Equivalences of K3 Surfaces and Orientation II
The Projective Case

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Joint work with: D. Huybrechts and E. Macrì (arXiv:0710.1645) and E. Macrì (arXiv:0804.2552)

Outline

1. **Derived Torelli Theorem**
   - Motivations
   - The statement

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2. **Ideas from the proof**
   - The strategy
   - The categorical setting
   - Deforming kernels
   - Concluding the argument

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2. **Ideas from the proof**
    - The strategy
    - The categorical setting
    - Deforming kernels
    - Concluding the argument
3. **Infinitesimal Derived Torelli Theorem**
    - First order deformations
    - The statement
    - Sketch of the proof

Outline

The problem

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Let X be a K3 surface.

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Main problem

Describe the group $\mathrm{Aut}\,(\mathrm{D^b}(X))$ of exact autoequivalences of the triangulated category

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Remark (Orlov)

Such a description is available when X is an abelian surface (actually an abelian variety).

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Remark

The automorphism is uniquely determined.

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Theorem (Borcea, Donaldson)

Consider the natural map

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Then $\mathrm{im}\,(\rho) = \mathrm{O}_+(H^2(X,\mathbb{Z}))$, where $\mathrm{O}_+(H^2(X,\mathbb{Z}))$ is the group of orientation preserving isometries.

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Remark

The kernel of ρ is not known!

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The equivalence Φ induces an action on cohomology

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(X) \arrow[r, "\Phi"] \arrow[d, "v(-)=\mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(X)}"'] & \mathrm{D}^{\mathrm{b}}(Y) \arrow[d, "v(-)=\mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(Y)}"] \\
\widetilde{H}(X,\mathbb{Z}) \arrow[r, "\Phi_H"] & \widetilde{H}(Y,\mathbb{Z})
\end{tikzcd}$$

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Orientation: Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class. Then $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1-\omega^2/2, \omega \rangle$ is a positive four-space in $\widetilde{H}(X,\mathbb{R})$ with a natural orientation.

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Problem

The isometry $j := (\mathrm{id})_{H^0\oplus H^4} \oplus (-\mathrm{id})_{H^2}$ is not orientation preserving. Is it induced by an autoequivalence?

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Main Theorem (Huybrechts–Macrì–S.)

Given a Hodge isometry $g : \widetilde{H}(X, \mathbb{Z}) \to \widetilde{H}(Y, \mathbb{Z})$, then there exists and equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ such that $g = \Phi_H$ if and only if g is orientation preserving.

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Szendroi's Conjecture is true: In terms of autoequivalences, this yields a surjective morphism

$$\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X)) \twoheadrightarrow \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})),$$

where $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$ is the group of orientation preserving Hodge isometries.

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Remark (Huybrechts-S.)

All known equivalences (and autoequivalences) are orientation preserving.

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Take any projective K3 surface X.

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- We proceed by contradiction assuming that there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X)$ such that $(\Phi_{\mathcal{E}})_H = j$.

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- Deform the K3 surface (along a line) in the moduli space such that generically we recover the behaviour of a generic K3 surface.
- Deform the kernel of the equivalence accordingly.
- Derive a contradiction using the generic case.

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For X a smooth projective variety, a formal deformation is a proper formal R-scheme

$$\pi : \mathcal{X} \to \mathrm{Spf}(R)$$

given by an inductive system of schemes $\mathcal{X}_n \to \mathrm{Spec}\,(R_n)$ (smooth and proper over R_n) and such that

$$\mathcal{X}_{n+1} \times_{R_{n+1}} \mathrm{Spec}\,(R_n) \cong \mathcal{X}_n.$$

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There exist sequences

$$\mathbf{Coh}_0(\mathcal{X}\times_R\mathcal{X}')\hookrightarrow\mathbf{Coh}(\mathcal{X}\times_R\mathcal{X}')\to\mathbf{Coh}((\mathcal{X}\times_R\mathcal{X}')_K)$$

$$\mathbf{Coh}_0(\mathcal{X})\hookrightarrow\mathbf{Coh}(\mathcal{X})\to\mathbf{Coh}((\mathcal{X})_K)$$

where $\mathbf{Coh}_0(\mathcal{X}\times_R\mathcal{X}')$ and $\mathbf{Coh}_0(\mathcal{X})$ are the abelian categories of sheaves supported on $X\times X$ and X respectively.

