Derived Categories in Algebraic Geometry

A first glance

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Outline

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1. What are they?

2. How to handle them?

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3. Stability

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3. Stability
4. Applications

The geometric setting

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in the projective space $\mathbb{P}^n_{\mathbb{K}}$, for an integer $d \geq 1$.

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Find a category, associated to X, which encodes important bits of its geometry!

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Example

The local description mentioned before can be thought as follows.

Let R be a noetherian ring. An R-module M is **coherent** if there exists an exact sequence

$$R^{\oplus k_1} \to R^{\oplus k_2} \to M \to 0,$$

for some non-negative integers k_1, k_2.

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- One of them is a moduli space on the other one (i.e. it parametrizes objects defined on the second one).

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- **Morphisms:** slightly more complicated then morphisms of complexes (...but we do not care here...).

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Important construction

For $\mathcal{A} \subseteq \mathrm{D}^{\mathrm{b}}(X)$, we take the category $\langle \mathcal{A} \rangle$ **generated** by $\mathcal{A}$ (i.e. the smallest full triang. subcat. of $\mathrm{D}^{\mathrm{b}}(X)$ containing $\mathcal{A}$ and closed under shifts, extensions, direct sums and summands).

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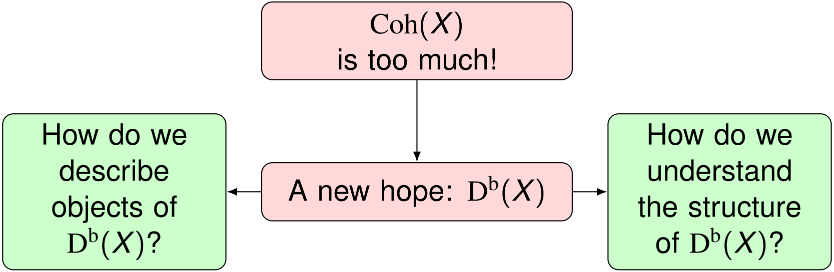
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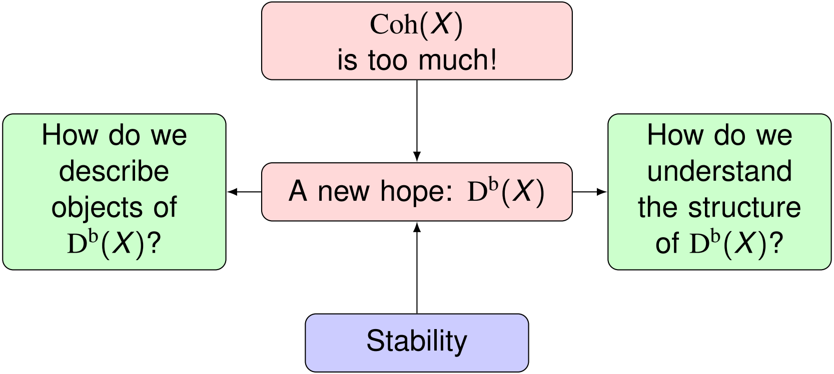
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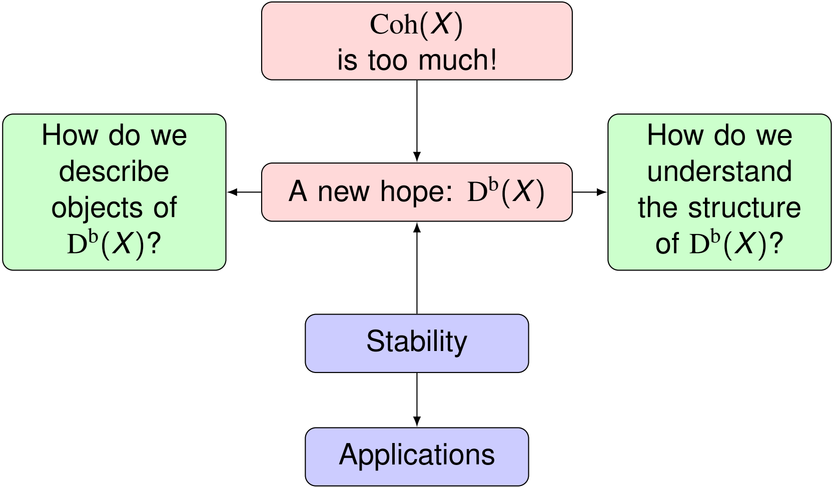

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- Take $E \in \mathrm{D}^{\mathrm{b}}(C)$ of the form

$$\ldots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \to \ldots$$

and define

$$\mathcal{H}^i(E) := \frac{\ker(d^i)}{\mathrm{Im}(d^{i-1})} \in \mathrm{Coh}(C).$$

How do we describe objects of $\mathrm{D}^{\mathrm{b}}(X)$?

