THE PASSAGE AMONG THE SUBCATEGORIES OF WEAKLY APPROXIMABLE TRIANGULATED CATEGORIES

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Abstract. In this article we prove that all the inclusions between the 'classical' and naturally defined full triangulated subcategories of a weakly approximable triangulated category are intrinsic (in one case under a technical condition). This extends all the existing results about subcategories of weakly approximable triangulated categories.

Together with a forthcoming paper about uniqueness of enhancements, our result allows us to generalize a celebrated theorem by Rickard which asserts that if R and S are left coherent rings, then a derived equivalence of R and S is "independent of the decorations". That is, if $D^?(R\text{–}\square)$ and $D^?(S\text{–}\square)$ are equivalent as triangulated categories for some choice of decorations ? and $\square$, then they are equivalent for every choice of decorations. But our theorem is much more general, and applies also to quasi-compact and quasi-separated schemes—even to the relative version, in which the derived categories consist of complexes with cohomology supported on a given closed subscheme with quasi-compact complement.

Contents

2020 *Mathematics Subject Classification.* Primary 18G80, secondary 14F08, 18N40, 18N60.

Key words and phrases. Triangulated categories,enhancements.

A. C. is a member of GNSAGA (INdAM) and was partially supported by the research project PRIN 2022 "Moduli spaces and special varieties". A. N. was partly supported by Australian Research Council Grants DP200102537 and DP210103397, and by ERC Advanced Grant 101095900-TriCatApp. P. S. was partially supported by the ERC Consolidator Grant ERC-2017-CoG-771507-StabCondEn, by the research project PRIN 2022 "Moduli spaces and special varieties", and by the research project FARE 2018 HighCaSt (grant number R18YA3ESPJ).

Introduction

Derived categories are by now old and well-established, but it helps occasionally to remember the difficulties that the people who first used them had to overcome. Let us therefore go back in time to the 1960s and 70s, when the subject was still new.

The first printed introduction to them was in Hartshorne's 1966 exposition of Grothendieck's duality theory [7], but the fuller and more thorough treatment in Verdier's PhD thesis circulated among the experts at the time—it was finally published in 1996 (see [33]). And the reader interested in the historical perspective is encouraged to also look at Illusie's exposés in SGA6 [12, 11, 10], which all address foundational questions about derived categories. There was an enormous effort in these early days to try to understand these objects, and figure out how best to work with them.

Recall that, associated to a scheme X, it is customary to attach three exact or abelian categories: the category of quasi-coherent sheaves on X, denoted here $\mathbf{Qcoh}(X)$, the subcategory of vector bundles on X, denoted here $\mathbf{Vect}(X)$, and (if X is noetherian) the category of coherent sheaves on X, which we will denote $\mathbf{Coh}(X)$. Now when we pass from abelian (or exact) categories to their derived categories, the objects of interest are cochain complexes of objects in the abelian (or exact) category, and the natural question becomes which cochain complexes should be permitted. And in the early days this was not clear—a great deal of thought and effort went into studying the advantages and disadvantages of the various options.

After several decades, the consensus is as follows. Let X be a scheme, and assume $Z \subseteq X$ is a Zariski-closed subset. For the derived categories of (quasi-)coherent sheaves to work well, it is best to impose finiteness hypotheses on the schemes that arise in the constructions we wish to make, and the finiteness hypotheses that yield a useful theory postulate that the scheme X is quasi-compact and quasi-separated[1], and that the open set $X \setminus Z$ is quasi-compact. And the objects of the seven useful (relative) derived categories, associated to the pair $Z \subseteq X$, should all be cochain complexes of sheaves of $\mathcal{O}_X$–modules, whose restriction to the open set $X \setminus Z$ is acyclic. And then we may wish to impose further restrictions, for example on the allowed cohomology sheaves. It so happens that the most useful of the various derived categories one could consider turn out to be the following seven:

(a) $\mathbf{D}_{\mathbf{qc},Z}(X)$. The only restriction is that all the cohomology sheaves must be quasi-coherent.
(b) $\mathbf{D}^-_{\mathbf{qc},Z}(X) \subseteq \mathbf{D}_{\mathbf{qc},Z}(X)$. This is the full subcategory of all complexes $\mathcal{C}$ with $\mathcal{H}^n(\mathcal{C}) = 0$ for $n \gg 0$.
(c) $\mathbf{D}^+_{\mathbf{qc},Z}(X) \subseteq \mathbf{D}_{\mathbf{qc},Z}(X)$. This is the full subcategory of all complexes $\mathcal{C}$ with $\mathcal{H}^n(\mathcal{C}) = 0$ for $n \ll 0$.
(d) $\mathbf{D}^b_{\mathbf{qc},Z}(X) \subseteq \mathbf{D}_{\mathbf{qc},Z}(X)$. This is the full subcategory of all complexes $\mathcal{C}$ with $\mathcal{H}^n(\mathcal{C}) = 0$ for all but finitely many $n \in \mathbb{Z}$.
(e) $\mathbf{D}^{\mathrm{perf}}_Z(X) \subseteq \mathbf{D}_{\mathbf{qc},Z}(X)$. This is the full subcategory of all *perfect* complexes supported on Z. A complex is perfect if it is locally isomorphic to a bounded complex of finite-rank vector bundles.

[1]Recall that a scheme X is quasi-separated if the intersection of any two quasi-compact open subsets of X is quasi-compact.

(f) $\mathbf{D}^p_{\mathbf{qc},Z}(X) \subseteq \mathbf{D}_{\mathbf{qc},Z}(X)$. This is the full subcategory of all *pseudocoherent* complexes supported on Z. A complex is pseudocoherent if it is locally isomorphic to a bounded-above complex of finite-rank vector bundles.

(g) $\mathbf{D}^{p,b}_{\mathbf{qc},Z}(X) \subseteq \mathbf{D}^p_{\mathbf{qc},Z}(X)$. This is the full subcategory of all objects $\mathcal{C} \in \mathbf{D}^p_{\mathbf{qc},Z}(X)$ such that $\mathcal{H}^n(\mathcal{C}) = 0$ for all but finitely many $n \in \mathbb{Z}$.

Note that the finiteness conditions on the complex, imposed in (e), (f) and (g), are all due to Illusie [10]. The definition of (f) in Illusie is slightly more complicated than what we presented, but the two definitions are equivalent—this can easily be seen by using [3, Theorem 5.1]. In fact, any of the three finiteness conditions (e), (f) and (g) is local in the flat topology. And for an affine scheme $X = \mathrm{Spec}\,(R)$, where by [3, Theorem 5.1] we know that the natural map $\mathbf{D}(R\text{–Mod}) \longrightarrow \mathbf{D}_{\mathbf{qc}}(X)$ is an equivalence, the complex $\mathcal{C} \in \mathbf{D}_{\mathbf{qc},Z}(X) \subseteq \mathbf{D}_{\mathbf{qc}}(X)$ will belong to the subcategories (e), (f) or (g) provided it is isomorphic in $\mathbf{D}(R\text{–Mod})$ to a complex of finitely generated, projective R–modules which is (respectively) bounded, bounded-above, or bounded-above with only finitely many nonvanishing cohomology groups.

And one issue that concerned the early workers in the field was that the derived Hom involved injective resolutions, while the derived tensor product involved flat resolutions. And in the early days injective resolutions were only known to exist in $\mathbf{D}^+_{\mathbf{qc},Z}(X)$, and flat resolutions only in $\mathbf{D}^-_{\mathbf{qc},Z}(X)$. To put things in historical perspective: it was not until Spaltenstein's 1988 article [32] that anyone came up with an adequate approach to forming derived tensor products and derived Homs in $\mathbf{D}_{\mathbf{qc},Z}(X)$ (although in some sense the homotopy theorists arrived at a different satisfactory approach to a parallel problem earlier, as discussed in [3]). Now especially in the case of a subject like Grothendieck's duality theory, where derived tensor products and derived Homs both occur and intermingle, this caused headaches.

To the early workers in the field it seemed crucial to find the right derived category for the problem at hand. Using derived categories was viewed as an art, and a good artist displayed her competence by choosing wisely the derived category to work with. In fact: to an extent this attitude persists to the present day, in birational geometry. The people using derived category techniques have been known to argue over the relative merits of the categories $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^b_{\mathrm{coh}}(X)$.

Against this background one can imagine how surprising was the work of Rickard's, which appeared in 1989 and 1991 in the two articles [27, 28], and proves:

Theorem A (Rickard). *Let R and S be two rings. Then, in the standard notation for the various derived categories associated to the two rings, the following are equivalent:*

(i) *There exists a triangulated equivalence* $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
(ii) *There exists a triangulated equivalence* $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
(iii) *There exists a triangulated equivalence* $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
(iv) *There exists a triangulated equivalence* $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.
(v) *There exists a triangulated equivalence* $\mathbf{D}^-(R\text{–proj}) \cong \mathbf{D}^-(S\text{–proj})$.
(vi) *There exists a triangulated equivalence* $\mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj})$.

If we assume further that the rings R and S are both left coherent, then the six equivalent conditions above are also equivalent to:

(vii) *There exists a triangulated equivalence* $\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod})$.[2]

If R and S are commutative this says that a triangulated equivalence $\mathbf{D}^?_{\square,W}(X) \cong \mathbf{D}^?_{\square,Z}(Y)$, in the case where $W = X = \mathrm{Spec}\,(R)$ and $Z = Y = \mathrm{Spec}\,(S)$ and $\square$ and ? are any of Illusie's decorations in the list (a)–(g) above, implies the equivalence of all other pairs.

It should be explained that Rickard's theory works by studying what an equivalence might look like. An equivalence $\mathbf{D}^?(R\text{–}\square) \cong \mathbf{D}^?(S\text{–}\square)$ must take the object $R \in \mathbf{D}^?(R\text{–}\square)$ to some cochain complex, an object in $\mathbf{D}^?(S\text{–}\square)$. And the basic idea of the theory is that this complex must be very special—it has come to be known as a *tilting complex.* And roughly the idea is that the tilting complexes are the same, independent of the decoration ? and $\square$.

The result. In this article and its sequel we give a vast improvement and a vast generalization of Rickard's remarkable result. First of all the improvement: we show that each of the seven derived categories on Rickard's list determines all the others. We mean this in the precise sense that, for each ordered pair $\mathcal{A}, \mathcal{B}$ of the derived categories, on Rickard's list of seven, there is an explicit recipe that takes the triangulated category $\mathcal{A}$ as input and outputs the triangulated category $\mathcal{B}$. Thus the seven derived categories are interchangeable, each of them knows all about the other six.

So much for the improvement, now the time has come for the generalization. Our result is not only about derived categories of rings, it is about weakly approximable triangulated categories with unique enhancements. And now it is time to remind the reader of the terminology.

We begin by recalling some work by the second author. In a series of recent articles he developed the notion of "approximable" and "weakly approximable" triangulated categories—there will be a brief review in Section 4. And for the purposes of this article, the important features are that every weakly approximable triangulated category $\mathcal{T}$ comes with intrinsically defined

(1) A preferred equivalence class of t–structures, see Remark 4.1.4 for more detail.
(2) Thick subcategories $\mathcal{T}^-$, $\mathcal{T}^+$, $\mathcal{T}^b$, $\mathcal{T}^-_c$, $\mathcal{T}^c$, $\mathcal{T}^b_c = \mathcal{T}^-_c \cap \mathcal{T}^b$, and $\mathcal{T}^{c,b} = \mathcal{T}^c \cap \mathcal{T}^b_c$. We will recall the definition of each of these subcategories in Remark 4.1.5.

And the relevance of all of this to Rickard's old theorem comes from two facts.

(3) If R is any ring, then the category $\mathcal{T} = \mathbf{D}(R\text{–Mod})$ is (weakly) approximable. Also, if X is a quasi-compact and quasi-separated scheme and $Z \subseteq X$ is a Zariski-closed subset with quasi-compact complement, then the category $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable. We will give references in Example 4.1.3.
(4) The general intrinsic subcategories of (2), of any weakly approximable triangulated category $\mathcal{T}$, can be computed when $\mathcal{T}$ is one the two special cases given in (3). They turn out to be

[2]Rickard's paper claims that it suffices for just *one* of the two rings R, S to be left coherent—with suitable adjustments made to the definition of $\mathbf{D}^b(R\text{–mod})$ for the non-left coherent ring. But the authors couldn't follow the argument given in the old, published paper, and neither could Rickard.

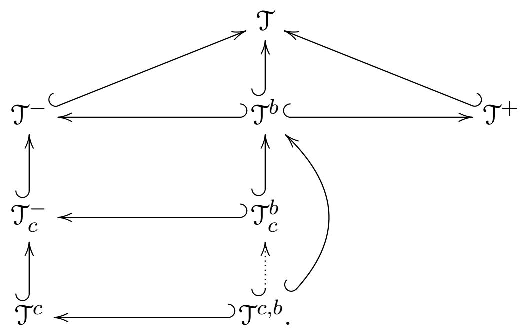
	$Z \subseteq X$ as in (a)–(g)	R a ring
$\mathcal{T}$	$\mathbf{D}_{\mathbf{qc},Z}(X)$	$\mathbf{D}(R\text{–Mod})$
$\mathcal{T}^-$	$\mathbf{D}^-_{\mathbf{qc},Z}(X)$	$\mathbf{D}^-(R\text{–Mod})$
$\mathcal{T}^+$	$\mathbf{D}^+_{\mathbf{qc},Z}(X)$	$\mathbf{D}^+(R\text{–Mod})$
$\mathcal{T}^b$	$\mathbf{D}^b_{\mathbf{qc},Z}(X)$	$\mathbf{D}^b(R\text{–Mod})$
$\mathcal{T}^c$	$\mathbf{D}^{\mathrm{perf}}_Z(X)$	$\mathbf{K}^b(R\text{–proj})$
$\mathcal{T}^-_c$	$\mathbf{D}^p_{\mathbf{qc},Z}(X)$	$\mathbf{K}^-(R\text{–proj})$
$\mathcal{T}^b_c$	$\mathbf{D}^{p,b}_{\mathbf{qc},Z}(X)$	$\mathbf{K}^{-,b}(R\text{–proj})$
$\mathcal{T}^{c,b}$	$\mathbf{D}^{\mathrm{perf}}_Z(X)$	$\mathbf{D}^{\mathrm{perf}}(R)$

In other words, in the case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$ we recover Illusie's old list (a)–(g), and in the case $\mathcal{T} = \mathbf{D}(R\text{–Mod})$ we recover the list of categories in Rickard's old Theorem A.

An immediate consequence, of the existence of a recipe that produces these subcategories out of $\mathcal{T}$, is that, in Theorem A, (i) implies all of (ii), (iii), (iv), (v), (vi) and (vii).

In summary, if $\mathcal{T}$ is a weakly approximable triangulated category, then the subcategories listed in (2) sit in the following commutative diagram:

(1)



The main aim of this paper is to show that all the solid inclusions $\mathcal{A} \hookrightarrow \mathcal{B}$ in (1) are intrinsic, in the sense that there is a recipe, depending only on the triangulated structure on $\mathcal{B}$, that describes which objects in $\mathcal{B}$ belong to the full subcategory $\mathcal{A}$. Since we found the notion of "recipe" difficult to formulate precisely, we state our main theorem as follows.

Theorem B. *All the inclusions in diagram* (1)*, represented by the solid arrows* $\mathcal{A} \hookrightarrow \mathcal{B}$*, are* invariant under triangulated equivalences. *By this we mean: given a pair of weakly approximable triangulated categories* $\mathcal{T}, \mathcal{T}'$*, as well as matching inclusions* $\mathcal{A} \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{T}$ *and* $\mathcal{A}' \hookrightarrow \mathcal{B}' \hookrightarrow \mathcal{T}'$ *from diagram* (1)*, then any triangulated equivalence* $\mathcal{B} \longrightarrow \mathcal{B}'$ *must restrict to a triangulated equivalence* $\mathcal{A} \longrightarrow \mathcal{A}'$*.*

Furthermore, the same is true for the (unique) inclusion in diagram (1) *represented by a dotted arrow, provided we further assume that one of the two conditions below holds.*

(i) $\mathcal{T}, \mathcal{T}'$ *are* coherent, *as in Definition 10.1.1, or*
(ii) $\mathcal{T}^c \subseteq \mathcal{T}^b_c$, $\mathcal{T}'^c \subseteq \mathcal{T}'^b_c$ *and* ${}^\perp(\mathcal{T}^b_c) \cap \mathcal{T}^-_c = \{0\} = {}^\perp(\mathcal{T}'^b_c) \cap \mathcal{T}'^-_c$.

Note that Theorem B does not mention enhancements, this entire article is enhancement-free and uses only triangulated category techniques.

The uniqueness of enhancements enters in the sequel, when we want to go back. This means that, when in the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ we want a recipe that constructs $\mathcal{B}$ out of $\mathcal{A}$, then to the best of the authors' knowledge such a recipe needs enhancements. And when the enhancements are unique, then so is the triangulated output. This will be discussed much more thoroughly and carefully in the sequel to the current article. Here we just observe that, for inclusions $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{A}' \hookrightarrow \mathcal{B}'$ as above, we can prove that the existence of a triangulated equivalence $\mathcal{A} \to \mathcal{A}'$ implies the existence of a triangulated equivalence $\mathcal{B} \to \mathcal{B}'$. But we do not know if every triangulated equivalence $\mathcal{A} \to \mathcal{A}'$ extends to a triangulated equivalence $\mathcal{B} \to \mathcal{B}'$, or if such extensions (when they exist) are unique. This problem is deeply related to the issue of comparing the autoequivalence groups of the various intrinsic subcategories which play a role in Theorem B. We will investigate this in future work.

Structure of the paper. We should finish the introduction by giving an outline of the structure of the article.

Section 1 is a brief reminder of the notation used to set up the theory of approximable triangulated categories. Section 2 is a less-brief reminder of compactly generated t–structures. The reason that the section on t–structures is less terse is that the results we refer to are scattered over an extensive literature, whose focus is applications irrelevant to us here. Therefore the portion of this vast literature that the reader needs, for the current manuscript, is tiny. Section 3 reproves Saorín and Šťovíček [30, Theorem 8.31], asserting that the heart of a compactly generated t–structure is always a locally finitely presented Grothendieck abelian category. The reason we go to the trouble of providing this new proof is that not only will we be using the result, but the lemmas in our proof will turn out to come up again in the course of later arguments. And the preliminaries of the paper end with Section 4, which outlines the parts of the theory of weakly approximable triangulated categories relevant to this manuscript.

And now we finally begin giving the recipes promised in Theorem B. We even go overboard: given any category $\mathcal{B}$ in the hierarchy of diagram in (1), we occasionally give explicit recipes for one of the subcategories not immediately below it in the hierarchy. This will happen for example in the case of $\mathcal{T}^-$. We have inclusions $\mathcal{T}^c \subseteq \mathcal{T}^-_c \subseteq \mathcal{T}^-$, but it so happens that we will first give a recipe for computing $\mathcal{T}^c \subseteq \mathcal{T}^-$, and then use it to concoct a recipe for $\mathcal{T}^-_c \subseteq \mathcal{T}^-$.

We note also that, when there are multiple paths in the hierarchy connecting a pair $\mathcal{A} \subseteq \mathcal{B}$, the recipes they yield are not necessarily equally complicated. We have inclusions

$$\begin{array}{ccc} \mathcal{T}^- & \hookleftarrow & \mathcal{T}^b \\ \uparrow & & \uparrow \\ \mathcal{T}^-_c & \hookleftarrow & \mathcal{T}^b_c \end{array}$$

and they combine to give two recipes for $\mathcal{T}^b_c \subseteq \mathcal{T}^-$. As it happens the path $\mathcal{T}^b_c \subseteq \mathcal{T}^-_c \subseteq \mathcal{T}^-$ is much less involved than the path $\mathcal{T}^b_c \subseteq \mathcal{T}^b \subseteq \mathcal{T}^-$.

Back to the structure of the paper, Section 5 studies the category $\mathcal{T}^-_c$ and its subcategories. The recipe for obtaining the subcategory $\mathcal{T}^c \subseteq \mathcal{T}^-_c$ is by a trick contained in Lemma 5.2.1, and the recipe for $\mathcal{T}^b_c \subseteq \mathcal{T}^-_c$ is not difficult once we know $\mathcal{T}^c$, see the proof of Proposition 5.2.2(ii). The same formula also works to give a recipe for $\mathcal{T}^{c,b}$ as a subcategory of $\mathcal{T}^c$, see Proposition 5.2.2(iii). This provides all the solid arrows in the bottom-left square of the hierarchy in the diagram (1).

Now for the category $\mathcal{T}^-$. In Proposition 6.1.3 we give a recipe for the preferred equivalence class of t–structures on $\mathcal{T}^-$, and Lemma 6.2.1 gives a recipe for $\mathcal{T}^c \subseteq \mathcal{T}^-$. Corollary 6.2.2 combines this information to give recipes for $\mathcal{T}^b \subseteq \mathcal{T}^-$ and $\mathcal{T}^-_c \subseteq \mathcal{T}^-$, completing our task with $\mathcal{T}^-$.

In the case of $\mathcal{T}^+$, the recipe for the preferred equivalence class of t–structures can be found in Proposition 6.1.7, and the recipe for $\mathcal{T}^b \subseteq \mathcal{T}^+$ that follows from it is in Corollary 6.2.2.

And now we come to the hardest part of the article, providing recipes for the subcategories of $\mathcal{T}^b$. In order to study this we introduce the notion of strongly pseudocompact objects in a triangulated category, the reader can see the subject developed in Section 7. And Corollary 8.1.6 tells us that, in the categories $\mathcal{T}^b$ and $\mathcal{T}^+$, the strongly pseudocompact objects are precisely $\mathcal{T}^b_c$. This delivers the promised recipe for $\mathcal{T}^b_c$ as a subcategory of $\mathcal{T}^b$ (and as it happens also as a subcategory of $\mathcal{T}^+$). And the recipe giving $\mathcal{T}^{c,b}$ as a subcategory of $\mathcal{T}^b$ is to be found in Lemma 8.2.2. Once again, this recipe also works to give $\mathcal{T}^{c,b}$ as a subcategory of $\mathcal{T}^+$.

And now for the dotted arrow in the hierarchy of the diagram (1). We have already said that we do not have a general recipe, we only know recipes that work in special cases. In this article we study the special case in which $\mathcal{T}^c$ is assumed to be contained in $\mathcal{T}^b$, and we find a recipe that works as long as ${}^\perp(\mathcal{T}^b_c) \cap \mathcal{T}^-_c = \{0\}$. See Proposition 9.2.3 for the precise recipe for $\mathcal{T}^c$.

In Section 10 we treat examples to which Proposition 9.2.3 applies. Any coherent, weakly approximable triangulated category satisfies ${}^\perp(\mathcal{T}^b_c) \cap \mathcal{T}^-_c = \{0\}$, in particular if R is a left coherent ring then $\mathbf{D}(R\text{–Mod})$ satisfies the hypothesis. Applying Proposition 9.2.3 to the situation of Rickard's Theorem A, this provides a new proof that, as long as the rings R, S are left coherent, then a triangulated equivalence as in Theorem A(vii) implies a triangulated equivalence as in Theorem A(vi). And this is one of the rare cases where going in the reverse direction does not require enhancements, see [16, Example 4.2 and Proposition 4.8].

But perhaps more remarkable is that ${}^\perp(\mathcal{T}^b_c) \cap \mathcal{T}^-_c = \{0\}$ holds for $\mathbf{D}_{\mathbf{qc},Z}(X)$ unconditionally, for any quasi-compact, quasi-separated X and any closed subset $Z \subseteq X$ with quasi-compact complement. Therefore in the category $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$ all the arrows of the diagram (1) are solid. For the proof see Proposition 10.2.1, which depends crucially on Haesemeyer's Theorem A.1.

Acknowledgements. It is our great pleasure to thank Lidia Angeleri Hügel and Rosanna Laking for inspiring conversations concerning the contents of Section 3, and to Jan Šťovíček for his careful reading and many improvements on an earlier version of the section. Thanks also to Sergio Pavon for pointing us to the beautiful recent article [9], which treats related topics from a completely different perspective. We are also grateful to Jeremy Rickard for patiently answering our questions and to Evgeny Shinder for pointing out possible future developments about the relations between the autoequivalence groups of the intrinsic subcategories. We also thank Rudradip Biswas, Kabeer Menali and Bregje Pauwels for suggestions, corrections and improvements to earlier versions.

The second author did much of his work during the trimester program Spectral Methods in Algebra, Geometry, and Topology at the Hausdorff Institute in Bonn. It is a pleasure to thank the Hausdorff Institute for its hospitality, and acknowledge the funding by the Deutsche Forschungsgemeinschaft (DFG) under Excellence Strategy EXC-2047/1-390685813. A later part of the work was carried out while the second author was Mercator Fellow at the University of Bielefeld and the first and third author were visiting the same institution. We are pleased to thank the institution for its warm hospitality and financial support, and also the Deutsche Forschungsgemeinschaft grant (SFB-TRR 358/1 2023 - 491392403) that helped underwrite the activity.

1. Reminder of some basic notation

In this short section we briefly summarize some basic constructions involving various notions of generation.

1.1. **Notation and constructions.**

We start by recalling that, if $\mathcal{T}$ is a triangulated category, then $\Sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$ denotes the shift functor.

Notation 1.1.1. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{A}, \mathcal{B}$ be full subcategories of $\mathcal{T}$. The following conventions will be used throughout:

(i) $\mathcal{A} * \mathcal{B} \subseteq \mathcal{T}$ is the full subcategory of all objects $x \in \mathcal{T}$ for which there exists a distinguished triangle $a \longrightarrow x \longrightarrow b$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.
(ii) $\mathrm{add}(\mathcal{A}) \subseteq \mathcal{T}$ is the full subcategory whose objects are all finite direct sums of objects of $\mathcal{A}$.
(iii) Assume $\mathcal{T}$ has coproducts. Then $\mathrm{Add}(\mathcal{A}) \subseteq \mathcal{T}$ is the full subcategory whose objects are all the small coproducts of objects of $\mathcal{A}$.
(iv) $\mathrm{smd}(\mathcal{A}) \subseteq \mathcal{T}$ is the full subcategory with objects all direct summands of objects of $\mathcal{A}$.

With this notation, we will furthermore adopt the conventions:

(v) $\mathrm{coprod}(\mathcal{A})$ is the smallest full subcategory $\mathcal{S} \subseteq \mathcal{T}$ satisfying

$$\mathcal{A} \subseteq \mathcal{S}, \qquad \mathcal{S} * \mathcal{S} \subseteq \mathcal{S}, \qquad \mathrm{add}(\mathcal{S}) \subseteq \mathcal{S}.$$

(vi) $\langle \mathcal{A} \rangle$ is given by the formula

$$\langle \mathcal{A} \rangle := \mathrm{smd}\left(\mathrm{coprod}(\mathcal{A})\right).$$

(vii) Assume $\mathcal{T}$ has coproducts. Then $\mathrm{Coprod}(\mathcal{A})$ is the smallest full subcategory $\mathcal{S} \subseteq \mathcal{T}$ satisfying

$$\mathcal{A} \subseteq \mathcal{S}, \qquad \mathcal{S} * \mathcal{S} \subseteq \mathcal{S}, \qquad \mathrm{Add}(\mathcal{S}) \subseteq \mathcal{S}.$$

(viii) Still assuming $\mathcal{T}$ has coproducts, then $\overline{\langle \mathcal{A} \rangle}$ is given by the formula

$$\overline{\langle \mathcal{A} \rangle} := \mathrm{smd}\left(\mathrm{Coprod}(\mathcal{A})\right).$$

The following special case will interest us later in the article.

Notation 1.1.2. Let $\mathcal{T}$ be a triangulated category, and let $G \in \mathcal{T}$ be an object.

(i) If $m \leqslant n$ are integers, possibly infinite, then $G[m,n]$ is defined to be the full subcategory of $\mathcal{T}$ whose objects are given by

$$G[m,n] := \{\Sigma^i G \mid i \in \mathbb{Z} \text{ and } m \leqslant -i \leqslant n\}.$$

In the rest of the paper when $m = -\infty$ (resp. $n = +\infty$) we use the notation $G(-\infty, n]$ (resp. $G[m, +\infty)$) instead of the above one.

(ii) The subcategory $\langle G \rangle^{[m,n]}$ is defined by the formula

$$\langle G \rangle^{[m,n]} := \mathrm{smd}\Big(\mathrm{coprod}\big(G[m,n]\big)\Big)\,.$$

(iii) Assuming $\mathcal{T}$ has coproducts, the subcategory $\overline{\langle G\rangle}^{[m,n]}$ is defined by the formula

$$\overline{\langle G\rangle}^{[m,n]} := \operatorname{smd}\Big(\operatorname{Coprod}\big(G[m,n]\big)\Big) .$$

The elementary properties of these constructions are discussed in [22, Section 1]. We will freely use the results proved there when needed—the reader may wish to have a quick look before proceeding with the rest of the current article.

1.2. **Basic properties.** We end the section with reminders of facts well-known to the experts. Since all of these facts are easy enough, we include the (short) proofs for the reader's convenience.

Recall that a subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is *strictly full* if it is full and, given an object $x \in \mathcal{C}$ and an isomorphism $x \longrightarrow y$ in $\mathcal{D}$, then y is in $\mathcal{C}$.

As a matter of notation, if $\mathcal{T}$ is a triangulated category and $\mathcal{A}$ is a full subcategory of $\mathcal{T}$, we will denote by $\mathcal{A}^\perp$ (resp. ${}^\perp\mathcal{A}$) the full subcategory of $\mathcal{T}$ consisting of all objects x such that $\operatorname{Hom}(a,x)=0$ (resp. $\operatorname{Hom}(x,a)=0$) for every $a \in \mathcal{A}$.

Lemma 1.2.1. *If $\mathcal{T}$ is a triangulated category with coproducts, and if $\mathcal{A} \subseteq \mathcal{T}$ is any full subcategory, then*

(i) *The subcategory* $\operatorname{Coprod}(\mathcal{A}) \subseteq \mathcal{T}$ *is strictly full.*
(ii) *The subcategory* $\operatorname{Coprod}(\mathcal{A})^\perp \subseteq \mathcal{T}$ *is also strictly full, and moreover* $\mathcal{A}^\perp = \operatorname{Coprod}(\mathcal{A})^\perp$*.*

Proof. We start with (i). By Notation 1.1.1(vii) the full subcategory $\mathcal{S} = \operatorname{Coprod}(\mathcal{A})$ is closed in $\mathcal{T}$ under all small coproducts, and coproducts are only defined up to isomorphism. Thus any isomorph B of an object $A \in \mathcal{S}$ is a coproduct of objects of $\mathcal{S}$, in this case B is the coproduct of all objects in the (singleton) set $\{A\}$.

Now for (ii), the fact that $\mathcal{B}^\perp$ is strictly full, for any full subcategory $\mathcal{B}$ of $\mathcal{T}$, is obvious by definition. Furthermore, Notation 1.1.1(vii) guarantees that $\mathcal{A} \subseteq \operatorname{Coprod}(\mathcal{A})$, and hence $\operatorname{Coprod}(\mathcal{A})^\perp \subseteq \mathcal{A}^\perp$. We need to prove the reverse inclusion.

Clearly $\operatorname{Hom}(\mathcal{A},\mathcal{A}^\perp)=0$, giving the inclusion $\mathcal{A} \subseteq {}^\perp\big(\mathcal{A}^\perp\big)$. But for any full subcategory $\mathcal{B}$ of $\mathcal{T}$, the subcategory $\mathcal{S} = {}^\perp\mathcal{B}$ is closed under coproducts and satisfies $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$. Applying this to $\mathcal{B} = \mathcal{A}^\perp$ we obtain, with $\mathcal{S} = {}^\perp\mathcal{B} = {}^\perp\big(\mathcal{A}^\perp\big)$,

$$\mathcal{A} \subseteq \mathcal{S}, \qquad \mathcal{S} * \mathcal{S} \subseteq \mathcal{S}, \qquad \operatorname{Add}(\mathcal{S}) \subseteq \mathcal{S}.$$

Hence ${}^\perp\big(\mathcal{A}^\perp\big)$ contains the minimal subcategory satisfying the conditions, that is $\operatorname{Coprod}(\mathcal{A}) \subseteq {}^\perp\big(\mathcal{A}^\perp\big)$. And this inclusion rewrites as $\operatorname{Hom}\big(\operatorname{Coprod}(\mathcal{A}),\mathcal{A}^\perp\big)=0$, that is $\mathcal{A}^\perp \subseteq \operatorname{Coprod}(\mathcal{A})^\perp$. □

Lemma 1.2.2. *Let $\mathcal{T}$ be a triangulated category with coproducts, let $\mathcal{A} \subseteq \mathcal{T}$ be a full subcategory, and assume $\Sigma\mathcal{A} \subseteq \mathcal{A}$. Then*

(i) $\Sigma\operatorname{Coprod}(\mathcal{A}) \subseteq \operatorname{Coprod}(\mathcal{A})$.
(ii) $\Sigma^{-1}\operatorname{Coprod}(\mathcal{A})^\perp \subseteq \operatorname{Coprod}(\mathcal{A})^\perp$.
(iii) *The subcategory* $\operatorname{Coprod}(\mathcal{A})$ *contains the homotopy colimit of any countable sequence of its morphisms.*

Proof. To see (i), observe that $\Sigma\operatorname{Coprod}(\mathcal{A}) = \operatorname{Coprod}(\Sigma\mathcal{A}) \subseteq \operatorname{Coprod}(\mathcal{A})$, where the equality is because Σ is an autoequivalence of the category $\mathcal{T}$ preserving extensions and (of course) coproducts, and the containment is by applying $\operatorname{Coprod}(-)$ to the inclusion $\Sigma\mathcal{A} \subseteq \mathcal{A}$.

We deduce (ii) from (i) formally, in the following easy way. By combining (i) with the fact that $\mathrm{Hom}\big(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})^\perp\big) = 0$, we get

$$\mathrm{Hom}\big(\Sigma\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})^\perp\big) = \mathrm{Hom}\big(\mathrm{Coprod}(\mathcal{A}), \Sigma^{-1}\mathrm{Coprod}(\mathcal{A})^\perp\big) = 0.$$

This proves the inclusion in (ii).

For (iii) recall that the homotopy colimit of a sequence of composable morphisms in $\mathcal{T}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots$$

is defined (up to noncanonical isomorphism) by the distinguished triangle

$$\coprod_{n=1}^{\infty} X_n \xrightarrow{1-shift} \coprod_{n=1}^{\infty} X_n \longrightarrow \underrightarrow{\mathrm{Hocolim}}\, X_n \longrightarrow \Sigma\left(\coprod_{n=1}^{\infty} X_i\right)$$

where $1\colon \coprod_{n=1}^\infty X_n \longrightarrow \coprod_{n=1}^\infty X_n$ is the identity map and *shift*: $\coprod_{n=1}^\infty X_n \longrightarrow \coprod_{n=1}^\infty X_n$ is the map whose only nonzero components are $f_n\colon X_n \longrightarrow X_{n+1}$.

Since X_n is assumed to belong to $\mathrm{Coprod}(\mathcal{A})$ for all $n > 0$, we deduce that $\coprod_{n=1}^\infty X_n$ belongs to $\mathrm{Coprod}(\mathcal{A})$. By (i) we have that $\Sigma\mathrm{Coprod}(\mathcal{A}) \subseteq \mathrm{Coprod}(\mathcal{A})$, and therefore $\Sigma\big(\coprod_{n=1}^\infty X_n\big)$ belongs to $\mathrm{Coprod}(\mathcal{A})$. And now the fact that $\underrightarrow{\mathrm{Hocolim}}\, X_n$ belongs to $\mathrm{Coprod}(\mathcal{A})$ is from its triangle of definition and because $\mathrm{Coprod}(\mathcal{A}) * \mathrm{Coprod}(\mathcal{A}) \subseteq \mathrm{Coprod}(\mathcal{A})$. □

2. The basics of t–structures

This section introduces the notion of t–structure and some of its basic properties. The presentation in Section 2.1 is somewhat non-traditional and we devote Section 2.2 to a comparison with the 'standard' presentation of the subject going back to Beĭlinson, Bernstein and Deligne [2]. Finally, in Section 2.3 we introduce the main example of interest in this paper: compactly generated t–structures.

