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STABILITY CONDITIONS ON GENERIC K3 SURFACES

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CONTENTS

A generic analytic K3 surface is a K3 surface X such that $\mathrm{Pic}(X) = \{0\}$.

(A) Describe the space of stability conditions on the derived category of these surfaces.

Motivation: Very few examples of a complete description of this variety are available (curves, A_n-singularities). In general one just gets a connected component.

(B) Describe the group of autoequivalences for K3 surfaces of this type.

Motivation: Find evidence for the truth of Bridgeland's conjecture and possibly find a way to prove the conjecture in the algebraic case.

Part 1
PRELIMINARIES

Coherent sheaves

In general, the abelian category $\mathbf{Coh}(X)$ of a smooth projective variety X is a very strong invariant.

Theorem (Gabriel). Let X and Y be smooth projective varieties such that

$$\mathbf{Coh}(X) \cong \mathbf{Coh}(Y).$$

Then there exists an isomorphism $X \cong Y$.

This is not the case for generic analytic K3 surfaces.

Theorem (Verbitsky). Let X and Y be K3 surfaces such that $\rho(X) = \rho(Y) = 0$. Then $\mathbf{Coh}(X) \cong \mathbf{Coh}(Y)$.

The same result was proved by Verbitsky for generic (non-projective) complex tori.

Derived categories

In the algebraic case the derived categories are good invariants: they preserve some deep geometric relationships!

Orlov: **Let X and Y be smooth projective varieties then any equivalence**

$$\Phi : \mathrm{D^b}(\mathbf{Coh}(X)) \xrightarrow{\sim} \mathrm{D^b}(\mathbf{Coh}(Y))$$

is of Fourier-Mukai type.

(There is a more general statement due to Canonaco-S.)

Warning: Verbitsky result implies that not all equivalences are of Fourier-Mukai type for generic analytic K3 surfaces.

So for the rest of this talk an equivalence or an autoequivalence will be always meant to be of Fourier-Mukai type.

The (bounded) derived category of X, denoted by $\mathsf{D}^{\mathsf{b}}(X)$ is defined as the triangulated subcategory

$$\mathsf{D}^{\mathsf{b}}_{\mathbf{Coh}}(\mathcal{O}_X\text{-}\mathbf{Mod})$$

of $\mathsf{D}(\mathcal{O}_X\text{-}\mathbf{Mod})$ whose objects are bounded complexes of $\mathcal{O}_X$-modules whose cohomologies are in $\mathbf{Coh}(X)$.

The reasons to choose this category are mainly the following:

- All geometric functors can be derived in this triangulated category: f_*, f^*, $\otimes$, Hom functors,... (Spalstein).

- Bondal and Van den Bergh: Serre duality is well-defined.

The choice of this category is not particularly painful in our case.

There is a natural functor

$$F : \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X)) \to \mathrm{D}^{\mathrm{b}}(X).$$

The following holds:

Theorem (Illusie, Bondal-Van den Bergh). **If X is a smooth compact complex surface then $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$.**

The same is possibly not true for the product $X \times X$.

Warning: When useful (stability conditions) think of $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ and just think of $\mathrm{D}^{\mathrm{b}}(X)$ when dealing with sophisticated questions involving derived functors!

Examples of functors

Example 1. Let $f : X \xrightarrow{\sim} X$ be an isomorphism. Then $f^* : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$ is an equivalence.

Example 2. The shift functor $[1] : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$ is obviously an equivalence.

Example 3. Let $\mathcal{E}$ be a spherical object, i.e.

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, \dim X\} \\ 0 & \text{otherwise.} \end{cases}$$

Consider the spherical twist $T_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$ that sends $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ to the cone of

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}.$$

The kernel of $T_{\mathcal{E}}$ is given by the cone of the natural map

$$\mathcal{E}^{\vee} \boxtimes \mathcal{E} \to \mathcal{O}_{\Delta}.$$

All these functors are orientation preserving!

Part 2
THE FIRST RESULT:
stability conditions

The statement

The space of all locally finite numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ is denoted by

$$\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X)).$$

To shorten the notation the locally finite numerical stability conditions will be simply called stability conditions.

Theorem 1. (H.-M.-S.) If X is a generic analytic K3 surface, then the space $\mathrm{Stab}(X)$ is connected and simply-connected.

Remark. It is completely unclear how to prove a similar result for projective K3 surfaces.

The strategy of the proof

(1) Describe all the spherical objects in the derived category $\mathrm{D}^{\mathrm{b}}(X)$.

No hope to do the same for projective K3 surfaces!

(2) Construct examples of stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$.

Here the procedure is slightly simpler than in the projective case.

(3) Control the stability of the skyscraper sheaves (...using spherical objects).

(4) Patch everything together using some topological argument.

