Derived Categories in Algebraic Geometry
A first glance

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Outline

Outline

The geometric setting

Let X be a smooth projective variety defined over an algebraically closed field $\mathbb{K}$.

Example

Consider, for example, the zero locus of

$$x_0^d + \ldots + x_n^d$$

in the projective space $\mathbb{P}^n_{\mathbb{K}}$, for an integer $d \geq 1$.

(Vague) Question

How do we *categorize* X?

Find a category, associated to X, which encodes important bits of its geometry!

The geometric setting

In a more precise form, consider the (abelian!) category $\mathrm{Coh}(X)$ of **coherent sheaves** on X:

- They have locally a finite presentation;
- The abelian structure on $\mathrm{Coh}(X)$ provides the relevant notion of **short exact sequence**:

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

Example

The local description mentioned before can be thought as follows.

Let R be a noetherian ring. An R-module M is **coherent** if there exists an exact sequence

$$R^{\oplus k_1} \to R^{\oplus k_2} \to M \to 0,$$

for some non-negative integers k_1, k_2.

Coherent sheaves are too much

We then have the following (simplified version of a) classical result:

Theorem (Gabriel)

Let X_1 and X_2 be smooth projective schemes over $\mathbb{K}$. Then $X_1 \cong X_2$ if and only if there is an equivalence of abelian categories $\mathrm{Coh}(X_1) \cong \mathrm{Coh}(X_2)$.

It has been generalized in various contexts by Antieau, Canonaco-S., Perego, ...

$\mathrm{Coh}(X)$ encodes too much of the geometry of X!

Looking for alternatives

Aim

We would like to associate to X a category **with a nice structure** which **weakly encodes** the geometry of X (not just up to isomorphism).

For example, the categories associated to X_1 and to X_2 must be close if

- X_1 and X_2 are nicely related by birational transformations;
- One of them is a moduli space on the other one (i.e. it parametrizes objects defined on the second one).

Looking for alternatives

Let us enlarge the category and look for the **bounded derived category of coherent sheaves**

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$$

on X.

- **Objects:** bounded complexes of coherent sheaves

$$\ldots \to 0 \to E^{-i} \to E^{-i+1} \to \ldots \to E^{k-1} \to E^{k} \to 0 \to \ldots$$

- **Morphisms:** slightly more complicated then morphisms of complexes (...but we do not care here...).

Basic operations

- Given a complex $E \in \mathrm{D}^{\mathrm{b}}(X)$, we can shift it to the left ($E \rightsquigarrow E[1]$) or to the right ($E \rightsquigarrow E[-1]$).
- We can take direct sums $E_1 \oplus E_2$ and direct summands of $E \in \mathrm{D}^{\mathrm{b}}(X)$.
- $\mathrm{D}^{\mathrm{b}}(X)$ is **not abelian** but it is **triangulated**: short exact sequences are replaced by (non functorial!) **distinguished triangles**

  $$E_1 \to E_2 \to E_3 \to E_1[1].$$

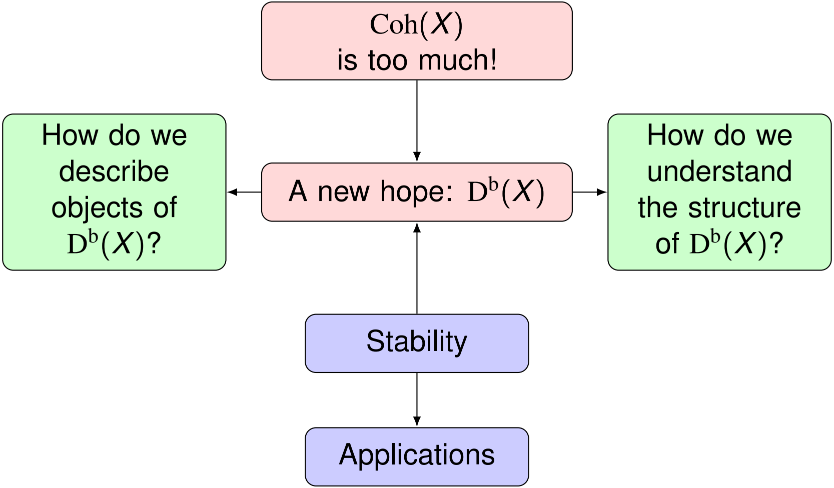
  We say that E_2 is an extension of E_1 and E_3.