2.1. Definitions and basic properties.

We begin with the following.

Definition 2.1.1. *Let $\mathcal{T}$ be a triangulated category. A strictly full subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called a* pre-aisle *if*

$$\Sigma\mathcal{S} \subseteq \mathcal{S} \qquad \text{and} \qquad \mathcal{S} * \mathcal{S} \subseteq \mathcal{S}\ .$$

The pre-aisle $\mathcal{S} \subseteq \mathcal{T}$ is called an aisle *if the inclusion $\mathsf{l}\colon \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint $\mathsf{l}_\rho\colon \mathcal{T} \longrightarrow \mathcal{S}$.*

Remark 2.1.2. Let $\mathcal{T}$ be a triangulated category. A strictly full subcategory $\mathcal{S} \subseteq \mathcal{T}$ is a *co-pre-aisle* if $\mathcal{S}^{\mathrm{op}} \subseteq \mathcal{T}^{\mathrm{op}}$ is a pre-aisle. And $\mathcal{S} \subseteq \mathcal{T}$ is a *co-aisle* if $\mathcal{S}^{\mathrm{op}} \subseteq \mathcal{T}^{\mathrm{op}}$ is an aisle.

Lemma 2.1.3. *Let $\mathcal{T}$ be a triangulated category, let $\mathcal{S} \subseteq \mathcal{T}$ be an aisle, let $\mathsf{l}\colon \mathcal{S} \longrightarrow \mathcal{T}$ be the inclusion, let $\mathsf{l}_\rho\colon \mathcal{T} \longrightarrow \mathcal{S}$ be its right adjoint and let $\varepsilon\colon \mathsf{l}\circ\mathsf{l}_\rho \longrightarrow \mathrm{id}$ be the counit of the adjunction.*

Let $B \in \mathcal{T}$ be any object and complete the morphism $\varepsilon_B\colon \mathsf{l}\circ\mathsf{l}_\rho(B) \longrightarrow B$ to a distinguished triangle $\mathsf{l}\circ\mathsf{l}_\rho(B) \xrightarrow{\varepsilon_B} B \xrightarrow{g} C \xrightarrow{h} \Sigma\circ\mathsf{l}\circ\mathsf{l}_\rho(B)$. Then C must belong to the subcategory $\mathcal{S}^\perp \subseteq \mathcal{T}$.

Proof. Consider the morphism $\varepsilon_C\colon \mathsf{I}\circ\mathsf{I}_\rho(C)\longrightarrow C$. It allows us first to build up a diagram

$$
\begin{array}{ccccccc}
 & & & & \mathsf{I}\circ\mathsf{I}_\rho(C) & \xrightarrow{\;h\circ\varepsilon_C\;} & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B) \\
 & & & & \Big\downarrow{\scriptstyle \varepsilon_C} & & \Big\| \\
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\;\varepsilon_B\;} & B & \xrightarrow{\;g\;} & C & \xrightarrow{\;h\;} & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)
\end{array}
$$

where the square is commutative, and then extend to a morphism of distinguished triangles

$$
\begin{array}{ccccccc}
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\;\alpha\;} & E & \xrightarrow{\;\gamma\;} & \mathsf{I}\circ\mathsf{I}_\rho(C) & \xrightarrow{\;h\circ\varepsilon_C\;} & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B) \\
\Big\| & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \varepsilon_C} & & \Big\| \\
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\;\varepsilon_B\;} & B & \xrightarrow{\;g\;} & C & \xrightarrow{\;h\;} & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B).
\end{array} \tag{2}
$$

From the triangle on the top row, coupled with the facts that both $\mathsf{I}\circ\mathsf{I}_\rho(B)$ and $\mathsf{I}\circ\mathsf{I}_\rho(C)$ belong to $\mathcal{S}$ and that $\mathcal{S}*\mathcal{S}\subseteq\mathcal{S}$, we learn that E must belong to $\mathcal{S}$. Therefore the morphism $\beta\colon E\longrightarrow B$ must factor (uniquely) as $E\xrightarrow{\sigma}\mathsf{I}\circ\mathsf{I}_\rho(B)\xrightarrow{\varepsilon_B}B$. But then the equality $\beta\circ\alpha=\varepsilon_B$ becomes $\varepsilon_B\circ\sigma\circ\alpha=\varepsilon_B$, that is the two composites

$$
\mathsf{I}\circ\mathsf{I}_\rho(B)\ \underset{\mathrm{id}}{\overset{\sigma\circ\alpha}{\rightrightarrows}}\ \mathsf{I}\circ\mathsf{I}_\rho(B)\xrightarrow{\;\varepsilon_B\;}B
$$

must be equal. But any morphism $S\longrightarrow B$, where $S\in\mathcal{S}$, has a *unique* factorization as $S\longrightarrow\mathsf{I}\circ\mathsf{I}_\rho(B)\xrightarrow{\varepsilon_B}B$, and applying this uniqueness to $S=\mathsf{I}\circ\mathsf{I}_\rho(B)$, we deduce that $\sigma\circ\alpha=\mathrm{id}$. That is the map $\alpha\colon\mathsf{I}\circ\mathsf{I}_\rho(B)\longrightarrow E$ must be a split monomorphism, and then from the fact that the top row in (2) is a distinguished triangle we deduce $h\circ\varepsilon_C=0$. Since also the bottom row in (2) is a distinguished triangle, ε_C must factor (in some way) as $\mathsf{I}\circ\mathsf{I}_\rho(C)\xrightarrow{f}B\xrightarrow{g}C$. But as $\mathsf{I}\circ\mathsf{I}_\rho(C)$ belongs to $\mathcal{S}$ the map $f\colon\mathsf{I}\circ\mathsf{I}_\rho(C)\longrightarrow B$ must factor (uniquely) as $\mathsf{I}\circ\mathsf{I}_\rho(C)\xrightarrow{\tau}\mathsf{I}\circ\mathsf{I}_\rho(B)\xrightarrow{\varepsilon_B}B$. Combining these factorizations we have that $\varepsilon_C\colon\mathsf{I}\circ\mathsf{I}_\rho(C)\longrightarrow C$ factors as

$$
\mathsf{I}\circ\mathsf{I}_\rho(C)\xrightarrow{\;\;\tau\;\;}\mathsf{I}\circ\mathsf{I}_\rho(B)\xrightarrow{\;\;\varepsilon_B\;\;}B\xrightarrow{\;\;g\;\;}C\ ,
$$

and, as $g\circ\varepsilon_B=0$, this composite must vanish.

Finally, every morphism $S\longrightarrow C$, where $S\in\mathcal{S}$ is any object, factors (uniquely) as $S\longrightarrow\mathsf{I}\circ\mathsf{I}_\rho(C)\xrightarrow{\varepsilon_C}C$. And now that we know the vanishing of $\varepsilon_C\colon\mathsf{I}\circ\mathsf{I}_\rho(C)\longrightarrow C$, it follows that any map $S\longrightarrow C$ must vanish. That is C belongs to the subcategory $\mathcal{S}^\perp\subseteq\mathcal{T}$. $\square$

And now we give the older, more symmetric version of the same.

Definition 2.1.4. *Let $\mathcal{T}$ be a triangulated category. A t–*structure *on $\mathcal{T}$ is a pair of strictly full subcategories $(\mathcal{S},\mathcal{S}')$ satisfying:*

(i) $\Sigma\mathcal{S}\subseteq\mathcal{S}$ *and* $\Sigma^{-1}\mathcal{S}'\subseteq\mathcal{S}'$.
(ii) $\mathrm{Hom}(\mathcal{S},\mathcal{S}')=0$.
(iii) *For any object $B\in\mathcal{T}$ there exists a distinguished triangle $A\longrightarrow B\longrightarrow C\longrightarrow\Sigma A$, with $A\in\mathcal{S}$ and $C\in\mathcal{S}'$.*

Remark 2.1.5. We note that the definition is self-dual: the pair $(\mathcal{S},\mathcal{S}')$ is a t–structure on $\mathcal{T}$ if and only if the pair $\big((\mathcal{S}')^{\mathrm{op}},\mathcal{S}^{\mathrm{op}}\big)$ is a t–structure on $\mathcal{T}^{\mathrm{op}}$.

Example 2.1.6. If $\mathcal{T}$ is a triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ is an aisle, then the pair $(\mathcal{S}, \mathcal{S}^{\perp})$ is a *t*–structure on $\mathcal{T}$.

Indeed, because $\mathcal{S}$ is an aisle, we are given that $\mathcal{S} \subseteq \mathcal{T}$ is a strictly full subcategory. The fact that $\mathcal{S}^{\perp} \subseteq \mathcal{T}$ is strictly full is by its definition. Because $\mathcal{S} \subseteq \mathcal{T}$ is an aisle, we are also given the inclusion $\Sigma\mathcal{S} \subseteq \mathcal{S}$. The other inclusion in Definition 2.1.4(i) follows from the same formal argument as in the proof of Lemma 1.2.2(ii). The fact that $\mathrm{Hom}(\mathcal{S}, \mathcal{S}^{\perp}) = 0$ is obvious, giving that the pair $(\mathcal{S}, \mathcal{S}^{\perp})$ satisfies Definition 2.1.4(ii). The fact that the pair $(\mathcal{S}, \mathcal{S}^{\perp})$ satisfies Definition 2.1.4(iii) was proved in Lemma 2.1.3.

The next proposition shows, among other things, that all *t*–structures are as in Example 2.1.6. This justifies the assertion we made, just before Definition 2.1.4, that *t*–structures are just a more symmetric presentation of the information contained in giving an aisle.

Proposition 2.1.7. *Let $\mathcal{T}$ be a triangulated category, and let $(\mathcal{S}, \mathcal{S}')$ be a t–structure on $\mathcal{T}$. Then the following assertions are all true:*

(i) *$\mathcal{S} \subseteq \mathcal{T}$ is an aisle and $\mathcal{S}' \subseteq \mathcal{T}$ is a co-aisle.*
(ii) *$\mathcal{S}' = \mathcal{S}^{\perp}$ and $\mathcal{S} = {}^{\perp}\mathcal{S}'$.*

Furthermore, let $\mathsf{I}\colon \mathcal{S} \longrightarrow \mathcal{T}$ and $\mathsf{J}\colon \mathcal{S}' \longrightarrow \mathcal{T}$ be the inclusion functors, let $\mathsf{I}_\rho\colon \mathcal{T} \longrightarrow \mathcal{S}$ be the right adjoint of I, let $\mathsf{J}_\lambda\colon \mathcal{T} \longrightarrow \mathcal{S}'$ be the left adjoint of J, let $\varepsilon\colon \mathsf{I}\circ\mathsf{I}_\rho \longrightarrow \mathrm{id}$ be the counit of the adjunction $\mathsf{I} \dashv \mathsf{I}_\rho$, and let $\eta\colon \mathrm{id} \longrightarrow \mathsf{J}\circ\mathsf{J}_\lambda$ be the unit of the adjunction $\mathsf{J}_\lambda \dashv \mathsf{J}$. Then we have:

(iii) *For each object $B \in \mathcal{T}$, there exists a unique morphism $\varphi_B\colon \mathsf{J}\circ\mathsf{J}_\lambda(B) \longrightarrow \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)$ such that*
$$\mathsf{I}\circ\mathsf{I}_\rho(B) \xrightarrow{\ \varepsilon_B\ } B \xrightarrow{\ \eta_B\ } \mathsf{J}\circ\mathsf{J}_\lambda(B) \xrightarrow{\ \varphi_B\ } \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)$$
is a distinguished triangle.
(iv) *The assignment taking an object $B \in \mathcal{T}$, to the unique morphism φ_B satisfying the condition in* (iii)*, delivers a natural transformation $\varphi\colon \mathsf{J}\circ\mathsf{J}_\lambda \longrightarrow \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho$.*

Proof. Let us begin by proving (ii), and note that it suffices to prove one of the equalities in (ii) as the other is its dual. Hence, we are reduced to proving the equality ${}^{\perp}\mathcal{S}' = \mathcal{S}$.

In Definition 2.1.4(ii) we are given that $\mathrm{Hom}(\mathcal{S}, \mathcal{S}') = 0$, that is the inclusion $\mathcal{S} \subseteq {}^{\perp}\mathcal{S}'$ is given. What needs proof is the reverse inclusion—we need to show that ${}^{\perp}\mathcal{S}' \subseteq \mathcal{S}$.

Choose therefore any object $B \in {}^{\perp}\mathcal{S}'$. Definition 2.1.4(iii) gives the existence of a distinguished triangle $A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow \Sigma A$ with $A \in \mathcal{S}$ and $C \in \mathcal{S}'$. Note that
$$\Sigma A \in \Sigma\mathcal{S} \subseteq \mathcal{S} \subseteq {}^{\perp}\mathcal{S}',$$
where the first inclusion is by Definition 2.1.4(i) and the second by Definition 2.1.4(ii). The part $B \longrightarrow C \longrightarrow \Sigma A$ of the triangle, coupled with the fact that we now know that both B and ΣA belong to ${}^{\perp}\mathcal{S}'$, tells us that C must belong to $({}^{\perp}\mathcal{S}') * ({}^{\perp}\mathcal{S}') \subseteq {}^{\perp}\mathcal{S}'$. Thus C must belong to ${}^{\perp}\mathcal{S}' \cap \mathcal{S}'$, and hence must be isomorphic to zero. Thus the map α must be an isomorphism. Since $A \in \mathcal{S}$ and the subcategory $\mathcal{S} \subseteq \mathcal{T}$ is strictly full, we have that $B \in \mathcal{S}$. This completes the proof of (ii).

Next we prove (i), and note that by duality it suffices to prove that $\mathcal{S}' \subseteq \mathcal{T}$ is a co-aisle.

The inclusion $\Sigma^{-1}\mathcal{S}' \subseteq \mathcal{S}'$ is part of Definition 2.1.4(i). By part (ii) of the current proposition we know that $\mathcal{S}' = \mathcal{S}^{\perp}$, and the inclusion $\mathcal{S}' * \mathcal{S}' \subseteq \mathcal{S}'$ follows immediately. Thus the information we have so far easily shows that $\mathcal{S}' \subseteq \mathcal{T}$ is a co-pre-aisle. What needs proof is that the inclusion $\mathsf{J}\colon \mathcal{S}' \longrightarrow \mathcal{T}$ has a left adjoint. We have to show that, for every object $B \in \mathcal{T}$, the functor $\mathrm{Hom}_{\mathcal{T}}\big(B, \mathsf{J}(-)\big)$ is a representable functor $\mathcal{S}' \longrightarrow \mathcal{A}b$. Choose therefore an object $B \in \mathcal{T}$.

Definition 2.1.4(iii) allows us to find some distinguished triangle $A \longrightarrow B \xrightarrow{\eta} \mathsf{J}(C) \longrightarrow \Sigma A$ with $A \in \mathcal{S}$ and $C \in \mathcal{S}'$. Now choose any object $X \in \mathcal{S}$, and apply $\mathrm{Hom}_{\mathcal{T}}\big(-, \mathsf{J}(X)\big)$ to this triangle. This gives us a long exact sequence, but as $\mathrm{Hom}\big(-, \mathsf{J}(X)\big)$ kills $\mathcal{S} \subseteq \mathcal{T}$ and A and ΣA both belong to $\mathcal{S}$, this long exact sequence simplifies to the second isomorphism in

$$\mathrm{Hom}_{\mathcal{S}'}(C, X) \xrightarrow{\quad \mathsf{J} \quad} \mathrm{Hom}_{\mathcal{T}}\big(\mathsf{J}(C), \mathsf{J}(X)\big) \xrightarrow{\mathrm{Hom}\left(\eta, \mathsf{J}(X)\right)} \mathrm{Hom}_{\mathcal{T}}\big(B, \mathsf{J}(X)\big) \tag{3}$$

The first isomorphism is by the full faithfulness of J.

So far we have produced a natural isomorphism $\mathrm{Hom}_{\mathcal{S}'}(C, -) \cong \mathrm{Hom}_{\mathcal{T}}\big(B, \mathsf{J}(-)\big)$. The functor $\mathrm{Hom}_{\mathcal{T}}\big(B, \mathsf{J}(-)\big)$ is therefore representable for every $B \in \mathcal{T}$, allowing us to form the left adjoint $\mathsf{J}_\lambda\colon \mathcal{T} \longrightarrow \mathcal{S}'$ to the natural inclusion. In fact, for $B \in \mathcal{T}$ we can choose $\mathsf{J}_\lambda(B) \in \mathcal{S}'$ to be $\mathsf{J}_\lambda(B) = C$, and the extension of this to a functor is unique. This completes the proof of (i).

But while we are at it let us also compute the counit of adjunction. By definition, given any object $B \in \mathcal{T}$, the counit of adjunction (evaluated at the object B) is the image of $1 \in \mathrm{Hom}_{\mathcal{S}'}\big(\mathsf{J}_\lambda(B), \mathsf{J}_\lambda(B)\big)$ under the natural isomorphism

$$\mathrm{Hom}_{\mathcal{S}'}\big(\mathsf{J}_\lambda(B), \mathsf{J}_\lambda(B)\big) \xrightarrow{\quad \cong \quad} \mathrm{Hom}_{\mathcal{T}}\big(B, \mathsf{J} \circ \mathsf{J}_\lambda(B)\big)$$

By our choice we have that $\mathsf{J}_\lambda(B) = C$, and the isomorphism above comes down to the composite in (3) with $X = C$. To put it concretely, given an object $B \in \mathcal{T}$, we choose an object $C \in \mathcal{S}'$ and a distinguished triangle $A \longrightarrow B \xrightarrow{\eta} \mathsf{J}(C) \longrightarrow \Sigma A$ with $A \in \mathcal{S}$. The existence of such a triangle comes from Definition 2.1.4(iii). And, when we choose $\mathsf{J}_\lambda(B)$ to be $\mathsf{J}_\lambda(B) = C$ as above, the image of $1 \in \mathrm{Hom}_{\mathcal{S}'}(C, C)$ under the natural composite computes to be $\eta\colon B \longrightarrow \mathsf{J}(C)$.

Now we proceed with the proof of (iii) and (iv). We begin by reminding the reader that the right adjoint I_ρ of I is only defined up to canonical isomorphism, as is the left adjoint J_λ of J. That is, given an object $B \in \mathcal{T}$, the objects $\mathsf{I}_\rho(B)$ and $\mathsf{J}_\lambda(B)$ are defined up to canonical isomorphism. If $\widetilde{\mathsf{I}}_\rho(B)$ and $\widetilde{\mathsf{J}}_\lambda(B)$ are a different pair of choices, then there are canonical isomorphisms $\alpha\colon \mathsf{I}_\rho(B) \longrightarrow \widetilde{\mathsf{I}}_\rho(B)$ and $\beta\colon \mathsf{J}_\lambda(B) \longrightarrow \widetilde{\mathsf{J}}_\lambda(B)$, rendering commutative the diagram

$$\begin{array}{ccccc}
\mathsf{I} \circ \mathsf{I}_\rho(B) & \xrightarrow{\varepsilon_B} & B & \xrightarrow{\eta_C} & \mathsf{J} \circ \mathsf{J}_\lambda(B) \\
\Big\downarrow{\scriptstyle \mathsf{I}(\alpha)} & & \Big\| & & \Big\downarrow{\scriptstyle \mathsf{J}(\beta)} \\
\mathsf{I} \circ \widetilde{\mathsf{I}}_\rho(B) & \xrightarrow{\widetilde{\varepsilon}_B} & B & \xrightarrow{\widetilde{\eta}_C} & \mathsf{J} \circ \widetilde{\mathsf{J}}_\lambda(B)
\end{array}$$

where $\varepsilon, \widetilde{\varepsilon}$ are the counits and $\eta, \widetilde{\eta}$ the units of the respective adjunctions. But the existence assertion in (iii) is clearly independent of the choices.

Let B be any object of $\mathcal{T}$; Definition 2.1.4(iii) allows us to choose objects $A \in \mathcal{S}$ and $C \in \mathcal{S}'$ and in $\mathcal{T}$ a distinguished triangle $\mathsf{I}(A) \xrightarrow{\varepsilon} B \xrightarrow{\eta} \mathsf{J}(C) \xrightarrow{\varphi} \Sigma \circ \mathsf{I}(A)$. We saw above that the left adjoint $\mathsf{J}_\lambda\colon \mathcal{T} \longrightarrow \mathcal{S}'$ of the inclusion $\mathsf{J}\colon \mathcal{S}' \longrightarrow \mathcal{T}$ can be chosen so that $\mathsf{J}_\lambda(B) = C$, and with this choice the unit of adjunction $\eta_B\colon B \longrightarrow \mathsf{J} \circ \mathsf{J}_\lambda(B)$ becomes the map $\eta\colon B \longrightarrow \mathsf{J}(C)$. Dually, the right adjoint $\mathsf{I}_\rho\colon \mathcal{T} \longrightarrow \mathcal{S}$ to the inclusion $\mathsf{I}\colon \mathcal{S} \longrightarrow \mathcal{T}$ can be chosen so that $\mathsf{I}_\rho(B) = A$, and with that choice the counit of adjunction $\varepsilon_B\colon \mathsf{I}\circ\mathsf{I}_\rho(B)\longrightarrow B$ becomes the map $\varepsilon\colon \mathsf{I}(A)\longrightarrow B$. Thus the triangle $\mathsf{I}(A)\xrightarrow{\varepsilon} B\xrightarrow{\eta}\mathsf{J}(C)\xrightarrow{\varphi}\Sigma\circ\mathsf{I}(A)$ rewrites as

$$\mathsf{I}\circ\mathsf{I}_\rho(B)\xrightarrow{\ \varepsilon_B\ } B\xrightarrow{\ \eta_B\ }\mathsf{J}\circ\mathsf{J}_\lambda(B)\xrightarrow{\ \varphi_B\ }\Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)\ ,$$

proving the existence assertion of (iii).

It remains to prove the uniqueness in (iii), and to prove (iv). Choose therefore any morphism $f\colon B\longrightarrow\widetilde{B}$ in $\mathcal{T}$. By the existence part of (iii) we can choose morphisms $\varphi_B\colon \mathsf{J}\circ\mathsf{J}_\lambda(B)\longrightarrow\Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)$ and $\varphi_{\widetilde{B}}\colon\mathsf{J}\circ\mathsf{J}_\lambda(\widetilde{B})\longrightarrow\Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(\widetilde{B})$ so that in the diagram

$$\begin{array}{ccccccc}
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\ \varepsilon_B\ } & B & \xrightarrow{\ \eta_B\ } & \mathsf{J}\circ\mathsf{J}_\lambda(B) & \xrightarrow{\ \varphi_B\ } & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)\\
{\scriptstyle \mathsf{I}\circ\mathsf{I}_\rho(f)}\downarrow & & \downarrow{\scriptstyle f} & & & & \\
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\ \varepsilon_{\widetilde{B}}\ } & \widetilde{B} & \xrightarrow{\ \eta_{\widetilde{B}}\ } & \mathsf{J}\circ\mathsf{J}_\lambda(\widetilde{B}) & \xrightarrow{\ \varphi_{\widetilde{B}}\ } & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)
\end{array}$$

the rows are distinguished triangles. The square commutes by the naturality of $\varepsilon\colon\mathsf{I}\circ\mathsf{I}_\rho\longrightarrow\mathrm{id}$. The axioms of triangulated categories allow us to extend this to some morphism of triangles

$$\begin{array}{ccccccc}
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\ \varepsilon_B\ } & B & \xrightarrow{\ \eta_B\ } & \mathsf{J}\circ\mathsf{J}_\lambda(B) & \xrightarrow{\ \varphi_B\ } & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)\\
{\scriptstyle \mathsf{I}\circ\mathsf{I}_\rho(f)}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(f)}\\
\mathsf{I}\circ\mathsf{I}_\rho(B) & \xrightarrow{\ \varepsilon_{\widetilde{B}}\ } & \widetilde{B} & \xrightarrow{\ \eta_{\widetilde{B}}\ } & \mathsf{J}\circ\mathsf{J}_\lambda(\widetilde{B}) & \xrightarrow{\ \varphi_{\widetilde{B}}\ } & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)
\end{array}$$

Next observe that both of the squares in the diagram below

$$\begin{array}{ccc}
B & \xrightarrow{\ \eta_B\ } & \mathsf{J}\circ\mathsf{J}_\lambda(B)\\
\downarrow{\scriptstyle f} & & {\scriptstyle \pi}\downarrow\ \downarrow{\scriptstyle \mathsf{J}\circ\mathsf{J}_\lambda(f)}\\
\widetilde{B} & \xrightarrow{\ \eta_{\widetilde{B}}\ } & \mathsf{J}\circ\mathsf{J}_\lambda(\widetilde{B})
\end{array}$$

commute; the one involving π by the above, and the one involving $\mathsf{J}\circ\mathsf{J}_\lambda(f)$ by the naturality of η. This gives two factorizations of the composite $B\xrightarrow{f}\widetilde{B}\xrightarrow{\eta_{\widetilde{B}}}\mathsf{J}\circ\mathsf{J}_\lambda(\widetilde{B})$ through $\eta_B\colon B\longrightarrow\mathsf{J}\circ\mathsf{J}_\lambda(B)$, which must agree. Thus $\pi=\mathsf{J}\circ\mathsf{J}_\lambda(f)$, and we deduce the commutativity of the square

$$\begin{array}{ccc}
\mathsf{J}\circ\mathsf{J}_\lambda(B) & \xrightarrow{\ \varphi_B\ } & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)\\
{\scriptstyle \mathsf{J}\circ\mathsf{J}_\lambda(f)}\downarrow & & \downarrow{\scriptstyle \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(f)}\\
\mathsf{J}\circ\mathsf{J}_\lambda(\widetilde{B}) & \xrightarrow{\ \varphi_{\widetilde{B}}\ } & \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(\widetilde{B}).
\end{array}$$

If we consider the special case where $f\colon B\longrightarrow\widetilde{B}$ is the identity map $\mathrm{id}\colon B\longrightarrow B$, the commutativity of the square above tells us that any two choices $\varphi_B,\varphi_{\widetilde{B}}$ for the map $\mathsf{J}\circ\mathsf{J}_\lambda(B)\longrightarrow\Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)$ must agree—that is we have proved the uniqueness part of (iii). And now that we know both the existence and the uniqueness assertions in (iii), the commutativity of the square above, for any morphism $f\colon B\longrightarrow\widetilde{B}$, completes the proof of (iv). □

Corollary 2.1.8. *Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{S}\subseteq\mathcal{T}$ be an aisle. Then $\mathcal{S}$ is closed in $\mathcal{T}$ under direct summands and coproducts. Dually, any co-aisle $\mathcal{S}'\subseteq\mathcal{T}$ is closed in $\mathcal{T}$ under direct summands and products.*

Just to clarify the statement above, let us stress that it says that any direct summand in $\mathcal{T}$ of an object of $\mathcal{S}$ must belong to $\mathcal{S}$, and given any collection of objects in $\mathcal{S}$ whose coproduct exists in $\mathcal{T}$, that coproduct must belong to $\mathcal{S}$.

Proof. It clearly suffices to prove the assertion about aisles. Let $\mathcal{S} \subseteq \mathcal{T}$ be an aisle; by Example 2.1.6 there exists a subcategory $\mathcal{S}' \subseteq \mathcal{T}$ such that the pair $(\mathcal{S}, \mathcal{S}')$ is a t–structure on $\mathcal{T}$; more explicitly Example 2.1.6 tells us that $\mathcal{S}' = \mathcal{S}^{\perp}$ works. But what is important for us here is Proposition 2.1.7(ii); it gives us the equality $\mathcal{S} = {}^{\perp}\mathcal{S}'$. The corollary now follows because ${}^{\perp}\mathcal{S}'$ is always closed in $\mathcal{T}$ under direct summands and coproducts. □

2.2. **The more traditional notation for t–structures.** The notion of t–structure originated with Beĭlinson, Bernstein and Deligne [2, Section 1.3]. The original motivation came from applications irrelevant to this manuscript, and hence in [2, Section 1.3] the subject is pursued from a perspective different from our Section 2.1—our treatment is much closer in spirit to Keller and Vossieck [13]. The very least we owe the reader is a glossary, explaining how to pass back and forth between our notation and the traditional one.

Notation 2.2.1. Let $\mathcal{T}$ be a triangulated category, and let $(\mathcal{S}, \mathcal{S}')$ be a t–structure on $\mathcal{T}$ as in Definition 2.1.4. One sets

(i) $\mathcal{T}^{\leqslant n} := \Sigma^{-n}\mathcal{S}$.
(ii) $\mathcal{T}^{\geqslant n} := \Sigma^{-n+1}\mathcal{S}'$.
(iii) The triangle $\mathsf{I}\circ\mathsf{I}_\rho(B) \xrightarrow{\varepsilon_B} B \xrightarrow{\eta_B} \mathsf{J}\circ\mathsf{J}_\lambda(B) \xrightarrow{\varphi_B} \Sigma\circ\mathsf{I}\circ\mathsf{I}_\rho(B)$, of Proposition 2.1.7, is traditionally written
$$B^{\leqslant 0} \xrightarrow{\ \varepsilon\ } B \xrightarrow{\ \eta\ } B^{\geqslant 1} \xrightarrow{\ \varphi\ } \Sigma B^{\leqslant 0} .$$
In other words, the functor which we have written as $\mathsf{I}\circ\mathsf{I}_\rho\colon \mathcal{T} \longrightarrow \mathcal{T}$ is normally written $(-)^{\leqslant 0}$, and the functor which we have written as $\mathsf{J}\circ\mathsf{J}_\lambda\colon \mathcal{T} \longrightarrow \mathcal{T}$ is normally written $(-)^{\geqslant 1}$.
(iv) More generally, for any integer $n \in \mathbb{Z}$ we define the functors $(-)^{\leqslant n}$ and $(-)^{\geqslant n}$ by the formulas
$$(-)^{\leqslant n} := \Sigma^{-n}\circ\mathsf{I}\circ\mathsf{I}_\rho\circ\Sigma^n, \qquad (-)^{\geqslant n} := \Sigma^{-n+1}\circ\mathsf{J}\circ\mathsf{J}_\lambda\circ\Sigma^{n-1}.$$
(v) As a pure matter of notation, in Definition 2.1.4 a t–structure was defined to be a pair of subcategories $(\mathcal{S}, \mathcal{S}')$ satisfying some conditions. In the comparison with the standard notation, we let $\mathcal{S} = \mathcal{T}^{\leqslant 0}$ and $\mathcal{S}' = \mathcal{T}^{\geqslant 1}$. In reading the literature the reader should just note that it is traditional to give a t–structure as a pair of subcategories $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$. To be consistent with Definition 2.1.4, in the rest of the paper, a t–structure will be denoted $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$.

As the reader can easily check, with the definitions as above, we have, for any integer $n \in \mathbb{Z}$ and any object $B \in \mathcal{T}$, a distinguished triangle
$$B^{\leqslant n} \xrightarrow{\ \varepsilon_n\ } B \xrightarrow{\ \eta_n\ } B^{\geqslant n+1} \xrightarrow{\ \varphi_n\ } \Sigma B^{\leqslant n} ,$$
with $B^{\leqslant n} \in \mathcal{T}^{\leqslant n} \subseteq \mathcal{T}$ and with $B^{\geqslant n+1} \in \mathcal{T}^{\geqslant n+1} \subseteq \mathcal{T}$, and this triangle is functorial in B. This is just the appropriate translation of the triangle of (iii).

Next, we recall the following definition. The reader can find a more complete treatment in Beĭlinson, Bernstein and Deligne [2, Section 1.2].

Definition 2.2.2. *Let $\mathcal{T}$ be a triangulated category. A strictly full subcategory $\mathcal{A} \subseteq \mathcal{T}$ is called an* admissible abelian subcategory of $\mathcal{T}$ *if the following conditions hold:*

(i) $\mathrm{Hom}(\mathcal{A}, \Sigma^n \mathcal{A}) = 0$ *for all* $n < 0$.

(ii) *The category $\mathcal{A}$ is abelian, and the short exact sequences in $\mathcal{A}$ are precisely the pairs of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ for which there exists a morphism $h\colon C \longrightarrow \Sigma A$ rendering $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ a distinguished triangle in $\mathcal{T}$.*

Next we want to refer the reader to known facts we will use, without providing our own proofs. In the case of these particular known facts, the results and proofs are available in a short section of a single manuscript, and we deemed that giving a reference was the way to go.

Reminder 2.2.3. Let $\mathcal{T}$ be a triangulated category, and let $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$ be a t–structure on $\mathcal{T}$, where the notation is as in Notation 2.2.1(v). Then the following is true:

(i) Let $m \leqslant n$ be integers. Then the functors $(-)^{\leqslant n}$ and $(-)^{\geqslant m}$ commute with each other up to natural isomorphism. Hence the composite functor

$$\left((-)^{\leqslant n}\right)^{\geqslant m} \cong \left((-)^{\geqslant m}\right)^{\leqslant n}$$

can be viewed as a functor $\tau^{[m,n]}\colon \mathcal{T} \longrightarrow \mathcal{T}^{\geqslant m} \cap \mathcal{T}^{\leqslant n}$.

(ii) The category $\mathcal{T}^{\heartsuit} := \mathcal{T}^{\geqslant 0} \cap \mathcal{T}^{\leqslant 0}$ is an admissible abelian subcategory of $\mathcal{T}$ called the *heart* of the t–structure $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$, and the functor $\tau^{[0,0]}\colon \mathcal{T} \longrightarrow \mathcal{T}^{\heartsuit}$ is homological. It will be often denoted $\mathcal{H}^0\colon \mathcal{T} \longrightarrow \mathcal{T}^{\heartsuit}$. As usual, we write $\mathcal{H}^n = \mathcal{H}^0 \circ \Sigma^n$, for every integer n.

We refer to [2, Proposition 1.3.5] for (i) and to [2, Théorème 1.3.6] for (ii).

2.3. **Compactly generated t–structures.**

As in [29], we introduce the following:

Definition 2.3.1. *Let $\mathcal{T}$ be a triangulated category. An object $c \in \mathcal{T}$ is declared to be* compact *if* $\mathrm{Hom}(c, -)$ *respects those coproducts that exist in $\mathcal{T}$.*

When $\mathcal{T}$ has coproducts this goes back to [19, Definition 0.1], and there are powerful techniques allowing us to work out what the compact objects are. But we will care about the more general situation.

Remark 2.3.2. Let $\mathcal{T}$ be a triangulated category. Then the full subcateory $\mathcal{T}^c \subseteq \mathcal{T}$ is defined to have for objects all the compacts in $\mathcal{T}$. Clearly $\mathcal{T}^c$ is strictly full, and one easily checks that

(i) $\mathcal{T}^c * \mathcal{T}^c \subseteq \mathcal{T}^c$.

(ii) $\Sigma \mathcal{T}^c = \mathcal{T}^c$.

(iii) $\mathrm{smd}(\mathcal{T}^c) = \mathcal{T}^c$

Or, to summarize the above properties more densely, $\mathcal{T}^c$ is a thick, triangulated subcategory of $\mathcal{T}$. But we also remind the reader of

(iv) For any compact object $c \in \mathcal{T}$ and any sequence of objects and composable morphisms in $\mathcal{T}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots$$

the natural map

$$\underrightarrow{\mathrm{colim}}\, \mathrm{Hom}(c, X_n) \longrightarrow \mathrm{Hom}(c, \underrightarrow{\mathrm{Hocolim}}\, X_n)$$

is an isomorphism.

