$ on $\mathrm{O}(A_S)$. On the other hand, taken $g \in G$, and given a marking φ for X, $\varphi \circ g \circ \varphi^{-1}$ induces an isometry on the lattice $T := \varphi(T_X)$, thus φ defines a homomorphism $G \hookrightarrow \mathrm{O}(T)$. The composition of this map and the map $\mathrm{O}(T) \to \mathrm{O}(A_T)$ gives an action of G on $\mathrm{O}(A_T) \cong \mathrm{O}(A_S)$.

Theorem 2.1. [4, Theorem 2.3]. *Let X be a K3 surface and let $\mathcal{G}(\mathrm{NS}(X)) = \mathcal{G}(S) = \{S_1, \cdots, S_m\}$. Then*

$$|FM(X)| = \sum_{j=1}^{m} |\mathrm{O}(S_j)\backslash \mathrm{O}(A_{S_j})/G|,$$

where the actions of the groups G and $\mathrm{O}(S_j)$ are defined as before.

The following corollary (which is Theorem 1.10 in [13]) determines the number q of FM-partners of a surface with Picard number 1.

Corollary 2.2. *Let X be a K3 surface with $\rho(X) = 1$ and such that $\mathrm{NS}(X) = \langle h \rangle$, with $h^2 = 2d$.*

(i) *The group $\mathrm{O}(A_S)$ is trivial if $d = 1$ while, if $d > 1$, $\mathrm{O}(A_S) \cong (\mathbb{Z}/2\mathbb{Z})^{p(d)}$, where $p(d)$ is the number of distinct primes q such that $q|d$. In particular, if $d \geq 2$, then $|\mathrm{O}(A_S)| = 2^{p(d)}$.*

(ii) *For all markings φ of X, the image of $H_{X,\varphi} := \{\varphi \circ g \circ \varphi^{-1} : g \in G\} \subseteq \mathrm{O}(T)$ in $\mathrm{O}(A_T)$ by the map $\mathrm{O}(T) \to \mathrm{O}(A_T)$ is $\{\pm \overline{id}\}$.*

In particular, $|FM(X)| = 2^{p(d)-1}$, where now we set $p(1) = 1$.

Assertion (i) is known and it can also be found in [15] (Lemma 3.6.1).

Using the notation of [10], we put $H^*(X,\mathbb{Z}) := H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$. Given $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta := (\beta_1, \beta_2, \beta_3)$ in $H^*(X,\mathbb{Z})$, using the cup product we define the bilinear form

$$\alpha \cdot \beta := -\alpha_1 \cup \beta_3 + \alpha_2 \cup \beta_2 - \alpha_3 \cup \beta_1.$$

From now on, depending on the context, $\alpha \cdot \beta$ will mean the bilinear form defined above or the cup product on $H^2(X,\mathbb{Z})$.

We give to $H^*(X,\mathbb{Z})$ an Hodge structure considering

$$\begin{aligned} H^*(X,\mathbb{C})^{2,0} &:= H^{2,0}(X), \\ H^*(X,\mathbb{C})^{0,2} &:= H^{0,2}(X), \\ H^*(X,\mathbb{C})^{1,1} &:= H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}). \end{aligned}$$

$\tilde{H}(X,\mathbb{Z})$ is the group $H^*(X,\mathbb{Z})$ with the bilinear form and the Hodge structure defined before.

For $v = (r,h,s) \in \tilde{H}(X,\mathbb{Z})$ with $r \in H^0(X,\mathbb{Z}) \cong \mathbb{Z}$, $s \in H^4(X,\mathbb{Z}) \cong \mathbb{Z}$ and $h \in H^2(X,\mathbb{Z})$, $M(v)$ is the moduli space of stable sheaves E on X such that $\mathrm{rk}E = r$, $c_1(E) = h$ and $s = c_1(E)^2/2 - c_2(E) + r$. If the stability is defined with respect to $A \in H^2(X,\mathbb{Z})$ we write $M_A(v)$. The vector v is *isotropic* if $v \cdot v = 0$. The vector $v \in \tilde{H}(X,\mathbb{Z})$ is primitive if $\tilde{H}(X,\mathbb{Z})/\mathbb{Z}v$ is free.

