Derived Torelli Theorem and Orientation

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Joint work with D. Huybrechts and E. Macrì (math.AG/0608430 + work in progress)

Outline

1. **Derived Torelli Theorem**
   - Motivations
   - The statement
   - Ideas form the proof
   - The conjecture

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   - The statement
   - Ideas form the proof
   - The conjecture
2. **The generic case**
   - The result
   - Sketch of the proof

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Let X be a K3 surface (i.e. a smooth complex compact simply connected surface with trivial canonical bundle).

Main problem

Describe the group $\mathrm{Aut}\,(\mathrm{D^b}(X))$ of exact autoequivalences of the triangulated category

$$\mathrm{D^b}(X) := \mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_X\text{-}\mathbf{Mod}).$$

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Remark (Orlov)

Such a description is available when X is an abelian surface (actually an abelian variety).

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1. If $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ is an equivalence, then there exists a naturally defined Hodge isometry

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2. Suppose there exists a Hodge isometry $g : \widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(Y, \mathbb{Z})$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ such that $\Phi_* = g$.

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Hodge structure: The weight-2 Hodge structure on $H^*(X,\mathbb{Z})$ is

$$\begin{aligned}\widetilde{H}^{2,0}(X) &:= H^{2,0}(X),\\ \widetilde{H}^{0,2}(X) &:= H^{0,2}(X),\\ \widetilde{H}^{1,1}(X) &:= H^0(X,\mathbb{C})\oplus H^{1,1}(X)\oplus H^4(X,\mathbb{C}).\end{aligned}$$

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3. It is easy to see that the isometry

$$j := (\mathrm{id})_{H^0 \oplus H^4} \oplus (-\mathrm{id})_{H^2}$$

is not orientation preserving.

Problem

According to the Derived Torelli Theorem, is the isometry j induced by an autoequivalence?

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Ideas from the proof

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Definition

$F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ is of Fourier–Mukai type if there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

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The complex $\mathcal{E}$ is called the kernel of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

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$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \text{ if } j < 0.$$

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Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$.

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Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniq. det. up to isomorphism.

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Using the Chern character one gets the commutative diagram:

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}(X) & \xrightarrow{\Phi} & \mathrm{D}^{\mathrm{b}}(Y) \\
\downarrow [-] & & \downarrow [-] \\
K(X) & \longrightarrow & K(Y) \\
\downarrow \mathrm{ch}\,(-)\cdot\sqrt{\mathrm{td}(X)} & & \downarrow \mathrm{ch}\,(-)\cdot\sqrt{\mathrm{td}(Y)} \\
\widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\Phi_*} & \widetilde{H}(Y,\mathbb{Z})
\end{array}$$

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Let $\mathrm{O}_+ := \mathrm{O}_+(\widetilde{H}(X, \mathbb{Z}))$ be the group of orientation preserving Hodge isometries of $\widetilde{H}(X, \mathbb{Z})$.

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Using the conjecture, we would get

$$1 \to ? \to \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X)) \xrightarrow{\Pi} \mathrm{O}_+ \to 1.$$

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Theorem (Huybrechts-Macrì-S.)

Let X and Y be generic analytic K3 surfaces (i.e. $\mathrm{Pic}\,(X) = \mathrm{Pic}\,(Y) = 0$). If

$$\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$$

is an equivalence of Fourier-Mukai type, then up to shift

$$\Phi_{\mathcal{E}} \cong T^{n}_{\mathcal{O}_Y} \circ f_*$$

for some $n \in \mathbb{Z}$ and an isomorphism

$$f : X \xrightarrow{\sim} Y.$$

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Definition

An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a spherical if

$$\mathrm{Hom}\,(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

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In particular, $\mathcal{O}_X$ is spherical.

The spherical twist $T_{\mathcal{O}_X} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ that sends $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ to the cone of

$$\bigoplus_i (\mathrm{Hom}\,(\mathcal{O}_X, \mathcal{F}[i]) \otimes \mathcal{O}_X[-i]) \to \mathcal{F}$$

is an orientation preserving equivalence.

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- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(X)$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

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(a) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

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with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

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- The minimal objects of $\mathcal{P}(\phi)$ are called stable of phase ϕ.
- The category $\mathcal{P}((0,1])$ is called the heart of σ.

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The group $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X))$ of exact autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ acts on $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$.

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Given $z = u + iv \in R$, take the subcategories

$$\mathcal{F}(z), \mathcal{T}(z) \subset \mathbf{Coh}(X)$$

defined as follows:

Stability conditions: the generic case

1. If $z \in R_+ \cup R_0$ then $\mathcal{F}(z)$ and $\mathcal{T}(z)$ are respectively the full subcategories of all torsion free sheaves and torsion sheaves.
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$$\mathcal{A}(z) := \left\{ \mathcal{E} \in \mathrm{D^b}(X) : \begin{array}{l} \bullet\ H^0(\mathcal{E}) \in \mathcal{T}(z) \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathcal{F}(z) \\ \bullet\ H^i(\mathcal{E}) = 0 \text{ oth.} \end{array} \right\}.$$

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For any $z = u + iv \in R$ we define the function

$$\begin{array}{rcl} Z : \mathcal{A}(z) & \to & \mathbb{C} \\ \mathcal{E} & \mapsto & \langle v(\mathcal{E}), (1, 0, z) \rangle = -u \cdot r - s - i(r \cdot v), \end{array}$$

where $v(\mathcal{E}) = (r, 0, s)$ is the Mukai vector of $\mathcal{E}$.