- In our example, we then have an isomorphism in $\mathrm{D}^{\mathrm{b}}(C)$

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Warning 1

For higher dimensional varieties, this is false!

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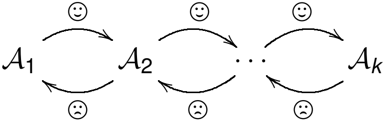
- There are no Homs from right to left between the k subcategories:



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$d = 1, 2$
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We have

$$\mathrm{D^b}(C) = \langle \mathcal{O}_C, \mathcal{O}_C(1) \rangle$$

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Genus $g \geq 2$

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Warning 2

Semiorthogonal decompositions are in general **non-canonical**!

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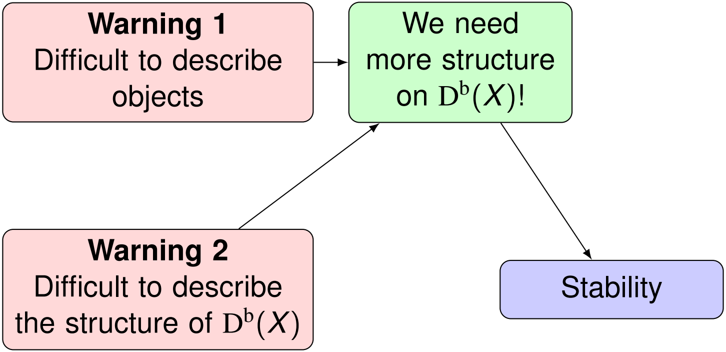
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- We have an abelian category $\mathrm{Coh}(C)$.
- We define a function

$$\mu_{\mathrm{slope}}(-) := \frac{\deg(-)}{\mathrm{rk}\,(-)}$$

(or $+\infty$ when the denominator is 0), defined on $\mathrm{Coh}(C)$.

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- The quotient E_{i+1}/E_i is semistable, for all i;
- $\mu_{\mathrm{slope}}(E_1/E_0) > \ldots > \mu_{\mathrm{slope}}(E_n/E_{n-1})$.

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Birational geometry
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(Bondal, Orlov, Kawamata,...)

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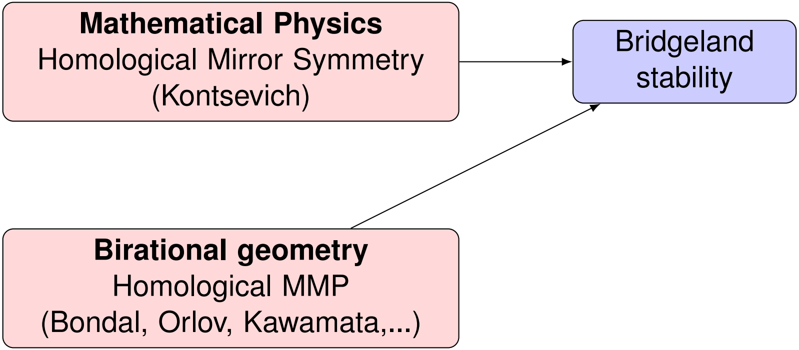

Stability conditions

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3. Support property (**Kontsevich-Soibelman**): wall and chamber structure with locally finitely many walls.

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The following is a remarkable result:

Theorem (Bridgeland)

If non-empty, the space $\mathrm{Stab}(\mathbf{T})$ parametrizing stability conditions on $\mathbf{T}$ is a complex manifold of dimension $\mathrm{rk}(\Gamma)$.

Stability conditions

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- Fourfolds: in general is a big mistery!

Outline

1. What are they?
2. How to handle them?
3. Stability
4. **Applications**

Main idea

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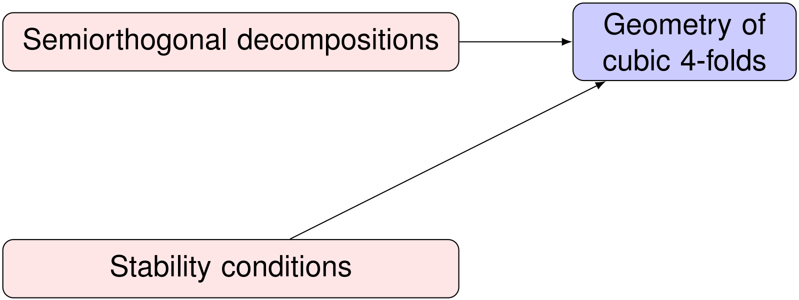

The setting

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Example ($d = 3$ and $n = 5$)

Consider, for example the zero locus of

$$x_0^3 + \ldots + x_5^3 = 0$$

in the projective space $\mathbb{P}^5$.