As we have observed, the results of Orlov in [14] imply that each FM-partner of X is isomorphic to an $M_h(v)$. We determine a set of Mukai vectors which corresponds bijectively to the isomorphism classes of the FM-partners of X in $FM(X)$. First of all, we recall the following theorem due to Mukai ([10]).

Theorem 2.3. [10, Theorem 1.5 3]. *If X is a K3 surface, $v = (r, h, s)$ is an isotropic vector in $\tilde{H}^{1,1}(X, \mathbb{Z}) = H^*(X, \mathbb{C})^{1,1} \cap H^*(X, \mathbb{Z})$ and $M_A(v)$ is non-empty and compact, then there is an isometry $\varphi : v^\perp/\mathbb{Z}v \to H^2(M_A(v), \mathbb{Z})$ which respects the Hodge structure.*

If $\mathrm{NS}(X) \cong \mathbb{Z}h$ with $h^2 = 2d = 2p_1^{e_1} \dots p_m^{e_m}$, where $k \geq 0$, $e_i \geq 1$ and p_i primes with $p_i \neq p_j$ if $i \neq j$, then we consider the Mukai vectors

$$v_J^I = v_{j_{s+1},...,j_m}^{j_1,...,j_s} = (p_{j_1}^{e_{j_1}} \dots p_{j_s}^{e_{j_s}}, h, p_{j_{s+1}}^{e_{j_{s+1}}} \cdots p_{j_m}^{e_{j_m}}),$$

where $I = \{j_1, \dots, j_s\}$ and $J = \{j_{s+1}, \dots, j_m\}$ are a partition of $\{1, \dots, m\}$ such that $I \amalg J = \{1, \dots, m\}$. The following theorem shows how to determine $|FM(X)|$ of them corresponding to non-isomorphic moduli spaces of stable sheaves.

Theorem 2.4. *Let X be a K3 surface with $\mathrm{NS}(X) = \mathbb{Z}h$ such that $h^2 = 2d = 2p_1^{e_1} \dots p_m^{e_m}$. Then, for all v_J^I as above, $M_h(v_J^I)$ is a 2-dimensional compact fine moduli space of stable sheaves on X. Moreover, if $M_h(v_{J_1}^{I_1}) \cong M_h(v_{J_2}^{I_2})$, then $v_{J_1}^{I_1} = v_{J_2}^{I_2}$ or $v_{J_2}^{I_2} = (s_1, h, r_1)$, with $v_{J_1}^{I_1} = (r_1, h, s_1)$, where the multindexes I_k and J_k vary over all the partitions of $\{1, \dots, m\}$.*

Proof. The vectors v_J^I are all isotropic and they are primitive in $\tilde{H}(X, \mathbb{Z})$, so, by Theorem 5.4 in [10] $M_h(v_J^I)$ is non-empty. Moreover the hypothesis of Theorem 4.1 in [10], are satisfied and so the moduli spaces are compact. By Corollary 0.2 in [11] they are 2-dimensional, while they are fine by the results in the appendix of [10].

If $m = 0$ or $m = 1$ then, by Corollary 2.2, we have only one moduli space with respectively $v = (1, h, 1)$ in the first case and $v = (1, h, p^e)$ in the second case.

Otherwise we must prove that if

$$v_1 = (r_1, h, s_1) = v_{J_1}^{I_1} \neq v_{J_2}^{I_2} = (r_2, h, s_2) = v_2,$$

with $v_2 \neq (s_1, h, r_1)$, then

$$M_h(v_1) \ncong M_h(v_2)$$

But by Theorem 2.3 and Torelli theorem, if we put

$$M_1 := v_1^\perp/\mathbb{Z}v_1 \quad \text{and} \quad M_2 := v_2^\perp/\mathbb{Z}v_2$$

then it suffices to show that there are no Hodge isometries between M_1 and M_2. Obviously, it suffices to show that there are no Hodge isometries between the transcendental lattices which lifts to an isometry of the second cohomology groups.

By definition, a representative of a class in M_i ($i = 1, 2$) is a vector (a, b, c) such that $bh = as_i + cr_i$, hence

$$bh \equiv as_i \pmod{r_i},$$

for $i = 1, 2$. From now on we will write (a, b, c) for the equivalence class or for a representative of the class. In fact, all the arguments we are going to propose are independent from the choice of a representative.

