Fourier–Mukai functors: existence

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Outline

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Derived categories (...roughly...)

Let **A** be an abelian category (e.g., **mod**-R, right R-modules, R an ass. ring with unity, and **Coh**(X)).

Definition

The **bounded derived category** $\mathrm{D}^b(\mathbf{A})$ of the abelian category **A** is such that:

- Objects: complexes of objects in **A**;
- Morphisms (roughly speaking): morphisms of complexes $+$ morphisms which are iso on cohomology are iso in $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$.

It is a **triangulated** category.

Triangulated categories (...roughly...)

Definition

A category **T** is **triangulated** if it is has an automorphism (called **shift**) $[1] : \mathbf{T} \to \mathbf{T}$, and a family of distinguished triangles $A \to B \to C \to A[1]$ satisfying certain axioms.

Definition

A functor $\mathrm{F} : \mathbf{T} \to \mathbf{T}'$ between triangulated categories is **exact** if it preserves shifts and distinguished triangles, up to isomorphism.

Given X, Y smooth projective varieties, a morphism $f : X \to Y$ and $\mathcal{E} \in \mathrm{D^b}(X)$ one has the exact (derived!) functors:

- $f_* : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ and $f^* : \mathrm{D^b}(Y) \to \mathrm{D^b}(X)$;
- $\mathcal{E} \otimes (-) : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$.

Mukai's example (1981)

Mukai studied a **duality** between $\mathrm{D}^{\mathrm{b}}(A)$ and $\mathrm{D}^{\mathrm{b}}(\hat{A})$ (here A is an abelian variety).

This is an equivalence

$$\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(A) \longrightarrow \mathrm{D}^{\mathrm{b}}(\hat{A})$$

such that $\mathsf{F}(-) := p_*(\mathcal{P} \otimes q^*(-))$ where $\mathcal{P} \in \mathbf{Coh}(A \times \hat{A})$ is the universal Picard sheaf.

The inverse of F sends a skyscraper sheaf $\mathcal{O}_{\mathrm{p}}$ (here p is a closed point of $\hat{A}$) on $\hat{A}$ to the degree 0 line bundle $L_{\mathrm{p}} \in \mathrm{Pic}^0(A)$ parametrized by p.

Fourier–Mukai functors

For X_1 and X_2 smooth projective varieties, we define the exact functor $\Phi_{\mathcal{E}} \colon \mathrm{D^b}(X_1) \to \mathrm{D^b}(X_2)$ as

$$\Phi_{\mathcal{E}}(-) := (p_2)_*(\mathcal{E} \otimes p_1^*(-)),$$

where $p_i \colon X_1 \times X_2 \to X_i$ is the natural projection and $\mathcal{E} \in \mathrm{D^b}(X_1 \times X_2)$.

Definition

An exact functor $\mathsf{F} \colon \mathrm{D^b}(X_1) \to \mathrm{D^b}(X_2)$ is a **Fourier–Mukai functor** (or of **Fourier–Mukai type**) if there exist $\mathcal{E} \in \mathrm{D^b}(X_1 \times X_2)$ and an isomorphism of functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$. The complex $\mathcal{E}$ is called a **kernel** of F.

Motivations

Assume that the base field is $\mathbb{C}$.

1. Fourier–Mukai functors (and equivalences) act on singular cohomology and preserve several additional structures (special Hodge decompositions and a special pairing).
2. They also act on Hochschild homology and cohomology. Hence one may control (first order) deformations of the varieties and of the Fourier–Mukai kernel at the same time.

Example

(1) and (2) allowed to give a partial description of the group of autoequivalences for K3 surfaces as conjectured by Szendroi (Huybrechts–Macrì–S.).

Two basic questions

1. Are all exact functors between the bounded derived categories of coherent sheaves on smooth projective varieties of Fourier–Mukai type?
2. Is the kernel of a Fourier–Mukai functor unique (up to isomorphism)?

Remark

A positive answer to the first one was conjectured by Bondal–Larsen–Lunts (and Orlov).

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Orlov's result

The following partly answers the above questions.

Theorem (Olov, 1997)

Let X_1 and X_2 be smooth projective varieties and let $\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ be an exact fully faithful functor admitting a left adjoint. Then there exists a unique (up to isomorphim) $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$ such that $\mathsf{F} \cong \Phi_{\mathcal{E}}$.

Bondal–van den Bergh: the adjoints always exist in this special setting (i.e. X_i smooth projective)!