As we already said: (i), (ii) and (iii) are easy and left to the reader. To prove (iv) apply the homological functor $\mathrm{Hom}(c,-)$ to the triangle defining $\underrightarrow{\mathrm{Hocolim}}\, X_n$ (see the proof of Lemma 1.2.2(iii) for a reminder of this triangle), and then use the fact that the compactness of c allows us to compute what $\mathrm{Hom}(c,-)$ does to the coproduct terms in this triangle.

The next result is the combination of [1, Theorem A.1 and Proposition A.2].

Theorem 2.3.3. *Let $\mathcal{T}$ be a triangulated category with coproducts, let $\mathcal{A}\subseteq\mathcal{T}^c$ be an essentially small full subcategory, and assume $\Sigma\mathcal{A}\subseteq\mathcal{A}$. Then the pair $\tau_{\mathcal{A}}:=\big(\mathrm{Coprod}(\mathcal{A}),\mathrm{Coprod}(\mathcal{A})^\perp\big)$ is a t–structure on $\mathcal{T}$, and the subcategory $\mathrm{Coprod}(\mathcal{A})^\perp$ is closed in $\mathcal{T}$ under coproducts.*

Proof. The fact that $\mathrm{Coprod}(\mathcal{A})$ and $\mathrm{Coprod}(\mathcal{A})^\perp$ are strictly full was proved in Lemma 1.2.1, the fact that $\Sigma\mathrm{Coprod}(\mathcal{A})\subseteq\mathrm{Coprod}(\mathcal{A})$ and $\Sigma^{-1}\mathrm{Coprod}(\mathcal{A})^\perp\subseteq\mathrm{Coprod}(\mathcal{A})^\perp$ was proved in Lemma 1.2.2, and the fact that $\mathrm{Hom}\big(\mathrm{Coprod}(\mathcal{A}),\mathrm{Coprod}(\mathcal{A})^\perp\big)=0$ is obvious.

To show that $\tau_{\mathcal{A}}$ is a t–structure on $\mathcal{T}$, it remains to check Definition 2.1.4(iii): for every $X\in\mathcal{T}$ we must produce a distinguished triangle $A\longrightarrow X\longrightarrow C\longrightarrow\Sigma A$, with $A\in\mathrm{Coprod}(\mathcal{A})$ and $C\in\mathrm{Coprod}(\mathcal{A})^\perp$. We propose to do it as follows: starting with $X\in\mathcal{T}$ we will produce a sequence

$$X_1\xrightarrow{f_1}X_2\xrightarrow{f_2}X_3\xrightarrow{f_3}X_4\xrightarrow{f_4}\cdots$$

of objects and morphisms in $\mathrm{Coprod}(\mathcal{A})$, all mapping to X, and we will then let $A\longrightarrow X$ be any compatible choice of the map $\underrightarrow{\mathrm{Hocolim}}\, X_n\longrightarrow X$.

Let us start. Because the subcategory $\mathcal{A}$ is essentially small, there is (up to isomorphism) only a set Λ of morphisms $f_\lambda\colon A_\lambda\longrightarrow X$ with $A_\lambda\in\mathcal{A}$. We begin with

(i) Let $X_1=\coprod_{\lambda\in\Lambda}A_\lambda$, and let the morphism $h_1\colon X_1\longrightarrow X$ be the obvious map. We observe that, for every object $A\in\mathcal{A}$, the functor $\mathrm{Hom}(A,-)$ takes the map h_1 to an epimorphism.

Now suppose we are given a morphism $h_n\colon X_n\longrightarrow X$, with $X_n\in\mathrm{Coprod}(\mathcal{A})$. Complete this to a distinguished triangle $Y_n\xrightarrow{\alpha_n}X_n\xrightarrow{h_n}X$, and let Λ_n be a set containing isomorphism classes of all possible maps $k\colon B\longrightarrow Y_n$ with $B\in\Sigma^{-1}\mathcal{A}$. Next we define

(ii) The object $Z_n\in\Sigma^{-1}\mathrm{Coprod}(\mathcal{A})$ is given by the formula $Z_n:=\coprod_{\lambda\in\Lambda_n}B_\lambda$, and the map $\beta_n\colon Z_n\longrightarrow Y_n$ is the obvious. By construction $\mathrm{Hom}(B,-)$ takes the map $\beta_n\colon Z_n\longrightarrow Y_n$ to an epimorphism whenever $B\in\Sigma^{-1}\mathcal{A}$.

Now we have composable morphisms $Z_n\xrightarrow{\beta_n}Y_n\xrightarrow{\alpha_n}X_n$; then we can complete the morphism $\gamma_n=\alpha_n\circ\beta_n\colon Z_n\longrightarrow X_n$ to a distinguished triangle $Z_n\xrightarrow{\gamma_n}X_n\xrightarrow{f_n}X_{n+1}\longrightarrow\Sigma Z_n$. As both X_n and ΣZ_n lie in $\mathrm{Coprod}(\mathcal{A})$, the triangle gives that $X_{n+1}\in\mathrm{Coprod}(\mathcal{A})$. And now the composite

$$\begin{array}{c} \overset{\gamma_n}{\overbrace{\qquad\qquad\qquad}} \\ Z_n\xrightarrow[\beta_n]{}Y_n\xrightarrow[\alpha_n]{}X_n\xrightarrow[h_n]{}X \end{array}$$

vanishes because $h_n\circ\alpha_n=0$, allowing us to factor the map $h_n\colon X_n\longrightarrow X$ as $X_n\xrightarrow{f_n}X_{n+1}\xrightarrow{h_{n+1}}X$. And the notable features of the construction are

(iii) We have factored the map $h_n\colon X_n \longrightarrow X$ as $X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{h_{n+1}} X$, with $X_{n+1} \in \mathrm{Coprod}(\mathcal{A})$, and in such a way that, for every $B \in \Sigma^{-1}\mathcal{A}$, the kernel of $\mathrm{Hom}(B, h_n)\colon \mathrm{Hom}(B, X_n) \longrightarrow \mathrm{Hom}(B, X)$ is annihilated by the morphism $\mathrm{Hom}(B, f_n)\colon \mathrm{Hom}(B, X_n) \longrightarrow \mathrm{Hom}(B, X_{n+1})$.

To put it more succinctly: for every $B \in \Sigma^{-1}\mathcal{A}$, the sequence

$$\mathrm{Hom}(B, X_1) \xrightarrow{f_1} \mathrm{Hom}(B, X_2) \xrightarrow{f_2} \mathrm{Hom}(B, X_3) \xrightarrow{f_3} \mathrm{Hom}(B, X_4) \xrightarrow{f_4} \cdots$$

is Ind-isomorphic to the sequence of monomorphisms

$$I_1 \xrightarrow{\widetilde{f}_1} I_2 \xrightarrow{\widetilde{f}_2} I_3 \xrightarrow{\widetilde{f}_3} I_4 \xrightarrow{\widetilde{f}_4} \cdots$$

where I_n is the image of the map $\mathrm{Hom}(B, h_n)\colon \mathrm{Hom}(B, X_n) \longrightarrow \mathrm{Hom}(B, X)$. If B belongs to $\mathcal{A} \subseteq \Sigma^{-1}\mathcal{A}$, this is actually a sequence of isomorphisms; after all by (i) the map $I_1 \longrightarrow \mathrm{Hom}(B, X)$ is epi, and hence so are all the maps $I_n \longrightarrow \mathrm{Hom}(B, X)$. But being both epi and mono they must all be isomorphisms. Now let $A := \underrightarrow{\mathrm{Hocolim}}\, X_n$. By Lemma 1.2.2(iii) the object A belongs to $\mathrm{Coprod}(\mathcal{A})$ and, with any choice of the morphism $h\colon A \longrightarrow X$ compatible with all the h_n, the computation above (taking into account also Remark 2.3.2(iv)) tells us that $\mathrm{Hom}(B, h)$ is a monomorphism for all $B \in \Sigma^{-1}\mathcal{A}$ and an isomorphism if $B \in \mathcal{A}$. Now complete to a distinguished triangle $A \xrightarrow{h} X \longrightarrow C \longrightarrow \Sigma A$. Take any object $G \in \mathcal{A}$ and apply the homological functor $\mathrm{Hom}(G, -)$ to this triangle. We deduce an exact sequence

$$\mathrm{Hom}(G, A) \xrightarrow{\alpha} \mathrm{Hom}(G, X) \longrightarrow \mathrm{Hom}(G, C) \longrightarrow \mathrm{Hom}(G, \Sigma A) \xrightarrow{\beta} \mathrm{Hom}(G, \Sigma X)$$

where, by the above, α is an isomorphism and β is a monomorphism. Hence $\mathrm{Hom}(G, C) = 0$, and as this is true for all $G \in \mathcal{A}$ we deduce that $C \in \mathcal{A}^\perp$, which is equal to $\mathrm{Coprod}(\mathcal{A})^\perp$ by Lemma 1.2.1(ii). This completes the proof that $\tau_{\mathcal{A}}$ is a t-structure on $\mathcal{T}$.

And now the equality $\mathrm{Coprod}(\mathcal{A})^\perp = \mathcal{A}^\perp$ of Lemma 1.2.1(ii), coupled with the fact that $\mathcal{A} \subseteq \mathcal{T}^c$, immediately tells us that $\mathrm{Coprod}(\mathcal{A})^\perp$ is closed in $\mathcal{T}$ under coproducts. □

Remark 2.3.4. As we have already said, Theorem 2.3.3 is the union of [1, Theorem A.1 and Proposition A.2]. The reason we have gone to the trouble of giving a self-contained, complete proof is the following.

In the proof of [1, Theorem A.1], the triangle $A \longrightarrow X \longrightarrow C \longrightarrow \Sigma A$ with $A \in \mathrm{Coprod}(\mathcal{A})$ and $C \in \mathrm{Coprod}(\mathcal{A})^\perp$ is produced as a homotopy colimit of distinguished triangles $X_n \xrightarrow{h_n} X \longrightarrow C_n \longrightarrow \Sigma X_n$, where as it happens the maps $h_n\colon X_n \longrightarrow X$ are the same as those of our proof of Theorem 2.3.3. But without enhancements it is not known that this homotopy limit can always be assumed to be a distinguished triangle, and hence the argument of [1, Theorem A.1] has a gap.

Our argument fixes the problem by letting $h\colon A \longrightarrow X$ be any choice of the homotopy colimit of the maps $h_n\colon X_n \longrightarrow X$, computing what the functor $\mathrm{Hom}(B, -)$ does to h for any $B \in \Sigma^{-1}\mathcal{A}$, and then, after completing $h\colon A \longrightarrow X$ to a distinguished triangle $A \longrightarrow X \longrightarrow C \longrightarrow \Sigma A$, using the information to show that $C \in \mathrm{Coprod}(\mathcal{A})^\perp$.

Of course, in the presence of enhancements the proof of [1, Theorem A.1] works just fine. But in this article the results are all enhancement-free, and hence we deemed it worthwhile going to a little trouble to give a proof of Theorem 2.3.3 that is evidently enhancement-free.

While still on the subject of enhancement-free proofs of Theorem 2.3.3: an alternative approach goes as follows. Still starting with the above sequence of triangles $X_n \xrightarrow{h_n} X \xrightarrow{k_n} C_n \longrightarrow \Sigma X_n$, we can focus instead on the map k_n. Let $C := \underrightarrow{\mathrm{Hocolim}}\, C_n$, and as in the proof of [1, Theorem A.1] it is easy to see that $C \in \mathrm{Coprod}(\mathcal{A})^\perp$. The challenge becomes to show that the homotopy colimit map $k\colon X \longrightarrow C$ can be so chosen that, when we complete to a distinguished triangle $A \longrightarrow X \xrightarrow{k} C \longrightarrow \Sigma A$, the object A belongs to $\mathrm{Coprod}(\mathcal{A})$. This can be done, but we have not included it here. One complete, enhancement-free proof of Theorem 2.3.3 was deemed enough.

Remark 2.3.5. We should also mention that Theorem 2.3.3 is only a baby case of a whole genre of results. There are several results in the literature giving other $\mathcal{A}$, with $\Sigma\mathcal{A} \subseteq \mathcal{A}$, for which it is known that $\big(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})^\perp\big)$ is a t–structure on $\mathcal{T}$. The basic idea is that, by the results of Section 2.1, this is equivalent to showing that the inclusion $\mathrm{Coprod}(\mathcal{A}) \longrightarrow \mathcal{T}$ has a right adjoint, and representability theorems can be used to produce the right adjoint.

The t–structures constructed in Theorem 2.3.3 deserve a special name.

Definition 2.3.6. *Let $\mathcal{T}$ be a triangulated category with coproducts. The t–structure of the form $\tau_{\mathcal{A}} := \big(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})^\perp\big)$, where $\mathcal{A} \subseteq \mathcal{T}^c$ is an essentially small full subcategory satisfying $\Sigma\mathcal{A} \subseteq \mathcal{A}$, will be called the* t–structure on $\mathcal{T}$ compactly generated by $\mathcal{A}$.

Remark 2.3.7. Following Notation 2.2.1(iv), we will denote by $(-)^{\leqslant n}_{\mathcal{A}}$ the functors associated to such a t–structure. Note that, since $\mathrm{Coprod}(\mathcal{A})^\perp$ is closed under coproducts by Theorem 2.3.3, such a functor commutes with coproducts. Furthermore we denote by $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$ its heart and by $\mathcal{H}^0_{\mathcal{A}}$ the corresponding homological functor, in accordance with Reminder 2.2.3(ii).

Furthermore, one can consider the following subcategories:

$$\mathcal{T}^{\leqslant 0}_{\mathcal{A},c} := \mathcal{T}^c \cap \mathcal{T}^{\leqslant 0}_{\mathcal{A}} \qquad \mathcal{T}^{\heartsuit}_{\mathcal{A},c} := \mathcal{H}^0_{\mathcal{A}}\big(\mathcal{T}^{\leqslant 0}_{\mathcal{A},c}\big) \subseteq \mathcal{T}^{\heartsuit}_{\mathcal{A}}.$$

Then [22, Proposition 1.9(ii)] gives us the final equality in the string

$$\mathcal{T}^{\leqslant 0}_{\mathcal{A},c} = \mathcal{T}^c \cap \mathcal{T}^{\leqslant 0}_{\mathcal{A}} = \mathcal{T}^c \cap \mathrm{Coprod}(\mathcal{A}) = \mathrm{smd}\big(\mathrm{coprod}(\mathcal{A})\big).$$

This clearly implies that $\mathcal{T}^{\leqslant 0}_{\mathcal{A},c}$ is essentially small, hence the same is true for $\mathcal{T}^{\heartsuit}_{\mathcal{A},c} = \mathcal{H}^0_{\mathcal{A}}\big(\mathcal{T}^{\leqslant 0}_{\mathcal{A},c}\big)$.

As the reader will see, the existence and basic properties of these t–structures will play an important role in the rest of the article.

3. The heart of a compactly generated t–structure

This section is devoted to proving important properties of hearts of t–structures generated by sets of compact objects as in Definition 2.3.6. The main theorem of this section can be found in Saorín and Šťovíček [30, Theorem 8.31]. The proof there is different—we present here a self-contained argument. The main reason is that the technical lemmas in our proof will be needed later in the paper.

Theorem 3.0.1. *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{A} \subseteq \mathcal{T}^c$ be an essentially small full subcategory satisfying $\Sigma\mathcal{A} \subseteq \mathcal{A}$. Then the heart $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$ of the t–structure $\tau_{\mathcal{A}} = \big(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})^\perp\big)$ is a locally finitely presented Grothendieck abelian category. Moreover: $\mathcal{T}^{\heartsuit}_{\mathcal{A},c} \subseteq \mathcal{T}^{\heartsuit}_{\mathcal{A}}$ is precisely the full subcategory of finitely presented objects in $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$.*

Recall that an abelian category $\mathcal{B}$ is a *Grothendieck category* if it satisfies axiom [AB5] (meaning that $\mathcal{B}$ has small coproducts and filtered colimits of exact sequences are exact in $\mathcal{B}$), and it has a set of generators. An object $b \in \mathcal{B}$ is *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{B}}(b, -)$ commutes with filtered colimits, and $\mathcal{B}$ is *locally finitely presented* if the full subcategory of finitely presented objects in $\mathcal{B}$ is essentially small and all objects of $\mathcal{B}$ are filtered colimits of finitely presented objects.

3.1. **Some technical results.** In this section, the standing assumption is that $\mathcal{T}$ is a triangulated category with coproducts and $\mathcal{A} \subseteq \mathcal{T}^c$ is an essentially small full subcategory satisfying $\Sigma\mathcal{A} \subseteq \mathcal{A}$.

The following little lemma may also be found in Saorín and Šťovíček [30, part 2b of Proposition 8.27]. The interested reader is encouraged to compare the approaches—the short summary is that in [30] the ideas are explored much more fully, including a converse to the lemma below.

Lemma 3.1.1. *If $f\colon c \longrightarrow X$ is a morphism in $\mathcal{T}$ with $c \in \mathcal{T}^c$ and $X \in \mathcal{T}_{\mathcal{A}}^{\leqslant 0}$, then there is a factorization of f as $c \longrightarrow t \longrightarrow X$ with $t \in \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$.*

Proof. By construction, f is a morphism from $c \in \mathcal{T}^c$ to $X \in \mathrm{Coprod}(\mathcal{A})$. By [22, Lemma 1.8(ii)] this map must factor through an object $t \in \mathrm{coprod}(\mathcal{A}) \subseteq \mathcal{T}^c \cap \mathcal{T}_{\mathcal{A}}^{\leqslant 0} = \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$. □

Lemma 3.1.2. *Let $f\colon c \longrightarrow X^{\geq 0}$ be a morphism in $\mathcal{T}$, where $c \in \mathcal{T}^c$ and the truncation $X^{\geq 0}$ of $X \in \mathcal{T}$ is with respect to the t–structure $\left(\mathcal{T}_{\mathcal{A}}^{\leqslant 0}, \mathcal{T}_{\mathcal{A}}^{\geqslant 1}\right)$. Then there are morphisms $\varphi\colon b \longrightarrow c$ and $g\colon b \longrightarrow X$, with $b \in \mathcal{T}^c$, such that*

(i) *The map $\varphi^{\geqslant 0}\colon b^{\geqslant 0} \longrightarrow c^{\geqslant 0}$ is an isomorphism;*
(ii) *The triangle below commutes*

$$\begin{array}{ccc} & & c^{\geqslant 0} \\ & \nearrow^{\varphi^{\geqslant 0}} & \big\downarrow{\scriptstyle f^{\geqslant 0}} \\ b^{\geqslant 0} & \xrightarrow{\ g^{\geqslant 0}\ } & X^{\geqslant 0}. \end{array}$$

Proof. Consider the distinguished triangle $X^{\leqslant -1} \longrightarrow X \longrightarrow X^{\geqslant 0} \xrightarrow{\delta} \Sigma X^{\leqslant -1}$. The composite $c \xrightarrow{f} X^{\geqslant 0} \xrightarrow{\delta} \Sigma X^{\leqslant -1}$ is a morphism from $c \in \mathcal{T}^c$ to the object $\Sigma X^{\leqslant -1} \in \mathcal{T}_{\mathcal{A}}^{\leqslant -2}$, and Lemma 3.1.1 permits us to factor it as $c \longrightarrow d \longrightarrow \Sigma X^{\leqslant -1}$ with $d \in \Sigma^2\mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$. And now form the distinguished triangle $\Sigma^{-1}d \longrightarrow b \xrightarrow{\varphi} c \longrightarrow d$. Since c and d are both compact so is b, and as d and $\Sigma^{-1}d$ both belong to $\mathcal{T}_{\mathcal{A}}^{\leqslant -1}$ the functor $(-)^{\geqslant 0}$ must take $\varphi\colon b \longrightarrow c$ to an isomorphism.

Next consider the commutative diagram where the rows are distinguished triangles

$$\begin{array}{ccccccc} b & \xrightarrow{\ \varphi\ } & c & \longrightarrow & d & \longrightarrow & \Sigma b \\ & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & \\ X & \longrightarrow & X^{\geqslant 0} & \xrightarrow{\ \delta\ } & \Sigma X^{\leqslant -1} & \longrightarrow & \Sigma X. \end{array}$$

We may complete it to a morphism of triangles, and in particular this allows us to produce a commutative square

$$\begin{array}{ccc} b & \xrightarrow{\ \varphi\ } & c \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \longrightarrow & X^{\geqslant 0} \end{array}$$

We have now constructed the object $b \in \mathcal{T}^c$ and the morphisms $\varphi\colon b \longrightarrow c$ and $g\colon b \longrightarrow X$. The last paragraph showed that $\varphi^{\geqslant 0}$ is an isomorphism, proving (i). And (ii) follows from applying the functor $(-)^{\geqslant 0}$ to the commutative square above. □

Lemma 3.1.3. *In the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$, the full subcategory $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ satisfies the following*

(i) *$\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ is closed in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ under extensions.*
(ii) *$\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ is closed in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ under cokernels.*
(iii) *For any object $X \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, the functors $\mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}^{\heartsuit}}(X,-)$ and $\mathrm{Ext}^1_{\mathcal{T}_{\mathcal{A}}^{\heartsuit}}(X,-)$ commute with coproducts in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$.*

Proof. We begin with (i): let $X, Z \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ and suppose we are given in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ a short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$. This means that in $\mathcal{T}$ there must be a morphism $Z \longrightarrow \Sigma X$ such that $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a distinguished triangle (see Reminder 2.2.3).

Now because X and Z belong to $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, there must exist objects a, c in $\mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ with $X = a^{\geqslant 0}$ and $Z = c^{\geqslant 0}$. In particular the t–structure truncation provides us with a morphism $c \longrightarrow c^{\geqslant 0} = Z$, and composing with the morphism $Z \longrightarrow \Sigma X$ produces a map $c \longrightarrow c^{\geqslant 0} \longrightarrow \Sigma a^{\geqslant 0} = (\Sigma a)^{\geqslant -1}$ which we will denote $f\colon c \longrightarrow (\Sigma a)^{\geqslant -1}$. Our construction so far has produced a commutative diagram

$$\begin{array}{ccccccc} c & \longrightarrow & c^{\geqslant -1} & \longrightarrow & c^{\geqslant 0} & = & Z \\ & \searrow^{f} & \downarrow{\scriptstyle f^{\geqslant -1}} & & \downarrow & & \downarrow \\ & & (\Sigma a)^{\geqslant -1} & = & (\Sigma a)^{\geqslant -1} & = & \Sigma X. \end{array}$$

To the morphism f we now apply Lemma 3.1.2 and conclude that there exists a compact object $\widetilde{c} \in \mathcal{T}^c$ and morphisms $\varphi\colon \widetilde{c} \longrightarrow c$ and $g\colon \widetilde{c} \longrightarrow \Sigma a$, satisfying

(iv) The map $\varphi^{\geqslant -1}\colon \widetilde{c}^{\geqslant -1} \longrightarrow c^{\geqslant -1}$ is an isomorphism, and hence $\widetilde{c}$ must belong to $\mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ and $\mathcal{H}^0_{\mathcal{A}}(\varphi)\colon \mathcal{H}^0_{\mathcal{A}}(\widetilde{c}) \longrightarrow \mathcal{H}^0_{\mathcal{A}}(c)$ is an isomorphism.
(v) The triangle below commutes

$$\begin{array}{ccc} & & c^{\geqslant -1} \\ & \nearrow^{\varphi^{\geqslant -1}} & \downarrow{\scriptstyle f^{\geqslant -1}} \\ \widetilde{c}^{\geqslant -1} & \xrightarrow[g^{\geqslant -1}]{} & (\Sigma a)^{\geqslant -1} \end{array}$$

In view of this we deduce the following commutative diagram

$$\begin{array}{ccccccc} \widetilde{c} & \longrightarrow & \widetilde{c}^{\geqslant -1} & \longrightarrow & c^{\geqslant 0} & = & Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^{\geqslant -1}} & & \downarrow & & \downarrow \\ \Sigma a & \longrightarrow & (\Sigma a)^{\geqslant -1} & = & (\Sigma a)^{\geqslant -1} & = & \Sigma X. \end{array}$$

Now complete the commutative square

$$\begin{array}{ccc} \widetilde{c} & \xrightarrow{g} & \Sigma a \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \Sigma X \end{array}$$

to a morphism of distinguished triangles

$$\begin{array}{ccccccc}
a & \longrightarrow & b & \longrightarrow & \widetilde{c} & \xrightarrow{\ g\ } & \Sigma a \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X
\end{array}$$

As a and $\widetilde{c}$ both belong to $\mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ so does b. And, by construction, the morphism $g^{\geqslant -1}\colon \widetilde{c}^{\geqslant -1} \longrightarrow (\Sigma a)^{\geqslant -1}$ factors as $\widetilde{c}^{\geqslant -1} \longrightarrow \widetilde{c}^{\geqslant 0} \longrightarrow (\Sigma a)^{\geqslant -1}$, forcing the composite $\widetilde{c}^{\leqslant -1} \longrightarrow \widetilde{c} \longrightarrow (\Sigma a)^{\geqslant -1}$ to vanish. Hence the morphism $\mathcal{H}_{\mathcal{A}}^{-1}(g)\colon \mathcal{H}_{\mathcal{A}}^{-1}(\widetilde{c}) \longrightarrow \mathcal{H}_{\mathcal{A}}^{-1}(\Sigma a) = \mathcal{H}_{\mathcal{A}}^{0}(a)$ vanishes. Thus the functor $\mathcal{H}_{\mathcal{A}}^{0}$ takes the morphism of triangles to a commutative diagram in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_{\mathcal{A}}^{0}(a) & \longrightarrow & \mathcal{H}_{\mathcal{A}}^{0}(b) & \longrightarrow & \mathcal{H}_{\mathcal{A}}^{0}(\widetilde{c}) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

and the fact that the outside maps are isomorphisms forces the map $\mathcal{H}_{\mathcal{A}}^{0}(b) \longrightarrow Y$ to be an isomorphism. Hence Y is isomorphic to $\mathcal{H}_{\mathcal{A}}^{0}(b) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, proving (i).

Now for (ii). Take a morphism $X \longrightarrow Y$ in $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. There exist objects $a, b \in \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ with $X = a^{\geqslant 0}$ and $Y = b^{\geqslant 0}$. The map $a \longrightarrow a^{\geqslant 0} = X$ can be composed with the morphism $X \longrightarrow Y$ to produce a map $a \longrightarrow a^{\geqslant 0} \longrightarrow b^{\geqslant 0}$ which we denote $f\colon a \longrightarrow b^{\geqslant 0}$. And now Lemma 3.1.2 allows us to find an object $\widetilde{a} \in \mathcal{T}^c$ and morphisms $\varphi\colon \widetilde{a} \longrightarrow a$ and $g\colon \widetilde{a} \longrightarrow b$ such that

(vi) The map $\varphi^{\geqslant 0}\colon \widetilde{a}^{\geqslant 0} \longrightarrow a^{\geqslant 0}$ is an isomorphism, and hence $\widetilde{a}$ must belong to $\mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ and $\mathcal{H}_{\mathcal{A}}^{0}(\varphi)\colon \mathcal{H}_{\mathcal{A}}^{0}(\widetilde{a}) \longrightarrow \mathcal{H}_{\mathcal{A}}^{0}(a)$ is an isomorphism.

(vii) The triangle below commutes

$$\begin{array}{ccc}
 & \overset{\varphi^{\geqslant 0}}{\nearrow} & a^{\geqslant 0} \\
\widetilde{a}^{\geqslant 0} & & \downarrow f^{\geqslant 0} \\
 & \underset{g^{\geqslant 0}}{\searrow} & b^{\geqslant 0}
\end{array}$$

Completing $g\colon \widetilde{a} \longrightarrow b$ to a distinguished triangle $\widetilde{a} \longrightarrow b \longrightarrow c \longrightarrow \Sigma\widetilde{a}$ produces for us an object c which is compact, and as b and $\Sigma\widetilde{a}$ both lie in $\mathcal{T}_{\mathcal{A}}^{\leqslant 0}$ we have that so does c. Thus $c \in \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$, and in cohomology we have an exact sequence $\mathcal{H}_{\mathcal{A}}^{0}(\widetilde{a}) \longrightarrow \mathcal{H}_{\mathcal{A}}^{0}(b) \longrightarrow \mathcal{H}_{\mathcal{A}}^{0}(c) \longrightarrow \mathcal{H}_{\mathcal{A}}^{1}(\widetilde{a}) = 0$. This identifies with $X \longrightarrow Y \longrightarrow \mathcal{H}_{\mathcal{A}}^{0}(c) \longrightarrow 0$, proving that the cokernel in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ of the map $X \longrightarrow Y$ is isomorphic to $\mathcal{H}_{\mathcal{A}}^{0}(c) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, establishing (ii).

It remains to prove (iii). Let c be an object in $\mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ and let $\{X_\lambda,\ \lambda \in \Lambda\}$ be any set of objects in $\mathcal{T}^{\geqslant -1}$. Suppose we are given a map

$$c^{\geqslant 0} \xrightarrow{\ f\ } \coprod_{\lambda \in \Lambda} X_\lambda\ .$$

The composite

$$c \longrightarrow c^{\geqslant 0} \xrightarrow{\ f\ } \coprod_{\lambda \in \Lambda} X_\lambda$$

is a morphism from the compact object c to a coproduct, and must factor through a finite subcoproduct. Therefore there is a subset $\Lambda' \subseteq \Lambda$, with $\Lambda \setminus \Lambda'$ finite and such that the longer composite

$$c \longrightarrow c^{\geqslant 0} \xrightarrow{\ f\ } \coprod_{\lambda\in\Lambda} X_\lambda \xrightarrow{\ \pi\ } \coprod_{\lambda\in\Lambda'} X_\lambda$$

vanishes, where π is the projection to the direct summand. But the vanishing of $c \longrightarrow c^{\geqslant 0} \stackrel{\pi\circ f}{\longrightarrow} \coprod_{\lambda\in\Lambda'} X_\lambda$ coupled with the distinguished triangle $c \longrightarrow c^{\geqslant 0} \longrightarrow \Sigma c^{\leqslant -1}$ allows us to produce a commutative square

$$\begin{array}{ccc} c^{\geqslant 0} & \xrightarrow{\ f\ } & \coprod_{\lambda\in\Lambda} X_\lambda \\ \downarrow & & \downarrow \pi \\ \Sigma c^{\leqslant -1} & \xrightarrow{\ \varphi\ } & \coprod_{\lambda\in\Lambda'} X_\lambda. \end{array}$$

Now because $\mathcal{T}_{\mathcal{A}}^{\geqslant -1}$ is closed under coproducts (see Theorem 2.3.3), the object $\coprod_{\lambda\in\Lambda'} X_\lambda$ belongs to $\mathcal{T}_{\mathcal{A}}^{\geqslant -1}$, while $\Sigma c^{\leqslant -1}$ clearly belongs to $\mathcal{T}_{\mathcal{A}}^{\leqslant -2}$. Hence the map φ must vanish, and thus so does the composite from top left to bottom right. This forces the map f to factor through $\coprod_{\lambda\in\Lambda\setminus\Lambda'} X_\lambda$.

The assertions in (iii) are the special cases where either all the X_λ belong to $\mathcal{T}_{\mathcal{A}}^{\heartsuit} \subseteq \mathcal{T}_{\mathcal{A}}^{\geqslant -1}$ or they all belong to $\Sigma\mathcal{T}_{\mathcal{A}}^{\heartsuit} \subseteq \mathcal{T}_{\mathcal{A}}^{\geqslant -1}$. ☐

Corollary 3.1.4. *The subcategory $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ is closed in $\mathcal{T}$ under direct summands.*

Proof. Let X be a direct summand of an object $Y \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Then there must exist an idempotent $e\colon Y \longrightarrow Y$ and an exact sequence in the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ of the form $Y \stackrel{e}{\longrightarrow} Y \longrightarrow X \longrightarrow 0$. But then Lemma 3.1.3(ii) tells us that X must belong to $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. ☐

Lemma 3.1.5. *Suppose we are given in the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ a diagram*

$$\begin{array}{ccccccc} & & & & c & & \\ & & & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

in which the row is exact and c belongs to the subcategory $\mathcal{T}_{\mathcal{A},c}^{\heartsuit} \subseteq \mathcal{T}_{\mathcal{A}}^{\heartsuit}$. Then we may extend it in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ to a commutative diagram with exact rows

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

with $a, b \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$.

Proof. Complete the morphism $X \longrightarrow Y$ to a distinguished triangle $X \longrightarrow Y \longrightarrow M \longrightarrow \Sigma X$ in $\mathcal{T}$. Then M is an object of $\mathcal{T}_{\mathcal{A}}^{\leqslant 0}$ and $\mathcal{H}_{\mathcal{A}}^0(M) = Z$. The vertical morphism in the given diagram therefore produces a map $c \longrightarrow Z = M^{\geqslant 0}$. Since c is assumed to belong to $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ there must exist an object $z \in \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ with $c = \mathcal{H}_{\mathcal{A}}^0(z) = z^{\geqslant 0}$. Thus we can form the composite $z \longrightarrow z^{\geqslant 0} = c \longrightarrow Z = M^{\geqslant 0}$.

Call this map $f\colon z \longrightarrow M^{\geqslant 0}$, and note that $f^{\geqslant 0}$ identifies with the given map $c \longrightarrow Z$. Lemma 3.1.2 now applies: there exist morphisms $\varphi\colon \widetilde{z} \longrightarrow z$ and $g\colon \widetilde{z} \longrightarrow M$ with $\widetilde{z} \in \mathcal{T}^c$ such that

(i) The map $\varphi^{\geqslant 0}\colon \widetilde{z}^{\geqslant 0} \longrightarrow z^{\geqslant 0}$ is an isomorphism. This implies that $\widetilde{z} \in \mathcal{T}^{\leqslant 0}_{\mathcal{A},c}$ and that $\widetilde{z}^{\geqslant 0} \cong z^{\geqslant 0} \cong c$.

(ii) The triangle below commutes

$$\begin{array}{ccc} & & z^{\geqslant 0} \\ & \nearrow^{\varphi^{\geqslant 0}} & \\ \widetilde{z}^{\geqslant 0} & & \Big\downarrow{\scriptstyle f^{\geqslant 0}} \\ & \searrow_{g^{\geqslant 0}} & \\ & & M^{\geqslant 0} \end{array}$$

Next apply Lemma 3.1.1 to the composite $\widetilde{z} \longrightarrow M \longrightarrow \Sigma X$: since $\widetilde{z} \in \mathcal{T}^c$ and $\Sigma X \in \mathcal{T}^{\leqslant -1}_{\mathcal{A}}$, there is a commutative square

$$\begin{array}{ccc} \widetilde{z} & \longrightarrow & \Sigma x \\ \downarrow & & \downarrow \\ M & \longrightarrow & \Sigma X \end{array}$$

with Σx a compact object in $\mathcal{T}^{\leqslant -1}_{\mathcal{A}}$. Now we complete the commutative square to a morphism of distinguished triangles

$$\begin{array}{ccccccc} x & \longrightarrow & y & \longrightarrow & \widetilde{z} & \longrightarrow & \Sigma x \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & \Sigma X. \end{array}$$

Since x and $\widetilde{z}$ both belong to $\mathcal{T}^{\leqslant 0}_{\mathcal{A},c}$ so does y. Therefore applying the functor $\mathcal{H}^0_{\mathcal{A}}$ we deduce a commutative diagram in $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$ with exact rows

$$\begin{array}{ccccccc} \mathcal{H}^0_{\mathcal{A}}(x) & \longrightarrow & \mathcal{H}^0_{\mathcal{A}}(y) & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

with $\mathcal{H}^0_{\mathcal{A}}(x)$ and $\mathcal{H}^0_{\mathcal{A}}(y)$ both in $\mathcal{T}^{\heartsuit}_{\mathcal{A},c}$. □

3.2. **The proof of Theorem 3.0.1.** We are now ready to combine the technical results of the previous section and prove our first main theorem. We keep the assumptions in Theorem 3.0.1, that is $\mathcal{T}$ is a triangulated category with coproducts and $\mathcal{A} \subseteq \mathcal{T}^c$ is an essentially small full subcategory satisfying $\Sigma\mathcal{A} \subseteq \mathcal{A}$.