The Hodge structures on M_1 and M_2 are induced by the ones defined on $\tilde{H}(X, \mathbb{Z})$, so, up to an isometry, we identify $\mathrm{NS}(M_h(v_1))$ and $\mathrm{NS}(M_h(v_2))$ with

$$S_1 := \langle (0, h, 2s_1) \rangle \subset M_1 \quad \text{and} \quad S_2 := \langle (0, h, 2s_2) \rangle \subset M_2$$

respectively.

Now, we can describe the transcendental lattices $T_1 := S_1^\perp$ and $T_2 := S_2^\perp$ of $M_h(v_1)$ and $M_h(v_2)$ respectively.

If $(a, b, c) \cdot (0, h, 2s_1) = 0$ then $bh \equiv 0 \pmod{r_1}$. Indeed, let us suppose that $bh \equiv K \pmod{r_1}$ where $K \not\equiv 0 \pmod{r_1}$. Then, by simple calculations, we obtain

$$(a, b, c) = (L, 0, H) + \left(0, n, \frac{n \cdot h - K}{r_1}\right)$$

as equivalence classes. Here $n = b - kh$, for a particular $k \in \mathbb{Z}$, $bh \equiv Ls_1 \pmod{r_1}$ and H is an integer. But now

$$0 = (a, b, c) \cdot (0, h, 2s_1) = -2Ls_1 + nh = -2Ls_1 + (b - kh)h =$$
$$= -2bh + 2wr_1 + bh - kh^2,$$

with $w \in \mathbb{Z}$. So

$$bh \equiv 0 \pmod{r_1}.$$

This is a contradiction. By these remarks and simple calculations, a class y in T_1, as an element of the quotient M_1, has representative $(0, n, nh/r_1)$. But $(0, n, nh/r_1) \cdot (0, h, 2s_1) = nh = 0$. So $y = (0, n, 0)$ and

$$T_1 = \{(0, n, 0) : n \in T_X\}.$$

Analogously we have

$$T_2 = \{(0, n, 0) : n \in T_X\}.$$

By Lemma 4.1 in [13] (see also point (ii) of Corollary 2.2), if $f : (T_1, \mathbb{C}\omega_1) \to (T_2, \mathbb{C}\omega_2)$ is a Hodge isometry, then all the Hodge isometries from T_1 into T_2 are f and $-f$. But in this case M_1 and M_2 inherit their Hodge structure from $\tilde{H}(X, \mathbb{Z})$. Hence the two Hodge isometries $f, g : T_1 \to T_2$ are

$$(0, n, 0) \overset{f}{\longmapsto} (0, n, 0) \quad \text{or} \quad (0, n, 0) \overset{g}{\longmapsto} (0, -n, 0).$$

Let us show that f cannot be lifted to an isometry from M_1 into M_2. Equivalently, this means that there are no isomorphisms between $M_h(v_1)$ and $M_h(v_2)$ which induces f.

We start by observing that, if $(a, b, c) \in M_i$ with $i = 1, 2$, then

$$(a, b, c) \cdot (0, h, 2s_i) \equiv -bh \pmod{r_i}.$$

Indeed, if $bh \equiv K \pmod{r_i}$ then $a \equiv L \pmod{r_i}$ and so $(a, b, c) = (L, 0, H) + (0, n, \frac{n \cdot h - K}{r_i})$ with n and H as before. So, $(a, b, c) \cdot (0, h, 2s_i) = -2bh + 2wr_i + bh - kh^2 \equiv -bh \pmod{r_i}$ and $nh \equiv bh \pmod{r_i}$.

Now let us suppose that there is an isometry $\varphi : M_1 \to M_2$ which induces f. We can prove that there is $(a, b, c) \in M_1$, with $bh \equiv 0 \pmod{r_1}$, such that $\varphi(a, b, c) = (d, e, f) \in M_2$ with $eh \not\equiv 0 \pmod{r_2}$. First of all, by our hypotheses about r_1 and r_2, we can suppose that there is a prime p which divides r_2 but which does not divide r_1 (otherwise we can change the roles of M_1 and M_2 in the following argument). By Theorem 1.14.4 in [12], there is an isometry