Controlling spherical objects

Let us start with an easy calculation:

Lemma. **The trivial line bundle $\mathcal{O}_X$ is the only spherical object in $\mathrm{Coh}(X)$.**

Proof. Suppose $\mathcal{E} \in \mathbf{Coh}(X)$ spherical. We know that

$$\begin{aligned}\langle v(\mathcal{E}^\vee), v(\mathcal{E})\rangle &= -\chi(\mathcal{E},\mathcal{E})\\ &= -\mathrm{hom} + \mathrm{ext}^1 - \mathrm{ext}^2\\ &= -1 + 0 - 1\\ &= -2.\end{aligned}$$

Since $\mathrm{Pic}(X) = \{0\}$, $v(\mathcal{E}) = (r, 0, s)$ with $r \cdot s = 1$. Clearly, $r \geq 0$ and thus $r = s = 1$.

Let $\mathcal{E}_{\mathrm{tor}}$ be the torsion part of $\mathcal{E}$. Then $\mathcal{E}_{\mathrm{tor}}$ is concentrated in dimension zero, since there are no curves in X. Let ℓ be its length.

Case $\ell = 0$. Then $\mathcal{E}$ is torsion free with Mukai vector $(1, 0, 1)$ and hence $\mathcal{E} \cong \mathcal{O}_X$.

Case $\ell > 0$. Since $\chi(\mathcal{E}/\mathcal{E}_{\text{tor}}, \mathcal{E}_{\text{tor}}) = \ell$, there would be a non-trivial homomorphism

$$\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}_{\text{tor}} \to \mathcal{E}_{\text{tor}} \hookrightarrow \mathcal{E}.$$

This would contradict the fact that, by definition, $\text{Hom}(\mathcal{E}, \mathcal{E}) \cong \mathbb{C}$. $\square$

No hope in the algebraic case: any line bundle is a spherical object!

An object $\mathcal{E} \in \text{D}^{\text{b}}(X)$ is rigid if $\text{Hom}(\mathcal{E}, \mathcal{E}[1]) = 0$.

Using induction one can prove the following:

Lemma. The rigid objects in $\mathrm{Coh}(X)$ are $\mathcal{O}_X^{\oplus n}$ for some $n \in \mathbb{N}$.

A K3 category is by definition a triangulated category $\mathbf{T}$ which

- is $\mathbb{C}$-linear;
- has functorial isomorphisms

$$\mathsf{Hom}(\mathcal{E},\mathcal{F}) \cong \mathsf{Hom}(\mathcal{F},\mathcal{E}[2])^\vee$$

  for all objects $\mathcal{E},\mathcal{F} \in \mathbf{T}$

With much more effort, one can prove the following:

Proposition. Let $\mathbf{A}$ be an abelian category which contains a spherical object $\mathcal{E} \in \mathbf{A}$ which is the only indecomposable rigid object in $\mathbf{A}$. Assume moreover that $\mathsf{D}^{\mathsf{b}}(\mathbf{A})$ is a K3 category. Then $\mathcal{E}$ is up to shift the only spherical object in $\mathsf{D}^{\mathsf{b}}(\mathbf{A})$

Hence $\mathcal{O}_X$ is, up to shift, the unique spherical object in $\mathsf{D}^{\mathsf{b}}(X)$.

Constructing stability conditions

Bridgeland: Give a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ with heart $\mathbf{A}$ and a stability function $Z : K(\mathbf{A}) \to \mathbb{C}$ which has the Harder-Narasimhan property.

A stability function is a $\mathbb{C}$-linear function which takes values in $\mathbb{H} \cup \mathbb{R}_{\leq 0}$, where $\mathbb{H}$ is the complex upper half plane).

In our specific case, $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ is non-empty!

Consider the open subset

$$R := \mathbb{C} \setminus \mathbb{R}_{\geq -1} = R_+ \cup R_- \cup R_0,$$

where the sets are defined in the natural way:

- $R_+ := R \cap \mathbb{H}$,
- $R_- := R \cap (-\mathbb{H})$,
- $R_0 := R \cap \mathbb{R}$.

Given $z = u + iv \in R$, take the subcategories $\mathcal{F}(z), \mathcal{T}(z) \subset \mathbf{Coh}(X)$ defined as follows:

- If $z \in R_{+} \cup R_{0}$ then $\mathcal{F}(z)$ and $\mathcal{T}(z)$ are respectively the full subcategories of all torsion free sheaves and torsion sheaves.

- If $z \in R_{-}$ then $\mathcal{F}(z)$ is trivial and $\mathcal{T}(z) = \mathbf{Coh}(X)$.

This is a special case of the *tilting construction* in Bridgeland's approach to the algebraic case.

