Equivalences of K3 Surfaces: Deformations and Orientation

Paolo Stellari

Dipartimento di Matematica “F. Enriques”
Università degli Studi di Milano

Joint work with:
E. Macrì (arXiv:0804.2552), D. Huybrechts and E. Macrì (arXiv:0710.1645)
and E. Macrì and M. Nieper-Wisskirchen (preprint)

Outline

1. **Motivations**
   - The problem
   - The analogies

Outline

1. **Motivations**
   - The problem
   - The analogies
2. **Infinitesimal Derived Torelli Theorem**
   - The setting
   - The statement
   - Sketch of the proof

Outline

Outline

1 Motivations
- The problem
- The analogies

2 Infinitesimal Derived Torelli Theorem
- The setting
- The statement
- Sketch of the proof

3 Orientation
- The statement
- The strategy
- The categorical setting
- Deforming kernels
- Concluding the argument

The problem

Let X be a K3 surface.

Main problem

Describe the group of exact autoequivalences of the triangulated category

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_X\text{-}\mathbf{Mod})$$

or of a first order deformation of it.

Remark (Orlov)

Such a description is available (in the non-deformed context) when X is an abelian surface (actually an abelian variety).

Outline

Geometry: automorphisms

Theorem (Torelli Theorem)

Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$$

which maps the class of an ample line bundle on X into the ample cone of Y. Then there exists a unique isomorphism $f : X \xrightarrow{\sim} Y$ such that $f_* = g$.

Lattice theory + Hodge structures + ample cone

Remark

The automorphism is uniquely determined.

Geometry: diffeomorphisms

Theorem (Borcea, Donaldson)

Consider the natural map

$$\rho : \mathrm{Diff}(X) \longrightarrow \mathrm{O}(H^2(X,\mathbb{Z})).$$

Then $\mathrm{im}\,(\rho) = \mathrm{O}_+(H^2(X,\mathbb{Z}))$, where $\mathrm{O}_+(H^2(X,\mathbb{Z}))$ is the group of orientation preserving isometries.

The orientation is given by the choice of a basis for the 3-dimensional positive space in $H^2(X,\mathbb{R})$.

Remark

The kernel of ρ is not known!

Orlov's result

Derived Torelli Theorem (Mukai, Orlov)

Let X and Y be smooth projective K3 surfaces. Then the following are equivalent:

1. There exists an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$.
2. There exists a Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(Y,\mathbb{Z})$.

The equivalence Φ induces an action on cohomology

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(X) & \xrightarrow{\Phi} & \mathrm{D}^{\mathrm{b}}(Y) \\ {\scriptstyle v(-)=\mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(X)}}\downarrow & & \downarrow{\scriptstyle v(-)=\mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(Y)}} \\ \widetilde{H}(X,\mathbb{Z}) & \xrightarrow{\Phi_H} & \widetilde{H}(Y,\mathbb{Z}) \end{array}$$

Main problem

Question

Can we understand better the action induced on cohomology by an equivalence?

Orientation: Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class. Then $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1-\omega^2/2, \omega\rangle$ is a positive four-space in $\widetilde{H}(X,\mathbb{R})$ with a natural orientation.

Problem

The isometry $j := (\mathrm{id})_{H^0\oplus H^4} \oplus (-\mathrm{id})_{H^2}$ is not orientation preserving. Is it induced by an autoequivalence?

Motivation

There exists an explicit description of the first order deformations of the abelian category of coherent sheaves on a smooth projective variety (Toda).

The existence of equivalences between the derived categories of smooth projective K3 surfaces is detected by the existence of special isometries of the total cohomologies.

Question

Can we get the same result for derived categories of first order deformations of K3 surfaces using special isometries between 'deformations' of the Hodge and lattice structures on the total cohomologies?