$$\psi : H^2(X, \mathbb{Z}) \longrightarrow U^3 \oplus E_8(-1)^2 = \Lambda$$

such that $k_1 := \psi(h) = (1, d, 0, \dots, 0)$, where $h^2 = 2d$. Let $k_2 := (0, r_1, 0, \dots, 0)$. Now $k_1 \cdot k_2 = r_1$ and we can take $n := \psi^{-1}(k_2)$. Obviously, the vector $(0, n, n \cdot h/r_1) \in M_1$ is such that $n \cdot h \equiv 0 \pmod{r_1}$. Let us suppose that $\varphi((0, n, n \cdot h/r_1)) = (d, e, f)$ with $e \cdot h \equiv 0 \pmod{r_2}$. By the previous remark, this is equivalent to say that

$$\varphi\left(\left(0, n, \frac{n \cdot h}{r_1}\right)\right) = \left(0, m, \frac{m \cdot h}{r_2}\right),$$

for a given $m \in H^2(X, \mathbb{Z})$.

Since $\mathrm{rk} S_1 = \mathrm{rk} S_2 = 1$, either $\varphi((0, h, 2s_1)) = (0, h, 2s_2)$ or $\varphi((0, h, 2s_1)) = -(0, h, 2s_2)$. In particular, if φ correspond to case (1) (the same argument holds if φ is as in case (2)), then

$$n \cdot h = \left(0, n, \frac{n \cdot h}{r_1}\right) \cdot (0, h, 2s_1) = \varphi\left(\left(0, n, \frac{n \cdot h}{r_1}\right) \cdot (0, h, 2s_1)\right) =$$
$$= \left(0, m, \frac{m \cdot h}{r_2}\right) \cdot (0, h, 2s_2) = m \cdot h.$$

In particular, $m \cdot h = n \cdot h = r_1$ which is not divisible by r_2. This gives a contradiction and thus $eh \not\equiv 0 \pmod{r_2}$.

The previous remarks show that if

$$(a,b,c)=\left(0,n,\frac{nh}{r_1}\right)$$

then

$$\varphi((a,b,c))=(d,e,f)=(L,0,H)+\left(0,m,\frac{m\cdot h-K}{r_2}\right),$$

with $L\not\equiv 0 \pmod{r_2}$. Let us take $(0,N,0)\in T_1$ and

$$\varphi(0,N,0)=f(0,N,0)=(0,N,0)\in T_2.$$

Then

$$(*)\quad nN=\left(0,n,\frac{nh}{r_1}\right)\cdot(0,N,0)=$$
$$=\left[(L,0,H)+\left(0,m,\frac{m\cdot h-K}{r_i}\right)\right]\cdot(0,N,0)=mN.$$

Because $H^2(X,\mathbb{Z})$ is unimodular and (*) is true for every $N\in T_X$, we have $m-n=kh\in \mathrm{NS}(X)$, where $k\in\mathbb{Z}$. But now $nh=(0,n,\frac{nh}{r_1})\cdot(0,h,2s_1)=[(L,0,H)+(0,m,\frac{m\cdot h-K}{r_i})]\cdot(0,h,2s_2)=mh-2Ls_2=nh+kh^2-2Ls_2=nh+2kr_2s_2-2Ls_2$. So $L\equiv 0 \pmod{r_2}$ which is contradictory.

Repeating the same arguments for g, we see that neither f nor g lifts to an isometry of the second cohomology groups. So, by Torelli Theorem, $M_h(v_1)\not\cong M_h(v_2)$. □

3. Genus and polarizations when $\rho=2$

In this paragraph we are interested in the number of non isomorphic FM-partners of K3 surfaces with a given polarization and Picard number 2.

Our main result is Theorem 3.3. First of all, we recall the following lemma which is an easy corollary of Nikulin's Theorem 1.14.2 in [12] and whose hypotheses are trivially verified if $\rho=2$.

Lemma 3.1. *Let L be an even unimodular lattice and let T_1 and T_2 be two even sublattice with the same signature $(t_{(+)},t_{(-)})$, where $t_{(+)}>0$ and $t_{(-)}>0$. Let the corresponding discriminant groups (A_{T_1},q_{T_1}) and (A_{T_2},q_{T_2}) be isometric and let $\mathrm{rk}T_1\geq 2+\ell(A_{T_1})$, where $\ell(A_{T_1})$ is the minimal number of generators of A_{T_1}. Then $T_1\cong T_2$.*

We prove the following lemma.