We first prove the existence of generators in the abelian category $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$.

Lemma 3.2.1. *The subcategory $\mathcal{T}^{\heartsuit}_{\mathcal{A},c}$ is essentially small and it generates the abelian category $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$.*

Proof. The fact that $\mathcal{T}^{\heartsuit}_{\mathcal{A},c}$ is essentially small was observed Remark 2.3.7. The next step in the proof is to note

(i) Every nonzero object X in the abelian category $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$ admits a nonzero map $x \longrightarrow X$ with $x \in \mathcal{A}$.

Assume $X \in \mathcal{A}^\perp$. Now use the equalities

$$\mathcal{A}^\perp = \text{Coprod}(\mathcal{A})^\perp = \mathcal{T}_{\mathcal{A}}^{\geqslant 1} ,$$

where the first equality is by Lemma 1.2.1(ii) and the second is by combining Definition 2.3.6 with Notation 2.2.1(ii). We deduce that $X \in \mathcal{T}_{\mathcal{A}}^{\geqslant 1}$, but as X is assumed to also belong to $\mathcal{T}_{\mathcal{A}}^{\heartsuit} \subseteq \mathcal{T}_{\mathcal{A}}^{\leqslant 0}$ this forces $X = 0$, completing the proof of (i).

Our next easy observation is

(ii) Every nonzero object X in the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ admits a nonzero map $a \longrightarrow X$ with $a \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$.

Take a nonzero object $X \in \mathcal{T}_{\mathcal{A}}^{\heartsuit}$. By (i) we have a nonzero morphism $x \longrightarrow X$ with $x \in \mathcal{A} \subseteq \mathcal{T}^c \cap \mathcal{T}_{\mathcal{A}}^{\leqslant 0} = \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$. This morphism admits a factorization as $x \longrightarrow x^{\geqslant 0} \longrightarrow X$, and as $\mathcal{H}_{\mathcal{A}}^0(x) = x^{\geqslant 0}$ belongs to $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ we have produced our nonzero map $x^{\geqslant 0} \longrightarrow X$, proving (ii).

And now for the proof of the Lemma. Let Y be an arbitrary object of the category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$. Because the subcategory $\mathcal{T}_{\mathcal{A},c}^{\heartsuit} \subseteq \mathcal{T}_{\mathcal{A}}^{\heartsuit}$ is essentially small, there is only a set Λ of maps $y \longrightarrow Y$ with $y \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. We need to prove that the morphism

$$X := \coprod_{\lambda \in \Lambda} y_\lambda \xrightarrow{\quad\varphi\quad} Y$$

is an epimorphism. In the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ we can complete the morphism $\varphi\colon X \longrightarrow Y$ to an exact sequence $X \xrightarrow{\varphi} Y \longrightarrow Z \longrightarrow 0$. We will assume $Z \neq 0$ and prove a contradiction. By (ii) there must exist a nonzero morphism $c \longrightarrow Z$ with $c \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, and we have a diagram

$$\begin{array}{ccccccc} & & & & c & & \\ & & & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

to which we can apply Lemma 3.1.5. It produces for us a commutative diagram with exact rows

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0, \end{array}$$

with $a, b \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Now the morphism $b \longrightarrow Y$ is a morphism from $b \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ to Y, and by construction it must factor through X. Therefore the composite $b \longrightarrow Y \longrightarrow Z$ must vanish. Hence so does the equal composite $b \longrightarrow c \longrightarrow Z$, but as $b \longrightarrow c$ is an epimorphism we deduce that $c \longrightarrow Z$ must vanish. This is a contradication. ☐

The next lemma is known. From Positselski and Šťovíček [25, Corollary 9.6 and Remark 9.7] the reader will learn that, if $\mathcal{A}$ is an abelian category with coproducts, if $G \subseteq \mathcal{A}$ is a set of generators, and if for all objects $g \in G$ the functor $\text{Hom}(g, -)$ respects coproducts, then $\mathcal{A}$ satisfies [AB5].

In Lemma 3.2.2 below we include the short proof—we present it in the special case where $\mathcal{A} = \mathcal{T}_{\mathcal{A}}^{\heartsuit}$, but the argument is easy to modify to cover the more general statement. The proof we give is included both for the reader's convenience, and because the technique will be useful. We will use it again in the proof of Lemma 3.2.3, and it will play a key role in Section 7.2, culminating in the proof of Proposition 7.2.6.

Lemma 3.2.2. *The abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ is a Grothendieck abelian category.*

Proof. Lemma 3.2.1 tells us that the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ has a set of generators $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Axiom [AB4] is easy: the categories $\mathcal{T}_{\mathcal{A}}^{\leqslant 0}$ and $\mathcal{T}_{\mathcal{A}}^{\geqslant 0}$ are both closed in $\mathcal{T}$ under coproducts, and hence so is their intersection $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$. And as coproducts of distinguished triangles are distinguished triangles, if we have a set of distinguished triangles $X_\lambda \longrightarrow Y_\lambda \longrightarrow Z_\lambda \longrightarrow \Sigma X_\lambda$ with $X_\lambda, Y_\lambda, Z_\lambda \in \mathcal{T}_{\mathcal{A}}^{\heartsuit}$, then the coproduct is a triangle with the same property. That is the coproduct in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ of short exact sequences $0 \longrightarrow X_\lambda \longrightarrow Y_\lambda \longrightarrow Z_\lambda \longrightarrow 0$ is a short exact sequence in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$.

Now let us prove [AB5]. Let I be a small filtered category and let $\mathsf{F}\colon I \longrightarrow \mathcal{T}_{\mathcal{A}}^{\heartsuit}$ be a functor; we need to prove that left derived functors $\underrightarrow{\operatorname{colim}}_j \mathsf{F}$ vanish when $j > 0$. Because the category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ satisfies [AB4] these left derived functors are computed as the cohomology of the complex obtained from realizing the nerve of the category I; there is a standard cochain complex $\mathcal{F}$ of the form

$$\cdots \longrightarrow \mathcal{F}^{-3} \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^{0} \longrightarrow 0$$

whose $(-j)^{\text{th}}$ cohomology is $\underrightarrow{\operatorname{colim}}_j \mathsf{F}$, and where $\mathcal{F}^{-n}$ is the coproduct over sequences of composable morphisms in I

$$i_0 \longrightarrow i_1 \longrightarrow \cdots \longrightarrow i_{n-1} \longrightarrow i_n$$

of $\mathsf{F}(i_0)$. And the differentials of this complex are standard.

Now pick any object $a \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, and apply the functor $\operatorname{Hom}(a,-)$ to $\mathcal{F}$. Lemma 3.1.3(iii) tells us that $\operatorname{Hom}(a,-)$ respects coproducts, and therefore it takes the cochain complex $\mathcal{F}$ to the standard cochain complex computing $\underrightarrow{\operatorname{colim}}_j \operatorname{Hom}\big(a, \mathsf{F}(-)\big)$, for the functor $\operatorname{Hom}\big(a, \mathsf{F}(-)\big)\colon I \longrightarrow \mathcal{A}b$. As the category $\mathcal{A}b$ satisfies [AB5], this filtered colimit has vanishing $\underrightarrow{\operatorname{colim}}_j$ for $j > 0$. Therefore the only nonvanishing cohomology of the complex $\operatorname{Hom}(a, \mathcal{F})$ is in degree 0.

Lemma 3.2.1 tells us that the subcategory $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ generates $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$. Hence any nonzero cohomology of the complex $\mathcal{F}$ would be detected by $\operatorname{Hom}(a,-)$ for some $a \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Thus it certainly follows that the cohomology of $\mathcal{F}$ is concentrated in degree 0, and the category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ satisfies [AB5]. □

The two lemmas above clearly prove the first part of the statement of Theorem 3.0.1: $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ is a Grothendieck abelian category with a set of generators given by the objects of $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$.

Lemma 3.2.3. *Every object in $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ is finitely presented in the abelian category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$*

One comment before proving the lemma above is in order here. Let I be a filtered category, let $F\colon I \longrightarrow \mathcal{T}_{\mathcal{A}}^{\heartsuit}$ be a functor, and let $a \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ be an object. To establish Lemma 3.2.3 we need to prove two things.

(i) Every morphism $a \longrightarrow \underrightarrow{\operatorname{colim}} F$ factors through $F(i)$ for some $i \in I$.
(ii) If $a \longrightarrow F(i)$ is a morphism such that the composite $a \longrightarrow F(i) \longrightarrow \underrightarrow{\operatorname{colim}} F$ vanishes, then there exists in I a morphism $i \longrightarrow j$ with $a \longrightarrow F(i) \longrightarrow F(j)$ already vanishing.

Using an argument similar to that in Positselski and Šťovíček [26, proof of Lemma 9.2(ii)] it is possible to unify the two parts. We leave this to the interested reader, observing that in the proof given below the two halves are similar.

Proof. Let I be a small filtered category and let $F\colon I \longrightarrow \mathcal{T}_{\mathcal{A}}^{\heartsuit}$ be a functor. With the complex $\mathcal{F}$ as in the proof of Lemma 3.2.2, meaning it is the standard complex

$$\cdots \longrightarrow \mathcal{F}^{-3} \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^{0} \longrightarrow 0$$

computing $\varinjlim_j F$, the fact that the category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ is Grothendieck gives the exactness in the category $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ of the sequence

$$\mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^{0} \longrightarrow \varinjlim F \longrightarrow 0.$$

Now let c be an object of $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ and suppose we are given a morphism $c \longrightarrow \varinjlim F$. Applying Lemma 3.1.5 to the diagram

$$\begin{array}{ccccccc} & & & & c & & \\ & & & & \downarrow & & \\ \mathcal{F}^{-1} & \longrightarrow & \mathcal{F}^{0} & \longrightarrow & \varinjlim F & \longrightarrow & 0 \end{array}$$

permits us to construct in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ a commutative diagram with exact rows

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}^{-1} & \longrightarrow & \mathcal{F}^{0} & \longrightarrow & \varinjlim F & \longrightarrow & 0 \end{array}$$

with $a, b \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. The maps $a \longrightarrow \mathcal{F}^{-1}$ and $b \longrightarrow \mathcal{F}^{0}$ are morphisms from $a, b \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ to coproducts of objects in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$, and Lemma 3.1.3(iii) tells us that both maps must factor through finite subcoproducts. There is a finite subcategory $I' \subseteq I$ such that the maps from a, b only see coproducts over I', and as I is filtered we may assume that the category I' has a terminal object t. Letting G be the composite functor $I' \hookrightarrow I \xrightarrow{F} \mathcal{T}_{\mathcal{A}}^{\heartsuit}$, we deduce a factorization

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{G}^{-1} & \longrightarrow & \mathcal{G}^{0} & \longrightarrow & \varinjlim G & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}^{-1} & \longrightarrow & \mathcal{F}^{0} & \longrightarrow & \varinjlim F & \longrightarrow & 0 \end{array}$$

where $\mathcal{G}$ is the standard complex computing $\varinjlim_j G$. But $\varinjlim G = G(t) = F(t)$, and we have factored $c \longrightarrow \varinjlim F$ as $c \longrightarrow F(t) \longrightarrow \varinjlim F$.

Next suppose we are given an object $c \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, an object $s \in I$, and a morphism $c \longrightarrow F(s)$ such that the composite $c \longrightarrow F(s) \longrightarrow \varinjlim F$ vanishes. As $F(s)$ is a direct summand of $\mathcal{F}^0$, the vanishing of the composite $c \longrightarrow \mathcal{F}^0 \longrightarrow \varinjlim F$ allows us to factor the map $c \longrightarrow \mathcal{F}^0$ as

$c \longrightarrow \mathcal{K} \longrightarrow \mathcal{F}^0$, where $\mathcal{K} = \ker(\mathcal{F}^0 \longrightarrow \underrightarrow{\operatorname{colim}}\, F)$. Combining this with the exact sequence

$$\mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \underrightarrow{\operatorname{colim}}\, F \longrightarrow 0$$

produces for us a diagram

$$\begin{array}{ccccccc} & & & & c & & \\ & & & & \downarrow & & \\ \mathcal{F}^{-2} & \longrightarrow & \mathcal{F}^{-1} & \longrightarrow & \mathcal{K} & \longrightarrow & 0 \end{array}$$

to which we can apply Lemma 3.1.5. Thus we construct a commutative diagram with exact rows

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}^{-2} & \longrightarrow & \mathcal{F}^{-1} & \longrightarrow & \mathcal{K} & \longrightarrow & 0 \end{array}$$

with $a, b \in \mathcal{T}^{\heartsuit}_{\mathcal{A},c}$. The maps $a \longrightarrow \mathcal{F}^{-2}$ and $b \longrightarrow \mathcal{F}^{-1}$ are morphisms from $a, b \in \mathcal{T}^{\heartsuit}_{\mathcal{A},c}$ to coproducts of objects in $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$, and Lemma 3.1.3(iii) tells us that both maps must factor through finite subcoproducts. There is a finite subcategory $I' \subseteq I$ such that the maps from a, b only see coproducts over I', and as I is filtered we may assume that the category I' has a terminal object t and that $s \in I'$. If G is the composite functor $I' \hookrightarrow I \xrightarrow{F} \mathcal{T}^{\heartsuit}_{\mathcal{A}}$, then we obtain a factorization through

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{G}^{-2} & \longrightarrow & \mathcal{G}^{-1} & \longrightarrow & \mathcal{K}' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}^{-2} & \longrightarrow & \mathcal{F}^{-1} & \longrightarrow & \mathcal{K} & \longrightarrow & 0 \end{array}$$

where

$$\mathcal{G}^{-2} \longrightarrow \mathcal{G}^{-1} \longrightarrow \mathcal{G}^0 \longrightarrow \underrightarrow{\operatorname{colim}}\, G \longrightarrow 0$$

is the standard cochain complex computing $\underrightarrow{\operatorname{colim}}_j$ for the functor G, and the sequence $0 \longrightarrow \mathcal{K}' \longrightarrow \mathcal{G}^0 \longrightarrow \underrightarrow{\operatorname{colim}}\, G \longrightarrow 0$ is exact. But $\underrightarrow{\operatorname{colim}}\, G = G(t) = F(t)$, and we deduce a commutative diagram with exact rows

$$\begin{array}{ccccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{G}^{-2} & \longrightarrow & \mathcal{G}^{-1} & \longrightarrow & \mathcal{G}^0 & \longrightarrow & G(t) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}^{-2} & \longrightarrow & \mathcal{F}^{-1} & \longrightarrow & \mathcal{F}^0 & \longrightarrow & \underrightarrow{\operatorname{colim}}\, F & \longrightarrow & 0. \end{array}$$

Thus we have found a morphism $s \longrightarrow t$ in I such that the composite $c \longrightarrow F(s) \longrightarrow F(t)$ vanishes.

This completes the proof that every object in $\mathcal{T}^{\heartsuit}_{\mathcal{A},c}$ is finitely presented in $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$. □

Next, we establish the following result.

Lemma 3.2.4. *Every object $Z \in \mathcal{T}_{\mathcal{A}}^{\heartsuit}$ is the filtered colimit of the objects in $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ mapping to it.*

Proof. Let I be the category whose objects are morphisms $f\colon c \longrightarrow Z$ with $c \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} c & \xrightarrow{\ f\ } & \\ {\scriptstyle\varphi}\big\downarrow & & Z \\ c'. & \xrightarrow{\ g\ } & \end{array}$$

Using Lemma 3.1.3 it is easy to show that the category I is filtered. Let $F\colon I \longrightarrow \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ be the functor taking the object $f\colon c \longrightarrow Z$ in I to the object $c \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, put $Y = \underrightarrow{\mathrm{colim}}\, F$ and let $\rho\colon Y \longrightarrow Z$ be the obvious map. Lemma 3.2.1 guarantees that ρ is an epimorphism. We assert that ρ is an isomorphism.

The proof is by contradiction. Let X be the kernel of ρ and assume X nonzero. Lemma 3.2.1 establishes that there must exist a nonzero map $a \longrightarrow X$ with $a \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Since $X \longrightarrow Y$ is a monomorphism the composite $a \longrightarrow X \longrightarrow Y = \underrightarrow{\mathrm{colim}}\, F$ is nonzero, but Lemma 3.2.3 tells us that $\mathrm{Hom}(a,-)$ commutes with filtered colimits. Therefore the map $a \longrightarrow \underrightarrow{\mathrm{colim}}\, F$ must factor as $a \longrightarrow F(i) \longrightarrow \underrightarrow{\mathrm{colim}}\, F$ for some $i \in I$. If $i \in I$ is the object $f\colon b \longrightarrow Z$, then we deduce in $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ a morphism $a \longrightarrow b$, and the composite $a \longrightarrow b \longrightarrow Z$ must vanish since it is equal to the vanishing composite $a \longrightarrow X \longrightarrow Y \longrightarrow Z$. If we complete $a \longrightarrow b$ to the exact sequence $a \longrightarrow b \stackrel{\varphi}{\longrightarrow} c \longrightarrow 0$, then Lemma 3.1.3(ii) tells us that $c \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ and by the above the map $f\colon b \longrightarrow Z$ must factor through $\varphi\colon b \longrightarrow c$. We deduce in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$ a commutative triangle

$$\begin{array}{ccc} b & \xrightarrow{\ f\ } & \\ {\scriptstyle\varphi}\big\downarrow & & Z \\ c & \xrightarrow{\ g\ } & \end{array}$$

This can be viewed as a morphism in I from $\{f\colon b \longrightarrow Z\}$ to $\{g\colon c \longrightarrow Z\}$, whose image under the functor F is $\varphi\colon b \longrightarrow c$. Thus we have produced in I a morphism $i \longrightarrow j$ such that the composite $a \longrightarrow F(i) \longrightarrow F(j)$ vanishes, proving the vanishing of the composite $a \longrightarrow X \longrightarrow Y$. This completes our contradiction. □

The proof of Theorem 3.0.1 is complete if we show that the objects of $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ are the only finitely presented objects.

Assume therefore that P is a finitely presented object in $\mathcal{T}_{\mathcal{A}}^{\heartsuit}$. By Lemma 3.2.4 we can find a filtered category I, a functor $F\colon I \longrightarrow \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, and an isomorphism $P \cong \underrightarrow{\mathrm{colim}}\, F$.

But P is finitely presented, and hence the identity map $P \stackrel{\mathrm{id}}{\longrightarrow} P = \underrightarrow{\mathrm{colim}}\, F$ must factor through some $F(i) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Thus P is a direct summand of an object $F(i) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, and Corollary 3.1.4 establishes that $P \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$.

4. Weakly approximable triangulated categories and their subcategories

In this section we introduce weakly approximable triangulated categories and some relevant full triangulated subcategories. This is carried out in Section 4.1. In the rest of the paper, we will need a general discussion about products and coproducts in weakly approximable triangulated categories. This is the content of Section 4.2.

4.1. **Definitions, examples and the relevant subcategories.** We recall the main object of study in this paper, first introduced in [18, Definition 0.21].

Definition 4.1.1. *A triangulated category with coproducts* $\mathcal{T}$ *is* weakly approximable *if there exist an integer* $A > 0$*, a compact generator* $G \in \mathcal{T}^c$ *and a t–structure* $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ *on* $\mathcal{T}$*, such that*

(i) $G \in \mathcal{T}^{\leqslant A}$ *and* $\mathrm{Hom}(G, \mathcal{T}^{\leqslant -A}) = 0$.

(ii) *For any object* $F \in \mathcal{T}^{\leqslant 0}$ *there exists in* $\mathcal{T}$ *a distinguished triangle* $E \longrightarrow F \longrightarrow D \longrightarrow \Sigma E$ *with* $E \in \overline{\langle G\rangle}^{[-A,A]}$ *and with* $D \in \mathcal{T}^{\leqslant -1}$*, where* $\overline{\langle G\rangle}^{[-A,A]}$ *is as in Notation 1.1.2 (iii).*

Axiom (i) tells us that the compact generator G must be bounded above (i.e. $G \in \mathcal{T}^-$) for the given t–structure, while axiom (ii) provides the reason for the name 'weakly approximable triangulated category': given a bounded above object in $\mathcal{T}$, its top nontrivial cohomology can be approximated by objects generated by G and its shifts in a prescribed interval.

Remark 4.1.2. Let $\mathsf{F}\colon \mathcal{T} \longrightarrow \mathcal{T}'$ be an exact equivalence such that $\mathcal{T}$ is weakly approximable with integer A, generator G and t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$. Then it is very easy to see from the definition that $\mathcal{T}'$ is weakly approximable with integer $A' := A$, generator $G' := \mathsf{F}(G)$ and t–structure $\Big((\mathcal{T}')^{\leqslant 0}, (\mathcal{T}')^{\geqslant 1}\Big)$ with $(\mathcal{T}')^{\leqslant 0} := \mathsf{F}(\mathcal{T}^{\leqslant 0})$ and $(\mathcal{T}')^{\geqslant 1} := \mathsf{F}(\mathcal{T}^{\geqslant 1})$.

Example 4.1.3. The reader should keep in mind the following examples of weakly approximable triangulated categories. In particular, those in (i) and (ii) will be discussed in Section 10.

(i) Let R be a ring. Then it is not difficult to prove that $\mathbf{D}(R\text{–Mod})$ is weakly approximable (see [18, Example 3.1] for a reference).

(ii) It is a deeper result that, if X is a quasi-compact and quasi-separated scheme and $Z \subseteq X$ is a closed subset such that $X \setminus Z$ is quasi-compact, then $\mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable. This can be found in [15, Theorem 3.2(iv)].

(iii) If $\mathcal{T}$ is the homotopy category of spectra, then [18, Example 3.2] shows that $\mathcal{T}$ is weakly approximable. Actually more is known: this category, as well as any of the ones in (i), is not only weakly approximable but *approximable.* We will not deal with the stronger version of approximability in this paper.

Remark 4.1.4. A weakly approximable triangulated category $\mathcal{T}$ comes with two t–structures. The first $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ is the one forming part of Definition 4.1.1. The second is obtained from the given compact generator G as follows: let $\mathcal{A} = G(-\infty, 0]$ as in Notation 1.1.2(i), that is take the minimal $\mathcal{A}$ with $G \in \mathcal{A}$ and such that $\Sigma\mathcal{A} \subseteq \mathcal{A}$. And then form the t–structure $\big(\mathcal{T}_G^{\leqslant 0}, \mathcal{T}_G^{\geqslant 1}\big)$ with $\mathcal{T}_G^{\leqslant 0} = \mathrm{Coprod}(\mathcal{A})$, using Theorem 2.3.3. And [18, Proposition 2.4] shows that the given t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ is in the same equivalence class as $\big(\mathcal{T}_G^{\leqslant 0}, \mathcal{T}_G^{\geqslant 1}\big)$, meaning that there exists a positive integer $N > 0$ such that $\mathcal{T}^{\leqslant -N} \subseteq \mathcal{T}_G^{\leqslant 0} \subseteq \mathcal{T}^{\leqslant N}$.

Note: in [18, Remark 0.15] it is shown that, if G and H are both compact generators for a triangulated category $\mathcal{T}$ with coproducts, then the t–structures $\big(\mathcal{T}_G^{\leqslant 0}, \mathcal{T}_G^{\geqslant 1}\big)$ and $\big(\mathcal{T}_H^{\leqslant 0}, \mathcal{T}_H^{\geqslant 1}\big)$ are equivalent. Thus for any triangulated category $\mathcal{T}$, with a single compact generator, we obtain a *preferred equivalence class of t–structures*—it is the equivalence class that contains $\big(\mathcal{T}_G^{\leqslant 0}, \mathcal{T}_G^{\geqslant 1}\big)$ for every compact generator G. See [18, Definition 0.14]. And in the case of a weakly approximable triangulated category $\mathcal{T}$, the t–structure given in Definition 4.1.1, for which Definition 4.1.1 (i) and (ii) both hold, must belong to the preferred equivalence class.

Remark 4.1.5. Let $\mathcal{T}$ be a weakly approximable triangulated category, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure in the preferred equivalence class. Then we can define the following full subcategories of $\mathcal{T}$, which are clearly independent of the choice of t–structure in the preferred equivalence class.

- *Bounded above objects*: $\mathcal{T}^- := \bigcup_{m=1}^{\infty} \mathcal{T}^{\leqslant m}$;
- *Bounded below objects*: $\mathcal{T}^+ := \bigcup_{m=1}^{\infty} \mathcal{T}^{\geqslant -m}$;
- *Bounded objects*: $\mathcal{T}^b := \mathcal{T}^- \cap \mathcal{T}^+$,
- *Compact objects*: $\mathcal{T}^c$, as defined in Definition 2.3.1 and Remark 2.3.2;
- *Pseudo-compact objects*: $\mathcal{T}_c^-$, where an object $F \in \mathcal{T}$ belongs to $\mathcal{T}_c^-$ if, for any integer $m > 0$, there exists in $\mathcal{T}$ a distinguished triangle $E \longrightarrow F \longrightarrow D$ with $E \in \mathcal{T}^c$ and with $D \in \mathcal{T}^{\leqslant -m}$. Thus
$$\mathcal{T}_c^- = \bigcap_{m=1}^{\infty} \big(\mathcal{T}^c * \mathcal{T}^{\leqslant -m}\big);$$
- *Bounded pseudo-compact objects*: $\mathcal{T}_c^b := \mathcal{T}_c^- \cap \mathcal{T}^b$.
- *Bounded compact objects*: $\mathcal{T}^{c,b} := \mathcal{T}^c \cap \mathcal{T}^b = \mathcal{T}^c \cap \mathcal{T}_c^b$.

The very definitions, listed above, tell us how to obtain out of $\mathcal{T}$ all of the subcatergories on the list. In other words: the part of Theorem B that says there are recipes for passing from $\mathcal{T}$ to its various subcategories is by the very nature of their definition, as above.

The following easy result clarifies the relation between $\mathcal{T}^-$, $\mathcal{T}^c$ and $\mathcal{T}_c^-$.

Lemma 4.1.6. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then $\mathcal{T}^c \subseteq \mathcal{T}_c^- \subseteq \mathcal{T}^-$.*

Proof. By definition of $\mathcal{T}_c^-$ it is clearly enough to show $\mathcal{T}^c \subseteq \mathcal{T}^-$. But this inclusion holds because $\mathcal{T}^c$ is classically generated by a compact generator G of $\mathcal{T}$ and $G \in \mathcal{T}_G^{\leqslant 0} \subseteq \mathcal{T}_G^- = \mathcal{T}^-$. □

In general, it is not always true that compact objects are bounded below as well, as illustrated by the example below. On the other hand, it is clear that $\mathcal{T}^{c,b} = \mathcal{T}^c \subseteq \mathcal{T}_c^b$ whenever $\mathcal{T}^c \subseteq \mathcal{T}^+$.

Example 4.1.7. If $\mathcal{T}$ is either $\mathbf{D}(R\text{–Mod})$ as in Example 4.1.3(i) or $\mathbf{D}_{\mathbf{qc},Z}(X)$ as in Example 4.1.3(ii), we have $\mathcal{T}^c \subseteq \mathcal{T}_c^b$. Indeed, $\mathbf{D}(R\text{–Mod})^c = \mathbf{D}^{\mathrm{perf}}(R)$ and $\mathbf{D}_{\mathbf{qc},Z}(X)^c = \mathbf{D}_Z^{\mathrm{perf}}(X)$, and in both cases perfect complexes are (cohomologically) bounded.

But the inclusion $\mathcal{T}^c \subseteq \mathcal{T}_c^b$ does not hold when $\mathcal{T}$ is the homotopy category of spectra as in Example 4.1.3(iii). In this case it is known that $\mathcal{T}^c$ is the homotopy category of finite spectra, and it is not difficult to compute the subcategory $\mathcal{T}_c^b$. It turns out that in this example we have $\mathcal{T}_c^b \neq \{0\} \neq \mathcal{T}^c$ but $\mathcal{T}^{c,b} = \{0\}$. There are also many examples of algebraic triangulated categories $\mathcal{T}$ such that $\mathcal{T}^{c,b}$ is a proper full subcategory of both $\mathcal{T}_c^b$ and $\mathcal{T}^c$.

Lemma 4.1.8. *If $\mathcal{T}$ is a weakly approximable triangulated category, then all the subcategories listed in Remark 4.1.5 are thick.*

Proof. By [18, Observation 0.12], the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are thick. It is straightforward to prove that $\mathcal{T}^c$ is thick using the definition. By [18, Proposition 2.10], $\mathcal{T}_c^-$ is thick. It then follows that $\mathcal{T}_c^b$ and $\mathcal{T}^{c,b}$ are thick as well. □

The fact that each of these subcategories is thick will play a key role in passing between them.

Finally now, once all the terms have been explained, the reader might wish to glance back at the table in the Introduction, which gives what the intrisic subcategories of Remark 4.1.5 turn out to be when $\mathcal{T} = \mathbf{D}(R\text{–Mod})$ and when $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$.

4.2. **Products and coproducts in weakly approximable triangulated categories.** This section collects some nice but technical properties of products and coproducts in weakly approximable triangulated categories which will be used in the rest of the paper. This is the first instance where axiom (ii) in Definition 4.1.1 plays a crucial role.

Lemma 4.2.1. *Let $\mathcal{T}$ be a compactly generated triangulated category, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be any t–structure. If $\{t_\lambda \mid \lambda \in \Lambda\}$ is any set of objects in $\mathcal{T}^{\geqslant 0}$, then the product $\prod_{\lambda\in\Lambda} t_\lambda$ belongs to $\mathcal{T}^{\geqslant 0}$.*

Now assume that $\mathcal{T}$ is a weakly approximable triangulated category, and that the t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ belongs to the preferred equivalence class. There exists an integer $B > 0$ such that, if $\{t_\lambda \mid \lambda \in \Lambda\}$ is any set of objects in $\mathcal{T}^{\leqslant 0}$, then the product $\prod_{\lambda\in\Lambda} t_\lambda$ belongs to $\mathcal{T}^{\leqslant B}$.

Proof. First observe that products exists in $\mathcal{T}$ (see [14, Remark 5.1.2]). Now, for any t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ we have that $\mathcal{T}^{\geqslant 0}$ is equal to $\big(\mathcal{T}^{\leqslant -1}\big)^{\perp}$, and is therefore closed under whatever products exist in $\mathcal{T}$ by Corollary 2.1.8. Thus the first part of the statement follows.

Assume that the category $\mathcal{T}$ is weakly approximable, and that the t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ belongs to the preferred equivalence class. Let G be a compact generator for $\mathcal{T}$. By [18, Proposition 2.6] we may choose an integer $A > 0$ such that $\mathrm{Hom}(\Sigma^{-A}G, \mathcal{T}^{\leqslant 0}) = 0$, and therefore $\mathrm{Hom}(\Sigma^{-\ell}G, -)$ is zero on $\mathcal{T}^{\leqslant 0}$, for every $\ell \geqslant A$. In other words,

$$\mathcal{T}^{\leqslant 0} \subseteq G[A, \infty)^{\perp}$$

where $G[A, \infty)$ is as in Notation 1.1.2. On the other hand, [6, Lemma 3.9(iv)], which uses (ii) of Definition 4.1.1, allows us to choose an integer $B > A$ such that

$$G[A, \infty)^{\perp} \subseteq \mathcal{T}^{\leqslant B}.$$

Given any collection of objects $\{t_\lambda \mid \lambda \in \Lambda\}$, all belonging to $\mathcal{T}^{\leqslant 0}$, they belong to the bigger $G[A, \infty)^{\perp}$, which is closed under products. Hence $\prod_{\lambda\in\Lambda} t_\lambda$ belongs to $G[A, \infty)^{\perp}$, and therefore to the bigger $\mathcal{T}^{\leqslant B}$. ☐

Lemma 4.2.2. *Let $\mathcal{T}$ be a weakly approximable triangulated category, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure on $\mathcal{T}$ in the preferred equivalence class. Suppose $\{t_m \mid 1 \leqslant m < \infty\}$ is a countable sequence of objects in $\mathcal{T}$ such that, for any integer $n > 0$, all but finitely many t_m lie in $\mathcal{T}^{\geqslant n} \cup \mathcal{T}^{\leqslant -n}$. Then the natural map*

$$\coprod_{m=1}^{\infty} t_m \longrightarrow \prod_{m=1}^{\infty} t_m$$

is an isomorphism.

Proof. Let $G \in \mathcal{T}$ be a compact generator, and let $\ell \in \mathbb{Z}$ be any integer. As $\Sigma^{\ell}G$ is a compact generator for $\mathcal{T}$ and $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ is a t–structure in the preferred equivalence class, [18, Proposition 2.6] allows us to find an integer $A > 0$ such that $\Sigma^{\ell}G \in \mathcal{T}^{\leqslant A}$ and $\mathrm{Hom}(\Sigma^{\ell}G, -)$ is zero on $\mathcal{T}^{\leqslant -A}$. Combining these, we have that $\mathrm{Hom}(\Sigma^{\ell}G, -)$ is trivial both on $\mathcal{T}^{\leqslant -A-1}$ and on $\mathcal{T}^{\geqslant A+1}$. And, by assumption, this means that $\mathrm{Hom}(\Sigma^\ell G,-)$ is zero on all but finitely many of the t_m. So we may choose an integer $N>0$ such that $\mathrm{Hom}(\Sigma^\ell G,t_m)=0$ for all $m>N$. But then $\mathrm{Hom}(\Sigma^\ell G,-)$ is trivial both on $\coprod_{m=N+1}^\infty t_m$ and on $\prod_{m=N+1}^\infty t_m$. And since the map

$$\coprod_{m=1}^{N} t_m \longrightarrow \prod_{m=1}^{N} t_m$$

is an isomorphism, we discover that $\mathrm{Hom}(\Sigma^\ell G,-)$ must take the map

$$\coprod_{m=1}^{\infty} t_m \xrightarrow{\ \varphi\ } \prod_{m=1}^{\infty} t_m$$

to an isomorphism. Since G is a compact generator and the above is true for every ℓ, the map φ must be an isomorphism. □

Lemma 4.2.3. *Let $\mathcal{T}$ be a weakly approximable triangulated category, and let $\big(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 1}\big)$ be a t–structure in the preferred equivalence class. Suppose $\{t_m\mid 1\leqslant m<\infty\}$ is a countable sequence of objects in $\mathcal{T}_c^-$. Assume further that, for any integer $n>0$, all but finitely many t_m lie in $\mathcal{T}^{\leqslant -n}$. Then the isomorphic objects*

$$Q=\coprod_{m=1}^{\infty} t_m \cong \prod_{m=1}^{\infty} t_m$$

lie in $\mathcal{T}_c^-$.

Proof. The fact that the natural map in an isomorphism, from the coproduct to the product, is by Lemma 4.2.2. What needs proof is that one of the objects belongs to $\mathcal{T}_c^-$.

Pick any integer $n>0$, and then choose an integer $N>0$ such that, for all $m\geqslant N$, the object t_m belongs to $\mathcal{T}^{\leqslant -n}$. The finite coproduct $\overline{Q}_n=\coprod_{m=1}^{N-1} t_m$ belongs to $\mathcal{T}_c^-$, and hence there exists a distinguished triangle $\overline{E}_n\longrightarrow\overline{Q}_n\longrightarrow\overline{D}_n$ with $\overline{E}_n\in\mathcal{T}^c$ and with $\overline{D}_n\in\mathcal{T}^{\leqslant -n}$.

