Inducing stability conditions

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Joint work with E. Macrì and S. Mehrotra
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Outline

1. **Stability conditions**
   - Motivations
   - Bridgeland's definition

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2. **Inducing stability conditions**
   - General technique
   - The equivariant case
   - Examples

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Aim 1: Attack and solve this problem in some interesting special cases.

Aim 2: Relate some connected component of the stability manifold to the description of the group of autoequivalences.

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- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(X)$ are full additive subcategories for each $\phi \in \mathbb{R}$

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with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

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- The universal cover $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ of $\mathrm{Gl}_2^+(\mathbb{R})$.

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1. For each connected component $\Sigma \subseteq \mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$ there is a linear subspace $V(\Sigma) \subseteq (K(\mathrm{D}^{\mathrm{b}}(X)) \otimes \mathbb{C})^{\vee}$ with a well-defined linear topology such that the natural map

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3. The manifold $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(X))$ parametrizing numerical stability conditions is finite dimensional.

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Also Polishchuk!

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Lemma

$$\mathrm{Dom}(F^{-1}) := \{\sigma' \in \mathrm{Stab}\,(\mathrm{D^b}(Y)) \,:\, \sigma = F^{-1}\sigma' \in \mathrm{Stab}\,(\mathrm{D^b}(X))\}$$

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Theorem A (M.-M.-S.)

The subset Γ_X of invariant stability conditions in $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X))$ is a closed submanifold with a closed embedding into $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}_G(X))$ via the forgetful functor.

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Theorem (Bridgeland)

The stability manifold $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D^b}(E))$ is naturally isomorphic to $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$.

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Proposition

$\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(E))$ is embedded as a closed submanifold into $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}_G(E))$.

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Proposition (Bridgeland)

If $\omega \cdot \omega > 2$, the pair $(Z_{\omega,\beta}, \mathcal{A}(\omega,\beta))$ defines a stability condition.

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Theorem (Bridgeland, Huybrechts-Macrì-S.)

If X is an abelian surface, then $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ is the unique connected component of maximal dimension. Moreover it is simply connected.

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Hence Γ_A is open and closed in $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ and, if non-empty, Γ_A is a connected component.

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Proposition

$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A))$ is realized as a closed submanifold of $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)))$.

Further perspectives (Toda, M.-M.-S.)

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Take a K3 surface or an abelian surface X with an involution $\iota_1 : X \to X$:

- Suppose that the derived category $\mathrm{D}^{\mathrm{b}}([X/\iota_1])$ of the quotient stack $[X/\iota_1]$ is equivalent to the derived category of a weighted projective space.

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Example

Take $X := E \times E$, with E elliptic curve and $\iota_1 := \iota \times \iota$.

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Apply the previous procedure of inducing stability conditions to construct stability conditions on $X \times E$ using stability conditions on $\mathrm{D}^{\mathrm{b}}([(X \times E)/(\iota_1 \times \iota_2)])$.

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Warning!

One may need to deform a bit the "easy" examples of stability conditions on $\mathrm{D}^{\mathrm{b}}([(X \times E)/(\iota_1 \times \iota_2)])$ to lift them to $X \times E$.

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In this special setting:

- $\mathbf{Coh}(Y)$ is naturally isomorphic to the abelian category $\mathbf{Coh}_G(X)$;
- $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_G(X)$.

Enriques surfaces: the second main result

Theorem B (M.-M.-S.)

There exist a connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ of $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ naturally embedded into $\mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ as a closed submanifold

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$$\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(Y)) \to \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$$

whose image contains the index-2 subgroup of G-equivariant orientation preserving Hodge isometries quotiented by G.

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whose image contains the index-2 subgroup of G-equivariant orientation preserving Hodge isometries quotiented by G.

Moreover, if Y is generic, the category $\mathrm{D}^{\mathrm{b}}(Y)$ does not contain spherical objects and $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$ is isomorphic to the distinguished connected component $\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$.

Outline

1. Stability conditions
   - Motivations
   - Bridgeland's definition
2. Inducing stability conditions
   - General technique
   - The equivariant case
   - Examples
3. Enriques surfaces
   - The connected component
   - A Derived Torelli Theorem
   - The generic case
4. The canonical bundle of $\mathbb{P}^1$
   - The setting
   - The result

A few ideas from the proof

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2. Given the map $\mathrm{Forg}_G^{-1} : \Gamma_X \to \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$, by Theorem A, $\Sigma(Y) := \mathrm{Forg}_G^{-1}(\Gamma_X \cap \mathrm{Stab}_{\mathcal{N}}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X)))$ is closed.