Lemma 3.2. *Let $L_{d,n}$ be the lattice $(\mathbb{Z}^2,M_{d,n})$, where*

$$M_{d,n}:=\begin{pmatrix}2d & n\\ n & 0\end{pmatrix},$$

with d and n positive integers such that $(2d,n)=1$. Then
(i) *the discriminant group $A_{L_{d,n}}$ is cyclic;*
(ii) *if d_1, d_2, n_1 and n_2 are positive integers such that $(2d_1,n_1)=(2d_2,n_2)=1$ then $A_{L_{d_1,n_1}}\cong A_{L_{d_2,n_2}}$ if and only if*
  (a.1) *$n_1=n_2$;*
  (b.1) *there is an integer α such that $(\alpha,n)=1$ and $d_1\alpha^2\equiv d_2 \pmod{n^2}$;*
(iii) *if $L_{d_1,n}\cong L_{d_2,n}$ then one of the following conditions holds*
  (a.2) *$d_1\equiv d_2 \pmod n$;*
  (b.2) *$d_1d_2\equiv 1 \pmod n$.*

Proof. Let $e_{d,n}=(1,0)^t$ and $f_{d,n}=(0,1)^t$ be generators of the lattice $L_{d,n}$. Under the hypothesis $(2d,n)=1$, (i) follows immediately because

$$A_{L_{d,n}}:=L^\vee_{d,n}/L_{d,n}$$

has order $|\det M_{d,n}| = n^2$ and it is cyclic with generator

$$\overline{f}_{d,n} := \frac{ne_{d,n} - 2df_{d,n}}{n^2}.$$

Indeed,

$$L^{\vee}_{d,n} = \left\langle \frac{ne_{d,n} - 2df_{d,n}}{n^2}, \frac{f_{d,n}}{n} \right\rangle$$

and $\overline{f}_{d,n}$ has order n^2 in $A_{L_{d,n}}$.

First we prove that the conditions (a.1) and (b.1) are necessary. The orders of $A_{L_{d_1,n_1}}$ and $A_{L_{d_2,n_2}}$ are n_1^2 and n_2^2 respectively with $n_1, n_2 > 0$, so $n := n_1 = n_2$ (which is (a.1)). If $A_{L_{d_1,n}}$ and $A_{L_{d_2,n}}$ are isomorphic as groups, there is an integer α prime with n such that the isomorphism is determined by

$$\overline{f}_{d_1,n} \mapsto \alpha \overline{f}_{d_2,n}.$$

But now

$$\overline{f}^2_{d_1,n} = \frac{-2d_1}{n^2} \qquad \overline{f}^2_{d_2,n} = \frac{-2d_2}{n^2},$$

and if we want $A_{L_{d_1,n}}$ and $A_{L_{d_2,n}}$ to be isometric as lattices, we must require

$$\frac{-2d_1}{n^2} \equiv \alpha^2 \frac{-2d_2}{n^2} \pmod 2.$$

This is true if and only if

$$d_1 \equiv \alpha^2 d_2 \pmod{n^2}.$$

So the necessity of condition (b.1) is proved. In the same way it follows that (a.1) and (b.1) are also sufficient.

Let us consider point (iii). The lattices $L_{d_1,n}$ and $L_{d_2,n}$ are isometric if and only if there is a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$(*) \quad A^t M_{d_1,n} A = M_{d_2,n}.$$

Let $L_{d_1,n}$ and $L_{d_2,n}$ be isometric and let

$$A := \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Then from (*) we obtain the two relations

(1) $d_2 = x^2 d_1 + xzn$;

(2) $2y(yd_1 + tn) = 0$.

By (2) we have only two possibilities: either $y = 0$ or $yd_1 = -tn$. Let $y = 0$. From the relation

$$1 = |\det(A)| = |xt - yz| = |xt|$$

it follows that $x = \pm 1$ and so, from (1), we have $d_1 \equiv d_2 \pmod n$, which is condition (a.2).