Outline

Hochschild homology and cohomology

For X any smooth projective variety, define the Hochschild homology

$$\mathrm{HH}_i(X) := \mathrm{Hom}_{\mathrm{D^b}(X\times X)}(\Delta_*\omega_X^\vee[i-\dim(X)], \mathcal{O}_{\Delta_X})$$

and the Hochschild cohomology

$$\mathrm{HH}^i(X) := \mathrm{Hom}_{\mathrm{D^b}(X\times X)}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[i]).$$

On the other hand we put

$$\mathrm{H}\Omega_i(X) := \bigoplus_{q-p=i} H^p(X, \Omega_X^q) \quad \mathrm{HT}^i(X) := \bigoplus_{p+q=i} H^p(X, \wedge^q \mathcal{T}_X).$$

Hochschild–Kostant–Rosenberg

There exist (the Hochschild–Kostant–Rosenberg) isomorphisms

$$I^X_{\mathrm{HKR}} : \mathrm{HH}_*(X) \to \mathrm{H}\Omega_*(X) := \bigoplus_i \mathrm{H}\Omega_i(X)$$

and

$$I_X^{\mathrm{HKR}} : \mathrm{HH}^*(X) \to \mathrm{HT}^*(X) := \bigoplus_i \mathrm{HT}^i(X).$$

One then defines the graded isomorphisms

$$I^X_K = (\mathrm{td}(X)^{1/2} \wedge (-)) \circ I^X_{\mathrm{HKR}} \qquad I_X^K = (\mathrm{td}(X)^{-1/2} \lrcorner (-)) \circ I_X^{\mathrm{HKR}}.$$

Toda's construction

1. Take a smooth projective variety X, $v \in \mathrm{HH}^2(X)$ and write
$$I_X^{\mathrm{HKR}}(v) = (\alpha, \beta, \gamma) \in \mathrm{HT}^2(X).$$

2. Define a sheaf $\mathcal{O}_X^{(\beta,\gamma)}$ of $\mathbb{C}[\epsilon]/(\epsilon^2)$-algebras on X depending only on β and γ.

3. Representing $\alpha \in H^2(X, \mathcal{O}_X)$ as a Cech 2-cocycle $\{\alpha_{ijk}\}$ one has an element $\widetilde{\alpha} := \{1 - \epsilon\alpha_{ijk}\}$ which is a Čech 2-cocycle with values in the invertible elements of the center of $\mathcal{O}_X^{(\beta,\gamma)}$.

Toda's construction

We get the abelian category

$$\mathbf{Coh}(\mathcal{O}_X^{(\beta,\gamma)}, \widetilde{\alpha})$$

of $\widetilde{\alpha}$-twisted coherent $\mathcal{O}_X^{(\beta,\gamma)}$-modules. Set

$$\mathbf{Coh}(X, v) := \mathbf{Coh}(\mathcal{O}_X^{(\beta,\gamma)}, \widetilde{\alpha}).$$

One also have an isomorphism $J : \mathrm{HH}^2(X_1) \to \mathrm{HH}^2(X_1)$ such that

$$(I_{X_1}^{\mathrm{HKR}} \circ J \circ (I_{X_1}^{\mathrm{HKR}})^{-1})(\alpha, \beta, \gamma) = (\alpha, -\beta, \gamma).$$

Outline

The Infinitesimal Derived Torelli Theorem

Theorem (Macrì–S.)

Let X_1 and X_2 be smooth complex projective K3 surfaces and let $v_i \in \mathrm{HH}^2(X_i)$, with $i = 1, 2$. Then the following are equivalent:

1. There exists a Fourier–Mukai equivalence
$$\Phi_{\widetilde{\mathcal{E}}} : \mathrm{D}^{\mathrm{b}}(X_1, v_1) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X_2, v_2)$$
with $\widetilde{\mathcal{E}} \in \mathrm{D}_{\mathrm{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$.