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Moreover, the following diagram commutes

$$\begin{array}{ccccc}
\Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) & \xrightarrow{\mathrm{Forg}_G^{-1}} & \Sigma(Y) & \xrightarrow{\mathrm{Inf}_G^{-1}} & \Gamma_X \cap \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) \\
\downarrow & & \downarrow{\scriptstyle \mathcal{Z}} & & \downarrow \\
(\mathcal{N}(X) \otimes \mathbb{C})^{\vee}_G & \xrightarrow{\mathrm{Forg}_{G*}^{\vee}} & (\mathcal{N}(Y) \otimes \mathbb{C})^{\vee} & \xrightarrow{\mathrm{Inf}_{G*}^{\vee}} & (\mathcal{N}(X) \otimes \mathbb{C})^{\vee}_G.
\end{array}$$

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We define

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y)) \subseteq \Sigma(Y)$$

to be the (non-empty) connected component containing the images of the stability conditions

$$(Z_{\omega,\beta}, \mathcal{A}(\omega, \beta))$$

with G-invariant $\omega, \beta \in \mathrm{NS}(X) \otimes \mathbb{Q}$ (previous example!).

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- The induced involution $\tilde{\iota} : \mathrm{Km}(A) \to \mathrm{Km}(A)$ has no fixed points.

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Proposition

There exist a connected component

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y))$$

and embeddings

$$\mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(A)) \hookrightarrow \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(Y)) \hookrightarrow \mathrm{Stab}^{\dagger}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)))$$

of closed submanifolds.

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The existence of the natural homomorphism

$$\Pi : \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(Y)) \to \mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G/G$$

in Theorem B is the analogue on the level of $\mathrm{D}^{\mathrm{b}}(Y)$.

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1. $\mathrm{Ker}^{G_\Delta}(\mathrm{D^b}(X)) := \{(\mathcal{G}, \lambda) \in \mathrm{D}^{\mathrm{b}}_{G_\Delta}(X \times X) : \Phi_{\mathcal{G}} \in \mathrm{Aut}\,(\mathrm{D^b}(X))\}$

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2. $\mathrm{Aut}\,(\mathrm{D^b}(X))_G := \{\Phi \in \mathrm{Aut}\,(\mathrm{D^b}(X)) : \iota^* \circ \Phi \circ \iota^* \cong \Phi\}$.

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Due to a remark by Ploog, the functors Forg_G and Inf_G are 2 : 1 and fit into the diagram

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This yields a natural surjective homomorphism

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Compose with the natural map

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Lattice structure: The Mukai pairing (Euler–Poincaré form up to sign). The lattice is denoted $\widetilde{H}(X,\mathbb{Z})$.

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Hodge structure: The weight-2 Hodge structure on $H^*(X,\mathbb{Z})$ is

$$\begin{aligned}\widetilde{H}^{2,0}(X) &:= H^{2,0}(X),\\ \widetilde{H}^{0,2}(X) &:= H^{0,2}(X),\\ \widetilde{H}^{1,1}(X) &:= H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}).\end{aligned}$$

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- We will denote by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$ ($\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$) the group of (orientation preserving) Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$.

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At the very end the proof boils down to the following:

Orientation

- We will denote by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))$ ($\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$) the group of (orientation preserving) Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$.
- The subgroups consisting of the equivariant isometries are denoted by $\mathrm{O}(\widetilde{H}(X,\mathbb{Z}))_G$ and $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))_G$.

At the very end the proof boils down to the following:

Proposition (Huybrechts-S.)

All known autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ are orientation preserving.

The connected component

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- Define $\mathcal{P}_0^+(Y) := \mathrm{Forg}_{G*}^\vee(\mathcal{P}_0^+(X))_G$.

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Proposition

The morphism $\mathcal{Z} : \Sigma(Y) \to \mathcal{N}(Y) \otimes \mathbb{C}$ defines a covering map onto $\mathcal{P}_0^+(Y)$ such that

$$\mathrm{Aut}^0(Y) := \mathrm{Aut}^0(\mathrm{D}^{\mathrm{b}}(Y))/\langle(-) \otimes \omega_Y\rangle$$

acts as the group of deck transformations.

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Remark: work in progress with Huybrechts and Macrì

Try to solve a similar problem for K3 surfaces (this would conclude also in the Enriques case).

Outline

Generic Enriques surfaces

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In the above setting, X does not contain rational curves. Hence Y does not contain rational curves neither.

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1. $\mathrm{Stab}^\dagger_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X)) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathrm{D}^{\mathrm{b}}(X))$ is isomorphic to $\Sigma(Y)$.
2. $\mathrm{D}^{\mathrm{b}}(Y)$ does not contain spherical objects.

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For K3 surfaces, spherical objects are always present (at least in the untwisted case).

As was proved in collaboration with Huybrechts and Macrì, the only way to reduce drastically the number of rigid and spherical objects is to pass to twisted or generic analytic K3 surfaces.

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- Let $i : \mathbb{P}^N \hookrightarrow X$ denote the zero-section and C its image.

- Let $\mathrm{D}^{\mathrm{b}}_0(X) := \mathrm{D}^{\mathrm{b}}_C(\mathbf{Coh}(X))$, the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ whose objects have cohomology sheaves supported on C.

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Remark

The functor i_* induces stability conditions from $\mathrm{Stab}\,(X)$ to $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^N))$ but the behaviour is not so nice.

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Remark

As a by-product we get a simple proof of the connectedness and simply-connectedness of the space $\mathrm{Stab}\,(X)$.

This was previously proved by Okada and, more generally, by Ishii-Uehara-Ueda.