Let us consider the case $yd_1 = -tn$. We know that $(d_1, n) = 1$ and hence $y = cn$ and $t = -cd_1$, with $c \in \mathbb{Z}$. From

$$1 = |\det(A)| = |-cxd_1 - cnz|$$

it follows that $c = \pm 1$. We suppose $c = 1$ (if $c = -1$ then the same arguments work by simple changes of signs). Multiplying both members of relation (1) by d_1 we have

$$d_2 d_1 \equiv x^2 d_1^2 \pmod n.$$

But we know that $\pm 1 = \det(A) = -xd_1 - nz$ and so $-xd_1 \equiv \pm 1 \pmod n$. Thus

$$1 \equiv x^2 d_1^2 \pmod n$$

and from this we obtain (b.2). □

Now we can prove the following theorem (note that point (iii) and (v) are exactly Theorem 1.7 in [13]).

Theorem 3.3. *Let N and d be positive integers. Then there are N K3 surfaces $X_1, \ldots, X_N$ with Picard number $\rho = 2$ such that*
(i) *X_i is elliptic, for every $i \in \{1, \ldots, N\}$;*
(ii) *there is $i \in \{1, \ldots, N\}$ such that X_i has a polarization of degree $2d$;*
(iii) *$\mathrm{NS}(X_i) \not\cong \mathrm{NS}(X_j)$ if $i \neq j$, where $i, j \in \{1, \ldots, N\}$;*
(iv) *$|det\mathrm{NS}(X_i)|$ is a square, for every $i \in \{1, \ldots, N\}$;*
(v) *there is an Hodge isometry between $(T_{X_i}, \mathbb{C}\omega_{X_i})$ and $(T_{X_j}, \mathbb{C}\omega_{X_j})$, for all $i, j \in \{1, \ldots, N\}$.*
In particular, X_i and X_j are non-isomorphic FM-partners, for all $i, j \in \{1, \ldots, N\}$.

Proof. The surjectivity of the period map for K3 surfaces implies that, given a sublattice S of Λ with rank 2 and signature $(1, 1)$, there is at least one K3 surface X such that its transcendental lattice T_X is isometric to $T := S^{\perp}$.

So the theorem follows if we can show that, for an arbitrary integer N, there are at least N sublattices of Λ with rank 2, signature $(1, 1)$ and representing zero which are non-isometric but whose orthogonal lattices are isometric in Λ.

Let us consider in $U \oplus U \hookrightarrow \Lambda$ the following sublattices

$$S_{d,n} := \langle \begin{pmatrix} 1 \\ q \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ n \\ 1 \\ 0 \end{pmatrix} \rangle,$$

with $(2q, n) = 1$ and $n > 0$.

We can observe that, when n and d vary, the lattices $S_{q,n}$ are primitive in Λ and the matrices associated to their quadratic forms are exactly the $M_{q,n}$. Since $M_{q,n}$ has negative determinant, the lattice has signature (1,1). Moreover, $S_{q,n}$ represents zero

Let $n > 2$ be a prime number such that $n > d^2N^4$. We choose $d_1 := d$, $d_2 := d2^2, \ldots, d_N = dN^2$. By definition, there is an integer α_i such that $(\alpha_i, n) = 1$ and

$$\alpha_i^2 d_1 \equiv d_i \pmod{n^2},$$

for every $i \in \{1, \ldots, N\}$. Thus the hypotheses (b.1) of Lemma 3.2 are satisfied and by point (ii) of the same lemma,

$$A_{S_{d_1,n}} \cong A_{S_{d_i,n}} \cong A_{S_{d_j,n}},$$

where $i, j \in \{1, ..., N\}$. By Lemma 3.2, there are isometries

$$\psi_i : S_{d_1,n}^{\perp} \to S_{d_i,n}^{\perp},$$

with $i \in \{2, ..., N\}$. Now let (X_1, φ_1) be a marked K3 surface associated to the lattice $S_{d_1,n}$. By the surjectivity of the period map we can consider the marked K3 surfaces (X_i, φ_i), with $i \in \{2, ..., N\}$, such that
(1) $\varphi_{i,\mathbb{C}}(\mathbb{C}\omega_{X_i}) = \psi_{i,\mathbb{C}}(\varphi_{1,\mathbb{C}}(\mathbb{C}\omega_{X_1}))$;
(2) $\varphi_i(\mathrm{NS}(X_i)) = S_{d_i,n}$;
(3) $\varphi_i(T_{X_i}) = S_{d_i,n}^{\perp}$.
Obviuosly, the surfaces X_i are FM-partners of X_1.