2. There exists an orientation preserving effective Hodge isometry
$$g : \widetilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X_2, v_2, \mathbb{Z}).$$

The structures

For X a K3, $v \in \mathrm{HH}^2(X)$ and σ_X is a generator for $\mathrm{HH}_2(X)$, let

$$w := I_K^X(\sigma_X) + \epsilon I_K^X(\sigma_X \circ v) \in \widetilde{H}(X,\mathbb{Z}) \otimes \mathbb{Z}[\epsilon]/(\epsilon^2).$$

The free $\mathbb{Z}[\epsilon]/(\epsilon^2)$-module of finite rank $\widetilde{H}(X,\mathbb{Z}) \otimes \mathbb{Z}[\epsilon]/(\epsilon^2)$ is endowed with:

1. The $\mathbb{Z}[\epsilon]/(\epsilon^2)$-linear extension of the generalized Mukai pairing $\langle -,- \rangle_M$.
2. A weight-2 decomposition on $\widetilde{H}(X,\mathbb{Z}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$

$$\widetilde{H}^{2,0}(X,v) := \mathbb{C}[\epsilon]/(\epsilon^2) \cdot w \qquad \widetilde{H}^{0,2}(X,v) := \overline{\widetilde{H}^{2,0}(X,v)}$$

and $\widetilde{H}^{1,1}(X,v) := (\widetilde{H}^{2,0}(X,v) \oplus \widetilde{H}^{0,2}(X,v))^{\perp}$.

The structures

This gives the infinitesimal Mukai lattice of X with respect to v, which is denoted by $\widetilde{H}(X, v, \mathbb{Z})$.

$$g : \widetilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X_2, v_2, \mathbb{Z})$$

which can be decomposed as $g = g_0 + \epsilon g_0$, where g_0 is an Hodge isometry of the Mukai lattices is called effective.

An effective isometry is orientation preserving if g_0 preserves the orientation of the four-space.

Outline

Deformations

We just sketch of the implication (i)⇒(ii).

- Let $\Phi_{\widetilde{\mathcal{E}}}:\mathrm{D}^{\mathrm{b}}(X_1,v_1)\xrightarrow{\sim}\mathrm{D}^{\mathrm{b}}(X_2,v_2)$ be an equivalence with kernel $\widetilde{\mathcal{E}}\in\mathrm{D}_{\mathrm{perf}}(X_1\times X_2,-J(v_1)\boxplus v_2)$.
- One shows that the restriction $\mathcal{E}\in\mathrm{D}^{\mathrm{b}}(X_1\times X_2)$ of $\widetilde{\mathcal{E}}$ is the kernel of a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}:\mathrm{D}^{\mathrm{b}}(X_1)\xrightarrow{\sim}\mathrm{D}^{\mathrm{b}}(X_2)$.
- Using Orlov's result, take the Hodge isometry $g_0:=(\Phi_{\mathcal{E}})_H:\widetilde{H}(X_1,\mathbb{Z})\to\widetilde{H}(X_2,\mathbb{Z})$.

The isometry

Toda: since $\widetilde{\mathcal{E}}$ is a first order deformation of $\mathcal{E}$,

$$(\Phi_{\mathcal{E}})^{\mathrm{HH}}(v_1) = v_2.$$

Important!

Assume we know that any Hodge isometry induced by an equivalence $\mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$ is orientation preserving.

To conclude and prove that

$$g := g_0 \otimes \mathbb{Z}[\epsilon]/(\epsilon^2) : \widetilde{H}(X_1, v_1, \mathbb{Z}) \to \widetilde{H}(X_2, v_2, \mathbb{Z})$$

is an effective orientation preserving Hodge isometry, we need two commutative diagrams.

Commutativity I

Any Fourier–Mukai functor acts on Hochschild homology.

Theorem (Macrì–S.)

Let X_1 and X_2 be smooth complex projective varieties and let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$. Then the following diagram

$$\begin{array}{ccc} \mathrm{HH}_*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})_{\mathrm{HH}}} & \mathrm{HH}_*(X_2) \\ \downarrow{\scriptstyle I_K^{X_1}} & & \downarrow{\scriptstyle I_K^{X_2}} \\ \widetilde{H}(X_1,\mathbb{C}) & \xrightarrow{(\Phi_{\mathcal{E}})_H} & \widetilde{H}(X_2,\mathbb{C}) \end{array}$$

commutes.