Homological algebra

Let us now look at the bounded derived category of coherent sheaves on X:

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Kuznetsov component of X

Exceptional objects:

$$\langle \mathcal{O}_X(iH) \rangle \cong \mathrm{D}^{\mathrm{b}}(\mathrm{pt})$$

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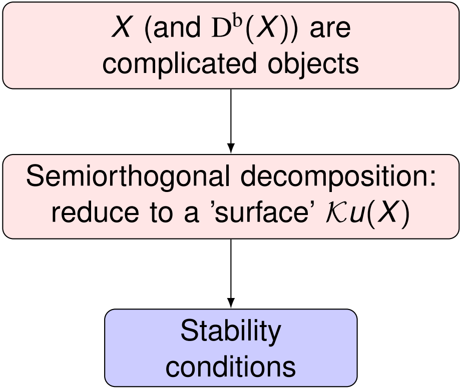
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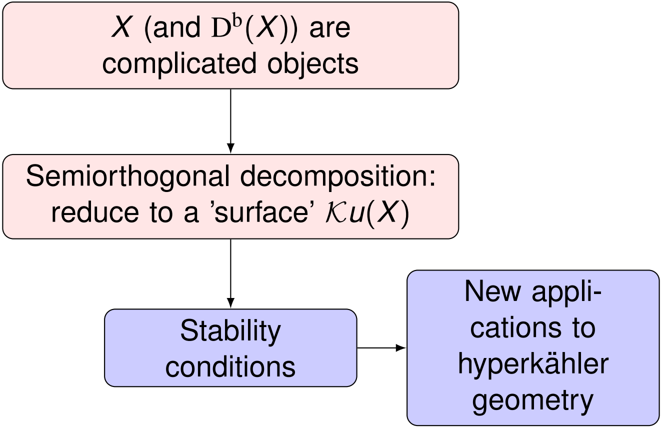
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New applications to hyperkähler geometry

Noncommutative K3s

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For every cubic 4-folds X, $\mathcal{K}u(X)$ behaves like $\mathrm{D}^{\mathrm{b}}(S)$, for S a **K3 surface** (i.e. a simply connected smooth projective variety of dimension 2 with trivial canonical bundle).

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But almost never, $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, for S a K3 surface.

This is related to the following:

Conjecture (Kuznetsov)

A cubic 4-fold X is **rational** (i.e. birational to $\mathbb{P}^4$ '$=$' a big open subset of X is isomorphic to an open subset of $\mathbb{P}^4$) if and only if $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, for some K3 surface S.

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If X is a cubic 4-fold, does $\mathcal{K}u(X)$ carry stability conditions?

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Question (Addington-Thomas, Huybrechts,...)

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Theorem 1 (Bayer-Lahoz-Macrì-S, BLMS+Nuer-Perry)

For any cubic fourfold X, we have $\mathrm{Stab}(\mathcal{K}u(X)) \neq \emptyset$. Moreover, we can explicitly describe a connected component $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ of $\mathrm{Stab}(\mathcal{K}u(X))$.

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We fix some topological invariants of the objects in $\mathcal{K}u(X)$ that we want to parametrize. These invariants are encoded by a vector v in cohomology that we usually call **Mukai vector.**

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Denote by

$$M_\sigma(\mathcal{K}u(X), v)$$

the space parametrizing σ-stable objects in $\mathcal{K}u(X)$ with Mukai vector v. Call it **moduli space.**

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What is the geometry of $M_\sigma(\mathcal{K}u(X), v)$?

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Theorem 2 (BLMNPS)

$M_\sigma(\mathcal{K}u(X), v)$ is non-empty if and only if $v^2 + 2 \geq 0$. Moreover, in this case, it is a smooth projective hyperkähler manifold of dimension $v^2 + 2$, deformation equivalent to a Hilbert scheme of points on a K3 surface.

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4. Two sporadic examples by O'Grady.

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Question

Can we recover these classical HK manifolds in our new framework?

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Theorem(s) (Li-Pertusi-Zhao, Lahoz-Lehn-Macrì-S.)

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New! (BLMNPS)

In these two cases, we can explicitely describe the birational models via variation of stability!

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For any pair (a, b) of coprime integers, there is a unirational locally complete 20-dimensional family, over an open subset of the moduli space of cubic fourfolds, of polarized smooth projective HKs of dimension $2n + 2$, where $n = a^2 - ab + b^2$.

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This follows from a new theory of Bridgeland stability in families + Theorem 2 in families + deformations of cubics in their 20-dim. family.