Now we show that, when $i \neq j$,

$$S_{d_i,n} \not\cong S_{d_j,n}.$$

First of all we know that, obviously, $d_j \not\equiv d_i \pmod{n}$ if $i \neq j$. On the other hand,

$$d_i d_j < d^2 N^4 < n,$$

so

$$1 \not\equiv d_i d_j \pmod{n}.$$

Hence, by point (iii) of Lemma 3.2, the lattices can not be isometric. The K3 surfaces $X_1, \dots, X_N$ are obviously elliptic and the discriminant of their Néron-Severi group is a square. Moreover X_1 has a polarization of degree $2d$.

This shows that it is possible to find N K3 surfaces which satisfy the hypotheses of the theorem. □

The previous theorem gives a new proof of the following result due to Oguiso ([13]).

Corollary 3.4. [13, Theorem 1.7]. *Let N be a natural number. Then there are N K3 surfaces $X_1, \dots, X_N$ with Picard number $\rho = 2$ such that*
(i) $\mathrm{NS}(X_i) \not\cong \mathrm{NS}(X_j)$ *if $i \neq j$, where $i, j \in \{1, \dots, N\}$;*
(ii) *there is an Hodge isometry between $(T_{X_i}, \mathbb{C}\omega_{X_i})$ and $(T_{X_j}, \mathbb{C}\omega_{X_j})$, for all $i, j \in \{1, \dots, N\}$.*

Remark 3.1. The proof proposed by Oguiso in [13] is based on deep results in number theory. In particular, it uses a result of Iwaniec [7] about the existence of infinitely many integers of type $4n^2+1$ which are product of two not necessarily distinct primes. Theorem 3.3 gives an elementary proof of Theorem 1.7 in [13] entirely based on simple remarks about lattices and quadratic forms.

Lemma 3.1 is true also when $L = U \oplus U \oplus U$. The period map is onto also for abelian surfaces (see [16]). Thus, using the lattices $S_{d,n}$ described before, the following proposition (similar to a result given in [6]) can be proved with the same techniques.

Proposition 3.5. *Let N and d be positive integers. Then there are N abelian surfaces $X_1, \dots, X_N$ with Picard number $\rho = 2$ such that*
(i) $\mathrm{NS}(X_i) \not\cong \mathrm{NS}(X_j)$ *if $i \neq j$, with $i, j \in \{1, \dots, N\}$;*
(ii) *there is $i \in \{1, \dots, N\}$ such that X_i has a polarization of degree $2d$;*
(iii) *there is an Hodge isometry between $(T_{X_i}, \mathbb{C}\omega_{X_i})$ and $(T_{X_j}, \mathbb{C}\omega_{X_j})$, for all $i, j \in \{1, \dots, N\}$.*

The following easy remark shows that it is possible to obtain an arbitrarily large number of M-polarizations on a $K3$ surface, for certain M.

Remark 3.2. Let N be a natural number. Then there are a primitive sublattice M of Λ with signature $(1,0)$ and a $K3$ surface X with $\rho(X) = 2$ such that X has at least N non-isomorphic M-polarizations. In particular X has at least N non-isomorphic M-polarized FM-partners.

In fact, let $S \cong U$, where U is, as usual, the hyperbolic lattice. Then, by the surjectivity of the period map, there is a $K3$ surface X such that $\mathrm{NS}(X) \cong S$.

Let d be a natural number with $d = p_1^{e_1} \dots p_n^{e_n}$. In S there are $2^{p(d)-1}$ primitive vectors with autointersection $2d$. Indeed they are all the vectors of type

$$f_J^I := (p_{j_1}^{e_{j_1}} \dots p_{j_s}^{e_{j_s}}, p_{j_{s+1}}^{e_{j_{s+1}}} \cdots p_{j_n}^{e_{j_n}}),$$

for I and J that vary in all possible partitions $I \amalg J = \{1, \dots, n\}$.

The group $\mathrm{O}(U)$ has only four elements (i.e. $\pm id$, the exchange of the vectors of the base and the composition of this map with $-id$). So it is easy to verify that all these polarizations are not isomorphic. Choosing d to be divisible by a sufficiently large number of distinct primes, we can find at least N non-isomorphic M-polarizations. The last assertion follows from Lemma 3.1.

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Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy

E-mail address: paolo.stellari@unimi.it

URL: https://sites.unimi.it/stellari