Consider the subcategories defined by means of $\mathcal{F}(z)$ and $\mathcal{T}(z)$ as follows:

- If $z \in R_+ \cup R_0$, we put

$$\mathcal{A}(z) := \left\{ \mathcal{E} \in \mathsf{D}^{\mathsf{b}}(X) : \begin{array}{l} \bullet\ H^0(\mathcal{E}) \in \mathcal{T}(z) \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathcal{F}(z) \\ \bullet\ H^i(\mathcal{E}) = 0 \text{ oth.} \end{array} \right\}.$$

- If $z \in R_-$, let $\mathcal{A}(z) = \mathbf{Coh}(X)$.

Bridgeland: $\mathcal{A}(z)$ is the heart of a bounded t-structure for any $z \in R$.

Now, for any $z = u + iv \in R$ we define the function

$$\begin{array}{rcl} Z : \mathcal{A}(z) & \to & \mathbb{C} \\ \mathcal{E} & \mapsto & \langle v(\mathcal{E}), (1, 0, z) \rangle \\ & & = -u \cdot r - s - i(r \cdot v), \end{array}$$

where $v(\mathcal{E}) = (r, 0, s)$ is the Mukai vector of $\mathcal{E}$.

The main properties of the pair $(\mathcal{A}(z), Z)$ are:

Lemma. **For any $z \in R$ the function Z defines a stability function on $\mathcal{A}(z)$ which has the Harder-Narasimhan property.**

Proof. The fact that Z is a stability function is an easy calculation with Mukai vectors and Bogomolov inequality.

The fact that Z has the HN-property is easy when the heart is $\mathbf{Coh}(X)$: use the standard one!

In the other case, Huybrechts proved that the heart is generated by shifted stable locally free sheaves and skyscraper sheaves. □

It is more difficult to prove the following result:

Proposition. **For any $\sigma \in \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$, there is $n \in \mathbb{Z}$ such that $T^n_{\mathcal{O}_X}(\mathcal{O}_x)$ is stable in σ, for any closed point $x \in X$.**

Controlling stability of skyscraper sheaves

The stability of skyscraper sheaves will be crucial in our proof.

Proposition. **Suppose that $\sigma = (Z, \mathcal{P})$ is a stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ for a K3 surface X with trivial Picard group. If all skyscraper sheaves $\mathcal{O}_x$ are stable of phase 1 with $Z(\mathcal{O}_x) = -1$, then $\sigma = \sigma_z$ for some $z \in R$.**

One can also determine other stable object in the special stability conditions previously identified.

Recall that an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is semirigid if $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[1]) \cong \mathbb{C} \oplus \mathbb{C}$.

Lemma. **Let $\sigma = (Z, \mathcal{P})$ be a stability condition associated to $z \in R_0$. Then the unique stable semirigid objects in σ are the skyscraper sheaves.**

The idea of the proof

We denote by T the twist by the spherical object $\mathcal{O}_X$.

Recall that the group $\widetilde{\mathrm{Gl}}_2^+(\mathbb{R})$ acts on the manifold Stab(X).

(A) Consider

$$W(X) := \widetilde{\mathrm{Gl}}_2^+(\mathbb{R})(R) \subset \mathrm{Stab}(X),$$

which can also be written as the union

$$W(X) = W_+ \cup W_- \cup W_0.$$

The previous results essentially prove

$$\mathrm{Stab}(X) = \bigcup_n T^n W(X).$$

(B) $W(X) \subset$ Stab(X) is an open connected subset.

First we show that the inclusion $R \subset$ Stab(X) is continuous.

Thus, R and hence

$$W(X) = \widetilde{\mathsf{Gl}}_2^+(\mathbb{R})(R)$$

are connected subsets of Stab(X).

Then one argues showing the openness of $W(X)$ in Stab(X).

(C) One proves that $T^n W(X)$ and $T^k W(X)$ are disjoint for

$$|n-k| \geq 2.$$

More precisely, we show

$$T^n W(X) \cap T^{n+1} W(X) = T^n W_-.$$

(D) As an immediate consequence of (C) and the connectedness of $W(X)$ proved in (B), one concludes that

$$\mathrm{Stab}(X) = \bigcup T^n W(X)$$

is connected!!

(E) Apply the van Kampen Theorem to the open cover in (D).

For what we proved, the intersections

$$T^n W(X) \cap T^k W(X) \subset \mathsf{Stab}(X)$$

are either empty for $|n-k| \geq 2$ or homeomorphic to the connected W_-.

Thus one simply verifies that the open sets

$$T^n W(X) \cong W(X)$$

are simply-connected.

This concludes the proof of Theorem 1.