Full implies faithful (in this case)

Aim: weaken the hypotheses of the theorem to get more general answers to (1)–(2).

Theorem (Canonaco–Orlov–S.)

Let X be a noetherian connected scheme, let $\mathbf{T}$ be a triangulated category and let $\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{T}$ be a full exact functor not isomorphic to the zero functor. Then F is also faithful.

Remark

- The result holds in much greater generality.
- The faithfulness assumption is redundant.

The improvement in the smooth case

Theorem (Canonaco–S., 2006)

Let X_1 and X_2 be smooth projective varieties and let $\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X_1)$,

$$(*) \qquad \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X_2)}(\mathsf{F}(\mathcal{F}), \mathsf{F}(\mathcal{G})[j]) = 0 \ \text{ if } j < 0.$$

Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$ and an isomorphism of functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

All exact functors $\mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ obtained by deriving an exact functor $\mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$ satisfy the assumption.

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Categories

Let X be a separated scheme of finite type over $\Bbbk$ and let Z be a subscheme of X which is proper over $\Bbbk$.

- $\mathrm{D}_Z(\mathbf{Qcoh}(X))$ is the derived category of unbounded complexes of quasi-coherent sheaves on X with cohomologies supported on Z.
- $\mathbf{Perf}(X)$ is the full subcategory of $\mathrm{D}(\mathbf{Qcoh}(X))$ consisting of complexes locally quasi-isomorphic to complexes of locally free sheaves of finite type over X.

We set

$$\mathbf{Perf}_Z(X) := \mathrm{D}_Z(\mathbf{Qcoh}(X)) \cap \mathbf{Perf}(X).$$

Assumptions

Let X_1 be a quasi-projective scheme containing a projective subscheme Z_1 such that $\mathcal{O}_{iZ_1} \in \mathbf{Perf}(X_1)$, for all $i > 0$ (e.g. either $Z_1 = X_1$ or X_1 is smooth), and let X_2 be a separated scheme of finite type over a field $\Bbbk$ with a proper subscheme Z_2.

$\mathsf{F} \colon \mathbf{Perf}_{Z_1}(X_1) \to \mathbf{Perf}_{Z_2}(X_2)$ is an exact functor such that

1. For any $\mathcal{A}, \mathcal{B} \in \mathbf{Coh}_{Z_1}(X_1) \cap \mathbf{Perf}_{Z_1}(X_1)$ and any integer $k < 0$, $\mathrm{Hom}\,(\mathsf{F}(\mathcal{A}), \mathsf{F}(\mathcal{B})[k]) = 0$;
2. For all $\mathcal{A} \in \mathbf{Perf}_{Z_1}(X_1)$ with trivial cohomologies in (strictly) positive degrees, there is $N \in \mathbb{Z}$ such that

$$\mathrm{Hom}\,(\mathsf{F}(\mathcal{A}), \mathsf{F}(\mathcal{O}_{|i|Z_1}(jH_1))) = 0,$$

for any $i < N$ and any $j << i$, where H_1 is an ample divisor on X_1.

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The statement

Theorem (Canonaco–S.)

If X_1, X_2, Z_1, Z_2 and F are as above, then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$ and an isomorphism of functors

$$\mathsf{F} \cong \Phi^s_{\mathcal{E}}.$$

Moreover, if X_i is smooth quasi-projective, for $i = 1, 2$, and $\mathbb{k}$ is perfect, then $\mathcal{E}$ is unique up to isomorphism.

Remark

$\Phi^s_{\mathcal{E}}$ is the natural generalization of the notion of Fourier–Mukai functor.

Remarks

If $Z_i = X_i$ and X_i is smooth, then the assumption (2) on the functor F is redundant. In particular we recover the previous generalization of Orlov's result involving only $(*)$.

If we just assume $X_i = Z_i$ (and no smoothness required!), we get a generalization of a very nice (and important) recent result by **Lunts–Orlov**.

Remark

As in Lunts–Orlov's case, we also get results about the (strong) uniqueness of dg-enhancements.

Applications

Using the theorem above, one proves that all autoequivalences of the following categories are of Fourier–Mukai type:

- **Fu–Yang** and **Keller–Yang**: the category generated by a 1-spherical object.
- **Ishii–Ueda–Uehara**: the category of A_n-singularities (already known; here we get a neat proof).
- **Bayer–Macrì**: local $\mathbb{P}^2$ (relevant for Mirror Symmetry: it is a 3-Calabi–Yau category).