Important construction

For $\mathcal{A} \subseteq \mathrm{D}^{\mathrm{b}}(X)$, we take the category $\langle \mathcal{A} \rangle$ **generated** by $\mathcal{A}$ (i.e. the smallest full triang. subcat. of $\mathrm{D}^{\mathrm{b}}(X)$ containing $\mathcal{A}$ and closed under shifts, extensions, direct sums and summands).

Outline

Summary



$\mathrm{Coh}(X)$ is too much!
A new hope: $\mathrm{D^b}(X)$
How do we describe objects of $\mathrm{D^b}(X)$?
How do we understand the structure of $\mathrm{D^b}(X)$?
Stability
Applications

How do we describe objects of $\mathrm{D^b}(X)$?

Let us go back to a special instance of the first example:

Example ($n = 2$, $d \geq 1$, $\mathbb{K} = \mathbb{C}$)

Consider the planar curve C which is the zero locus of

$$x_0^d + x_1^d + x_2^d$$

in the projective space $\mathbb{P}^2$.

- Take $E \in \mathrm{D^b}(C)$ of the form

$$\ldots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \to \ldots$$

and define

$$\mathcal{H}^i(E) := \frac{\ker(d^i)}{\mathrm{Im}(d^{i-1})} \in \mathrm{Coh}(C).$$

How do we describe objects of $\mathrm{D}^{\mathrm{b}}(X)$?

- In our example, we then have an isomorphism in $\mathrm{D}^{\mathrm{b}}(C)$

$$E \cong \bigoplus_i \mathcal{H}^i(E)[-i].$$

- On the other hand, each $F \in \mathrm{Coh}(C)$ splits as $F \cong F_{\mathrm{tf}} \oplus F_{\mathrm{tor}}$, where F_{tf} is locally free and F_{tor} is supported on points.

In conclusion: each $E \in \mathrm{D}^{\mathrm{b}}(C)$ is a direct sum of **shifted** locally free or torsion sheaves.

Warning 1

For higher dimensional varieties, this is false!

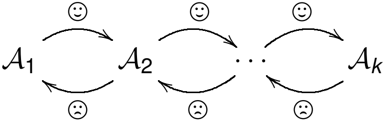
How do we understand the structure of $\mathrm{D}^{\mathrm{b}}(X)$?

We say that $\mathrm{D}^{\mathrm{b}}(X)$ has a **semiorthogonal decompisition**

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_k \rangle$$

if

- $\mathrm{D}^{\mathrm{b}}(X)$ is generated by extensions, shifts, direct sums and summands by the objects in $\mathcal{A}_1, \ldots, \mathcal{A}_k$;
- There are no Homs from right to left between the k subcategories:



How do we understand the structure of $\mathrm{D}^{\mathrm{b}}(X)$?

Consider again the case of the planar curve C

$$x_0^d + x_1^d + x_2^d = 0.$$

$d = 1, 2$
Genus $g = 0$

We have

$$\mathrm{D}^{\mathrm{b}}(C) = \langle \mathcal{O}_C, \mathcal{O}_C(1) \rangle$$

where

$$\langle \mathcal{O}_C(i) \rangle \cong \mathrm{D}^{\mathrm{b}}(\mathrm{pt})$$

for $i = 0, 1$.

$d = 3$
Genus $g = 1$

$\mathrm{D}^{\mathrm{b}}(C)$ is indec.

But interesting autoequivalece group.

$d \geq 4$
Genus $g \geq 2$

$\mathrm{D}^{\mathrm{b}}(C)$ is indec.

But uninteresting autoequivalece group.

How do we understand the structure of $\mathrm{D}^{\mathrm{b}}(X)$?

Conclusion

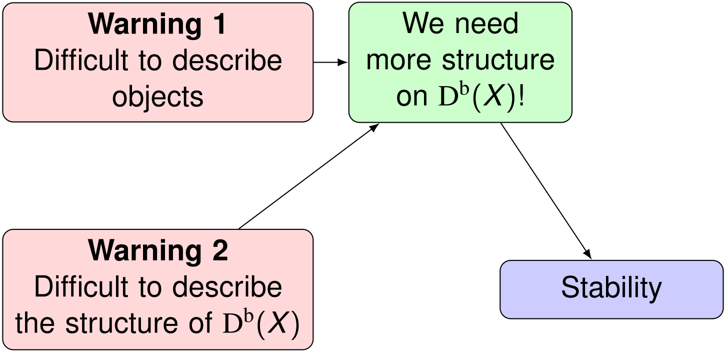
In general, when X is a Fano variety (in our standing example, we want $d - n - 1 < 0$) we look for interesting decompositions, hoping that:

- The components are simpler and of 'smaller dimension';
- Get a dimension reduction: a component encodes much of the geometry of X.

