Fourier–Mukai functors: existence

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Outline

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- $f_* : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ and $f^* : \mathrm{D^b}(Y) \to \mathrm{D^b}(X)$;
- $\mathcal{E} \otimes (-) : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$.

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The inverse of F sends a skyscraper sheaf $\mathcal{O}_{\mathrm{p}}$ (here p is a closed point of $\hat{A}$) on $\hat{A}$ to the degree 0 line bundle $L_{\mathrm{p}} \in \mathrm{Pic}^0(A)$ parametrized by p.

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$$\Phi_{\mathcal{E}}(-) := (p_2)_*(\mathcal{E} \otimes p_1^*(-)),$$

where $p_i \colon X_1 \times X_2 \to X_i$ is the natural projection and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$.

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An exact functor $\mathsf{F} \colon \mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ is a **Fourier–Mukai functor** (or of **Fourier–Mukai type**) if there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$ and an isomorphism of functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$. The complex $\mathcal{E}$ is called a **kernel** of F.

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Example

(1) and (2) allowed to give a partial description of the group of autoequivalences for K3 surfaces as conjectured by Szendroi (Huybrechts–Macrì–S.).

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Remark

A positive answer to the first one was conjectured by Bondal–Larsen–Lunts (and Orlov).

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Theorem (Olov, 1997)

Let X_1 and X_2 be smooth projective varieties and let $\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ be an exact fully faithful functor admitting a left adjoint. Then there exists a unique (up to isomorphim) $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$ such that $\mathsf{F} \cong \Phi_{\mathcal{E}}$.

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Bondal–van den Bergh: the adjoints always exist in this special setting (i.e. X_i smooth projective)!

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Theorem (Canonaco–Orlov–S.)

Let X be a noetherian connected scheme, let **T** be a triangulated category and let $\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{T}$ be a full exact functor not isomorphic to the zero functor. Then F is also faithful.

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Remark

- The result holds in much greater generality.
- The faithfulness assumption is redundant.

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Theorem (Canonaco–S., 2006)

Let X_1 and X_2 be smooth projective varieties and let $\mathsf{F}\colon \mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X_1)$,

$$(*) \qquad \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X_2)}(\mathsf{F}(\mathcal{F}), \mathsf{F}(\mathcal{G})[j]) = 0 \ \text{ if } j < 0.$$

Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$ and an isomorphism of functors $\mathsf{F} \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

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All exact functors $\mathrm{D}^{\mathrm{b}}(X_1) \to \mathrm{D}^{\mathrm{b}}(X_2)$ obtained by deriving an exact functor $\mathbf{Coh}(X_1) \to \mathbf{Coh}(X_2)$ satisfy the assumption.

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- $\mathrm{D}_Z(\mathbf{Qcoh}(X))$ is the derived category of unbounded complexes of quasi-coherent sheaves on X with cohomologies supported on Z.
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We set

$$\mathbf{Perf}_Z(X) := \mathrm{D}_Z(\mathbf{Qcoh}(X)) \cap \mathbf{Perf}(X).$$

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Let X_1 be a quasi-projective scheme containing a projective subscheme Z_1 such that $\mathcal{O}_{iZ_1} \in \mathbf{Perf}(X_1)$, for all $i > 0$ (e.g. either $Z_1 = X_1$ or X_1 is smooth), and let X_2 be a separated scheme of finite type over a field $\Bbbk$ with a proper subscheme Z_2.

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$\mathsf{F}\colon \mathbf{Perf}_{Z_1}(X_1) \to \mathbf{Perf}_{Z_2}(X_2)$ is an exact functor such that

1. For any $\mathcal{A}, \mathcal{B} \in \mathbf{Coh}_{Z_1}(X_1) \cap \mathbf{Perf}_{Z_1}(X_1)$ and any integer $k < 0$, $\mathrm{Hom}\,(\mathsf{F}(\mathcal{A}), \mathsf{F}(\mathcal{B})[k]) = 0$;
2. For all $\mathcal{A} \in \mathbf{Perf}_{Z_1}(X_1)$ with trivial cohomologies in (strictly) positive degrees, there is $N \in \mathbb{Z}$ such that

   $$\mathrm{Hom}\,(\mathsf{F}(\mathcal{A}), \mathsf{F}(\mathcal{O}_{|i|Z_1}(jH_1))) = 0,$$

   for any $i < N$ and any $j << i$, where H_1 is an ample divisor on X_1.

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Theorem (Canonaco–S.)

If X_1, X_2, Z_1, Z_2 and F are as above, then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$ and an isomorphism of functors

$$\mathsf{F} \cong \Phi^s_{\mathcal{E}}.$$

Moreover, if X_i is smooth quasi-projective, for $i = 1, 2$, and $\mathbb{k}$ is perfect, then $\mathcal{E}$ is unique up to isomorphism.

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Remark

$\Phi^s_{\mathcal{E}}$ is the natural generalization of the notion of Fourier–Mukai functor.

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Remark

As in Lunts–Orlov's case, we also get results about the (strong) uniqueness of dg-enhancements.

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- **Fu–Yang** and **Keller–Yang**: the category generated by a 1-spherical object.
- **Ishii–Ueda–Uehara**: the category of A_n-singularities (already known; here we get a neat proof).
- **Bayer–Macrì**: local $\mathbb{P}^2$ (relevant for Mirror Symmetry: it is a 3-Calabi–Yau category).