Since $\mathcal{T}^{\leqslant -n}$ contains t_m for all $m\geqslant N$ and is closed under coproducts, we have that $Q_n=\coprod_{m=N}^\infty t_m$ must belong to $\mathcal{T}^{\leqslant -n}$ But now form the direct sum of the two distinguished triangles

$$\begin{array}{ccccccc}\overline{E}_n & \longrightarrow & \overline{Q}_n & \longrightarrow & \overline{D}_n & \longrightarrow & \Sigma\overline{E}_n\\ 0 & \longrightarrow & Q_n & \longrightarrow & Q_n & \longrightarrow & 0\end{array}$$

to obtain the distinguished triangle $\overline{E}_n\longrightarrow\overline{Q}_n\oplus Q_n\longrightarrow\overline{D}_n\oplus Q_n\longrightarrow\Sigma\overline{E}_n$. This does the trick; we have that $\overline{E}_n\in\mathcal{T}^c$ and $\overline{D}_n\oplus Q\in\mathcal{T}^{\leqslant -n}$, hence $Q=\overline{Q}_n\oplus Q_n\in\mathcal{T}^c*\mathcal{T}^{\leqslant -n}$. And since $n>0$ is arbitrary this proves that $Q\in\mathcal{T}_c^-$. □

Lemma 4.2.4. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then the inclusion functors*

$$\mathcal{T}^-\hookrightarrow\mathcal{T},\qquad \mathcal{T}^b\hookrightarrow\mathcal{T},\qquad \mathcal{T}^+\hookrightarrow\mathcal{T}$$

respect both products and coproducts. More precisely, if $\{t_\lambda\mid\lambda\in\Lambda\}$ is a set of objects in one of the three subcategories $\mathcal{T}^b$, $\mathcal{T}^-$ or $\mathcal{T}^+$, and if either the coproduct or the product exists in the subcategory, then both the coproduct and the product exist in the subcategory and agree (respectively) with the coproduct and product in $\mathcal{T}$.

Proof. We will treat the inclusion $\mathcal{T}^b \hookrightarrow \mathcal{T}$, leaving to the reader the other two completely similar cases. Recall that if $\mathcal{T}$ is a weakly approximable triangulated category, then, in the preferred equivalence class of t–structures, we may choose a t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ such that $\mathcal{T}^{\geqslant 0}$ is closed under coproducts (by Theorem 2.3.3). Furthermore, any t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$, with $\mathcal{T}^{\geqslant 0}$ closed under coproducts, is such that the truncation functors $(-)^{\leqslant 0}$ and $(-)^{\geqslant 0}$ both respect coproducts. Thus, let us choose and fix a t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ in the preferred equivalence class, such that $\mathcal{T}^{\geqslant 0}$ is closed under coproducts.

Suppose $\{t_\lambda \mid \lambda \in \Lambda\}$ is a set of objects in $\mathcal{T}^b$, and suppose $P \in \mathcal{T}^b$ is either the coproduct or the product of the t_λ in the category $\mathcal{T}^b$. Then there must exist an integer $n > 0$ with $P \in \mathcal{T}^{\leqslant n} \cap \mathcal{T}^{\geqslant -n}$. But every t_λ is a direct summand of P, and hence every t_λ must belong to $\mathcal{T}^{\leqslant n} \cap \mathcal{T}^{\geqslant -n}$.

Because both $\mathcal{T}^{\leqslant n}$ and $\mathcal{T}^{\geqslant -n}$ are closed under coproducts in $\mathcal{T}$, the coproduct $t_1 = \coprod_{\lambda\in\Lambda} t_\lambda$ in $\mathcal{T}$ must belong to $\mathcal{T}^{\leqslant n} \cap \mathcal{T}^{\geqslant -n} \subseteq \mathcal{T}^b$. Now $\mathcal{T}^{\geqslant -n} = \big(\mathcal{T}^{< -n}\big)^{\perp}$ is closed under products in $\mathcal{T}$, while Lemma 4.2.1 produced an integer $B > 0$ such that the product in $\mathcal{T}$ of objects in $\mathcal{T}^{\leqslant n}$ must lie in $\mathcal{T}^{\leqslant n+B}$. This gives us that the product $t_2 = \prod_{\lambda\in\Lambda} t_\lambda$ in $\mathcal{T}$ must belong to $\mathcal{T}^{\leqslant n+B} \cap \mathcal{T}^{\geqslant -n} \subseteq \mathcal{T}^b$. □

5. The subcategories of $\mathcal{T}_c^-$ and of $\mathcal{T}^c$

In this section we give the recipes promised by the solid arrows of Theorem B, in the case of the subcategories of $\mathcal{T}_c^-$ and of $\mathcal{T}^c$. We give these recipes in Section 5.2. As a prelude we include the short Section 5.1, which clarifies the properties of $\mathcal{T}_c^-$ as a set of generators for $\mathcal{T}$.

5.1. **Preliminaries to understanding the subcategory $\mathcal{T}_c^- \subseteq \mathcal{T}$.** The results of this section are not essential to the article, but do help form a better understanding of the subcategory $\mathcal{T}_c^- \subseteq \mathcal{T}$.

We begin with a technical observation.

Lemma 5.1.1. *Let $\mathcal{T}$ be a triangulated category with countable coproducts, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure on $\mathcal{T}$. Suppose $F \in \mathcal{T}$ is an object, and assume we are given a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of objects of $\mathcal{T}$, with compatible maps $E_m \longrightarrow F$, and such that*

(i) *Any choice of a factorization through the homotopy colimit gives an isomorphism*

$$F \cong \underrightarrow{\text{Hocolim}}\, E_m \ .$$

(ii) *If we complete $E_m \longrightarrow F$ to a distinguished triangle $E_m \longrightarrow F \longrightarrow D_m$, then $D_m \in \mathcal{T}^{\leqslant -m}$.*

Then, in the distinguished triangle

$$\coprod_{m=1}^{\infty} E_m \xrightarrow{\ \text{id}-\textit{shift}\ } \coprod_{m=1}^{\infty} E_m \longrightarrow F \xrightarrow{\ \delta\ } \coprod_{m=1}^{\infty} \Sigma E_m$$

defining the homotopy colimit, we have that the morphism δ must factor as

$$F \longrightarrow \coprod_{m=1}^{\infty} \Sigma\big(E_m^{\leqslant -m+1}\big) \longrightarrow \coprod_{m=1}^{\infty} \Sigma E_m.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} F \xrightarrow{\ \delta\ } \coprod_{m=1}^{\infty} \Sigma E_m & \xrightarrow{\ \mathrm{id}-\mathit{shift}\ } & \coprod_{m=1}^{\infty} \Sigma E_m & & \\ \downarrow & & \downarrow & & \\ \coprod_{m=1}^{\infty} \Sigma(E_m^{\geqslant -m+2}) & \xrightarrow{\ \alpha\ } & \coprod_{m=1}^{\infty} \Sigma(E_m^{\geqslant -m+3}) & \xrightarrow{\ \beta\ } & \coprod_{m=2}^{\infty} \Sigma(E_m^{\geqslant -m+3}), \end{array}$$

where the horizontal maps α and β are the obvious ones. This means that β is the projection to the direct summand, where we discard the term corresponding to $m = 1$. And α is the map whose components $E_m^{\geqslant -m+2} \longrightarrow E_m^{\geqslant -m+3}$ and $E_m^{\geqslant -m+2} \longrightarrow E_{m+1}^{\geqslant -m+2}$ are the t–structure truncations of the obvious maps. As $(\mathrm{id} - \mathit{shift}) \circ \delta = 0$ (being the composite of two morphisms in a distinguished triangle), the composite from top left to bottom right in the diagram must vanish.

Now observe that the map $\beta \circ \alpha$ is an isomorphism. Indeed, we first observe that our hypotheses (i) and (ii) in the statement guarantee that the map $E_m^{\geqslant -m+2} \longrightarrow E_{m+1}^{\geqslant -m+2}$ must be an isomorphism, as both map isomorphically to $F^{\geqslant -m+2}$. Call the coproduct over m of these isomorphisms $-\varphi$. Then the map $\beta \circ \alpha$ can be written as $\varphi \circ (\mathrm{id} - \tau)$, where τ is some map we do not care about beyond the fact the infinite sum $(\mathrm{id} + \tau + \tau^2 + \cdots)$ exists, as the sum is finite on each summand of the first coproduct. Therefore $\beta \circ \alpha = \varphi \circ (\mathrm{id} - \tau)$ is an isomorphism as we claimed.

It then follows that the composite

$$F \xrightarrow{\ \delta\ } \coprod_{m=1}^{\infty} \Sigma E_m \longrightarrow \coprod_{m=1}^{\infty} \Sigma(E_m^{\geqslant -m+2})$$

must vanish. The conclusion of the lemma is then immediate. ☐

Lemma 5.1.2. *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure on $\mathcal{T}$. Suppose $F \in \mathcal{T}$ is an object, and assume we are given a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of objects of $\mathcal{T}$ as in Lemma 5.1.1. Assume further that all the objects E_m are compact in $\mathcal{T}$ and that we are given in $\mathcal{T}$ a set of objects $\{X_\lambda \mid \lambda \in \Lambda\}$ and a morphism*

$$F \xrightarrow{\ f\ } \coprod_{\lambda \in \Lambda} X_\lambda$$

such that, for each integer m, the composite $E_m \longrightarrow F \xrightarrow{f} \coprod_{\lambda \in \Lambda} X_\lambda$ vanishes.

Then there is a countable subset $\Lambda' \subseteq \Lambda$ and a factorization of f as

$$F \longrightarrow \coprod_{\lambda \in \Lambda'} X_\lambda^{\leqslant -n_\lambda} \longrightarrow \coprod_{\lambda \in \Lambda'} X_\lambda \hookrightarrow \coprod_{\lambda \in \Lambda} X_\lambda$$

where the inclusion is of the countable subcoproduct, the maps induced by the truncation are the obvious, and for any integer $m > 0$ there exist only finitely many $\lambda \in \Lambda'$ with $n_\lambda < m$.

Proof. The vanishing of each composite $E_m \longrightarrow F \xrightarrow{f} \coprod_{\lambda\in\Lambda} X_\lambda$ yields a vanishing composite

$$\coprod_{m=1}^{\infty} E_m \longrightarrow F \xrightarrow{\ f\ } \coprod_{\lambda\in\Lambda} X_\lambda$$

which, in the notation of Lemma 5.1.1, means that the morphism f must factor as

$$F \xrightarrow{\ \delta\ } \coprod_{m=1}^{\infty} \Sigma E_m \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda.$$

Since each E_m is compact, for each m the map from E_m to the coproduct must factor through a finite subcoproduct. Taking the union over m gives a countable subset $\Lambda' \subseteq \Lambda$.

And the remainder of the current Lemma comes from combining the above with the conclusion of Lemma 5.1.1, which allows us to factor the above further as

$$F \longrightarrow \coprod_{m=1}^{\infty} \Sigma(E_m^{\leqslant -m+1}) \longrightarrow \coprod_{m=1}^{\infty} \Sigma E_m \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda.$$

This concludes the proof. □

If we assume more about the t–structure on $\mathcal{T}$, and about its relation with compact objects, then the factorization obtained in Lemma 5.1.2 can be made without restrictions on the map f.

Lemma 5.1.3. *Let F and the sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ mapping to it be as in Lemma 5.1.2. Assume further that*

(i) *The t–structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1})$ is such that $\mathcal{T}^{\geqslant 0}$ is closed under coproducts.*
(ii) *For every compact object $C \in \mathcal{T}$ there exists an integer $m > 0$ with* $\mathrm{Hom}(C, \mathcal{T}^{\leqslant -m}) = 0$.
(iii) *Given a countable collection of objects $\{Y_i \in \mathcal{T} \mid 1 \leqslant i < \infty\}$, such that for every integer $n > 0$ we have that all but finitely many of the Y_i belong to $\mathcal{T}^{\leqslant -n}$, the natural map*

$$\coprod_{i=1}^{\infty} Y_i \longrightarrow \prod_{i=1}^{\infty} Y_i$$

is an isomorphism.

Suppose next that we are given in $\mathcal{T}$ a set of objects $\{X_\lambda \mid \lambda \in \Lambda\}$ and a morphism

$$F \xrightarrow{\ f\ } \coprod_{\lambda\in\Lambda} X_\lambda.$$

Then there is a countable subset $\Lambda' \subseteq \Lambda$ and a factorization of f as

$$F \longrightarrow \coprod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} \longrightarrow \coprod_{\lambda\in\Lambda'} X_\lambda \hookrightarrow \coprod_{\lambda\in\Lambda} X_\lambda,$$

where the inclusion is of the countable subcoproduct, the maps induced by the truncation are the obvious, and for any integer $m > 0$ there exist only finitely many $\lambda \in \Lambda'$ with $n_\lambda < m$.

Proof. For every integer $m > 0$ we have a distinguished triangle $E_m \longrightarrow F \longrightarrow D_m$ with $E_m \in \mathcal{T}^c$ and with $D_m \in \mathcal{T}^{\leqslant -m}$. Now consider the composite

$$E_m \longrightarrow F \xrightarrow{\ f\ } \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda^{\geqslant -m+1}.$$

As E_m is compact this composite factors through a finite subcoproduct, which we denote $\Lambda_m \subseteq \Lambda$. Without loss of generality we may assume $\Lambda_m \subseteq \Lambda_{m+1}$. It follows that composite

$$E_m \longrightarrow F \xrightarrow{\ f\ } \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda^{\geqslant -m+1} \longrightarrow \coprod_{\lambda\in\Lambda\setminus\Lambda_m} X_\lambda^{\geqslant -m+1}$$

vanishes. But then the map from F to the term on the right must factor as

$$F \longrightarrow D_m \longrightarrow \coprod_{\lambda\in\Lambda\setminus\Lambda_m} X_\lambda^{\geqslant -m+1} .$$

By hypothesis (i) in the statement, the category $\mathcal{T}^{\geqslant -m+1}$ is closed under coproducts, hence the coproduct on the right is an object in $\mathcal{T}^{\geqslant -m+1}$. But D_m belongs to $\mathcal{T}^{\leqslant -m}$, and the map $D_m \longrightarrow \coprod_{\lambda\in\Lambda\setminus\Lambda_m} X_\lambda^{\geqslant -m+1}$ must vanish. Hence, for each $\lambda \in \Lambda \setminus \Lambda_m$, the map $F \longrightarrow X_\lambda$ can be factored as $F \longrightarrow X_\lambda^{\leqslant -m} \longrightarrow X_\lambda$.

Now for each of the finitely many elements $\lambda \in \Lambda_{m+1} \setminus \Lambda_m$ choose a factorization of the map $F \longrightarrow X_\lambda$ as $F \xrightarrow{\varphi_\lambda} X_\lambda^{\leqslant -m} \longrightarrow X_\lambda$. We declare that $\Lambda_0 = \emptyset$, and when λ is in the finite set $\Lambda_1 = \Lambda_1 \setminus \Lambda_0$ we set $\varphi_\lambda \colon F \longrightarrow X_\lambda$ to be untruncated. The morphisms φ_λ, where λ is in the union $\cup_{m=1}^\infty(\Lambda_m \setminus \Lambda_{m-1})$, combine to a single morphism to the product of these countably many trucations. And by hypothesis (iii) in the statement this product agrees with the coproduct. We have produced a countable subset $\Lambda' \subseteq \Lambda$ and a composite

$$F \longrightarrow \coprod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} \longrightarrow \coprod_{\lambda\in\Lambda'} X_\lambda \hookrightarrow \coprod_{\lambda\in\Lambda} X_\lambda.$$

Call this composite g. The construction tells us that $(f-g)^{\geqslant -n+1} = 0$ for every $n > 0$. Hence $f - g$ must factor through an object in $\mathcal{T}^{\leqslant -n}$ for every integer n.

Now each E_m is compact, and hypothesis (ii) above says that there exists an integer n such that $\mathrm{Hom}(E_m, \mathcal{T}^{\leqslant -n}) = 0$. Hence the composite $E_m \longrightarrow F \xrightarrow{(f-g)} \coprod_{\lambda\in\Lambda} X_\lambda$ vanishes. This puts us in the situation of Lemma 5.1.2, and the map $f - g$ must have a factorization of the required form. □

We are about to specialize to the case of weakly approximable triangulated categories. And for this, the next little lemma will help.

Lemma 5.1.4. *Let $\mathcal{T}$ be a weakly approximable triangulated category, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a compactly generated t–structure in the preferred equivalence class. Then any morphism $F \longrightarrow X$, with $F \in \mathcal{T}_c^-$ and $X \in \mathcal{T}^{\leqslant 0}$, factors as $F \longrightarrow F' \longrightarrow X$ with $F' \in \mathcal{T}_c^- \cap \mathcal{T}^{\leqslant 0}$.*

Proof. Because F belongs to $\mathcal{T}_c^-$, there exists a distinguished triangle $E \longrightarrow F \longrightarrow D$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leqslant 0}$. And since $F \in \mathcal{T}_c^-$ and $E \in \mathcal{T}^c \subseteq \mathcal{T}_c^-$, the triangle tells us that $D \in \mathcal{T}_c^- \cap \mathcal{T}^{\leqslant 0}$.

Now form the composite $E \longrightarrow F \longrightarrow X$. It is a morphism from the compact object E to $X \in \mathcal{T}^{\leqslant 0}$, for the compactly generated t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$. By Lemma 3.1.1 we may factor the map $E \longrightarrow X$ as $E \longrightarrow E' \longrightarrow X$ with $E' \in \mathcal{T}^c \cap \mathcal{T}^{\leqslant 0}$. This gives us a commutative square

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ E' & \longrightarrow & X. \end{array}$$

Now form the homotopy pushout of the maps $E' \longleftarrow E \longrightarrow F$. By [21, Lemma 1.4.4] we obtain a morphism of distinguished triangles where the square on the left is homotopy cartesian

$$\begin{array}{ccccccc} E & \longrightarrow & F & \longrightarrow & D & \longrightarrow & \Sigma E \\ \downarrow & & \downarrow & & \| & & \downarrow \\ E' & \longrightarrow & F' & \longrightarrow & D & \longrightarrow & \Sigma E'. \end{array}$$

The fact that the bottom row is a distinguished triangle, with $E' \in \mathcal{T}^c \cap \mathcal{T}^{\leqslant 0} \subseteq \mathcal{T}^-_c \cap \mathcal{T}^{\leqslant 0}$ and with $D \in \mathcal{T}^-_c \cap \mathcal{T}^{\leqslant 0}$, tells us that $F' \in \mathcal{T}^-_c \cap \mathcal{T}^{\leqslant 0}$. And since the square on the left is a homotopy pushout, we may fill in the dotted arrow in the diagram

$$\begin{array}{ccccc} E & \longrightarrow & F & & \\ \downarrow & & \downarrow & \searrow & \\ E' & \longrightarrow & F' & & \\ & \searrow & & \ddots & \downarrow \\ & & & & X \end{array}$$

to give the required factorization $F \longrightarrow F' \longrightarrow X$. □

Following [21, Definitions 3.3.1 and 4.1.1], recall that if $\mathcal{T}$ is a triangulated category with small coproducts, a set $\mathcal{S}$ of objects in $\mathcal{T}$ is *an* $\aleph_1$*–perfect class of* $\aleph_1$*–small objects* if $0 \in \mathcal{S}$ and, given an object $F \in \mathcal{S}$, an arbitrary set of objects $\{X_\lambda \mid \lambda \in \Lambda\}$ in $\mathcal{T}$, and any morphism

$$F \xrightarrow{\ f\ } \coprod_{\lambda \in \Lambda} X_\lambda \tag{4}$$

then there exists a factorization

$$F \longrightarrow \coprod_{\lambda \in \Lambda'} F_\lambda \xrightarrow{\coprod_{\lambda \in \Lambda'} f_\lambda} \coprod_{\lambda \in \Lambda'} X_\lambda \hookrightarrow \coprod_{\lambda \in \Lambda} X_\lambda \tag{5}$$

with $\Lambda' \subseteq \Lambda$ a countable set, where F_λ are all objects of $\mathcal{S}$ with maps $f_\lambda \colon F_\lambda \longrightarrow X_\lambda$.

We are now ready to prove the main result of this section which clarifies the role of $\mathcal{T}^-_c$ as a set of generators for $\mathcal{T}$.

Theorem 5.1.5. *Let $\mathcal{T}$ be a weakly approximable triangulated category, and let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure in the preferred equivalence class. Then the full subcategory $\mathcal{T}^-_c$ is an $\aleph_1$–perfect class of $\aleph_1$–small objects. Moreover, for every morphism as in (4), a factorization as in (5) can be chosen such that, for every integer $n > 0$, all but finitely many of the F_λ lie in $\mathcal{T}^{\leqslant -n}$.*

Proof. Since $\mathcal{T}$ is weakly approximable, given $F \in \mathcal{T}_c^-$, the hypotheses of Lemma 5.1.3 are all fulfilled. Indeed, the existence of a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ as in the assumptions of Lemma 5.1.3 follows from [18, Corollary 2.14]. Moreover, by Theorem 2.3.3, the given t–structure on $\mathcal{T}$ can be replaced by a compactly generated equivalent one for which $\mathcal{T}^{\geqslant 0}$ is closed under coproducts, achieving hypothesis (i) of Lemma 5.1.3. Hypothesis (ii) of the same lemma follows from [18, Lemma 2.8]. Finally hypothesis (iii) of Lemma 5.1.3 is the content of Lemma 4.2.2. Thus Lemma 5.1.3 can be used and allows us to factor f as

$$F \xrightarrow{\ \varphi\ } \coprod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} \longrightarrow \coprod_{\lambda\in\Lambda'} X_\lambda \hookrightarrow \coprod_{\lambda\in\Lambda} X_\lambda .$$

Now consider the composite below

$$F \xrightarrow{\ \varphi\ } \coprod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} \xrightarrow{\ \sim\ } \prod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} .$$

Hypothesis (iii) of Lemma 5.1.3 tells us that the map from the coproduct to the product is an isomorphism. Now for each $\lambda \in \Lambda'$, Lemma 5.1.4 permits us to factor the morphism $F \longrightarrow X_\lambda^{\leqslant -n_\lambda}$ as $F \longrightarrow F_\lambda \longrightarrow X_\lambda^{\leqslant -n_\lambda}$ with $F_\lambda \in \mathcal{T}_c^- \cap \mathcal{T}^{\leqslant -n_\lambda}$. Since the coproduct and product agree for the collection $\{F_\lambda \mid \lambda \in \Lambda'\}$, this permits us to (uniquely) fill in the dotted arrow in the diagram

$$\begin{array}{ccccc} & & \varphi & & \\ F & \dashrightarrow & \coprod_{\lambda\in\Lambda'} F_\lambda & \longrightarrow & \coprod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} \\ & \searrow & \downarrow \wr & & \downarrow \wr \\ & & \prod_{\lambda\in\Lambda'} F_\lambda & \longrightarrow & \prod_{\lambda\in\Lambda'} X_\lambda^{\leqslant -n_\lambda} . \end{array}$$

The top row of the diagram provides the factorization φ which delivers the result. ☐

5.2. **Intrinsic descriptions of the subcategories of $\mathcal{T}_c^-$ and $\mathcal{T}^c$.** We are ready to deal with the main task of this section: the intrinsic description of the subcategories of $\mathcal{T}_c^-$. Let us start with a preliminary result.

Lemma 5.2.1. *Let $\mathcal{T}$ be a weakly approximable triangulated category. An object $F \in \mathcal{T}_c^-$ belongs to the thick subcategory $\mathcal{T}^c \subseteq \mathcal{T}_c^-$ if and only if, for each object $Q \in \mathcal{T}_c^-$, there exists an integer $N = N(Q) > 0$ such that* $\mathrm{Hom}(F, \Sigma^n Q) = 0$ *for all $n > N$.*

Proof. Choose an integer $A > 0$, a compact generator $G \in \mathcal{T}$, and a t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ as in Definition 4.1.1. Recall that such a t–structure is in the preferred equivalence class (by Remark 4.1.4). If F is compact then $F \in \langle G\rangle^{[-B,B]}$, for some integer $B > 0$ (see, for example, [18, Remark 0.15 and Lemma 0.9(ii)]). Combining this with the condition $\mathrm{Hom}(G, \mathcal{T}^{\leqslant -A}) = 0$ of Definition 4.1.1(i), we have that $\mathrm{Hom}(F, \mathcal{T}^{\leqslant -A-B}) = 0$. Let $Q \in \mathcal{T}_c^-$ be arbitrary. As $Q \in \mathcal{T}_c^- \subseteq \mathcal{T}^-$, there must exist an integer $m > 0$ with $Q \in \mathcal{T}^{\leqslant m}$. Hence $\Sigma^n Q \in \mathcal{T}^{\leqslant -A-B}$ for all $n > A + B + m$, and so $\mathrm{Hom}(F, \Sigma^n Q) = 0$.

Conversely, suppose we are given an object $F \in \mathcal{T}_c^-$ such that, for each $Q \in \mathcal{T}_c^-$, there exists an integer $N = N(Q) > 0$ with $\mathrm{Hom}(F, \Sigma^n Q) = 0$ for all $n > N$. We need to prove that F is compact. Because F belongs to $\mathcal{T}_c^-$ we may choose and fix, for any integer $m > 0$, a distinguished triangle $E_m \longrightarrow F \longrightarrow D_m$ with $E_m \in \mathcal{T}^c$ and with $D_m \in \mathcal{T}^{\leqslant -m}$. Since in this triangle E_m and F both belong to $\mathcal{T}_c^-$, also $D_m \in \mathcal{T}_c^-$. Next form, in the category $\mathcal{T}$, the object

$$Q = \coprod_{m=1}^{\infty} \Sigma^{-m} D_{2m}.$$

By the above we have that $\Sigma^{-m} D_{2m} \in \mathcal{T}^{\leqslant -m} \cap \mathcal{T}_c^-$, and Lemma 4.2.3 allows us to conclude that Q belongs to $\mathcal{T}_c^-$. By the hypothesis on F there must exist an integer $M > 0$ with $\mathrm{Hom}(F, \Sigma^n Q) = 0$ for all $n \geqslant M$. In particular $\mathrm{Hom}(F, \Sigma^M Q) = 0$. As D_{2M} is a direct summand of $\Sigma^M Q = \coprod_{m=1}^{\infty} \Sigma^{M-m} D_{2m}$, we have that $\mathrm{Hom}(F, D_{2M}) = 0$. But then in the distinguished triangle $E_{2M} \longrightarrow F \longrightarrow D_{2M}$ the map $F \longrightarrow D_{2M}$ must vanish, making F a direct summand of the compact object E_{2M}. Therefore F must be compact. □

We are now ready to prove the main result of this section.

Proposition 5.2.2. *Lat $\mathcal{T}$ be a weakly approximable triangulated category.*

(i) *The full subcategory $\mathcal{T}^c \subseteq \mathcal{T}_c^-$ has for objects all those $F \in \mathcal{T}_c^-$ such that, for any object $Q \in \mathcal{T}_c^-$, there exists an integer $N > 0$ with* $\mathrm{Hom}(F, \Sigma^n Q) = 0$*, for all $n > N$.*

(ii) *For any classical generator $G \in \mathcal{T}^c$, the full subcategory $\mathcal{T}_c^b \subseteq \mathcal{T}_c^-$ is given by the formula*

$$\mathcal{T}_c^b = \bigcup_{n=1}^{\infty} \big(G(-\infty, -n]\big)^{\perp}$$

where the perpendicular is taken in $\mathcal{T}_c^-$.

(iii) *For any classical generator $G \in \mathcal{T}^c$, the full subcategory $\mathcal{T}^{c,b} \subseteq \mathcal{T}^c$ is given by the same formula*

$$\mathcal{T}^{c,b} = \bigcup_{n=1}^{\infty} \big(G(-\infty, -n]\big)^{\perp}$$

where this time the perpendicular is taken in $\mathcal{T}^c$.

Proof. The characterization in (i) of $\mathcal{T}^c \subseteq \mathcal{T}_c^-$ was proved in Lemma 5.2.1. Thus we only need to prove (ii) and (iii).

Recall that a classical generator $G \in \mathcal{S} = \mathcal{T}^c$ is the same as a compact generator for $\mathcal{T}$. If $\mathcal{A} = G(-\infty, 0]$, then in the category $\mathcal{T}$ we have the equalities

$$\mathcal{A}^{\perp} = \mathrm{Coprod}(\mathcal{A})^{\perp} = \mathcal{T}_G^{\geqslant 1} \ ,$$

where the first equality is by Lemma 1.2.1(ii) and the second is by Theorem 2.3.3. Thus, still in the category $\mathcal{T}$, we have

$$\bigcup_{n=1}^{\infty} \big(G(-\infty, -n]\big)^{\perp} \quad = \quad \bigcup_{n=1}^{\infty} \mathcal{T}_G^{\geqslant -n+1} \quad = \quad \mathcal{T}^+ \ ,$$

and intersecting with $\mathcal{T}_c^-$ gives (ii) while intersecting with $\mathcal{T}^c$ yields (iii). □

6. The subcategories of $\mathcal{T}^-$, and a recipe for $\mathcal{T}^b$ as a subcategory of $\mathcal{T}^+$

This section is about the inclusions in the diagram (1) of the subcategories of $\mathcal{T}^-$, and about describing $\mathcal{T}^b$ intrinsically as a subcategory of $\mathcal{T}^+$. This will be elaborated in Section 6.2. But first, in Section 6.1, we discuss the intrinsic t–structure in $\mathcal{T}^?$, for $? = -, +, b$.

6.1. **The preferred equivalence class of t–structure on $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.** For the rest of this section, we assume $\mathcal{T}$ to be a triangulated category with small coproducts and a single compact generator G. We know that $\mathcal{T}$ is endowed with a preferred equivalence class of t–structures. And, by the definition of the "preferred" $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$, any t–structure on $\mathcal{T}$, in the preferred equivalence class, restricts to a t–structure on each of the three preferred subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$. Furthermore, the restrictions of any two equivalent t–structures are equivalent. This defines for us preferred equivalence classes of t–structures on each of full triangulated subcategories mentioned above. But this definition is in terms of the embedding into $\mathcal{T}$.

The aim of this section will be to show that, the preferred equivalence class of t–structures, on each of $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$, has an intrinsic description—by which we mean a description which doesn't mention the embedding into $\mathcal{T}$.

We begin with the following.

Definition 6.1.1. *Let $\mathcal{S}$ be a triangulated category, and consider the class $\mathcal{P}(\mathcal{S})$ of all full subcategories $P \subseteq \mathcal{T}$ satisfying $\Sigma P \subseteq P$.*

(i) *Two elements $P, Q \in \mathcal{P}(\mathcal{S})$ are* equivalent *if there exists an integer $A > 0$ with $\Sigma^A P \subseteq Q \subseteq \Sigma^{-A} P$.*
(ii) *Given two equivalence classes $[P]$ and $[Q]$ of elements of $\mathcal{P}(\mathcal{S})$, we set $[P] \leqslant [Q]$ if, for a choice of representatives $P \in [P]$ and $Q \in [Q]$, there exists an integer $A > 0$ with $\Sigma^A P \subseteq Q$.*

In the sequel, we will use the shorthand $\mathcal{P} := \mathcal{P}(\mathcal{S})$.

Remark 6.1.2. If we start with two t–structure $\left(\mathcal{S}_1^{\leqslant 0}, \mathcal{S}_1^{\geqslant 1}\right)$ and $\left(\mathcal{S}_2^{\leqslant 0}, \mathcal{S}_2^{\geqslant 1}\right)$ on a triangulated category $\mathcal{S}$ and we set $P := \mathcal{S}_1^{\leqslant 0}$ and $Q := \mathcal{S}_2^{\leqslant 0}$, then P and Q are equivalent as in Definition 6.1.1 if and only if the two t–structures are equivalent in the sense discussed in Remark 4.1.4. That is, there is a positive integer $N > 0$ such that $\mathcal{S}_1^{\leqslant -N} \subseteq \mathcal{S}_2^{\leqslant 0} \subseteq \mathcal{S}_1^{\leqslant N}$. The main point of Definition 6.1.1 is that we extend this equivalence relation beyond t–structures and, more importantly, we introduce the partial order of Definition 6.1.1(ii).

Let us consider the bounded above case.

Proposition 6.1.3. *Let $\mathcal{T}$ be a triangulated category with coproducts and a single compact generator, and put $\mathcal{S} = \mathcal{T}^-$, where $\mathcal{T}^-$ is understood to mean with respect to the preferred equivalence class of t–structures on $\mathcal{T}$. Then a t–structure $\left(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\right)$ belongs to the preferred equivalence class if and only if the following two conditions below are both satisfied:*

(i) $\mathcal{S} = \bigcup_{m=1}^{\infty} \mathcal{S}^{\leqslant m}$.
(ii) *The equivalence class $\left[\mathcal{S}^{\leqslant 0}\right]$ is the unique minimal one with respect to the partial ordering $\leqslant$ for aisles of t–structures satisfying (i).*

Proof. If $(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 1})$ is a t–structure on $\mathcal{T}$ in the preferred equivalence class, then by definition

$$\mathcal{S}=\mathcal{T}^{-}=\cup_{m=1}^{\infty}\mathcal{T}^{\leqslant m}.$$

The restriction of $(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 1})$ to a t–structure $(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1})$ on $\mathcal{S}$ is defined by the formulas $\mathcal{S}^{\leqslant 0}=\mathcal{S}\cap\mathcal{T}^{\leqslant 0}$ and $\mathcal{S}^{\geqslant 0}=\mathcal{S}\cap\mathcal{T}^{\geqslant 0}$, which in the case of $\mathcal{S}=\mathcal{T}^{-}$ yields $\mathcal{S}^{\leqslant 0}=\mathcal{T}^{\leqslant 0}$ and therefore $\mathcal{S}^{\leqslant m}=\mathcal{T}^{\leqslant m}$. Thus $\mathcal{S}=\cup_{m=1}^{\infty}\mathcal{S}^{\leqslant m}$ holds, and the restricted t–structure on $\mathcal{S}$ satisfies (i) in the statement.

What needs proof is that any t–structure $(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1})$ on $\mathcal{S}$, which satisfies (i), must have that $[\mathcal{S}^{\leqslant 0}]$ is bigger than $[\mathcal{T}^{\leqslant 0}]$ in the ordering $\leqslant$ of Definition 6.1.1(ii). For this purpose, pick a compact generator $G\in\mathcal{T}$. As $G\in\mathcal{T}_G^{\leqslant 0}\subseteq\mathcal{T}^{-}=\mathcal{S}$, and since $\mathcal{S}=\bigcup_{m=1}^{\infty}\mathcal{S}^{\leqslant m}$, we must have that G belongs to $\mathcal{S}^{\leqslant m}$ for some integer $m>0$. But then $G(-\infty,0]\subseteq\mathcal{S}^{\leqslant m}$, and as the category $\mathcal{S}^{\leqslant m}$ is the aisle of a t–structure it is closed under extensions and coproducts which exist in $\mathcal{T}^{-}$, giving

$$\mathcal{S}^{\leqslant m}\supseteq\operatorname{Coprod}\big(G(-\infty,0]\big)=\mathcal{T}_G^{\leqslant 0}.$$

This proves the inequality $[\mathcal{T}_G^{\leqslant 0}]\leqslant[\mathcal{S}^{\leqslant 0}]$ and, as $(\mathcal{T}_G^{\leqslant 0},\mathcal{T}_G^{\geqslant 1})$ is a representative of the preferred equivalence class of t–structures, the proof is complete. □

Let $\mathcal{T}$ be a triangulated category with coproducts and a single compact generator G. Then the functor $\operatorname{Hom}_{\mathcal{T}}(G,-)$ is a homological functor $\operatorname{Hom}_{\mathcal{T}}(G,-)\colon\mathcal{T}\longrightarrow\mathcal{A}b$ respecting coproducts. Because $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathcal{A}b$, the functor $\mathsf{H}\colon\mathcal{T}^{\mathrm{op}}\longrightarrow\mathcal{A}b$ given by the formula

$$\mathsf{H}(-):=\operatorname{Hom}_{\mathcal{A}b}\big(\operatorname{Hom}_{\mathcal{T}}(G,-)\,,\,\mathbb{Q}/\mathbb{Z}\big)$$

is homological and takes coproducts in $\mathcal{T}$ to products in $\mathcal{A}b$. Hence, by [20, Theorem 3.1], the functor H is representable, meaning that there exists an object $\operatorname{bc}(G)\in\mathcal{T}$ and a natural isomorphism $\mathsf{H}(-)\cong\operatorname{Hom}_{\mathcal{T}}\big(-,\operatorname{bc}(G)\big)$. The reason for the notation is that, in honor of the paper [4] where the construction was first used, the object $\operatorname{bc}(G)$ is called the *Brown-Comenetz dual* of G.