Commutativity II

Using that for K3 surfaces $H^{0,2}$ is 1-dimensional and the previous result, one get the following commutative diagram (for a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}$):

$$\begin{array}{ccc}
\mathrm{HH}^*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})^{\mathrm{HH}}} & \mathrm{HH}^*(X_2) \\
\Big\downarrow{\scriptstyle (-)\circ\sigma_{X_1}} & & \Big\downarrow{\scriptstyle (-)\circ(\Phi_{\mathcal{E}})_{\mathrm{HH}}(\sigma_{X_1})} \\
\mathrm{HH}_*(X_1) & \xrightarrow{(\Phi_{\mathcal{E}})_{\mathrm{HH}}} & \mathrm{HH}_*(X_2) \\
\Big\downarrow{\scriptstyle I_K^{X_1}} & & \Big\downarrow{\scriptstyle I_K^{X_2}} \\
\widetilde{H}(X_1,\mathbb{C}) & \xrightarrow{(\Phi_{\mathcal{E}})_H} & \widetilde{H}(X_2,\mathbb{C}),
\end{array}$$

where σ_{X_1} is a generator of $\mathrm{HH}_2(X_1)$.

Outline

The motivation

We go back to the original problem of describing the group of exact autoequivalences of the derived category of a K3 surface.

Remarks

1. To conclude the previous argument involving (first order) deformations, we need to prove that any equivalence induces an orientation preserving Hodge isometry.
2. The (quite involved) proof of this result will use deformation of kernels.

The statement

Main Theorem (Huybrechts–Macrì–S.)

Given a Hodge isometry $g : \widetilde{H}(X, \mathbb{Z}) \to \widetilde{H}(Y, \mathbb{Z})$, then there exists and equivalence $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ such that $g = \Phi_H$ if and only if g is orientation preserving.

Szendroi's Conjecture is true: In terms of autoequivalences, this yields a surjective morphism

$$\mathrm{Aut}\,(\mathrm{D^b}(X)) \twoheadrightarrow \mathrm{O}_+(\widetilde{H}(X, \mathbb{Z})),$$

where $\mathrm{O}_+(\widetilde{H}(X, \mathbb{Z}))$ is the group of orientation preserving Hodge isometries.

The 'easy' implication

The statement: If g is orientation preserving than it lifts to an equivance.

- A result of Hosono–Lian–Oguiso–Yau (heavily relaying on Mukai/Orlov's Derived Torelli Theorem) shows that, up to composing with the isometry j, every isometry can be lifted to an equivalence.
- Since we know that j is not orientation preserving we conclude using the following:

Remark (Huybrechts-S.)

All known equivalences (and autoequivalences) are orientation preserving.

Outline

The non-orientation Hodge isometry

Take any projective K3 surface X.

- Consider the non-orientation preserving Hodge isometry
$$j := (\mathrm{id})_{H^0 \oplus H^4} \oplus (-\mathrm{id})_{H^2}.$$
- Since one implication is already true, to prove the main theorem, it is enough to show that j is not induced by a Fourier–Mukai equivalence.
- We proceed by contradiction assuming that there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X)$ such that $(\Phi_{\mathcal{E}})_H = j$.

The idea of the proof

- **Huybrechts–Macrì–S.:** For some particular K3 surfaces we know that j is not induced by any Fourier–Mukai equivalence: K3 surfaces with trivial Picard group.
- Deform the K3 surface in the moduli space such that generically we recover the behaviour of a generic K3 surface.
- Deform the kernel of the equivalence accordingly.
- Derive a contradiction using the generic case.

Outline

Formal deformations

Take $R := \mathbb{C}[[t]]$ to be the ring of power series in t with field of fractions $K := \mathbb{C}((t))$.

Define $R_n := \mathbb{C}[[t]]/(t^{n+1})$. Then $\mathrm{Spec}\,(R_n) \subset \mathrm{Spec}\,(R_{n+1})$.