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Lemma

For any $z \in R$ the function Z defines a stability function on $\mathcal{A}(z)$ which has the Harder-Narasimhan property.

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Proposition

For any $\sigma \in \mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$, there is $n \in \mathbb{Z}$ such that $T^n_{\mathcal{O}_X}(\mathcal{O}_p)$ is stable in σ, for any closed point $p \in X$.

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Lemma

If $z \in \mathbb{R}_{<0}$, then the only semi-rigid stable objects in $\mathcal{A}(z)$ are the skyscraper sheaves.

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Thus, the heart $\mathcal{P}'((0, 1])$ of the t-structure associated to σ' (identified with $\mathcal{A}(z)$) contains as stable objects the images $\Psi(\mathcal{O}_p)$ of all closed points $p \in X$ and all skyscraper sheaves $\mathcal{O}_y$.

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(e) But there are no non-trivial line bundles on Y.

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The manifold parametrizing numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ is connected and simply-connected.

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The manifold parametrizing numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ is connected and simply-connected.

This proves a conjecture by Bridgeland in the generic analytic case.

Outline

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We proceed by contradiction assuming that there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X)$ such that $(\Phi_{\mathcal{E}})_* = j$.

The twistor space

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Definition

A Kähler class $\omega \in H^{1,1}(X,\mathbb{R})$ is called very general if there is no non-trivial integral class $0 \neq \alpha \in H^{1,1}(X,\mathbb{Z})$ orthogonal to ω, i.e. $\omega^{\perp} \cap H^{1,1}(X,\mathbb{Z}) = 0$.

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Take the twistor space $\mathbb{X}(\omega)$ of X determined by the choice of a very general Kähler class $\omega \in \mathcal{K}_X \cap \mathrm{Pic}\,(X) \otimes \mathbb{R}$. Hence we get a complex deformation

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Take $R := \mathbb{C}[[t]]$ to be the ring of power series in t with residue field $K := \mathbb{C}((t))$.

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If $R_n := k[[t]]/t^{n+1}$, then the infinitesimal neighbourhoods

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \mathrm{Spec}\,(R_n),$$

form an inductive system and give rise to a formal R-scheme

$$\pi : \mathcal{X} \to \mathrm{Spf}(R),$$

which is the formal neighbourhood of X in $\mathbb{X}(\omega)$.

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Proposition

Let $v_1 \in H^1(X, \mathcal{T}_X)$ be the Kodaira–Spencer class of first order deformation given by a twistor space $\mathbb{X}(\omega)$ as above. Then

$$v_1' := \Phi_{\mathcal{E}}^{HH}(v_1) \in H^1(X, \mathcal{T}_X).$$

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Proposition (Toda)

For v_1 and v'_1 as before, there exists $\mathcal{E}_1 \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_1 \times_{R_1} \mathcal{X}'_1)$ such that

$$i_1^* \mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}'_1 \times_{R_1} \mathcal{X}'_1$ is the natural inclusion.

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Hence there is a first order deformation of $\mathcal{E}$.

Higher order deformations

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Work in progress (... almost concluded)

Construct at any order n, an analytic deformation $\mathcal{X}'_n$ such that there exists $\mathcal{E}_n \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_n \times_{R_n} \mathcal{X}'_n)$, with

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Our approach imitates the first order case.

Outline

1. Derived Torelli Theorem
   - Motivations
   - The statement
   - Ideas form the proof
   - The conjecture
2. The generic case
   - The result
   - Sketch of the proof
3. The general projective case
   - The strategy
   - Deforming kernels
   - Concluding the argument

Equivalences go to equivalences

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There exists a sequence

$$\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}') \to \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K),$$

where $\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}')$ is the abelian category of sheaves on $\mathcal{X} \times_R \mathcal{X}'$ supported on $X \times X$.

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Let $\widetilde{\mathcal{E}} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X} \times_R \mathcal{X}')$ be such that $\mathcal{E} = i^*\widetilde{\mathcal{E}}$ (here $i : X \times X \to \mathcal{X} \times_R \mathcal{X}'$ is the inclusion). Then $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}_K$ are kernels of Fourier–Mukai equivalences.

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Here $\widetilde{\mathcal{E}}_K$ is the image via the natural functor in

$$\mathrm{D}^{\mathrm{b}}((\mathcal{X} \times_R \mathcal{X}')_K) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)).$$

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The triangulated category $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(\mathcal{X}_K))$ is a generic K3 category, i.e. [2] is the Serre functor and $(\mathcal{O}_{\mathcal{X}})_K$ is, up to shifts, the unique spherical object.

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Use the generic analytic case

Hence, reasoning as the analytic generic case, one can compose $\Phi_{\mathcal{E}_K}$ with some power of the spherical twist by $(\mathcal{O}_{\mathcal{X}})_K$ getting a Fourier–Mukai equivalence $\Phi_{\mathcal{G}_K}$ where $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$.

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$$(\Phi_{\mathcal{G}_0})_* : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

is such that $(\Phi_{\mathcal{G}_0})_* = (\Phi_{\mathcal{E}})_* = j$.

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Notice that $\mathcal{G}_0$ and $\mathcal{E}$ have the same Mukai vector!

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Open question

Which is the kernel of the map $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$?