UNIQUENESS OF ENHANCEMENTS OF TRIANGULATED CATEGORIES

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ABSTRACT. We survey recent progress on uniqueness of enhancements of triangulated categories, and existence of lifts of exact functors. We focus on the geometric context.

Classical homological algebra can roughly be thought of as the study of functors taking geometric objects (e.g. manifolds) to simple algebraic invariants (e.g. abelian groups). Around 1960 people began looking at refinements — such as functors taking manifolds to cochain complexes of abelian groups. And almost from the start there were several options considered, depending on how much information one wanted to retain about the morphisms between cochain complexes.

Triangulated categories are the minimalistic option: they remember only the *homotopy equivalence class* of the cochain maps. It is well-established what a triangulated category should be — the axioms, as well as the key examples, were formulated early on by Verdier. There is, by contrast, no consensus on what ought to be the "right" framework for *enhancements,* which keep more of the information: in the literature the reader will encounter the pretriangulated dg categories of Bondal and Kapranov, the pretriangulated A_∞-categories in the sense of Fukaya, the stable model categories of Hovey, Palmieri and Strickland, and (most recently) the stable infinity categories of Lurie.

Let **Tri** be the 2-category whose objects are triangulated categories, and let **Enh** be the 2-category whose objects are the reader's preferred flavor of enhancements. A 1-morphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is called an *equivalence* if there exists a 1-morphism $\psi : \mathcal{B} \longrightarrow \mathcal{A}$ and 2-isomorphisms $\mathrm{id} \cong \varphi\psi$ and $\mathrm{id} \cong \psi\varphi$. There is always a forgetful 2-functor $\mathsf{Fgt} : \mathbf{Enh} \longrightarrow \mathbf{Tri}$, and it is customary to adopt the following conventions:

Definition 1. (i) *A* 1*-morphism* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *in* **Enh** *is called a* quasi-equivalence *if* $\mathsf{Fgt}(\varphi)$ *is an equivalence in the* 2*-category* **Tri***.*

(ii) *An* enhancement $\mathcal{A} \in \mathbf{Enh}$ *for the triangulated category* $\mathcal{T} \in \mathbf{Tri}$ *means an object for which there exists an equivalence* $\varphi : \mathcal{T} \longrightarrow \mathsf{Fgt}(\mathcal{A})$*.*

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(iii) *A triangulated category* $\mathcal{T}$ *is said to* have a unique enhancement *if it has an enhancement, and for any pair of enhancements* $\mathcal{A}, \mathcal{B} \in \mathbf{Enh}$ *there is in the* 2*-category* **Enh** *a zigzag of quasi-equivalences connecting* $\mathcal{A}$ *and* $\mathcal{B}$*, meaning a diagram*

$$\mathcal{A} \longleftarrow \mathcal{C}_1 \longrightarrow \mathcal{C}'_1 \longleftarrow \mathcal{C}_2 \longrightarrow \mathcal{C}'_2 \longleftarrow \cdots \longrightarrow \mathcal{C}'_{n-1} \longleftarrow \mathcal{C}_n \longrightarrow \mathcal{B}$$

where all the arrows are quasi-equivalences as defined in (i)*.*

(iv) *Given two objects* $\mathcal{A}, \mathcal{B}$ *in* **Enh***, a* 1*-morphism* $\varphi : \mathsf{Fgt}(\mathcal{A}) \longrightarrow \mathsf{Fgt}(\mathcal{B})$ *in* **Tri** *is* liftable *if, in the* 2*-category* **Enh***, we have a zigzag as in* (iii) *whose image under the functor* Fgt *is a* 1*-morphism isomorphic to* φ*.*

Problem 2. (1) Does every triangulated category $\mathcal{T}$ have an enhancement? (2) Are the enhancements unique? and (3) Are all triangulated functors liftable?

The answer to Problem 2(1) is negative but it took a long time for anyone to come up with examples, and the phenomenon is not well-understood. The only two known counterexamples to date are due to Muro–Schwede–Strickland [6] and Rizzardo–Van den Bergh [9]. For Problem 2 (2) and (3), the early evidence was that there too the answer was negative. The first known counterexample to (3) was in the paper [7] by the second author. The first two counterexamples to (2) may be found are due to Schwede [12, Section 2.1] and Schlichting [11]. Looking at the evidence available in the early 2000s, one would have probably concluded that Problem 2(1) was impossibly hard, and the answer to Problems 2 (2) and (3) was a resounding No. The only positive evidence about these problems came from the paper [8] by Orlov, which proves that any fully faithful functor $\mathbf{D}^b_{\mathrm{coh}}(X_1) \longrightarrow \mathbf{D}^b_{\mathrm{coh}}(X_2)$, with each X_i a smooth, projective variety over a field, is liftable. In the light of this background comes the brave, inspired 2004 conjecture of Bondal, Larsen and Lunts [2]:

Conjecture 3. Natural, geometric categories like $\mathbf{D}^b_{\mathrm{coh}}(X)$ have unique enhancements, and the triangulated functors between them are all liftable.

It turns out that the half of the conjecture about functors is false in general. The first counterexamples are due to Vologodsky [13] and Rizzardo–Van den Bergh–Neeman [10]. But there is also good news about the conjecture: (a) The uniquencess of enhancements, in the sense of Definition 1(iii), is true for a wide class of "natural" triangulated categories. And (b) there are large classes of triangulated functors which are liftable.

In the remainder of the survey we will confine our attention to (a), we will now summarize the literature on the natural triangulated categories which are known to have unique enhancements. The highlights may be found in four articles: Lunts and Orlov [5], Canonaco and Stellari [4], Antieau [1], and Canonaco, Neeman and Stellari [3] The short summary is that the class of triangulated categories proved to have unique enhancements grows larger as we progress through the articles. The first in the series, [5], used the techniques of compact generation, the article [4] improved the results by employing the more powerful machinery of well generated triangulated categories, and [1] upped the ante by using ∞-category methods. And the much-improved results obtained in [1] seemed to offer compelling evidence of the superiority of the ∞-category machine.

The fourth article in the series [3] breaks the pattern. In terms of the machinery employed it represents a long step down: the main theorems rely on little more modern than a 1970 theorem of Auslander's. But not only is the article able to reproduce the results of [1] without the machinery, it ends up solving most of the open questions in [1] and in other parts of the literature.

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