Part 3
THE SECOND RESULT:
equivalences

The statement

Theorem 2. (H.-M.-S.) Let X and Y be generic analytic K3 surfaces. If

$$\Phi_{\mathcal{E}} : \mathsf{D}^{\mathsf{b}}(X) \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(Y)$$

is an equivalence of Fourier-Mukai type, then up to shift

$$\Phi_{\mathcal{E}} \cong T^{n}_{\mathcal{O}_Y} \circ f_*$$

for some $n \in \mathbb{Z}$ and an isomorphism

$$f : X \xrightarrow{\sim} Y.$$

For autoequivalences it reads:

Theorem 3. (H.-M.-S.) If X is an analytic generic K3 surface, then

$$\mathsf{Aut}(\mathsf{D}^{\mathsf{b}}(X)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathsf{Aut}(X).$$

The first two factors are generated respectively by the shift functor and the spherical twist $T_{\mathcal{O}_X}$.

Remarks

(1) First notice that an analytic K3 surface X does not have non-isomorphic Fourier-Mukai partners.

The algebraic case is very much different. In fact, it is proved (Oguiso and S.) that for any positive integer N, there are N non-isomorphic algebraic K3 surfaces $X_1, \ldots, X_N$ such that

$$\mathrm{D}^{\mathrm{b}}(X_i) \cong \mathrm{D}^{\mathrm{b}}(X_j),$$

for $i, j \in \{1, \ldots, N\}$.

(2) The part of the autoequivalence group which 'detects' the geometry of the K3 surface is the automorphism group.

We will go back to this issue later.

The proof

The proof is easy now that we have complete control on the stability conditions and the stability of skyscraper sheaves.

(1) Take the distinguished stability condition

$$\sigma = \sigma_{(u,v=0)}$$

constructed before. Let

$$\tilde{\sigma} := \Phi_{\mathcal{E}}(\sigma).$$

(2) Denote by T the spherical twist $T_{\mathcal{O}_Y}$.

We have seen that, there exists an integer n such that all skyscraper sheaves $\mathcal{O}_x$ are stable of the same phase in the stability condition $T^n(\tilde{\sigma})$.

(3) The composition $\Psi := T^n \circ \Phi_{\mathcal{E}}$ has the properties:

- It is again an equivalence of Fourier-Mukai type $\Psi := \Phi_{\mathcal{F}}$ (Orlov).
- It sends the stability condition σ to a stability condition σ' for which all skyscraper sheaves are stable of the same phase.
- Up to shifting the kernel $\mathcal{F}$ sufficiently, we can assume that $\phi_{\sigma'}(\mathcal{O}_y) \in (0,1]$ for all closed points $y \in Y$.

Thus, the heart $\mathcal{P}((0,1])$ of the t-structure associated to σ' (identified with $\mathcal{A}(z)$) contains as stable objects the images $\Psi(\mathcal{O}_x)$ of all closed points $x \in X$ and all skyscraper sheaves $\mathcal{O}_y$.

(4) We observed that the only semi-rigid stable objects in $\mathcal{A}(z)$ are the skyscraper sheaves.

Hence, for all $x \in X$ there exists a point $y \in Y$ such that $\Psi(\mathcal{O}_x) \cong \mathcal{O}_y$.

This suffices to conclude that the Fourier-Mukai equivalence $\Psi_{\mathcal{F}}$ is a composition of f_*, for some isomorphism

$$f : X \xrightarrow{\sim} Y,$$

and a line bundle twist.

(This heavily relies on the fact that the equivalences are of Fourier-Mukai type.)

But there are no non-trivial line bundles on Y.

An important remark

Our proof just depends on the notion of stability condition more than on the topology of the space of stability conditions.

In particular we have not used that $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ is connected and simply-connected.

If we believe in Bridgeland's conjecture this is not true for algebraic K3 surfaces. In that case, the description of a connected component of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ is important.

Back to geometry

We actually proved that the interesting part of the autoequivalence group is (in principle) encoded by the automorphism group.

We now have two questions:

(1) How do we construct examples of K3 surfaces with trivial Picard group?

(2) Can we completely describe the automorphism group of these K3 surfaces?

McMullen: He constructed examples of K3 surfaces with trivial Picard group starting from Salem polynomials of degree 22 (two special real(!) roots).

These K3 surfaces have an automorphism of infinite order.

Oguiso: For a K3 surface X the automorphism group $\mathrm{Aut}(X)$ is either trivial or isomorphic to $\mathbb{Z}$.

The last case is verified just for countably many K3 surfaces! Exactly the K3 surfaces constructed by McMullen.

Our result should read:

Given a K3 surface X with $\mathrm{Pic}(X) = \{0\}$, then $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ is isomorphic either to $\mathbb{Z}^{\oplus 2}$ or to $\mathbb{Z}^{\oplus 3}$.

So the group of autoequivalences of Fourier-Mukai type detects the generic analytic K3 surfaces of McMullen type.