For X a smooth projective variety, a formal deformation is a proper formal R-scheme

$$\pi : \mathcal{X} \to \mathrm{Spf}(R)$$

given by an inductive system of schemes $\mathcal{X}_n \to \mathrm{Spec}\,(R_n)$ (smooth and proper over R_n) and such that

$$\mathcal{X}_{n+1} \times_{R_{n+1}} \mathrm{Spec}\,(R_n) \cong \mathcal{X}_n.$$

The categories

There exist sequences

$$\mathbf{Coh}_0(\mathcal{X}\times_R\mathcal{X}')\hookrightarrow\mathbf{Coh}(\mathcal{X}\times_R\mathcal{X}')\to\mathbf{Coh}((\mathcal{X}\times_R\mathcal{X}')_K)$$

$$\mathbf{Coh}_0(\mathcal{X})\hookrightarrow\mathbf{Coh}(\mathcal{X})\to\mathbf{Coh}((\mathcal{X})_K)$$

where $\mathbf{Coh}_0(\mathcal{X}\times_R\mathcal{X}')$ and $\mathbf{Coh}_0(\mathcal{X})$ are the abelian categories of sheaves supported on $X\times X$ and X respectively.

In this setting we also have the sequences

$$\mathrm{D}^{\mathrm{b}}_0(\mathcal{X}\times_R\mathcal{X}')\hookrightarrow\mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_{\mathcal{X}\times_R\mathcal{X}'}\text{-Mod})\to\mathrm{D}^{\mathrm{b}}((\mathcal{X}\times_R\mathcal{X}')_K)$$

$$\mathrm{D}^{\mathrm{b}}_0(\mathcal{X})\hookrightarrow\mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_{\mathcal{X}}\text{-Mod})\to\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$$

The key example: the twistor space

Let us focus now on the case when X is a K3 surface.

Definition

A Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ is called very general if there is no non-trivial integral class $0 \neq \alpha \in H^{1,1}(X, \mathbb{Z})$ orthogonal to ω, i.e. $\omega^{\perp} \cap H^{1,1}(X, \mathbb{Z}) = 0$.

Take the twistor space $\mathbb{X}(\omega)$ of X determined by the choice of a very general Kähler class $\omega \in \mathcal{K}_X \cap \mathrm{Pic}\,(X) \otimes \mathbb{R}$:

$$\pi : \mathbb{X}(\omega) \to \mathbb{P}(\omega).$$

The key example: the twistor space

Remark

$\mathbb{X}(\omega)$ parametrizes the complex structures 'compatible' with ω.

Choosing a local parameter t around $0 \in \mathbb{P}(\omega)$ we get a formal deformation $\mathcal{X} \to \mathrm{Spf}(R)$.

More precisely:

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \mathrm{Spec}\,(R_n),$$

form an inductive system and give rise to a formal R-scheme

$$\pi : \mathcal{X} \to \mathrm{Spf}(R),$$

which is the formal neighbourhood of X in $\mathbb{X}(\omega)$.

The generic category

Proposition

If X is a K3 surface and $\mathcal{X}$ is as before, then $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K) \cong \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(\mathcal{X}_K))$. Moreover, $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ is a generic K-linear K3 category.

A K-linear category is a K3 category if it contains at least a spherical object and the shift by 2 is the Serre functor.

A K3 category is generic if, up to shift, it contains only one spherical object.

Remark

In this setting, the unique spherical object is $(\mathcal{O}_{\mathcal{X}})_K$, the image of $\mathcal{O}_{\mathcal{X}}$.

Equivalences

As before, given $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_{\mathcal{X}\times_R\mathcal{X}'}\text{-Mod})$, we denote by $\mathcal{F}_K$ the natural image in the category $\mathrm{D}^{\mathrm{b}}((\mathcal{X} \times_R \mathcal{X}')_K)$.

Proposition

Let $\widetilde{\mathcal{E}} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X} \times_R \mathcal{X}')$ be such that $\mathcal{E} = i^*\widetilde{\mathcal{E}}$. Then $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}_K$ are kernels of Fourier–Mukai equivalences.

Here we denoted by $i : X \times X \to \mathcal{X} \times_R \mathcal{X}'$ the natural inclusion.