Warning 2

Semiorthogonal decompositions are in general **non-canonical**!

Summary



Warning 1
Difficult to describe objects
We need more structure on $\mathrm{D}^{\mathrm{b}}(X)$!
Warning 2
Difficult to describe the structure of $\mathrm{D}^{\mathrm{b}}(X)$
Stability

Outline

Back to sheaves

Take $n = 2$ in our example (but this can make it work for any smooth projective variety!).

The idea is that we want to **filter** any coherent sheaf in a canonical way!

More precisely:

- We have an abelian category $\mathrm{Coh}(C)$.
- We define a function

$$\mu_{\text{slope}}(-) := \frac{\deg(-)}{\mathrm{rk}\,(-)}$$

(or $+\infty$ when the denominator is 0), defined on $\mathrm{Coh}(C)$.

Back to sheaves

Definition

A sheaf $E \in \mathrm{Coh}(C)$ is **(semi)stable** if, for all non-trivial subsheaves $F \hookrightarrow E$ such that $\mathrm{rk}(F) < \mathrm{rk}(E)$, we have

$$\mu_{\text{slope}}(F) < (\leq) \mu_{\text{slope}}(E)$$

Harder–Narasimhan filtration

Any sheaf E has a filtration

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_{n-1} \hookrightarrow E_n = E$$

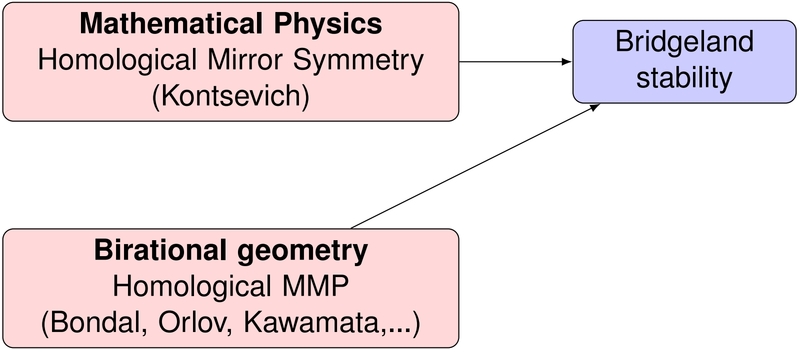
such that

- The quotient E_{i+1}/E_i is semistable, for all i;
- $\mu_{\text{slope}}(E_1/E_0) > \ldots > \mu_{\text{slope}}(E_n/E_{n-1})$.

From sheaves to complexes

Question

Can we have something similar for objects in $\mathrm{D}^{\mathrm{b}}(X)$?



Stability conditions

Let us start discussing the general setting: we do not even need to work with the specific triangulated category $\mathrm{D^b}(X)$!

- Let $\mathbf{T}$ be a triangulated category;

- Let Γ be a free abelian group of finite rank with a surjective map $v\colon K(\mathbf{T}) \to \Gamma$.

Example

$\mathbf{T} = \mathrm{D^b}(C)$, for C a smooth projective curve.

$$\Gamma = N(C) = H^0 \oplus H^2$$

with

$$v = (\mathrm{rk}, \deg)$$

A **Bridgeland stability condition** on $\mathbf{T}$ is a pair $\sigma = (\mathbf{A}, Z)$:

Stability conditions

- **A** is the heart of a bounded *t*-structure on **T**;
- $Z\colon \Gamma \to \mathbb{C}$ is a group homomorphism

Example

$\mathbf{A} = \mathrm{Coh}(C)$

$Z(v(-)) = -\deg + \sqrt{-1}\mathrm{rk}.$

such that, for any $0 \neq E \in \mathbf{A}$,

1. $Z(v(E)) \in \mathbb{R}_{>0} e^{(0,1]\pi\sqrt{-1}}$;
2. E has a Harder-Narasimhan filtration with respect to $\mu_\sigma(-) = -\frac{\mathrm{Re}(Z)(-)}{\mathrm{Im}(Z)(-)}$ (or $+\infty$);
3. Support property (**Kontsevich-Soibelman**): wall and chamber structure with locally finitely many walls.