Lemma 6.1.4. *Let $\mathcal{T}$ be a triangulated category with coproducts and a single compact generator G, and assume that $\operatorname{Hom}(G,\Sigma^n G)=0$ for $n\gg 0$. Then the object $\operatorname{bc}(G)$ belongs to $\mathcal{T}^{+}$, and $\operatorname{Hom}\big(X,\operatorname{bc}(G)\big)=0$ if and only if $\operatorname{Hom}(G,X)=0$.*

Proof. For any $X\in\mathcal{T}$ we have that $\operatorname{Hom}\big(X,\operatorname{bc}(G)\big)$ is canonically isomorphic to

$$\operatorname{Hom}_{\mathcal{A}b}\big(\operatorname{Hom}_{\mathcal{T}}(G,X)\,,\,\mathbb{Q}/\mathbb{Z}\big).$$

As $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of the category of abelian groups, this vanishes if and only if $\operatorname{Hom}_{\mathcal{T}}(G,X)=0$. This proves the second assertion of the lemma.

To prove the first one, we apply the second assertion to $X:=\Sigma^n G$. By hypothesis there exists an integer $B>0$ such that $\operatorname{Hom}(G,\Sigma^n G)$ vanishes for all $n>B$. Hence $\operatorname{Hom}\big(\Sigma^n G,\operatorname{bc}(G)\big)$ vanishes for all $n>B$, that is ${}^{\perp}\operatorname{bc}(G)$ contains $G(-\infty,-B-1]$. But then

$${}^{\perp}\operatorname{bc}(G)\supseteq\operatorname{Coprod}\big(G(-\infty,-B-1]\big)=\mathcal{T}_G^{\leqslant -B-1}.$$

From this we deduce that $\operatorname{bc}(G)$ must lie in $\mathcal{T}_G^{\geqslant -B}\subseteq\mathcal{T}^{+}$. □

Notation 6.1.5. Let $\mathcal{S}$ be any triangulated category, and let $\mathcal{P} = \mathcal{P}(\mathcal{S})$ be as in Definition 6.1.1. For any object $I \in \mathcal{S}$ we can form $I[0,\infty) \subseteq \mathcal{S}$ as in Notation 1.1.2(i), and then let $P(I) \in \mathcal{P}(\mathcal{S})$ be given by the formula $P(I) = {}^{\perp}I[0,\infty)$. Spelling this out, we obtain

$$P(I) := {}^{\perp}I[0,\infty) = \bigcap_{m=0}^{\infty} {}^{\perp}\big(\Sigma^{-m}I\big) \in \mathcal{P}.$$

We adopt this notation for the rest of this section.

Example 6.1.6. Assume $\mathcal{T}$ is a weakly approximable triangulated category and $G \in \mathcal{T}$ is a compact generator. Let us work out what $P(I)$ is, in the case $\mathcal{S} = \mathcal{T}^+$ and $I = \mathrm{bc}(G)$. The fact that I belongs to $\mathcal{S} = \mathcal{T}^+$ was proved in Lemma 6.1.4 (note that $\mathrm{Hom}(G, \Sigma^n G) = 0$ for $n \gg 0$ by (i) of Definition 4.1.1). On the other hand $\mathrm{Hom}(X, \Sigma^{-m}I) = 0$ if and only if $\mathrm{Hom}(\Sigma^m X, I) = 0$. By Lemma 6.1.4, the latter equality is equivalent to $\mathrm{Hom}(G, \Sigma^m X) = 0$. Thus with $I = \mathrm{bc}(G)$ we get the explicit description $P(I) = G[0,\infty)^{\perp}$. By [6, Lemma 3.9(iv)] we deduce that there exists an integer $A > 0$ with $P(I) \subseteq \mathcal{S} \cap \mathcal{T}^{\leqslant A} = \mathcal{S}^{\leqslant A}$.

This explicit description is useful in treating the bounded below case.

Proposition 6.1.7. *Let $\mathcal{T}$ be a weakly approximable triangulated category, and put $\mathcal{S} = \mathcal{T}^+$. Then, with $P(I)$ as in Notation 6.1.5, the collection of equivalence classes in $\mathcal{P}(\mathcal{S})$ of the form $\big[P(I)\big]$ has a unique minimal member, in the partial order of Definition 6.1.1. Moreover, a t–structure $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ belongs to the preferred equivalence class if and only if $\big[\mathcal{S}^{\leqslant 0}\big]$ is in the equivalence class of the minimal $\big[P(I)\big]$.*

Proof. Let $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ be the intersection with $\mathcal{S}$ of a t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ in the preferred equivalence class. We need to show that $\big[\mathcal{S}^{\leqslant 0}\big] \leqslant \big[P(J)\big]$ for every $J \in \mathcal{S}$, and that there exists an $I \in \mathcal{S}$ with $\big[P(I)\big] \leqslant \big[\mathcal{S}^{\leqslant 0}\big]$.

For any J belonging to $\mathcal{S}$ we may assume, after shifting, that $J \in \mathcal{S}^{\geqslant 1}$. Therefore $J[0,\infty) \subseteq \mathcal{S}^{\geqslant 1}$, and $P(J) = {}^{\perp}J[0,\infty) \supseteq \mathcal{S}^{\leqslant 0}$. Hence, in the partial order of Definition 6.1.1, we have

$$\big[\mathcal{S}^{\leqslant 0}\big] \leqslant \big[P(J)\big]$$

for every $J \in \mathcal{S}$.

It remains to observe that, if G is a compact generator for $\mathcal{T}$, then $I := \mathrm{bc}(G) \in \mathcal{S}$ and

$$\big[P(I)\big] \leqslant \big[\mathcal{S}^{\leqslant 0}\big] \ .$$

by the discussion in Example 6.1.6. ☐

We are now ready to treat the bounded case.

Proposition 6.1.8. *Suppose $\mathcal{T}$ is a weakly approximable triangulated category and put $\mathcal{S} = \mathcal{T}^b$. Then, with $P(I)$ as in Notation 6.1.5, the collection of equivalence classes in $\mathcal{P}(\mathcal{S})$ of the form $\big[P(I)\big]$ has a unique minimal member, in the partial order of Definition 6.1.1. Moreover, a t–structure $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ belongs to the preferred equivalence class if and only if $\big[\mathcal{S}^{\leqslant 0}\big]$ is in the equivalence class of the minimal $\big[P(I)\big]$.*

Proof. Let $\big(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1}\big)$ be the intersection with $\mathcal{S}$ of a t–structure $\big(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 1}\big)$ in the preferred equivalence class. As in the previous proof, for any object J belonging to $\mathcal{S}$ we may assume, after shifting, that $J\in\mathcal{S}^{\geqslant 1}$. Therefore $J[0,\infty)\subseteq\mathcal{S}^{\geqslant 1}$, hence $\mathcal{S}^{\leqslant 0}\subseteq {}^\perp J[0,\infty)=P(J)$, and this gives

$$\big[\mathcal{S}^{\leqslant 0}\big]\leqslant\big[P(J)\big]$$

for every J. Again, to conclude the proof, we need to exhibit an object $I\in\mathcal{S}=\mathcal{T}^b$ with

$$\big[P(I)\big]\leqslant\big[\mathcal{S}^{\leqslant 0}\big]\ .$$

Here we take

$$I:=\mathrm{bc}(G)^{\leqslant 0}\oplus\prod_{\ell=1}^{\infty}\Sigma^\ell\Big(\mathrm{bc}(G)^{\leqslant\ell}\Big)^{\geqslant\ell}\ .$$

First we need to show that $I\in\mathcal{T}^b$. As $\mathrm{bc}(G)$ belongs to $\mathcal{T}^+$ it follows that $\mathrm{bc}(G)^{\leqslant 0}$ must belong to $\mathcal{T}^b$. And as all the other terms, in the product defining I, belong to $\mathcal{T}^\heartsuit=\mathcal{T}^{\leqslant 0}\cap\mathcal{T}^{\geqslant 0}$, Lemma 4.2.1 guarantees that the product is in $\mathcal{S}=\mathcal{T}^b$.

Finally we claim that, with $A>0$ as in Example 6.1.6, we have that $P(I)\subseteq\mathcal{S}^{\leqslant A}$. Indeed, suppose $X\in\mathcal{S}=\mathcal{T}^b$ lies outside $\mathcal{S}^{\leqslant A}=\mathcal{T}^b\cap\mathcal{T}^{\leqslant A}$. Then it also lies outside $\mathcal{T}^+\cap\mathcal{T}^{\leqslant A}$, and by setting $J=\mathrm{bc}(G)$ and applying Example 6.1.6 we deduce that X cannot belong to $P(J)$. Therefore there must exist in $\mathcal{T}^+$ a nonzero map $X\longrightarrow\Sigma^{-m}J$ for some $m\geqslant 0$. Since X belongs to $\mathcal{T}^b$, this map must factor as $X\longrightarrow\Sigma^{-m}\big(J^{\leqslant\ell}\big)\longrightarrow\Sigma^{-m}J$, for some $\ell\geqslant 0$. As the composite is nonzero, the map $X\longrightarrow\Sigma^{-m}\big(J^{\leqslant\ell}\big)$ cannot be trivial. Let us choose $\ell\geqslant 0$ to be the smallest ℓ for which such a factorization exists.

If $\ell=0$ this gives a nonzero map $X\longrightarrow\Sigma^{-m}J^{\leqslant 0}$. But $J^{\leqslant 0}$ is a direct summand of I, and hence we have a nonzero map $X\longrightarrow\Sigma^{-m}I$.

If $\ell>0$ consider the composite $X\longrightarrow\Sigma^{-m}\big(J^{\leqslant\ell}\big)\longrightarrow\Sigma^{-m}\Big(\big(J^{\leqslant\ell}\big)^{\geqslant\ell}\Big)$. This composite cannot vanish by the minimality of ℓ. Hence, in this case, we get a nonzero map $X\longrightarrow\Sigma^{-m}\Big(\big(J^{\leqslant\ell}\big)^{\geqslant\ell}\Big)$. As $\Sigma^\ell\Big(\big(J^{\leqslant\ell}\big)^{\geqslant\ell}\Big)$ is a direct summand of I, this provides a nonzero map $X\longrightarrow\Sigma^{-m-\ell}I$.

Either way, X does not belong to $P(I)$. Since X was an arbitrary object outside $\mathcal{S}^{\leqslant A}$, we deduce that $P(I)\subseteq\mathcal{S}^{\leqslant A}$. $\square$

6.2. **The intrinsic inclusions.** We are now ready to give recipes for the subcategories of $\mathcal{T}^-$. Let us start with the case of compact objects.

Lemma 6.2.1. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then $(\mathcal{T}^-)^c=\mathcal{T}^c$.*

Proof. Since $\mathcal{T}^c$ is contained in $\mathcal{T}^-$ and since, by Lemma 4.2.4, the inclusion $\mathcal{T}^-\hookrightarrow\mathcal{T}$ preserves coproducts, we have the inclusion $\mathcal{T}^c\subseteq(\mathcal{T}^-)^c$. We need to prove the reverse inclusion.

For this choose a t–structure $\big(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 1}\big)$ in the preferred equivalence class, such that $\mathcal{T}^{\geqslant 0}$ is closed under coproducts and hence the truncation functor $(-)^{\leqslant 0}$ commutes with coproducts. This can be achieved due to the discussion in Section 2.3 (see, in particular, Remark 2.3.7).

Let $c\in(\mathcal{T}^-)^c$. Because c belongs to $\mathcal{T}^-$, we may (after shifting) assume $c\in\mathcal{T}^{\leqslant 0}$. And now let $\{t_\lambda\mid\lambda\in\Lambda\}$ be an arbitrary collection of objects in $\mathcal{T}$ and suppose we are given a map

$$c\xrightarrow{\ \psi\ }\coprod_{\lambda\in\Lambda}t_\lambda\ .$$

As c belongs to $\mathcal{T}^{\leqslant 0}$ the map ψ factors through $\left(\coprod_{\lambda\in\Lambda} t_\lambda\right)^{\leqslant 0}$, but as the functor $(-)^{\leqslant 0}$ commutes with coproducts we can write this factorization as

$$c \xrightarrow{\quad\varphi\quad} \coprod_{\lambda\in\Lambda} t_\lambda^{\leqslant 0} \longrightarrow \coprod_{\lambda\in\Lambda} t_\lambda$$

As $c \in (\mathcal{T}^-)^c$, the map φ must factor through a finite subcoproduct. But then the map ψ also factors through a finite subcoproduct. □

We conclude the task of this section, proving the following, additional parts of Theorem B.

Corollary 6.2.2. *If $\mathcal{T}$ is a weakly approximable triangulated category, then the inclusions in the diagram* (1) *of the subcategories $\mathcal{T}^b \subseteq \mathcal{T}^-$, $\mathcal{T}^b \subseteq \mathcal{T}^+$ and $\mathcal{T}_c^- \subseteq \mathcal{T}^-$ are intrinsic.*

Proof. Lemma 6.2.1 above provides a recipe for $\mathcal{T}^c$ as a subcategory of $\mathcal{T}^-$, while in Proposition 6.1.3 we gave a recipe for the preferred equivalence class of t–structures on $\mathcal{T}^-$, and in Proposition 6.1.7 we gave a recipe for the preferred equivalence class of t–structures on $\mathcal{T}^+$. The proof of the current corollary combines the information.

In terms of the preferred equivalence class of t–structures, we can describe $\mathcal{T}^b$ as a subcategory of $\mathcal{T}^-$ by the recipe

$$\mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+ = \bigcup_{m=1}^{\infty} (\mathcal{T}^-)^{\geqslant -m}$$

for any t–structure in the preferred equivalence class. The term on the far right has an intrinsic description in $\mathcal{T}^-$ by Proposition 6.1.3, hence $\mathcal{T}^b \subseteq \mathcal{T}^-$ is intrinsic. The formula

$$\mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+ = \bigcup_{m=1}^{\infty} (\mathcal{T}^+)^{\leqslant m}$$

gives a recipe for $\mathcal{T}^b$ as a subcategory of $\mathcal{T}^+$, and in view of Proposition 6.1.7 the term on the far right is intrinsic in $\mathcal{T}^+$.

The subcategory $\mathcal{T}_c^- \subseteq \mathcal{T}^-$ is given by the formula

$$\mathcal{T}_c^- = \bigcap_{m=1}^{\infty} \mathcal{T}^c * \mathcal{T}^{\leqslant -m} = \bigcap_{m=1}^{\infty} (\mathcal{T}^-)^c * (\mathcal{T}^-)^{\leqslant -m} \ ,$$

where the second equality is by Lemma 6.2.1, where $(\mathcal{T}^-)^c$ is obviously intrinsic in $\mathcal{T}^-$, and where $(\mathcal{T}^-)^{\leqslant -m}$ has (up to equivalence) an intrinsic description in $\mathcal{T}^-$ by Proposition 6.1.3. Hence the term on the far right gives an intrinsic recipe for $\mathcal{T}_c^- \subseteq \mathcal{T}^-$. □

7. Generalities about pseudocompact and strongly pseudocompact objects

Using the discussion in the previous sections, in this one we introduce and study two new classes of objects: the pseudocompact and the strongly pseudocompact ones. Later in the paper, we will specialize to the case of weakly approximable triangulated categories.

7.1. **Pseudocompact objects.** Let us first introduce a slight generalization of the notion of compact object. We always assume that $\mathcal{S}$ is a triangulated category and that P is an element of $\mathcal{P} := \mathcal{P}(\mathcal{S})$, with $\mathcal{P}(\mathcal{S})$ as in Definition 6.1.1.

Definition 7.1.1. (i) *An object $c \in \mathcal{S}$ is P–*pseudocompact *if for any set $\{s_\lambda \mid \lambda \in \Lambda\}$ of objects, which all belong to $P^\perp$ and whose coproduct exists in $\mathcal{S}$, any morphism*

$$c \longrightarrow \coprod_{\lambda\in\Lambda} s_\lambda$$

must factor through a finite subcoproduct.

(ii) *Given an equivalence class $[P]$ of elements in $\mathcal{P}$, an object is* $[P]$–pseudocompact *if it is Q–pseudocompact for every $Q \in [P]$.*

The full subcategory of all P–pseudocompact objects will be denoted $\mathcal{S}^{pc}_P$. And the full subcategory of all $[P]$–pseudocompact objects will be written $\mathcal{S}^{pc}_{[P]}$.

Example 7.1.2. There are some obvious examples of P-pseudocompact objects, for special Ps, and here we discuss a particularly simple one. Let $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ be a t–structure on $\mathcal{S}$ such that $\mathcal{S}^{\geqslant 0}$ is closed in $\mathcal{S}$ under coproducts. Then every object in $\mathcal{S}^{\leqslant 0}$ is $\mathcal{S}^{\leqslant 0}$–pseudocompact. To see this let $c \in (\mathcal{S}^{\leqslant 0})$ be any object. Assume $\{s_\lambda \mid \lambda \in \Lambda\}$ is a collection of objects, which all belong to $(\mathcal{S}^{\leqslant 0})^\perp = \mathcal{S}^{\geqslant 1}$, and whose coproduct exists in $\mathcal{S}$. By assumption $\mathcal{S}^{\geqslant 1}$ is closed in $\mathcal{S}$ under coproducts, hence a morphism $c \longrightarrow \coprod_{\lambda\in\Lambda} s_\lambda$ is a map from $c \in \mathcal{S}^{\leqslant 0}$ to an object in $\mathcal{S}^{\geqslant 1}$ and must vanish.

The first result is easy.

Lemma 7.1.3. *Let $\mathcal{S}$ be a triangulated category, and let $P \in \mathcal{P}$ be any element. Then the subcategory $\mathcal{S}^{pc}_P$ is closed under direct summands and extensions. To say it in symbols:*

$$\mathrm{smd}(\mathcal{S}^{pc}_P) = \mathcal{S}^{pc}_P \qquad \textit{and} \qquad \mathcal{S}^{pc}_P * \mathcal{S}^{pc}_P = \mathcal{S}^{pc}_P\ .$$

Proof. The assertion about direct summands is clear. To prove the equality $\mathcal{S}^{pc}_P * \mathcal{S}^{pc}_P = \mathcal{S}^{pc}_P$ let $A \longrightarrow B \longrightarrow C$ be a distinguished triangle in $\mathcal{S}$, with A and C both P–pseudocompact. Let $\{s_\lambda \mid \lambda \in \Lambda\}$ be a set of objects in $P^\perp \subseteq \mathcal{S}$, whose coproduct exists in $\mathcal{S}$. Choose any morphism

$$B \xrightarrow{\ \varphi\ } \coprod_{\lambda\in\Lambda} s_\lambda\ .$$

Because A is P–pseudocompact the composite

$$A \longrightarrow B \xrightarrow{\ \varphi\ } \coprod_{\lambda\in\Lambda} s_\lambda$$

must factor through a finite subcoproduct, which means that the natural projection to a subset $\Lambda' \subseteq \Lambda$ with a finite complement gives a vanishing composite

$$A \longrightarrow B \xrightarrow{\ \varphi\ } \coprod_{\lambda\in\Lambda} s_\lambda \longrightarrow \coprod_{\lambda\in\Lambda'} s_\lambda\ .$$

Hence there is a factorization

$$B \longrightarrow C \xrightarrow{\quad\psi\quad} \coprod_{\lambda\in\Lambda'} s_\lambda \ ,$$

and the P–pseudocompactness of C guarantees that ψ factors through a finite subcoproduct, meaning that by passing to a subset $\Lambda''\subseteq\Lambda'$ with finite complement we obtain the vanishing of

$$B \longrightarrow C \longrightarrow \coprod_{\lambda\in\Lambda'} s_\lambda \longrightarrow \coprod_{\lambda\in\Lambda''} s_\lambda \ .$$

Thus φ factors through the coproduct over $\Lambda\setminus\Lambda''$, which is a finite set. □

Corollary 7.1.4. *Let $\mathcal{S}$ be a triangulated category, and let $[P]\subseteq\mathcal{P}$ be an equivalence class of elements in $\mathcal{P}$. Then $\mathcal{S}^{pc}_{[P]}$ is a thick subcategory of $\mathcal{S}$.*

Proof. The closure under direct summands and extensions follows from Lemma 7.1.3. The closure under suspension is because we have an equality of equivalence classes $[P]=[\Sigma P]$. □

Example 7.1.5. Let $\mathcal{S}$ be a triangulated category, and let $\big(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1}\big)$ be a t–structure such that $\mathcal{S}^{\geqslant 0}$ is closed in $\mathcal{S}$ under coproducts. By Example 7.1.2 every object of $\mathcal{S}^{\leqslant 0}$ is $\mathcal{S}^{\leqslant 0}$–pseudocompact, and clearly every object in $\mathcal{S}^c$ is also $\mathcal{S}^{\leqslant 0}$–pseudocompact. By Lemma 7.1.3 every object of $\mathcal{S}^c*\mathcal{S}^{\leqslant 0}$ is $\mathcal{S}^{\leqslant 0}$–pseudocompact. It follows that every object in

$$\mathcal{S}^-_{c,[\mathcal{S}^{\leqslant 0}]} := \bigcap_{m=0}^{\infty} \mathcal{S}^c * \mathcal{S}^{\leqslant -m}$$

is $\mathcal{S}^{\leqslant -m}$–pseudocompact for every $m>0$, that is $\mathcal{S}^-_{c,[\mathcal{S}^{\leqslant 0}]}\subseteq\mathcal{S}^{pc}_{[\mathcal{S}^{\leqslant 0}]}$.

Now suppose that $\mathcal{T}$ is a weakly approximable triangulated category, and $\big(\mathcal{T}^{\leqslant 0},\mathcal{T}^{\geqslant 1}\big)$ is a t–structure in the preferred equivalence class. Then we get the easy equality $\mathcal{T}^-_{c,[\mathcal{T}^{\leqslant 0}]}=\mathcal{T}^-_c$. Hence we get the inclusion

$$\mathcal{T}^-_c\subseteq\mathcal{T}^{pc}_{[\mathcal{T}^{\leqslant 0}]} \ .$$

We will return to this in Section 8.1.

Lemma 7.1.6. *Let $\mathcal{S}$ be a triangulated category, let $\big(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1}\big)$ be a t–structure on $\mathcal{S}$, and assume $\mathcal{S}^{\geqslant 0}$ is closed in $\mathcal{S}$ under coproducts. If C is a $\mathcal{S}^{\leqslant 0}$–pseudocompact object in $\mathcal{S}$ then so is $C^{\geqslant 2}$.*

Proof. We have a distinguished triangle $C^{\leqslant 1}\longrightarrow C\longrightarrow C^{\geqslant 2}\longrightarrow\Sigma C^{\leqslant 1}$, with C in $\mathcal{S}^{pc}_{\mathcal{S}^{\leqslant 0}}$ by hypothesis. Now $\Sigma C^{\leqslant 1}\in\mathcal{S}^{\leqslant 0}\subseteq\mathcal{S}^{pc}_{\mathcal{S}^{\leqslant 0}}$, where the inclusion is by Example 7.1.2, and the result now follows from Lemma 7.1.3. □

7.2. **Strongly pseudocompact objects.** In this section we introduce the slightly more restrictive notion of strongly pseudocompact objects. The authors do not know whether all pseudocompact objects are strongly pseudocompact as well.

Let us start from the definition and assume, as in the previous section, that $\mathcal{S}$ is a triangulated category and that $\mathcal{P}=\mathcal{P}(\mathcal{S})$ is as in Definition 6.1.1.

Definition 7.2.1. *Given $P\in\mathcal{P}$, an object $c\in\mathcal{S}$ is* $[P]$–strongly pseudocompact *if the following two conditions are satisfied:*

(i) *c is $[P]$–pseudocompact.*
(ii) *Assume we are given any t–structure $\big(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1}\big)$ on the category $\mathcal{S}$, such that $\big[\mathcal{S}^{\leqslant 0}\big]=[P]$. Suppose that $\mathcal{S}^{\geqslant 0}$ is closed in $\mathcal{S}$ under coproducts and the heart $\mathcal{S}^{\heartsuit}$ satisfies [AB5]. Then the functor* $\mathrm{Hom}(c,-)$ *commutes with filtered colimits in $\mathcal{S}^{\heartsuit}$.*

The collection of all $[P]$–strongly pseudocompact objects will be written $\mathcal{S}^{spc}_{[P]}$.

Let us discuss a slightly technical construction that will be used in a short while. Fix a t–structure $\big(\mathcal{S}^{\leqslant 0},\mathcal{S}^{\geqslant 1}\big)$ on $\mathcal{S}$ and assume that we are given in $\mathcal{S}^{\heartsuit}$ a complex

$$\mathcal{F}^{\bullet}:=\{\ \cdots\longrightarrow\mathcal{F}^{-4}\longrightarrow\mathcal{F}^{-3}\longrightarrow\mathcal{F}^{-2}\longrightarrow\mathcal{F}^{-1}\longrightarrow\mathcal{F}^{0}\ \}\tag{6}$$

which is exact.

Example 7.2.2. We have already encountered an example of such a complex. Indeed, let $\mathcal{A}$ be an abelian category satisfying [AB4] and let I be a filtered category and let $\mathsf{F}\colon I\longrightarrow\mathcal{A}$ be a functor. Let $\mathcal{F}^{\bullet}$ be the standard cochain complex for computing $\underrightarrow{\mathrm{colim}}_j\, F$, which we introduced in the proof of Lemma 3.2.2. Let us briefly recall that $\mathcal{F}^{\bullet}$ has the form

$$\cdots\longrightarrow\mathcal{F}^{-3}\longrightarrow\mathcal{F}^{-2}\longrightarrow\mathcal{F}^{-1}\longrightarrow\mathcal{F}^{0}\longrightarrow 0\ ,$$

the $(-j)^{\text{th}}$ cohomology is $\underrightarrow{\mathrm{colim}}_j\,\mathsf{F}$, and $\mathcal{F}^{-n}$ is the coproduct over sequences of composable morphisms in I

$$i_0\longrightarrow i_1\longrightarrow\cdots\longrightarrow i_{n-1}\longrightarrow i_n$$

of $\mathsf{F}(i_0)$. If $\mathcal{A}$ satisfies [AB5], then this cochain complex is a resolution of $\underrightarrow{\mathrm{colim}}\,\mathsf{F}$.

Now let $\mathcal{K}^n$ be the cokernel of the map $\mathcal{F}^{n-1}\longrightarrow\mathcal{F}^n$. Thus $\mathcal{K}^0$ is the cokernel of $\mathcal{F}^{-1}\longrightarrow\mathcal{F}^0$. The acyclicity of the complex tells that $\mathcal{K}^{-1}$ is the kernel of $\mathcal{F}^0\longrightarrow\mathcal{K}^0$ and, for all $n<-1$, the object $\mathcal{K}^n$ is the kernel of the map $\mathcal{F}^{n+1}\longrightarrow\mathcal{F}^{n+2}$. For $i\in\mathbb{N}_{>0}$, we inductively construct the objects $C_i\in\mathcal{S}$ with natural maps $C_i\longrightarrow C_{i+1}\longrightarrow\Sigma^{i+1}\mathcal{F}^{-i-1}$ as follows:

- Define C_1 and the morphisms $\mathcal{F}^0\longrightarrow C_1\longrightarrow\Sigma\mathcal{F}^{-1}$ by completing $\mathcal{F}^{-1}\longrightarrow\mathcal{F}^0$ to a distinguished triangle $\mathcal{F}^{-1}\longrightarrow\mathcal{F}^0\longrightarrow C_1\longrightarrow\Sigma\mathcal{F}^{-1}$.

By construction we have that $C_1^{\leqslant -1}=\Sigma\mathcal{K}^{-2}$ and $C_1^{\geqslant 0}=\mathcal{K}^0$. Now let us continue our inductive constructions as follows:

- Assume the object C_n has been defined, and satisfies $C_n^{\geqslant 0}=\mathcal{K}^0$ and $C_n^{\leqslant -1}=\Sigma^n\mathcal{K}^{-n-1}$. Then we take the composite

  $$\varphi\colon\Sigma^n\mathcal{F}^{-n-1}\longrightarrow\Sigma^n\mathcal{K}^{-n-1}=\!=\!=C_n^{\leqslant -1}\longrightarrow C_n$$

  And the morphisms $C_n\longrightarrow C_{n+1}\longrightarrow\Sigma^{n+1}\mathcal{F}^{-n-1}$ are formed by completing the above composite to a distinguished triangle

$$\Sigma^n\mathcal{F}^{-n-1}\xrightarrow{\ \varphi\ }C_n\longrightarrow C_{n+1}\longrightarrow\Sigma^{n+1}\mathcal{F}^{-n-1}\tag{7}$$

By completing the composable morphisms $\Sigma^n\mathcal{F}^{-n-1}\longrightarrow\Sigma^n\mathcal{K}^{-n-1}\longrightarrow C_n$ to an octahedron the reader can easily show that $C_{n+1}^{\geqslant 0}=\mathcal{K}^0$ and $C_{n+1}^{\leqslant -1}=\Sigma^{n+1}\mathcal{K}^{-n-2}$.

Lemma 7.2.3. *In the setting above, let $\mathcal{F}^\bullet$ be a complex as in* (6) *and let $X \in \mathcal{S}$ be such that, for all $i \in \mathbb{Z}$,* $\mathrm{Hom}(\Sigma^i X, -)$ *takes $\mathcal{F}^\bullet$ to an exact sequence. Then any morphism $X \longrightarrow C_n$ must factor through the morphism $\Sigma^n \mathcal{F}^{-n-1} \oplus \mathcal{F}^0 \longrightarrow C_n$, where the morphism $\mathcal{F}^0 \longrightarrow C_n$ is the composite*

$$\mathcal{F}^0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots C_{n-1} \longrightarrow C_n \ ,$$

while the morphism $\Sigma^n \mathcal{F}^{-n-1} \longrightarrow C_n$ is the composite

$$\Sigma^n \mathcal{F}^{-n-1} \longrightarrow \Sigma^n \mathcal{K}^{-n-1} = C_n^{\leqslant -1} \longrightarrow C_n \ .$$

Proof. We prove the result by induction on $n > 0$, starting with $n = 1$. Suppose we are given a morphism $f \colon X \longrightarrow C_1$. The distinguished triangle $C_1 \longrightarrow \Sigma\mathcal{F}^{-1} \longrightarrow \Sigma\mathcal{F}^0$ tells us that the composite $X \longrightarrow C_1 \longrightarrow \Sigma\mathcal{F}^{-1}$ gives a morphism $X \longrightarrow \Sigma\mathcal{F}^{-1}$ such that the composite $X \longrightarrow \Sigma\mathcal{F}^{-1} \longrightarrow \Sigma\mathcal{F}^0$ vanishes.

The assumption in the statement implies that this morphism factors as $X \longrightarrow \Sigma\mathcal{F}^{-2} \longrightarrow \Sigma\mathcal{F}^{-1}$. This gives us a morphism $X \longrightarrow \Sigma\mathcal{F}^{-2}$, and we may now form the composite

$$g \colon X \longrightarrow \Sigma\mathcal{F}^{-2} \longrightarrow \Sigma\mathcal{K}^{-2} = C_1^{\leqslant -1} \longrightarrow C_1.$$

The construction guarantees that $(f - g) \colon X \longrightarrow C_1$ is annihilated by the morphism $C_1 \longrightarrow \Sigma\mathcal{F}^{-1}$, and, by the distinguished triangle $\mathcal{F}^0 \longrightarrow C_1 \longrightarrow \Sigma\mathcal{F}^{-1}$, we get that $(f - g)$ must factor as $X \longrightarrow \mathcal{F}^0 \longrightarrow C_1$. This proves the case $n = 1$.

Now assume the claim has been proved for $n > 0$, and let $f \colon X \longrightarrow C_{n+1}$ be a morphism. The distinguished triangle $C_{n+1} \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1} \longrightarrow \Sigma C_n$ in (7) gives the vanishing of the composite

$$X \xrightarrow{f} C_{n+1} \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1} \longrightarrow \Sigma C_n$$

and hence certainly also the vanishing of the longer composite

$$X \xrightarrow{f} C_{n+1} \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1} \longrightarrow \Sigma C_n \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n},$$

where the morphism $C_n \longrightarrow \Sigma^n\mathcal{F}^{-n}$ comes from the distinguished triangle defining C_n. Thus we have a vanishing composite $X \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1} \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n}$, and the assumption in the statement allows us to factor the morphism $X \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1}$ as $X \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-2} \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1}$. But the morphism $X \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-2}$ may now be used to form the composite

$$g \colon X \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-2} \longrightarrow \Sigma^{n+1}\mathcal{K}^{-n-2} = C_{n+1}^{\leqslant -1} \longrightarrow C_{n+1}.$$

The construction is such that the map $(f - g) \colon X \longrightarrow C_{n+1}$ composes to zero with the morphism $C_{n+1} \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-1}$. This vanishing, in combination with the distinguished triangle in (7), permits us to factor the morphism $(f - g) \colon X \longrightarrow C_{n+1}$ as $X \longrightarrow C_n \longrightarrow C_{n+1}$. By induction, the morphism $X \longrightarrow C_n$ must factor as $X \longrightarrow \Sigma^n\mathcal{F}^{-n-1} \oplus \mathcal{F}^0 \longrightarrow C_n$. Now we observe that the composite $\Sigma^n\mathcal{F}^{-n-1} \longrightarrow C_n \longrightarrow C_{n+1}$ vanishes by the distinguished triangle (7), and hence the morphism $f \colon X \longrightarrow C_{n+1}$ does indeed factor as $X \longrightarrow \Sigma^{n+1}\mathcal{F}^{-n-2} \oplus \mathcal{F}^0 \longrightarrow C_{n+1}$. ☐

Lemma 7.2.4. *In the setting above, let $\mathcal{F}^\bullet$ be a complex as in* (6) *and let $X \in \mathcal{S}$ be such that, for all $i \in \mathbb{Z}$,* $\mathrm{Hom}(\Sigma^i X, -)$ *takes $\mathcal{F}^\bullet$ to an exact sequence. Assume further that* $\mathrm{Hom}(X, \mathcal{S}^{\leqslant -n}) = 0$ *for $n \gg 0$. Then the sequence*

$$\mathrm{Hom}(X, \mathcal{F}^{-1}) \longrightarrow \mathrm{Hom}(X, \mathcal{F}^0) \longrightarrow \mathrm{Hom}(X, \mathcal{K}^0) \longrightarrow 0 \ .$$

is exact.