Outline

The first order deformation

The equivalence $\Phi_{\mathcal{E}}$ induces a morphim

$$\Phi_{\mathcal{E}}^{\mathrm{HH}} : \mathrm{HH}^2(X) \to \mathrm{HH}^2(X).$$

Proposition

Let $v_1 \in H^1(X, \mathcal{T}_X)$ be the Kodaira–Spencer class of first order deformation given by a twistor space $\mathbb{X}(\omega)$ as above. Then

$$v_1' := \Phi_{\mathcal{E}}^{\mathrm{HH}}(v_1) \in H^1(X, \mathcal{T}_X).$$

The first order deformation

Remark

$\mathrm{HH}_*(X)$ and $\mathrm{H\Omega}_*(X)$ have natural module structures over $\mathrm{HH}^*(X)$ and $\mathrm{HT}^*(X)$ respectively.

To prove this result we use the following:

Theorem (Macrì–Nieper-Wisskirchen–S.)

The isomorphisms $I^K_X \colon \mathrm{HH}^*(X) \xrightarrow{\sim} \mathrm{HT}^*(X)$ and $I^X_K \colon \mathrm{HH}_*(X) \xrightarrow{\sim} \mathrm{H\Omega}_*(X)$ are compatible with the module structures on $\mathrm{HH}_*(X)$ and $\mathrm{H\Omega}_*(X)$ when X

- has trivial canonical bundle or
- has dimension 1 or
- is a projective space.

The first order deformation

Let $\mathcal{X}'_1$ be the first order deformation corresponding to v'_1.

Using results of Toda one gets the following conclusion

Proposition (Toda)

For v_1 and v'_1 as before, there exists $\mathcal{E}_1 \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_1 \times_{R_1} \mathcal{X}'_1)$ such that

$$i_1^* \mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}'_1 \times_{R_1} \mathcal{X}'_1$ is the natural inclusion.

Hence there is a first order deformation of $\mathcal{E}$.

Higher order deformations

More generally

We construct, at any order n, a deformation $\mathcal{X}'_n$ such that there exists $\mathcal{E}_n \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_n \times_{R_n} \mathcal{X}'_n)$, with

$$i_n^* \mathcal{E}_n = \mathcal{E}_{n-1}.$$

Main difficulties

1. Write the obstruction to deforming complexes in terms of Atiyah–Kodaira classes (Huybrechts–Thomas).
2. Show that the obstruction is zero.

Our approach imitates the first order case (using relative Hochschild homology).

Outline

The generic fiber

Use the generic analytic case

There exist integers n and m such that the Fourier–Mukai equivalence

$$T^n_{(\mathcal{O}_{\mathcal{X}})_K} \circ \Phi_{\mathcal{E}_K}[m]$$

has kernel $\mathcal{G}_K \in \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)$, for $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$.

Remark

This shows that the autoequivalences of the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ behaves like the derived category of a complex K3 surface with trivial Picard group (Huybrechts–Macrì–S.).

Key ingredients

In the previous proof we use that $(\mathcal{O}_{\mathcal{X}})_K$ is the unique, up to shift, spherical object in $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$.

In particular, we use that given a locally finite stability condition σ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$, there exists an integer n such that in the stability condition $T^n_{(\mathcal{O}_{\mathcal{X}})_K}(\sigma)$ all K-rational points are stable with the same phase.

Remark

Notice that for our proof we use stability conditions in a very mild form. We just use a specific stability condition in which we can classify all semi-rigid stable objects.

The conclusion

Properties of $\mathcal{G}$

1. $\mathcal{G}_0 := i^*\mathcal{G}$ is a sheaf in **Coh**$(X \times X)$.
2. The natural morphism

$$(\Phi_{\mathcal{G}_0})_H : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

is such that $(\Phi_{\mathcal{G}_0})_H = (\Phi_{\mathcal{E}})_H = j$.

For the second part, we show that $\mathcal{G}_0$ and $\mathcal{E}$ induce the same action on the Grothendieck groups and have the same Mukai vector!

The conclusion

The contradiction is now obtained using the following lemma:

Lemma

If $\mathcal{G}_0 \in \mathbf{Coh}(X \times X)$, then $(\Phi_{\mathcal{G}_0})_H \neq j$.

Warning!

We have not proved that $\mathcal{E}$ is a (shift of a) sheaf! We have just proved that the action in cohomology is the same as the one of a sheaf!