Stability conditions

Warning 3

The example is somehow misleading: it only works in dimension 1!

The following is a remarkable result:

Theorem (Bridgeland)

If non-empty, the space $\mathrm{Stab}(\mathbf{T})$ parametrizing stability conditions on $\mathbf{T}$ is a complex manifold of dimension $\mathrm{rk}(\Gamma)$.

Stability conditions

Warning 4

It is a very difficult problem to construct stability conditions!

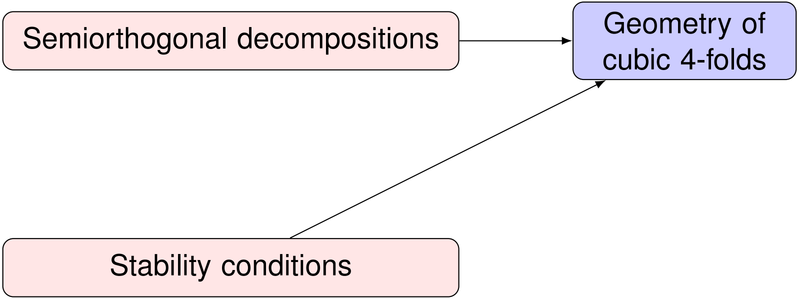
It is solved mainly in these cases:

- Curves (Bridgeland, Macrì);
- Surfaces over $\mathbb{C}$ (Bridgeland, Arcara Bertram), surface in positive characteristic (only partially solved);
- Fano threefolds (Bernardara-Macrì-Schmidt-Zhao, Li);
- Threefolds with trivial canonical bundle: abelian 3-folds (Maciocia-Piyaratne, Bayer-Macrì-S.), Calabi-Yau (Bayer-Macrì-S., Li);
- Fourfolds: in general is a big mistery!

Outline

Main idea

We want to combine all the ideas we discussed so far:



The setting

Let X be a **cubic fourfold** (i.e. a smooth hypersurface of degree 3 in $\mathbb{P}^5$). Let H be a hyperplane section.

Most of the time defined over $\mathbb{C}$ but, for some results, defined over a field $\mathbb{K} = \overline{\mathbb{K}}$ with $\mathrm{char}(\mathbb{K}) \neq 2$.

Example ($d = 3$ and $n = 5$)

Consider, for example the zero locus of

$$x_0^3 + \ldots + x_5^3 = 0$$

in the projective space $\mathbb{P}^5$.

Homological algebra

Let us now look at the bounded derived category of coherent sheaves on X:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

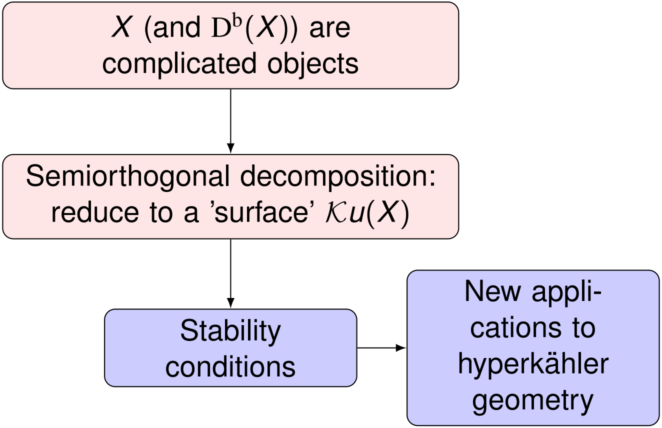
$$\mathcal{K}u(X) = \left\{ E \in \mathrm{D}^{\mathrm{b}}(X) : \begin{array}{l} \mathrm{Hom}\,(\mathcal{O}_X(iH), E[p]) = 0 \\ i = 0, 1, 2 \;\; \forall p \in \mathbb{Z} \end{array} \right\}$$

Kuznetsov component of X

Exceptional objects:

$$\langle \mathcal{O}_X(iH) \rangle \cong \mathrm{D}^{\mathrm{b}}(\mathrm{pt})$$

The idea


X (and $\mathrm{D}^{\mathrm{b}}(X)$) are complicated objects
Semiorthogonal decomposition: reduce to a 'surface' $\mathcal{K}u(X)$
Stability conditions
New applications to hyperkähler geometry

Noncommutative K3s

Remark

For every cubic 4-folds X, $\mathcal{K}u(X)$ behaves like $\mathrm{D}^{\mathrm{b}}(S)$, for S a **K3 surface** (i.e. a simply connected smooth projective variety of dimension 2 with trivial canonical bundle).