Proof. Choose an integer $n > 0$ with $\mathrm{Hom}(X, \mathcal{S}^{\leqslant -n}) = 0$, and consider the object C_n constructed above. Note that it is easy to see that the distinguished triangle $C_n^{\leqslant -1} \longrightarrow C_n \longrightarrow C_n^{\geqslant 0} \longrightarrow \Sigma C_n^{\leqslant -1}$ identifies with $C_n \longrightarrow \mathcal{K}^0 \longrightarrow \Sigma^{n+1}\mathcal{K}^{-n-1}$.

For any morphism $X \longrightarrow \mathcal{K}^0$ we have that the composite $X \longrightarrow \mathcal{K}^0 \longrightarrow \Sigma^{n+1}\mathcal{K}^{-n-1}$ must vanish, and hence the morphism must factor as $X \longrightarrow C_n \longrightarrow \mathcal{K}^0$. Lemma 7.2.3 permits us to further factor it through $X \longrightarrow \Sigma^n\mathcal{F}^{-n-1} \oplus \mathcal{F}^0 \longrightarrow C_n$. But by assumption any morphism $X \longrightarrow \Sigma^n\mathcal{F}^{-n-1}$ vanishes, giving us the factorization of $X \longrightarrow \mathcal{K}^0$ through $X \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{K}^0$. Thus the morphism $\mathrm{Hom}(X, \mathcal{F}^0) \longrightarrow \mathrm{Hom}(X, \mathcal{K}^0)$ is surjective.

Now in $\mathcal{S}^\heartsuit$ we have a short exact sequence $0 \longrightarrow \mathcal{K}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{K}^0 \longrightarrow 0$, which means that in $\mathcal{S}$ there is a distinguished triangle $\mathcal{K}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{K}^0 \longrightarrow \Sigma\mathcal{K}^{-1}$. If we are given a morphism $X \longrightarrow \mathcal{F}^0$ such that the composite $X \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{K}^0$ vanishes, then it must factor through $\mathcal{K}^{-1}$. But now we can apply the paragraph above to the complex

$$\cdots \longrightarrow \mathcal{F}^{-4} \longrightarrow \mathcal{F}^{-3} \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1}$$

to deduce the surjectivity of $\mathrm{Hom}(X, \mathcal{F}^{-1}) \longrightarrow \mathrm{Hom}(X, \mathcal{K}^{-1})$. Assembling this together gives the required exact sequence

$$\mathrm{Hom}(X, \mathcal{F}^{-1}) \longrightarrow \mathrm{Hom}(X, \mathcal{F}^0) \longrightarrow \mathrm{Hom}(X, \mathcal{K}^0) \longrightarrow 0.$$

□

The next result shows that, under mild assumptions, all compact objects in $\mathcal{S}$ are strongly pseudocompact.

Lemma 7.2.5. *Let $\mathcal{S}$ be a triangulated category, and let $P \in \mathcal{P}$. Assume that, for every compact object $c \in \mathcal{S}$, there exists an integer $n > 0$ such that* $\mathrm{Hom}(c, \Sigma^n P) = 0$. *Then* $\mathcal{S}^c \subseteq \mathcal{S}^{spc}_{[P]}$.

Proof. It is obvious from Definition 7.1.1 that every compact object is $[P]$–pesudocompact. What needs proof is part (ii) of Definition 7.2.1.

Assume therefore that we are given a t–structure $(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1})$ on the category $\mathcal{S}$, as in Definition 7.2.1. This means that $\mathcal{S}^{\leqslant 0}$ must belong to $[P]$, the subcategory $\mathcal{S}^{\geqslant 0}$ is closed in $\mathcal{S}$ under coproducts, the heart $\mathcal{S}^\heartsuit$ satisfies [AB5], and the inclusion $\mathcal{S}^\heartsuit \longrightarrow \mathcal{S}$ respects coproducts.

Take any filtered category I and any functor $\mathsf{F}\colon I \longrightarrow \mathcal{S}^\heartsuit$, and form the cochain complex $\mathcal{F}^\bullet$ as in Example 7.2.2. In the category $\mathcal{S}^\heartsuit$ the cohomology of the complex $\mathcal{F}^\bullet$ is concentrated in degree 0. If $c \in \mathcal{S}$ is any compact object and $i \in \mathbb{Z}$ is an integer, then the functor $\mathrm{Hom}(\Sigma^i c, -)$ takes the complex $\mathcal{F}^\bullet$ to a complex computing $\underrightarrow{\mathrm{colim}}_j$ for the functor $\mathrm{Hom}(\Sigma^i c, \mathsf{F}(-))\colon I \longrightarrow \mathcal{A}b$. As the category of abelian groups satisfies [AB5] these $\underrightarrow{\mathrm{colim}}_j$ vanish for every $j < 0$, and for $j = 0$ we obtain $\underrightarrow{\mathrm{colim}}\,\mathrm{Hom}\big(\Sigma^i c, \mathsf{F}(-)\big)$. The part about $j = 0$ gives us the exact sequence

$$\mathrm{Hom}(c, \mathcal{F}^{-1}) \longrightarrow \mathrm{Hom}(c, \mathcal{F}^0) \longrightarrow \underrightarrow{\mathrm{colim}}\,\mathrm{Hom}\big(c, \mathsf{F}(-)\big) \longrightarrow 0\ ,$$

while the assertion about $j < 0$ tells us that, for every integer $i \in \mathbb{Z}$, the functor $\mathrm{Hom}(\Sigma^i c, -)$ takes

$$\cdots \longrightarrow \mathcal{F}^{-3} \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0$$

to an exact sequence. Because $\mathcal{S}^{\leqslant 0} \in [P]$ and $\mathrm{Hom}(c, \Sigma^n P) = 0$ for $n \gg 0$, we have that $\mathrm{Hom}(c, \mathcal{S}^{\leqslant -n}) = 0$ for $n \gg 0$. Thus by Lemma 7.2.4 the sequence

$$\mathrm{Hom}(c, \mathcal{F}^{-1}) \longrightarrow \mathrm{Hom}(c, \mathcal{F}^{0}) \longrightarrow \mathrm{Hom}(c, \underrightarrow{\mathrm{colim}}\, \mathsf{F}) \longrightarrow 0\ .$$

is exact. Combining the above we deduce that the natural map

$$\underrightarrow{\mathrm{colim}}\, \mathrm{Hom}\big(c, \mathsf{F}(-)\big) \longrightarrow \mathrm{Hom}(c, \underrightarrow{\mathrm{colim}}\, \mathsf{F})$$

is an isomorphism. □

Proposition 7.2.6. *Let $\mathcal{S}$ be a triangulated category, and let $P \in \mathcal{P}$ be such that $\mathcal{S}$ admits a t–structure as in part (ii) of Definition 7.2.1. Assume moreover that, for every compact object $c \in \mathcal{S}$, there exists an integer $n > 0$ such that $\mathrm{Hom}(c, \Sigma^n P) = 0$. Then $\mathcal{S}^-_{c,[P]} \subseteq \mathcal{S}^{spc}_{[P]}$.*

Proof. In Example 7.1.5 we noted that $\mathcal{S}^-_{c,[P]} \subseteq \mathcal{S}^{pc}_{[P]}$, thus we need to verify Definition 7.2.1(ii). Assume therefore that we are given a t–structure $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ on the category $\mathcal{S}$, satisfying the hypotheses of Definition 7.2.1(ii). Take any filtered category I and any functor $\mathsf{F}\colon I \longrightarrow \mathcal{S}^{\heartsuit}$. If $x \in \mathcal{S}^-_{c,[P]}$, then there exists a distinguished triangle $b \longrightarrow c \longrightarrow x \longrightarrow \Sigma b$ in $\mathcal{S}$ with $c \in \mathcal{S}^c$ and $b \in \mathcal{S}^{\leqslant -1}$. But then, for every object $H \in \mathcal{S}^{\heartsuit} \subseteq \mathcal{S}^{\geqslant 0}$, we have that the map $c \longrightarrow x$ induces an isomorphism $\mathrm{Hom}(x, H) \longrightarrow \mathrm{Hom}(c, H)$. Lemma 7.2.5 tells us that $\mathrm{Hom}(c, -)$ commutes with filtered colimits in $\mathcal{S}^{\heartsuit}$, and hence so does $\mathrm{Hom}(x, -)$. □

8. The subcategories of $\mathcal{T}^b$

This section starts with a discussion of strongly pseudocompact objects in weakly approximable triangulated categories, see Section 8.1. This will be crucial in Section 8.2, where we prove that the subcategories of $\mathcal{T}^b$ in the diagram (1) are intrinsic. And, as the reader will see, the recipe that works for these subcategories in $\mathcal{T}^b$ also describes them in the larger $\mathcal{T}^+$.

8.1. **Strongly pseudocompactness in weakly approximable triangulated categories.** Let $\mathcal{T}$ be a weakly approximable triangulated category. In Section 6.1 we observed that the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ all have preferred equivalence classes of t–structures with intrinsic descriptions. Thus for $\mathcal{S}$ being any of $\mathcal{T}^b$, $\mathcal{T}^+$, $\mathcal{T}^-$ and $\mathcal{T}$, it becomes interesting to study the compact, $\big[\mathcal{S}^{\leqslant 0}\big]$–pseudocompact and $\big[\mathcal{S}^{\leqslant 0}\big]$–strongly pseudocompact objects in $\mathcal{S}$, where $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ is a t–structure in the preferred equivalence class.

To simplify the notation, when $\mathcal{S}$ is a above, we say that an object is *pseudocompact,* without specifying with respect to which class of objects in $\mathcal{P}(\mathcal{S})$, if it is $\big[\mathcal{S}^{\leqslant 0}\big]$–pseudocompact, where $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ is a t–structure in the preferred equivalence class. Similarly, an object is *strongly pseudocompact* if it is $\big[\mathcal{S}^{\leqslant 0}\big]$–strongly pseudocompact. The full subcategories of all pseudocompact (resp. strongly pseudocompact) objects in $\mathcal{S}$ will be denoted $\mathcal{S}^{pc}$ (resp. $\mathcal{S}^{spc}$).

Lemma 8.1.1. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then we have the equalities*

$$(\mathcal{T}^b)^c = (\mathcal{T}^b)^{pc} = \mathcal{T}^b \cap (\mathcal{T}^-)^{pc} = \mathcal{T}^+ \cap (\mathcal{T}^-)^{pc} \qquad \textit{and} \qquad (\mathcal{T}^+)^c = (\mathcal{T}^+)^{pc} = \mathcal{T}^+ \cap \mathcal{T}^{pc}.$$

Proof. First observe that, when $\mathcal{S} = \mathcal{T}^?$, where $? = b, +, -, \emptyset$, a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of objects of $\mathcal{S}$ is contained in $P^\perp$ for some $P \in \left[\mathcal{S}^{\leqslant 0}\right]$ if and only if it is contained in $\mathcal{S}^{\geqslant -n}$ for some integer $n > 0$.

On the other hand, when $\mathcal{S} = \mathcal{T}^?$, where $? = b, +$, by Lemma 4.2.4 if the coproduct of a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of objects of $\mathcal{S}$ exists in $\mathcal{S}$, then again the set must be contained in $\mathcal{S}^{\geqslant -n}$ for some integer $n > 0$.

All the assertions in the statement follow easily from the two facts above. $\square$

Not quite so trivial is the next Lemma.

Lemma 8.1.2. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then we have the equalities*

$$\mathcal{T}^{pc} = (\mathcal{T}^-)^{pc} \qquad \text{and} \qquad (\mathcal{T}^+)^c = (\mathcal{T}^b)^c = (\mathcal{T}^+)^{pc} = (\mathcal{T}^b)^{pc} = \mathcal{T}^+ \cap \mathcal{T}^{pc}$$

Proof. First of all, without loss of generality, we choose, in the preferred equivalence class, a t–structure $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$ such that $\mathcal{T}^{\geqslant 0}$ is closed under coproducts. Suppose $t \in \mathcal{T}$ is pseudocompact. Then the truncation morphisms $t \longrightarrow t^{\geqslant n}$ assemble to a single map $t \longrightarrow \prod_{n=0}^\infty t^{\geqslant n}$, but Lemma 4.2.2 tells us that the natural morphism

$$\coprod_{n=0}^\infty t^{\geqslant n} \longrightarrow \prod_{n=0}^\infty t^{\geqslant n}$$

is an isomorphism. Thus the map $t \longrightarrow \prod_{n=0}^\infty t^{\geqslant n}$ factors uniquely as

$$t \xrightarrow{\ \varphi\ } \coprod_{n=0}^\infty t^{\geqslant n} \longrightarrow \prod_{n=0}^\infty t^{\geqslant n} ,$$

and as t is pseudocompact the map φ must factor through a finite subcoproduct. Hence the truncation morphisms $t \longrightarrow t^{\geqslant n}$ must vanish for $n \gg 0$, forcing t to belong to $\mathcal{T}^-$. That is $\mathcal{T}^{pc} = \mathcal{T}^{pc} \cap \mathcal{T}^- \subseteq (\mathcal{T}^-)^{pc}$.

Now let c be an object in $(\mathcal{T}^-)^{pc}$, let m be an integer, let $\{t_\lambda \mid \lambda \in \Lambda\}$ be a set of objects in $\left(\mathcal{T}^{\leqslant m}\right)^\perp = \mathcal{T}^{\geqslant m+1}$, and let

$$c \xrightarrow{\ \psi\ } \coprod_{\lambda \in \Lambda} t_\lambda$$

be any morphism. Because c belongs to $\mathcal{T}^-$ we may choose an integer $n > 0$ with $c \in \mathcal{T}^{\leqslant n}$, and the morphism ψ will have to factor through $\left(\coprod_{\lambda \in \Lambda} t_\lambda\right)^{\leqslant n} = \coprod_{\lambda \in \Lambda} t_\lambda^{\leqslant n}$, where the equality follows from the fact that, as we explained in Section 2.3 (see, in particular, Remark 2.3.7), the truncation functors preserves coproducts. Hence ψ must factor as

$$c \xrightarrow{\ \rho\ } \coprod_{\lambda \in \Lambda} t_\lambda^{\leqslant n} \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda ,$$

and the $\left[\mathcal{T}^{\leqslant 0}\right]$–pseudocompactness of c in the category $\mathcal{T}^-$ forces ρ to factor through a finite subcoproduct. Hence ψ factors through a finite subcoproduct, and we deduce that $(\mathcal{T}^-)^{pc} \subseteq \mathcal{T}^{pc}$.

We have therefore proved that $(\mathcal{T}^-)^{pc} = \mathcal{T}^{pc}$. The remaining equalities in the statement come from intersecting $(\mathcal{T}^-)^{pc} = \mathcal{T}^{pc}$ with $\mathcal{T}^+$ and combining with the equalities of Lemma 8.1.1. $\square$

For this article the important lemma will be the following.

Lemma 8.1.3. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then we have the equalities*

$$\mathcal{T}^{spc} = (\mathcal{T}^-)^{spc} \qquad \textit{and} \qquad (\mathcal{T}^+)^{spc} = (\mathcal{T}^b)^{spc} = \mathcal{T}^+ \cap \mathcal{T}^{spc} \ .$$

Proof. For an object $c \in \mathcal{S}$ the extra restriction imposed by strong pseudocompactness, in addition to the requirements of ordinary pseudocompactness, is an assertion about the way $\mathrm{Hom}(c, -)$ behaves on filtered colimits of objects in the hearts of certain t–structures in the preferred equivalence class. The point here is that the collection of such hearts is the same, independent of which $\mathcal{S}$ we choose in the collection $\mathcal{S} \in \{\mathcal{T}^b, \mathcal{T}^-, \mathcal{T}^+, \mathcal{T}\}$. Thus the equalities of the current statement follow from the equalities

$$\mathcal{T}^{pc} = (\mathcal{T}^-)^{pc} \qquad \text{and} \qquad (\mathcal{T}^+)^{pc} = (\mathcal{T}^b)^{pc} = \mathcal{T}^+ \cap \mathcal{T}^{pc}$$

of Lemma 8.1.2 by just adding the "strongly" restriction in each case. □

In addition, we can prove the following.

Lemma 8.1.4. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then $\mathcal{T}^{spc}$ is a thick subcategory of $\mathcal{T}$.*

Proof. In Corollary 7.1.4 we saw that $\mathcal{T}^{pc} \subseteq \mathcal{T}$ is a thick subcategory. The extra restriction, required of objects in $\mathcal{T}^{spc} \subseteq \mathcal{T}^{pc}$, is clearly stable under shifts and direct summands, and it remains to show that it is stable under extensions.

Assume therefore that $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$ is a distinguished triangle in $\mathcal{T}$, with $a, c \in \mathcal{T}^{spc}$. We need to prove that $b \in \mathcal{T}^{spc}$. Let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure in the preferred equivalence class, with $\mathcal{T}^{\geq 0}$ closed under coproducts and $\mathcal{T}^{\heartsuit}$ an abelian category satisfying [AB5]. We need to show that the functor $\mathrm{Hom}(b, -)$ commutes with filtered colimits in $\mathcal{T}^{\heartsuit}$. Choose therefore a filtered category I and a functor $\mathsf{F} \colon I \longrightarrow \mathcal{T}^{\heartsuit}$. Now the functor $\mathrm{Hom}(-, \underrightarrow{\mathrm{colim}}\, \mathsf{F})$ is a homological functor on $\mathcal{T}$, as is the functor $\underrightarrow{\mathrm{colim}}\, \mathrm{Hom}\big(-, \mathsf{F}(i)\big)$. The natural transformation

$$\underrightarrow{\mathrm{colim}}\, \mathrm{Hom}\big(-, \mathsf{F}(i)\big) \longrightarrow \mathrm{Hom}(-, \underrightarrow{\mathrm{colim}}\, \mathsf{F})$$

is an isomorphism when evaluated on all suspensions of a and c, and the 5-Lemma tells us that it must also be an isomorphism on b. □

We are now ready to prove the main result of this section.

Proposition 8.1.5. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then $\mathcal{T}^{spc} = \mathcal{T}^-_c$.*

Proof. Proposition 7.2.6 proved the inclusion $\mathcal{T}^-_c \subseteq \mathcal{T}^{spc}$, in generality greater than we need. It remains to prove the inclusion $\mathcal{T}^{spc} \subseteq \mathcal{T}^-_c$.

Without loss of generality, we may choose and fix a compactly generated t–structure $\big(\mathcal{T}^{\leqslant 0}_{\mathcal{A}}, \mathcal{T}^{\geqslant 1}_{\mathcal{A}}\big)$ in the preferred equivalence class. Lemma 3.2.2 tells us that the heart $\mathcal{T}^{\heartsuit}_{\mathcal{A}}$ of this t–structure is an abelian category satisfying [AB5]. The key will be to prove the following:

Claim. With the notation as above, let c be an object in $\mathcal{T}^{spc} \cap \mathcal{T}^{\leqslant 0}_{\mathcal{A}}$. Then there exists a distinguished triangle $b \longrightarrow c \longrightarrow d \longrightarrow \Sigma b$ with $b \in \mathcal{T}^c$ and with $d \in \mathcal{T}^{spc} \cap \mathcal{T}^{\leqslant -1}_{\mathcal{A}}$.

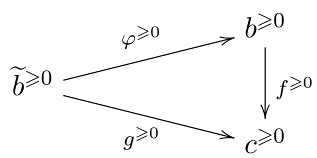
Assume that the claim holds and choose an object $c \in \mathcal{T}^{spc} = (\mathcal{T}^-)^{spc}$, where the equality is by Lemma 8.1.3. We wish to show that $c \in \mathcal{T}^-_c$. Because c belongs to $(\mathcal{T}^-)^{spc} \subseteq \mathcal{T}^-$, we have that

$c \in \mathcal{T}_{\mathcal{A}}^{\leqslant n}$ for some $n \geqslant 0$. Shifting if necessary we may assume that $c \in \mathcal{T}_{\mathcal{A}}^{\leqslant 0}$. Now we apply the claim iteratively. Setting $c = d_0$ we produce a sequence of morphisms $d_0 \longrightarrow d_1 \longrightarrow d_2 \longrightarrow \cdots$ in the category $\mathcal{T}^{spc}$, with $d_n \in \mathcal{T}_{\mathcal{A}}^{\leqslant -n}$. And by the claim this can be done in such a way that in the distinguished triangles $b_n \longrightarrow d_n \longrightarrow d_{n+1} \longrightarrow \Sigma b_n$ the object b_n is compact. The octahedral axiom implies that the distinguished triangle $b \longrightarrow c \longrightarrow d_n \longrightarrow \Sigma b$ has $b \in \mathcal{T}^c$ and $d_n \in \mathcal{T}_{\mathcal{A}}^{\leqslant -n}$, showing that $c \in \mathcal{T}_c^-$.

Thus it remains to prove the claim. Lemma 3.2.4 tells us that $c^{\geqslant 0} \in \mathcal{T}_{\mathcal{A}}^{\heartsuit}$ must be the filtered colimit of the finitely presented objects mapping to it. So there exists a filtered category I, and a functor $\mathsf{F}\colon I \longrightarrow \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$, such that $c^{\geqslant 0} = \underrightarrow{\operatorname{colim}}\,\mathsf{F}$. But because c is strongly pseudocompact the map $c \longrightarrow c^{\geqslant 0} = \underrightarrow{\operatorname{colim}}\,\mathsf{F}$ must factor through some $\mathsf{F}(i) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. But the map $c \longrightarrow \mathsf{F}(i)$ is a morphism from c to $\mathsf{F}(i) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit} \subseteq \mathcal{T}_{\mathcal{A}}^{\geqslant 0}$, and must therefore factor uniquely through $c^{\geqslant 0}$. We deduce that the identity map on $c^{\geqslant 0}$ factors through $\mathsf{F}(i)$, meaning that $c^{\geqslant 0}$ is a direct summand of the object $\mathsf{F}(i) \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Corollary 3.1.4 tells us that the category $\mathcal{T}_{\mathcal{A},c}^{\heartsuit}$ is closed under retracts, and we deduce that $c^{\geqslant 0} \in \mathcal{T}_{\mathcal{A},c}^{\heartsuit}$. Hence there exists a compact object $b \in \mathcal{T}_{\mathcal{A},c}^{\leqslant 0}$ with $b^{\geqslant 0} \cong c^{\geqslant 0}$.

Now consider the composite $f\colon b \longrightarrow b^{\geqslant 0} \cong c^{\geqslant 0}$. By construction $f^{\geqslant 0}\colon b^{\geqslant 0} \longrightarrow c^{\geqslant 0}$ is an isomorphism. Lemma 3.1.2 yields a compact object $\widetilde{b}$, and morphisms $\varphi\colon \widetilde{b} \longrightarrow b$ and $g\colon \widetilde{b} \longrightarrow c$ such that

- The map $\varphi^{\geqslant 0}\colon \widetilde{b}^{\geqslant 0} \longrightarrow b^{\geqslant 0}$ is an isomorphism.
- The triangle below commutes



The commutativity of the triangle, coupled with the fact that both $\varphi^{\geqslant 0}$ and $f^{\geqslant 0}$ are isomorphisms, forces $g^{\geqslant 0}$ to be an isomorphism. Completing g to a distinguished triangle $\widetilde{b} \longrightarrow c \longrightarrow d \longrightarrow \Sigma\widetilde{b}$ we deduce that d must belong to $\mathcal{T}^{\leqslant -1}$. Now $\widetilde{b} \in \mathcal{T}^c \subseteq \mathcal{T}^{psc}$ where the inclusion is by Lemma 7.2.5. And the fact that in the distinguished triangle both $\widetilde{b}$ and c belong to $\mathcal{T}^{spc}$, coupled with Lemma 8.1.4, tells us that $d \in \mathcal{T}^{spc}$. This completes the proof of the claim and hence of the Proposition. □

Corollary 8.1.6. *Let $\mathcal{T}$ be a weakly approximable triangulated category. Then*

$$\mathcal{T}^{spc} = (\mathcal{T}^-)^{spc} = \mathcal{T}_c^- \qquad \textit{and} \qquad (\mathcal{T}^+)^{spc} = (\mathcal{T}^b)^{spc} = \mathcal{T}_c^b\ .$$

Proof. It follows directly from Proposition 8.1.5 and Lemma 8.1.3. □

8.2. **The intrinsic subcategories of $\mathcal{T}^b$ and $\mathcal{T}^+$.** The discussion in the previous section allows us to prove the following.

Lemma 8.2.1. *Let $\mathcal{T}$ be a weakly approximable triangulated category and let $\mathcal{S} = \mathcal{T}^?$, with $? = +, b$. Then $\mathcal{T}_c^b$ is an intrinsic subcategory of $\mathcal{S}$*

Proof. Corollary 8.1.6 yields the equalities $\mathcal{T}_c^b = (\mathcal{T}^b)^{spc} = (\mathcal{T}^+)^{spc}$. The claim is then obvious. □

The following result shows that also $\mathcal{T}^{c,b}$ is an intrinsic subcategory both of $\mathcal{T}^b$ and $\mathcal{T}^+$.

Lemma 8.2.2. *Let $\mathcal{T}$ be a weakly approximable triangulated category and let $\mathcal{S} = \mathcal{T}^?$, with $? = +, b$. Let $\big(\mathcal{S}^{\leqslant 0}, \mathcal{S}^{\geqslant 1}\big)$ be a t–structure in the preferred equivalence class. An object $c \in \mathcal{T}^b_c \subseteq \mathcal{S}$ belongs to $\mathcal{T}^{c,b}$ if and only if there exists an integer $n > 0$ with* $\mathrm{Hom}(c, \mathcal{S}^{\leqslant -n}) = 0$.

Proof. If c belongs to $\mathcal{T}^c \cap \mathcal{T}^b_c \subseteq \mathcal{T}^c$ then there exists an integer $n > 0$ with $\mathrm{Hom}(c, \mathcal{T}^{\leqslant -n}) = 0$, whence $\mathrm{Hom}(c, \mathcal{S}^{\leqslant -n}) = 0$.

Now assume that we have an object $c \in \mathcal{T}^b_c \subseteq \mathcal{S}$, and that we are given an integer $n > 0$ with $\mathrm{Hom}(c, \mathcal{S}^{\leqslant -n}) = 0$. If $t \in \mathcal{T}^{\leqslant -n-1}$ is any object, then for each $\ell > n + 1$ the object $t^{\geqslant -\ell}$ belongs to $\mathcal{S}^{\leqslant -n-1}$, whether $\mathcal{S} = \mathcal{T}^b$ or $\mathcal{S} = \mathcal{T}^+$. Then, for all $\ell > n + 1$,

$$\mathrm{Hom}\Big(c, t^{\geqslant -\ell}\Big) \quad = \quad 0 \quad = \quad \mathrm{Hom}\Big(c, \Sigma^{-1}\big(t^{\geqslant -\ell}\big)\Big)\ .$$

By [17, Proposition 3.2] the weakly approximable triangulated category $\mathcal{T}$ is left-complete, meaning that t is isomorphic to $\underleftarrow{\mathrm{Holim}}\ t^{\geqslant -\ell}$. Hence there exists in $\mathcal{T}$ a distinguished triangle

$$\prod_{\ell=n+2}^{\infty} \Sigma^{-1}\big(t^{\geqslant -\ell}\big) \longrightarrow t \longrightarrow \prod_{\ell=n+2}^{\infty} t^{\geqslant -\ell} \longrightarrow \prod_{\ell=n+2}^{\infty} \big(t^{\geqslant -\ell}\big).$$

Since $\mathrm{Hom}(c, -)$ is trivial when evaluated on the product terms in the triangle, we must have $\mathrm{Hom}(c, t) = 0$. As $t \in \mathcal{T}^{\leqslant -n-1}$ is arbitrary we conclude that $\mathrm{Hom}(c, \mathcal{T}^{\leqslant -n-1}) = 0$.

But now c belongs to $\mathcal{T}^b_c \subseteq \mathcal{T}^-_c$, and Lemma 5.2.1 together with the vanishing of $\mathrm{Hom}(c, \mathcal{T}^{\leqslant -n-1})$ implies that $c \in \mathcal{T}^c$. □

9. The case $\mathcal{T}^c \subseteq \mathcal{T}^b_c$: a procedure to recover $\mathcal{T}^c$

Let $\mathcal{T}$ be a weakly approximable triangulated category. As we said in the introduction, we do not in general understand how to recognise in $\mathcal{T}^b_c$ which objects belong to the subcategory $\mathcal{T}^{c,b} = \mathcal{T}^c \cap \mathcal{T}^b_c$. There are, however, two special cases in which we can say something useful:

(a) If $\mathcal{T}$ is *coherent*.
(b) If $\mathcal{T}^c \subseteq \mathcal{T}^b_c$ and furthermore ${}^{\perp}(\mathcal{T}^b_c) \cap \mathcal{T}^-_c = \{0\}$.

For an exposition of the algorithm in (a), in full generality, the reader is referred to [16].

In Section 9.2 we will concern ourselves with the precise statement and proof of (b). The recipe for $\mathcal{T}^{c,b} = \mathcal{T}^c$ is in Proposition 9.2.3. An example of a $\mathcal{T}$, to which (b) applies, is $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a quasi-compact, quasi-separated scheme and with $Z \subseteq X$ a closed subset with quasi-compact complement (see Proposition 10.2.1). Furthermore, the hypothesis ${}^{\perp}(\mathcal{T}^b_c) \cap \mathcal{T}^-_c = \{0\}$ also holds for any coherent weakly approximable triangulated category. Therefore, if we specialize to the case where $\mathcal{T}^c \subseteq \mathcal{T}^b_c$, then (b) gives an approach to (a) different from [16, Propositions 5.6 and 6.5]. See Section 10.1 for further discussion.

9.1. Classical generators and special equivalence classes.

In this section we develop general results which will help to recognize the subcategory $\mathcal{T}^{c.b}$ of $\mathcal{T}^b_c$. We start from a definition which is very similar, in spirit, to Definition 6.1.1.

Definition 9.1.1. *Let $\mathcal{S}$ be a triangulated category, and consider the class $\mathcal{Q} = \mathcal{Q}(\mathcal{S})$ of all full subcategories $P \subseteq \mathcal{S}$ satisfying $P \subseteq \Sigma P$.*

(i) *Two elements* $P, Q \in \mathcal{Q}$ *are* equivalent *if there exists an integer* $A > 0$ *with* $\Sigma^{-A}P \subseteq Q \subseteq \Sigma^{A}P$.
(ii) *Given two equivalence classes* $[P]$ *and* $[Q]$ *of elements of* $\mathcal{Q}$*, then* $[P] \leqslant [Q]$ *if, for a choice of representatives* $P \in [P]$ *and* $Q \in [Q]$*, there exists an integer* $A > 0$ *with* $\Sigma^{-A}P \subseteq Q$.

The difference between the $\mathcal{P}(\mathcal{S})$ of Definition 6.1.1 and the $\mathcal{Q}(\mathcal{S})$ of Definition 9.1.1 is that the $P \in \mathcal{P}(\mathcal{S})$ are assumed to satisfy $\Sigma P \subseteq P$, while the $P \in \mathcal{Q}(\mathcal{S})$ must have $P \subseteq \Sigma P$. Next we focus on the $P \in \mathcal{P}(\mathcal{S})$ and $Q \in \mathcal{Q}(\mathcal{S})$ that will interest us in this section.

Given $H \in \mathcal{S}$, we define the following two subcategories of $\mathcal{S}$:

$$P_H(\mathcal{S}) := H[0,\infty)^{\perp} \qquad Q_H(\mathcal{S}) := H(-\infty,0]^{\perp}. \tag{8}$$

When there is no confusion about the triangulated category $\mathcal{S}$, we use the shorthands $P_H = P_H(\mathcal{S})$ and $Q_H = Q_H(\mathcal{S})$.

Remark 9.1.2. For any object $H \in \mathcal{S}$ we have the inclusions and equalities

$$\Sigma P_H \subseteq P_H, \quad P_H * P_H = P_H, \quad \mathrm{add}(P_H) = P_H, \quad \mathrm{smd}(P_H) = P_H$$

as well as

$$Q_H \subseteq \Sigma Q_H, \quad Q_H * Q_H = Q_H, \quad \mathrm{add}(Q_H) = Q_H, \quad \mathrm{smd}(Q_H) = Q_H.$$

If X is an object of P_H, it follows from the first set of inclusions that $\langle X\rangle^{(-\infty,0]} \subseteq P_H$. If X is an object of Q_H, the second set of inclusions gives that $\langle X\rangle^{[0,\infty)} \subseteq Q_H$.

For an object H of a triangulated category $\mathcal{S}$ we may ask the following hypothesis to be satisfied:

Hypothesis 9.1.3. *There exists an integer* $A > 0$ *such that:*
(i) *For every object* $X \in \mathcal{S}$ *there exists an integer* $B > 0$*, with* $X \in \Sigma^{-B}P_H \cap \Sigma^{B}Q_H$.
(ii) $\mathrm{Hom}(\Sigma^{A}P_H, Q_H) = 0$.
(iii) *For every object* $F \in P_H$ *and every integer* $m > 0$*, there exists a distinguished triangle* $E \longrightarrow F \longrightarrow D$ *in* $\mathcal{S}$ *with* $E \in \langle H\rangle^{[1-m-A,A]}$ *and with* $D \in \Sigma^{m}P_H$.

Remark 9.1.4. We take a bit of time to elaborate on Hypothesis 9.1.3(iii). As stated, this item starts with an object $F \in P_H$ and an integer $m > 0$. We claim that, in fact, the general case can be deduced from the special case where $m = 1$, as follows:

- Take $F \in P_H$. By the case $m = 1$ of Hypothesis 9.1.3(iii), we may construct a distinguished triangle $E_1 \longrightarrow F \longrightarrow D_1$ with $E_1 \in \langle H\rangle^{[-A,A]}$ and with $D_1 \in \Sigma P_H$.
- Now we proceed by induction. Suppose we have constructed the sequence

$$F = D_0 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \cdots \longrightarrow D_{n-1} \longrightarrow D_n,$$

  with $D_i \in \Sigma^i P_H$ and such that, in the distinguished triangle $\widetilde{E}_i \longrightarrow D_i \longrightarrow D_{i+1}$, we have $\widetilde{E}_i \in \langle H\rangle^{[-i-A,-i+A]}$. Then the previous step allows us to continue this a step further.

It is easy to see that, if we complete the composite $F \longrightarrow D_m$ to a distinguished triangle $E_m \longrightarrow F \longrightarrow D_m$, then $E_m \in \langle H\rangle^{[1-m-A,A]}$. Moreover, for any i in the interval $0 < i < m$, we have that the E_m we constructed satisfies

$$E_m \in \langle H\rangle^{[1-i-A,A]} * \langle H\rangle^{[1-m-A,-i+A]}.$$

More precisely, in the distinguished triangle $E_i \longrightarrow E_m \longrightarrow \widetilde{D}$ we have $E_i \in \langle H\rangle^{[1-i-A,A]}$ and $\widetilde{D} \in \langle H\rangle^{[1-m-A,-i+A]}$.

The hypothesis above looks technical at first sight. The first aim of this section is to study which objects do satisfy it.

Lemma 9.1.5. *Let $\mathcal{T}$ be a weakly approximable triangulated category, and let $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$ be a t–structure on $\mathcal{T}$ in the preferred equivalence class and assume that $\mathcal{T}^c \subseteq \mathcal{T}^b_c$.*
If $G \in \mathcal{T}^c \subseteq \mathcal{T}^b_c$ is such that $\mathcal{T}^c = \langle G\rangle$, then

$$[P_G(\mathcal{T}^b_c)] = \left[\mathcal{T}^b_c \cap \mathcal{T}^{\leqslant 0}\right] \qquad \textit{and} \qquad [Q_G(\mathcal{T}^b_c)] = \left[\mathcal{T}^b_c \cap \mathcal{T}^{\geqslant 0}\right] .$$

Let us remind that an object $G \in \mathcal{T}^c$ such that $\mathcal{T}^c = \langle G\rangle$ is called a *classical generator* for $\mathcal{T}^c$. And it is automatic that a classical generator $G \in \mathcal{T}^c$ compactly generates the larger category $\mathcal{T}$.