But almost never, $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, for S a K3 surface.

This is related to the following:

Conjecture (Kuznetsov)

A cubic 4-fold X is **rational** (i.e. birational to $\mathbb{P}^4$ '$=$' a big open subset of X is isomorphic to an open subset of $\mathbb{P}^4$) if and only if $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, for some K3 surface S.

The results: existence of stability conditions

We have seen that if S is a K3 surface, then $\mathrm{D}^{\mathrm{b}}(S)$ carries stability conditions.

Question (Addington-Thomas, Huybrechts,...)

If X is a cubic 4-fold, does $\mathcal{K}u(X)$ carry stability conditions?

Theorem 1 (Bayer-Lahoz-Macrì-S, BLMS+Nuer-Perry)

For any cubic fourfold X, we have $\mathrm{Stab}(\mathcal{K}u(X)) \neq \emptyset$. Moreover, we can explicitly describe a connected component $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ of $\mathrm{Stab}(\mathcal{K}u(X))$.

The results: moduli spaces

Whenever we have a good notion of stability, we would like to parametrize all objects which are (semi)stable with respect to it.

We fix some topological invariants of the objects in $\mathcal{K}u(X)$ that we want to parametrize. These invariants are encoded by a vector v in cohomology that we usually call **Mukai vector.**

Fix a stability condition $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ which is nice with respect to v (always possible!).

Denote by

$$M_\sigma(\mathcal{K}u(X), v)$$

the space parametrizing σ-stable objects in $\mathcal{K}u(X)$ with Mukai vector v. Call it **moduli space.**

The results: moduli spaces

Question

What is the geometry of $M_\sigma(\mathcal{K}u(X), v)$?

This is a non-trivial question as moduli spaces of Bridgeland stable objects are, in general, 'strange' objects.

Theorem 2 (BLMNPS)

$M_\sigma(\mathcal{K}u(X), v)$ is non-empty if and only if $v^2 + 2 \geq 0$. Moreover, in this case, it is a smooth projective hyperkähler manifold of dimension $v^2 + 2$, deformation equivalent to a Hilbert scheme of points on a K3 surface.

The results: moduli spaces

Definition

A **hyperkähler manifold** is a simply connected compact kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form.

There are very few examples (up to deformation):

1. K3 surfaces;
2. Hilbert schemes of points on K3 surface (denoted by $\mathrm{Hilb}^n(\mathrm{K3})$;
3. Generalized Kummer varieties (from abelian surfaces);
4. Two sporadic examples by O'Grady.

Hyperkähler geometry and cubic 4-folds

The fact that any cubic 4-fold X has a very interesting hyperkähler geometry is classical. This is related to rational curves in X:

- **Beauville-Donagi:** the variety parametrizing lines in X is a HK 4-fold;
- **Lehn-Lehn-Sorger-van Strated:** if X does not contain a plane, the moduli space of (generalized) twisted cubics in X is, after a fibration and the contraction of a divisor, a HK 8-fold.

Question

Can we recover these classical HK manifolds in our new framework?

Hyperkähler geometry and cubic 4-folds

Theorem(s) (Li-Pertusi-Zhao, Lahoz-Lehn-Macrì-S.)

The variety of lines and the one of twisted cubics are isomorphic to moduli spaces of stable objects in the Kuznetsov component.

New! (BLMNPS)

In these two cases, we can explicitely describe the birational models via variation of stability!

Hyperkähler geometry and cubic 4-folds

But the most striking application is the following:

Corollary (BLMNPS)

For any pair (a, b) of coprime integers, there is a unirational locally complete 20-dimensional family, over an open subset of the moduli space of cubic fourfolds, of polarized smooth projective HKs of dimension $2n + 2$, where $n = a^2 - ab + b^2$.

This follows from a new theory of Bridgeland stability in families $+$ Theorem 2 in families $+$ deformations of cubics in their 20-dim. family.

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