Proof. Since we only want to compute the equivalence classes of $P_G = P_G(\mathcal{T}^b_c)$ and $Q_G = Q_G(\mathcal{T}^b_c)$, we are free to replace the t–structure $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$ by the equivalent t–structure $\left(\mathcal{T}^{\leqslant 0}_G, \mathcal{T}^{\geqslant 1}_G\right)$. In this case $\mathcal{T}^{\leqslant 0}_G = \operatorname{Coprod}\big(G(-\infty, 0]\big)$, providing the second equality in

$$\mathcal{T}^{\geqslant 1}_G = \left(\mathcal{T}^{\leqslant 0}_G\right)^{\perp} = \operatorname{Coprod}\big(G(-\infty, 0]\big)^{\perp} = G(-\infty, 0]^{\perp},$$

where the orthogonals are taken in $\mathcal{T}$ and the last equality is due to Lemma 1.2.1(ii). Hence the equality $Q_G = \mathcal{T}^b_c \cap \mathcal{T}^{\geqslant 1}_G$ comes by intersecting with $\mathcal{T}^b_c$.

Now we turn to computing the equivalence class of P_G. By Definition 4.1.1(i) there exists an integer $A > 0$ such that $G \in {}^{\perp}\mathcal{T}^{\leqslant -A}$. Hence $G[0, \infty) \subseteq {}^{\perp}\mathcal{T}^{\leqslant -A}$, or equivalently

$$\mathcal{T}^{\leqslant 0} \subseteq G[A, \infty)^{\perp}$$

where the orthogonals are taken in $\mathcal{T}$. The inclusion $\mathcal{T}^b_c \cap \mathcal{T}^{\leqslant 0} \subseteq \Sigma^{-A} P_G$ comes from intersecting with $\mathcal{T}^b_c$. On the other hand, [6, Lemma 3.9(iv)] permits us to choose an integer $B > 0$ such that

$$G[-B, \infty)^{\perp} \subseteq \mathcal{T}^{\leqslant 0},$$

and intersecting with $\mathcal{T}^b_c$ gives the inclusion $\Sigma^B P_G \subseteq \mathcal{T}^b_c \cap \mathcal{T}^{\leqslant 0}$. ☐

Corollary 9.1.6. *Let $\mathcal{T}$ be a weakly approximable triangulated category and assume that $\mathcal{T}^c \subseteq \mathcal{T}^b_c$. If $G \in \mathcal{T}^c \subseteq \mathcal{T}^b_c$ is such that $\mathcal{T}^c = \langle G\rangle$, then $\operatorname{Hom}(\Sigma^A P_G, Q_G) = 0$ for $A \gg 0$.*

Proof. Let $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$ be a t–structure on $\mathcal{T}$ in the preferred equivalence class. By Lemma 9.1.5 there exists an integer $B > 0$ such that

$$\Sigma^B P_G \subseteq \mathcal{T}^b_c \cap \mathcal{T}^{\leqslant 0} \qquad \text{and} \qquad \Sigma^{-B} Q_G \subseteq \mathcal{T}^b \cap \mathcal{T}^{\geqslant 0}.$$

But as $\operatorname{Hom}\big(\Sigma^n(\mathcal{T}^{\leqslant 0}), \mathcal{T}^{\geqslant 0}\big) = 0$ for $n > 0$, we deduce that $\operatorname{Hom}(\Sigma^A P_G, Q_G) = 0$ for $A > 2B$. ☐

Putting all these results together, we get that classical generators of $\mathcal{T}^c$ satisfy our technical assumption.

Proposition 9.1.7. *Let $\mathcal{T}$ be a weakly approximable triangulated category and assume that $\mathcal{T}^c \subseteq \mathcal{T}^b_c$. If $G \in \mathcal{T}^c \subseteq \mathcal{T}^b_c$ is such that $\mathcal{T}^c = \langle G\rangle$, then G satisfies Hypothesis 9.1.3 (with $\mathcal{S} = \mathcal{T}^b_c$).*

Proof. We begin with (i) in Hypothesis 9.1.3. Let $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1})$ be a t–structure in the preferred equivalence class, and let $X \in \mathcal{T}_c^b$ be an object. Because $\mathcal{T}_c^b$ is contained in $\mathcal{T}^b$, the object $X \in \mathcal{T}_c^b$ must lie in $\mathcal{T}^{\leqslant n} \cap \mathcal{T}^{\geqslant -n}$ for some $n \geq 0$. By Lemma 9.1.5 there exists an integer $B > 0$ with

$$\mathcal{T}_c^b \cap \mathcal{T}^{\leqslant n} \subseteq \Sigma^{-n-B} P_G \qquad \text{and} \qquad \mathcal{T}_c^b \cap \mathcal{T}^{\geqslant -n} \subseteq \Sigma^{n+B} Q_G$$

and hence

$$X \in \mathcal{T}_c^b \cap \mathcal{T}^{\leqslant n} \cap \mathcal{T}^{\geqslant -n} \subseteq \Sigma^{-n-B} P_G \cap \Sigma^{n+B} Q_G.$$

Item (ii) of Hypothesis 9.1.3 directly follows from Corollary 9.1.6. So let us deal with item (iii) in the hypothesis. Replacing the t–structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1})$ by an equivalent one if necessary, we may choose an integer A such that $\mathcal{T}_c^b \cap \mathcal{T}^{\leqslant -A} \subseteq P_G \subseteq \mathcal{T}_c^b \cap \mathcal{T}^{\leqslant 0}$. And now [18, Corollary 2.14] allows us to choose an integer $A' > 0$ such that, for any integer $m > 0$, any object $F \in P_G \subseteq \mathcal{T}_c^b \cap \mathcal{T}^{\leqslant 0}$ admits a distinguished triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}_c^b \cap \mathcal{T}^{\leqslant -A-m} \subseteq \Sigma^m P_G$ and with $E \in \langle G \rangle^{[1-m-A-A', A']}$. □

9.2. **The general criterion.** Let us put ourselves in the following context. Let $\mathcal{T}$ be a weakly approximable triangulated category, assume that $\mathcal{T}^c$ is contained in $\mathcal{T}_c^b$, and let $H \in \mathcal{S} = \mathcal{T}_c^b$ be an object satisfying Hypothesis 9.1.3. Choose a compactly generated t–structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1})$ in the preferred equivalence class, and let $\mathcal{H}\colon \mathcal{T} \longrightarrow \mathcal{T}^{\heartsuit}$ be the induced homological functor.

Let $F \in P_H$ be any object and, for $m > 0$, let

$$E_m \longrightarrow F \longrightarrow D_m \longrightarrow \Sigma E_m \tag{9}$$

be the distinguished triangles in Hypothesis 9.1.3(iii). We assume we constructed them according to Remark 9.1.4. In particular, they form sequences which allow us to define

$$E := \underrightarrow{\mathrm{Hocolim}}\, E_m \qquad \text{and} \qquad D := \underrightarrow{\mathrm{Hocolim}}\, D_m.$$

Again, for each $m > 0$, we are given the natural morphisms $E_m \longrightarrow E$ and $D_m \longrightarrow D$ which we can complete in $\mathcal{T}$ to the distinguished triangles

$$E_m \longrightarrow E \longrightarrow \widetilde{E}_m \qquad \text{and} \qquad D_m \longrightarrow D \longrightarrow \widetilde{D}_m.$$

Lemma 9.2.1. *In the setting above, the following hold:*

(i) *There exists an integer $B > 0$ such that, for all $m > 0$, both $\widetilde{E}_m$ and $\widetilde{D}_m$ lie in $\mathcal{T}^{\leqslant -m+B}$.*
(ii) *The objects E and D belong to $\mathcal{T}_c^-$.*
(iii) *There is in $\mathcal{T}_c^-$ a distinguished triangle $E \longrightarrow F \longrightarrow D \longrightarrow \Sigma E$.*

Proof. We are assuming that H belongs to $\mathcal{T}_c^b \subseteq \mathcal{T}^-$, and hence we may assume, up to shift, that $H \in \mathcal{T}^{\leqslant 0}$. Let $F \in P_H$.

Take the triangle (9) constructed in $\mathcal{T}_c^b$ in Remark 9.1.4. By the same remark, for any pair of integers $m < n$, the morphism $E_m \longrightarrow E_n$ may be completed to a distinguished triangle $E_m \longrightarrow E_n \longrightarrow \widetilde{D}_{m,n}$ with $\widetilde{D}_{m,n} \in \langle H \rangle^{[1-n-A,-m+A]} \subseteq \mathcal{T}^{\leqslant -m+A}$. Fix the integer $m > 0$; the long exact sequence in cohomology associated to such a triangle tells us that $\mathcal{H}^i(E_m) \longrightarrow \mathcal{H}^i(E_n)$ is an isomorphism for all $i > -m + A + 1$. Now form the morphism of triangles

$$\begin{array}{ccccccc}
E_m & \longrightarrow & F & \longrightarrow & D_m & \longrightarrow & \Sigma E_m \\
\downarrow & & \| & & \downarrow & & \downarrow \\
E_n & \longrightarrow & F & \longrightarrow & D_n & \longrightarrow & \Sigma E_n
\end{array}$$

and, by applying $\mathcal{H}$ to it, we learn that the induced morphism $\mathcal{H}^i(D_m) \longrightarrow \mathcal{H}^i(D_n)$ is also an isomorphism in the range $i > -m + A + 1$. Taking the colimit as $n \longrightarrow \infty$ and combining with [18, Lemma 1.4(iv)] we deduce that, for $i > -m + A + 1$, the functor $\mathcal{H}^i$ takes the natural maps $E_m \longrightarrow E$ and $D_m \longrightarrow D$ to isomorphisms. The long exact sequence in cohomology, applied to the distinguished triangles $E_m \longrightarrow E \longrightarrow \widetilde{E}_m$ and $D_m \longrightarrow D \longrightarrow \widetilde{D}_m$, tells us that $\widetilde{E}_m$ and $\widetilde{D}_m$ both belong to $\mathcal{T}^{\leqslant -m+A+1}$, proving (i).

By construction, the objects E_m and D_m both belong to $\mathcal{T}^b_c \subseteq \mathcal{T}^-_c$, and from the distinguished triangles $E_m \longrightarrow E \longrightarrow \widetilde{E}_m$ and $D_m \longrightarrow D \longrightarrow \widetilde{D}_m$ we now deduce that E and D belong to $\mathcal{T}^-_c * \mathcal{T}^{\leqslant -m+A+1}$. On the other hand $\mathcal{T}^-_c = \bigcap_{n=1}^{\infty} \big(\mathcal{T}^c * \mathcal{T}^{\leqslant -n}\big)$, and hence E and D both belong to

$$\mathcal{T}^-_c * \mathcal{T}^{\leqslant -m+A+1} \subseteq \big(\mathcal{T}^c * \mathcal{T}^{\leqslant -m+A+1}\big) * \mathcal{T}^{\leqslant -m+A+1} = \mathcal{T}^c * \big(\mathcal{T}^{\leqslant -m+A+1} * \mathcal{T}^{\leqslant -m+A+1}\big) = \mathcal{T}^c * \mathcal{T}^{\leqslant -m+A+1}.$$

As $m > 0$ is arbitrary, we deduce that $D, E \in \mathcal{T}^-_c$, proving (ii).

Now consider the commutative square

$$\begin{array}{ccc}
\coprod_{n=1}^{\infty} D_n & \xrightarrow{\;1-shift\;} & \coprod_{n=1}^{\infty} D_n \\
\downarrow & & \downarrow \\
\coprod_{n=1}^{\infty} \Sigma E_n & \xrightarrow[\;1-shift\;]{} & \coprod_{n=1}^{\infty} \Sigma E_n
\end{array}$$

and complete to a morphism of triangles

$$\begin{array}{ccccc}
\coprod_{n=1}^{\infty} D_n & \xrightarrow{\;1-shift\;} & \coprod_{n=1}^{\infty} D_n & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} \\
\coprod_{n=1}^{\infty} \Sigma E_n & \xrightarrow[\;1-shift\;]{} & \coprod_{n=1}^{\infty} \Sigma E_n & \longrightarrow & \Sigma E.
\end{array}$$

Next we may complete φ to a distinguished triangle $E \longrightarrow \widetilde{F} \longrightarrow D \xrightarrow{\varphi} \Sigma E$. For every integer $m > 0$, we have the commutative diagram

$$\begin{array}{ccccccc}
E_m & \longrightarrow & F & \longrightarrow & D_m & \longrightarrow & \Sigma E_m \\
 & & & & \downarrow & & \downarrow \\
E & \longrightarrow & \widetilde{F} & \longrightarrow & D & \xrightarrow{\;\varphi\;} & \Sigma E
\end{array}$$

which we can complete to a morphism of triangles

$$\begin{array}{ccccccc} E_m & \longrightarrow & F & \longrightarrow & D_m & \longrightarrow & \Sigma E_m \\ \downarrow & & \downarrow{\scriptstyle \lambda_m} & & \downarrow & & \downarrow \\ E & \longrightarrow & \widetilde{F} & \longrightarrow & D & \xrightarrow{\varphi} & \Sigma E. \end{array}$$

Applying the functor $\mathcal{H}$ gives a map of long exact sequences, and as $\mathcal{H}^i$ takes $E_m \longrightarrow E$ and $D_m \longrightarrow D$ to isomorphisms for all $i > -m + A + 1$, we deduce that $\mathcal{H}^i$ takes the map λ_m to an isomorphism whenever $i > -m + A + 2$. Since F is assumed to belong to $\mathcal{T}_c^b \subseteq \mathcal{T}^b$, there exists an integer $B > 0$ with $\mathcal{H}^i(F) = 0$, for all $i < -B$. For any integer $i < -B$, choose an integer $m > 0$ with $-m + A + 2 < i$, and the isomorphism $\mathcal{H}^i(\lambda_m)\colon \mathcal{H}^i(F) \to \mathcal{H}^i(\widetilde{F})$ establishes that $\mathcal{H}^i(\widetilde{F}) \cong \mathcal{H}^i(F) = 0$. Thus $\mathcal{H}^i(\widetilde{F})$ also vanishes for all $i < -B$.

Therefore for any integer $m > B + A + 2$ we have that, for every $i \in \mathbb{Z}$, the functor $\mathcal{H}^i$ takes the map $\lambda_m\colon F \longrightarrow \widetilde{F}$ to an isomorphism. For $i \geqslant -B$ this is because m is chosen big enough, and for $i < -B$ it is because $\mathcal{H}^i(F) \cong 0 \cong \mathcal{H}^i(\widetilde{F})$. By [6, Lemma 3.6] the t–structure $\left(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\right)$ is non-degenerate (meaning that the intersections of all the $\mathcal{T}^{\leqslant n}$ and of all the $\mathcal{T}^{\geqslant n}$ are trivial), and hence the morphism $\lambda_m\colon F \longrightarrow \widetilde{F}$ must be an isomorphism in $\mathcal{T}$ (see [2, Proposition 1.3.7]). Replacing the triangle $E \longrightarrow \widetilde{F} \longrightarrow D \xrightarrow{\varphi} \Sigma E$ by an isomorphic triangle $E \longrightarrow F \longrightarrow D \xrightarrow{\varphi} \Sigma E$ completes the proof of (iii). □

In conclusion, for every $F \in \mathcal{T}_c^b$, we obtain a distinguished triangle $E \longrightarrow F \longrightarrow D \longrightarrow \Sigma E$ in $\mathcal{T}_c^-$. Since we need to remember the dependence on H and on F, we will write $E_{(F,H)}$ and $D_{(F,H)}$ for the objects E and D.

Lemma 9.2.2. *In the setting at the beginning of this section, if $F \in P_H$, then $\mathcal{T}_c^b \subseteq \left(D_{(F,H)}\right)^{\perp}$.*

Proof. Let $X \in \mathcal{T}_c^b$ be any object, and we need to prove the vanishing of $\mathrm{Hom}(D, X)$, where $D = D_{(F,H)}$. Because X belongs to $\mathcal{T}_c^b$ and H satisfies Hypothesis 9.1.3(i), there exists an integer $B > 0$ with $X \in \Sigma^B Q_H$. Because Hypothesis 9.1.3(ii) holds for H, we may (after increasing the integer $B > 0$) assume that $\mathrm{Hom}(\Sigma^B P_H, X) = 0$. The construction of the triangle (9) is such that $D_m \in \Sigma^m P_H$, and hence for all $m > B$ we have $\mathrm{Hom}(D_m, X) = 0$.

Complete the morphism $D_m \longrightarrow D$ to a distinguished triangle

$$D_m \longrightarrow D \longrightarrow \widetilde{D}_m \longrightarrow \Sigma D_m. \tag{10}$$

By Lemma 9.2.1(i) we may, after increasing the integer B, assume that $\widetilde{D}_m \in \mathcal{T}^{\leqslant -m+B}$. But X belongs to $\mathcal{T}_c^b \subseteq \mathcal{T}^+$, and hence $X \in \mathcal{T}^{\geqslant -N}$, for some $N > 0$. Hence, for all $m > B + N$, we have $\widetilde{D}_m \in \mathcal{T}^{<-B-N+B} = \mathcal{T}^{<-N}$ and so $\mathrm{Hom}(\widetilde{D}_m, X) = 0$.

Choose therefore any $m > B + N$. In the distinguished triangle (10) we have $\mathrm{Hom}(D_m, X) = 0 = \mathrm{Hom}(\widetilde{D}_m, X)$. Hence $\mathrm{Hom}(D, X) = 0$. □

We can now prove the main result of this section which gives an intrinsic description of $\mathcal{T}^c$ in $\mathcal{T}_c^b$ under a technical assumption.

Proposition 9.2.3. *Let $\mathcal{T}$ be a weakly approximable triangulated category such that $\mathcal{T}^c \subseteq \mathcal{T}_c^b$. Suppose that ${}^{\perp}(\mathcal{T}_c^b) \cap \mathcal{T}_c^- = \{0\}$. Assume $G \in \mathcal{T}_c^b$ belongs to the subcategory $\mathcal{T}^c$ and classically generates it. Then, in the category $\mathcal{S} = \mathcal{T}_c^b$, the equivalence class $[P_G]$ is the unique maximal* *one with respect to the partial order $\leqslant$, among the classes $[P_H]$ with $H \in \mathcal{S} = \mathcal{T}_c^b$ satisfying Hypothesis 9.1.3.*

Furthermore, an object $X \in \mathcal{T}_c^b$ belongs to the subcategory $\mathcal{T}^c \subseteq \mathcal{T}_c^b$ if and only if, for some object $H \in \mathcal{T}_c^b$ satisfying Hypothesis 9.1.3 and with $[P_H]$ maximal, we have $\mathrm{Hom}(X, P_H) = 0$.

Proof. Let $H \in \mathcal{T}_c^b$ be an object satisfying Hypothesis 9.1.3. Up to shift, we can assume $G \in P_H$, thanks to Hypothesis 9.1.3(i). So we can form in $\mathcal{T}_c^-$ the distinguished triangle

$$E_{(G,H)} \longrightarrow G \longrightarrow D_{(G,H)} \longrightarrow \Sigma E_{(G,H)}$$

of Lemma 9.2.1(iii). By Lemma 9.2.2 we have that $D_{(G,H)} \in \mathcal{T}_c^-$ is an object with $\big(D_{(G,H)}\big)^\perp$ containing $\mathcal{T}_c^b$. By hypothesis ${}^\perp(\mathcal{T}_c^b) \cap \mathcal{T}_c^- = \{0\}$, and hence $D_{(G,H)} = 0$. Therefore the map from $E = E_{(G,H)} \longrightarrow G$ is an isomorphism. Thus we can factor the identity of G as the composite $G \longrightarrow E \longrightarrow G$.

On the other hand, by assumption, we have that $E = \underrightarrow{\mathrm{Hocolim}}\, E_m$, with each E_m in some $\langle H\rangle^{[1-m-A,A]}$. The map from the compact object G to $E = \underrightarrow{\mathrm{Hocolim}}\, E_m$ factors through some $E_m \longrightarrow E$, allowing us to write the identity on G as a composite $G \longrightarrow E_m \longrightarrow G$. Therefore G is a direct summand of an object E_m, and hence belongs to $\langle H\rangle^{[-B,B]}$ for some integer $B > 0$. We deduce that $G[0,\infty) \subseteq \langle H\rangle^{[-B,\infty)}$. Taking orthogonals in $\mathcal{T}_c^b$ now gives the inclusion

$$\Sigma^B P_H = H[-B,\infty)^\perp = \Big(\langle H\rangle^{[-B,\infty)}\Big)^\perp \subseteq G[0,\infty)^\perp = P_G.$$

This shows $[P_H] \leqslant [P_G]$. The first part of the statement is then proven as Proposition 9.1.7 shows that G satisfies Hypothesis 9.1.3.

To prove the second part of the statement, suppose that $X \in \mathcal{T}_c^b$ is such that $\mathrm{Hom}(X, P_H) = 0$ for some H with $[P_H]$ maximal. Combining the first part of the statement with Lemma 9.1.5 we have that $X^\perp$ contains $\mathcal{T}_c^b \cap \mathcal{T}^{\leqslant -B}$ for some $B > 0$. Since $G \in \mathcal{T}^c$ is a classical generator, [18, Corollary 2.14] gives the existence of a distinguished triangle

$$E \longrightarrow X \longrightarrow D \longrightarrow \Sigma E, \tag{11}$$

with $E \in \langle G\rangle$ and $D \in \mathcal{T}^{\leqslant -B}$. As we are assuming that $\mathcal{T}^c \subseteq \mathcal{T}_c^b$ we have that $E \in \mathcal{T}^c$ must belong to $\mathcal{T}_c^b$, and the distinguished triangle (11), coupled with the fact that both E and X belong to $\mathcal{T}_c^b$, tells us that $D \in \mathcal{T}_c^b$ as well. Hence D belongs to $\mathcal{T}_c^b \cap \mathcal{T}^{\leqslant -B}$, and so the map $X \longrightarrow D$ in (11) must vanish. Thus X is a direct summand of the object $E \in \mathcal{T}^c$ and X must belong to $\mathcal{T}^c$.

To complete the proof note that, if $X \in \mathcal{T}^c$ and $G \in \mathcal{T}^c \subseteq \mathcal{T}_c^b$ is a classical generator, then $X \in \langle G\rangle^{[-B,B]}$ for some $B > 0$. Now, by definition, ${}^\perp P_{\Sigma^B G}$ contains $\langle G\rangle^{[-B,B]}$, and hence $X^\perp$ contains $P_{\Sigma^B G}$, with $[P_{\Sigma^B G}] = [P_G]$ maximal by the first part. □

10. Examples and applications

The first part of this section is about two geometric examples where the assumption ${}^\perp(\mathcal{T}_c^b) \cap \mathcal{T}_c^- = \{0\}$ is automatically verified: $\mathcal{T} = \mathbf{D}(R\text{–Mod})$ for R a coherent ring or, more generally, $\mathcal{T}$ a coherent weakly approximable triangulated category, and $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X quasi-compact and quasi-separated with $Z \subseteq X$ a closed subset with quasi-compact complement. This is done in Section 10.1 and Section 10.2, respectively.

10.1. **The coherent case.** The first situation where Proposition 9.2.3 applies is provided by weakly approximable triangulated categories of the following special type.

Definition 10.1.1. *A weakly approximable triangulated category* $\mathcal{T}$ *is* coherent *if, for any t–structure* $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ *in the preferred equivalence class, there exists an integer* $N > 0$ *such that every object* $Y \in \mathcal{T}_c^-$ *admits a distinguished triangle* $X \longrightarrow Y \longrightarrow Z$ *with* $X \in \mathcal{T}_c^- \cap \mathcal{T}^{\leqslant N}$ *and with* $Z \in \mathcal{T}_c^b \cap \mathcal{T}^{\geqslant 0}$.

Example 10.1.2. Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose there exists, in the preferred equivalence class, a t–structure $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ such that $\big(\mathcal{T}_c^- \cap \mathcal{T}^{\leqslant 0}, \mathcal{T}_c^- \cap \mathcal{T}^{\geqslant 0}\big)$ is a t–structure on $\mathcal{T}_c^-$. Then it is automatic that $\mathcal{T}$ is coherent. After all: for every object $Y \in \mathcal{T}_c^-$, the truncation triangles $Y^{\leqslant 0} \longrightarrow Y \longrightarrow Y^{\geqslant 1}$ will satisfy $Y^{\leqslant 0} \in \mathcal{T}_c^- \cap \mathcal{T}^{\leqslant 0}$ and $Y^{\geqslant 1} \in \mathcal{T}_c^- \cap \mathcal{T}^{\geqslant 1} = \mathcal{T}_c^b \cap \mathcal{T}^{\geqslant 1}$

Concrete examples abound. If there exists a left coherent ring R and a triangulated equivalence of categories $\mathcal{T} \cong \mathbf{D}(R\text{–Mod})$, then $\mathcal{T}$ is weakly approximable, the standard t–structure on $\mathbf{D}(R\text{–Mod})$ is in the preferred equivalence class, and it restricts to a t–structure on $\mathcal{T}_c^- = \mathbf{D}^-(R\text{–mod})$. Similarly: if there exists a coherent and quasi-separated scheme X, with $Z \subseteq X$ a closed subscheme such that $X \setminus Z$ is quasi-compact, and a triangulated equivalence of categories $\mathcal{T} \cong \mathbf{D}_{\mathbf{qc},Z}(X)$, then $\mathcal{T}$ is a weakly approximable, the standard t–structure on $\mathbf{D}_{\mathbf{qc},Z}(X)$ is in the preferred equivalence class, and it restricts to a t–structure on the subcategory $\mathcal{T}_c^- = \mathbf{D}^-_{\mathrm{coh},Z}(X)$.

Proposition 10.1.3. *If* $\mathcal{T}$ *is a weakly approximable and coherent triangulated category such that* $\mathcal{T}^c \subseteq \mathcal{T}_c^b$*, then the condition* ${}^\perp(\mathcal{T}_c^b) \cap \mathcal{T}_c^- = \{0\}$ *of Proposition 9.2.3 is satisfied.*

Proof. Let $\big(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 1}\big)$ be a t–structure in the preferred equivalence class and assume that $D \in \mathcal{T}_c^-$ is an object with $D^\perp \supseteq \mathcal{T}_c^b$. By the coherence assumption, we get distinguished triangles

$$X_m \longrightarrow \Sigma^m D \longrightarrow Z_m \longrightarrow \Sigma X_m,$$

with $X_m \in \mathcal{T}_c^- \cap \mathcal{T}^{\leqslant N}$ and $Z_m \in \mathcal{T}_c^- \cap \mathcal{T}^{\geqslant 0} \subseteq \mathcal{T}_c^b$. But then the morphisms $\Sigma^m D \longrightarrow Z_m$ vanish, for all $m \in \mathbb{Z}$ and $\Sigma^m D$ is a direct summand of $X_m \in \mathcal{T}^{\leqslant N}$. Therefore D must belong to $\bigcap_{m=0}^\infty \mathcal{T}^{\leqslant -m} = \{0\}$, where the vanishing is by [6, Lemma 3.6]. Thus $D = 0$ and we are done. □

10.2. **The case of $\mathbf{D}_{\mathbf{qc},Z}(X)$.** Let X be a quasi-compact, quasi-separated scheme, and let $Z \subseteq X$ be a closed subset with quasi-compact complement. The category $\mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable by Example 4.1.3(ii). On the other hand, [15, Theorem 3.2(i)] says that the compact objects in $\mathbf{D}_{\mathbf{qc},Z}(X)$ are all perfect complexes. In particular they have bounded cohomology. Thus we are in the situation where $\mathcal{T}^c \subseteq \mathcal{T}_c^b$

We can now prove the following key result.

Proposition 10.2.1. *The assumption* ${}^\perp(\mathcal{T}_c^b) \cap \mathcal{T}_c^- = \{0\}$ *holds for* $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$.

Proof. Let $D \in \mathcal{T}_c^-$ be an object such that $D^\perp$ contains $\mathcal{T}_c^b$, and hence $\mathcal{T}^c$. In order to prove that $D = 0$, it is enough to show that, for all open immersions $i\colon W \hookrightarrow X$ with $W \subseteq X$ an affine open subset, we have $i^*D = 0$.

For $i\colon W \hookrightarrow X$ as above, [15, Lemma 7.5(i)] shows that, for any compact object $F \in \mathbf{D}_{\mathbf{qc},Z\cap W}(W)$ and any compact generator $G \in \mathbf{D}_{\mathbf{qc},Z}(X)$, there exists an integer $M > 0$ with $i_*F \in \overline{\langle G\rangle}^{[-M,M]}$. But then any morphism $D \longrightarrow i_*F$ is a map from $D \in \mathcal{T}_c^-$ to $i_*F \in \overline{\langle G\rangle}^{[-M,M]}$, and [17, Lemma 2.7] tells us that it must factor as $D \longrightarrow E \longrightarrow i_*F$ for some object $E \in \langle G \rangle^{[-M,M]}$. On the other hand, the map $D \longrightarrow E$ must vanish because $D^\perp$ contains $\langle G \rangle^{[-M,M]} \subseteq \mathcal{T}^c$. Hence, for every compact object $F \in \mathbf{D}_{\mathbf{qc},Z\cap W}(W)$, we have

$$\mathrm{Hom}(i^*D, F) \cong \mathrm{Hom}(D, i_*F) \cong 0.$$

The output is that we can reduce to the case where X is an affine scheme.

Let $j \colon X \setminus Z \longrightarrow X = \mathrm{Spec}\,(R)$ be the open immersion, and complete the unit of adjunction $\mathcal{O}_X \longrightarrow j_*j^*\mathcal{O}_X$ to a distinguished triangle

$$L' \longrightarrow \mathcal{O}_X \longrightarrow j_*j^*\mathcal{O}_X \longrightarrow \Sigma L'. \tag{12}$$

Choose any compact generator $H \in \mathbf{D}_{\mathbf{qc},Z}(X)$. By [15, Lemma 7.4] there exists an integer $r > 0$ such that $L' \in \overline{\langle H \rangle}^{[-r,r]} \subseteq \mathbf{D}_{\mathbf{qc},Z}(X)$. Suppose $D \longrightarrow \Sigma^n L'$ is some morphism. Since it is a morphism from an object in $\mathcal{T}_c^-$ to an object in $\overline{\langle H \rangle}^{[-r-n,r-n]}$, [17, Lemma 2.7] allows us to factor it through an object $F \in \langle H \rangle^{[-r-n,r-n]} \subseteq \mathcal{T}^c$. As $D^\perp$ contains $\mathcal{T}^c$, the map $D \longrightarrow F$ and therefore the composite $D \longrightarrow F \longrightarrow \Sigma^n L'$ must vanish. As a consequence, the functor $\mathrm{Hom}(D, -)$ applied to $\Sigma^n L'$ is trivial, for all $n \in \mathbb{Z}$. On the other hand,

$$\mathrm{Hom}(D, \Sigma^n j_*j^*\mathcal{O}_X) \cong \mathrm{Hom}(j^*D, \Sigma^n j^*\mathcal{O}_X) = 0$$

because $D \in \mathbf{D}_{\mathbf{qc},Z}(X)$ and hence $j^*D \in j^*\mathbf{D}_{\mathbf{qc},Z}(X) = \{0\}$. From (12), we deduce that $\mathrm{Hom}(D, -)$ applied to $\Sigma^n \mathcal{O}_X$ is trivial, for all $n \in \mathbb{Z}$. Under the equivalence $\mathcal{T} \subseteq \mathbf{D}_{\mathbf{qc}}(X) \cong \mathbf{D}(R)$, the complex $D \in \mathcal{T}_c^-$ is a bounded above cochain complex of finitely generated projective R-modules and we have just proved that $\mathrm{Hom}(D, \Sigma^n R) = 0$ for all $n \in \mathbb{Z}$. Applying Theorem A.1 to the complex $P^\bullet := \mathrm{Hom}(D, R)$, we deduce that $D = 0$. □

Appendix A. Appendix by Christian Haesemeyer

In this appendix, we prove the following:

Theorem A.1. *Let R be a commutative ring, and $P^\bullet$ a bounded below cochain complex of finitely generated projective R-modules. If $P^\bullet$ is acyclic, then it is chain contractible.*

Remark A.2. *The statement also holds for (not necessarily commutative) rings with bounded finitistic dimension. (Noetherian local commutative rings fall into this class, by the Auslander - Buchsbaum formula.) For more on this, see* [31]*. However, the statement is false, in general, over non-commutative rings (see* [24]*).*

In order to prove the theorem, it suffices to show that the cocycle modules $Z^n(P^\bullet)$ are projective for all n. Since they are finitely presented by hypothesis, we can and will assume that R is local and we are dealing with a complex of free modules. We employ the following definition, due to Hochster [8]; we will use the version of the definition given by Northcott in [23, Section 5.5]. We remind the reader that the term "grade" is often called "depth" in recent literature; we will stick with grade to align with the references we use.

Definition A.3. *Let I be an ideal in R, and M an R-module. The* true *or* polynomial *grade of I on M, denoted* $\mathrm{Gr}(I; M)$*, is the limit (possibly ∞) of the increasing sequence*

$$\mathrm{gr}(IR[x_1, \ldots, x_n]; R[x_1, \ldots, x_n] \otimes_R M),$$

where $\mathrm{gr}(IR[x_1, \dots, x_n]; R[x_1, \dots, x_n] \otimes_R M)$ *denotes the (classical) grade, that is the upper bound on the length of* $R[x_1, \dots, x_n] \otimes_R M$*-regular sequences in* $IR[x_1, \dots, x_n]$ *(again, possibly infinite). Following usual practice, we write* $\mathrm{Gr}(I)$ *for* $\mathrm{Gr}(I; R)$*.*

Northcott proves (see [23, Chapter 5, Theorems 9 and 13]):

Lemma A.4. *Let* $I \subseteq R$ *be a proper ideal that can be generated by* n *elements. Then* $\mathrm{Gr}(I) \leqslant n$*.*

Given an R-homomorphism $\phi \colon E \to F$ of finite rank free R-modules, write $\mathfrak{A}_\nu(\phi)$ for the determinantal ideal of R generated by the $\nu \times \nu$-minors of ϕ (this ideal does not depend on a choice of basis of the modules E and F). The rank of ϕ is defined as the maximal ν such that $\mathfrak{A}_\nu(\phi) \neq 0$ (see [23, Equation (3.2.1)]); we denote the determinantal ideal with ν the rank of ϕ by $\mathfrak{A}(\phi)$.

Now suppose we are given a (chain) complex $0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} F_0$ of finite rank free R-modules which is exact except, possibly, at F_0. Northcott proves the following result (see [23, Chapter 6, Theorem 14]); the noetherian version of this theorem is due to Buchsbaum and Eisenbud [5].

Theorem A.5. *For* $0 < i \leqslant n$*, we have the inequality* $\mathrm{Gr}(\mathfrak{A}(d_i)) \geqslant i$*.*

With this in hand, we are ready to prove Theorem A.1.

Proof. (of Theorem A.1) As discussed above, we may assume that R is local, and the complex $P^\bullet$ is a complex of free modules of finite rank; it then suffices to show show that $Z^k(P^\bullet)$ is free for all integers k. Consider the differential $d^{k-2} \colon P^{k-2} \to P^{k-1}$. Its determinantal ideal $\mathfrak{A}(d^{k-2})$ is finitely generated, say, by s generators. Applying Theorem A.5 to the complex $P^\bullet$, brutally truncated in degree $k+s$, we conclude that $\mathfrak{A}(d^{k-2})$ must have true grade greater than s, and hence, cannot be a proper ideal by Lemma A.4. By [23, Chapter 6, Exercise 11], the cokernel of d^{k-2}, that is, the cocycle module $Z^k(P^\bullet)$, is free. □

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