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In this setting we also have the sequences

$$\mathrm{D}^{\mathrm{b}}_0(\mathcal{X}\times_R\mathcal{X}')\hookrightarrow\mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_{\mathcal{X}\times_R\mathcal{X}'}\text{-Mod})\to\mathrm{D}^{\mathrm{b}}((\mathcal{X}\times_R\mathcal{X}')_K)$$

$$\mathrm{D}^{\mathrm{b}}_0(\mathcal{X})\hookrightarrow\mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_{\mathcal{X}}\text{-Mod})\to\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$$

The key example: the twistor space

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Definition

A Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ is called very general if there is no non-trivial integral class $0 \neq \alpha \in H^{1,1}(X, \mathbb{Z})$ orthogonal to ω, i.e. $\omega^{\perp} \cap H^{1,1}(X, \mathbb{Z}) = 0$.

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Take the twistor space $\mathbb{X}(\omega)$ of X determined by the choice of a very general Kähler class $\omega \in \mathcal{K}_X \cap \operatorname{Pic}(X) \otimes \mathbb{R}$:

$$\pi : \mathbb{X}(\omega) \to \mathbb{P}(\omega).$$

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Remark

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Choosing a local parameter t around $0 \in \mathbb{P}(\omega)$ we get a formal deformation $\mathcal{X} \to \mathrm{Spf}(R)$.

More precisely:

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \mathrm{Spec}\,(R_n),$$

form an inductive system and give rise to a formal R-scheme

$$\pi : \mathcal{X} \to \mathrm{Spf}(R),$$

which is the formal neighbourhood of X in $\mathbb{X}(\omega)$.

The generic category

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Proposition

If X is a K3 surface and $\mathcal{X}$ is as before, then $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K) \cong \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(\mathcal{X}_K))$. Moreover, $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ is a generic K-linear K3 category.

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Remark

In this setting, the unique spherical object is $(\mathcal{O}_{\mathcal{X}})_K$, the image of $\mathcal{O}_{\mathcal{X}}$.

Equivalences

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As before, given $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_{\mathcal{X}\times_R\mathcal{X}'}\text{-Mod})$, we denote by $\mathcal{F}_K$ the natural image in the category $\mathrm{D}^{\mathrm{b}}((\mathcal{X}\times_R\mathcal{X}')_K)$.

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Let $\widetilde{\mathcal{E}} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}\times_R\mathcal{X}')$ be such that $\mathcal{E} = i^*\widetilde{\mathcal{E}}$. Then $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}_K$ are kernels of Fourier–Mukai equivalences.

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Here we denoted by $i : X \times X \to \mathcal{X}\times_R\mathcal{X}'$ the natural inclusion.

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Proposition

Let $v_1 \in H^1(X, \mathcal{T}_X)$ be the Kodaira–Spencer class of first order deformation given by a twistor space $\mathbb{X}(\omega)$ as above. Then

$$v_1' := \Phi_{\mathcal{E}}^{\mathrm{HH}}(v_1) \in H^1(X, \mathcal{T}_X).$$

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Proposition (Toda)

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$$i_1^*\mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}_1' \times_{R_1} \mathcal{X}_1'$ is the natural inclusion.

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Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}_1' \times_{R_1} \mathcal{X}_1'$ is the natural inclusion.

Hence there is a first order deformation of $\mathcal{E}$.

Higher order deformations

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More generally

We construct, at any order n, an analytic deformation $\mathcal{X}'_n$ such that there exists $\mathcal{E}_n \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_n \times_{R_n} \mathcal{X}'_n)$, with

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2. Show that the obstruction is zero.

Our approach imitates the first order case (using relative Hochschild homology).

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Use the generic analytic case

There exist integers *n* and *m* such that the Fourier–Mukai equivalence

$$T^n_{(\mathcal{O}_{\mathcal{X}})_K} \circ \Phi_{\mathcal{E}_K}[m]$$

has kernel $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$.

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Remark

This shows that the autoequivalences of the derived category $\mathrm{D}^\mathrm{b}(\mathcal{X}_K)$ behaves like the derived category of a complex K3 surface with trivial Picard group.

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Definition

A *K*-rational point of $\pi : \mathcal{X} \to \mathrm{Spf}(R)$ is an integral formal subscheme $\mathcal{Z} \subseteq \mathcal{X}$ which is flat of relative dimension zero and such that $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathrm{Spf}(R)$ is an isomorphism.

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- One constructs a locally finite stability condition σ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ such that, if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ is σ-stable and semi-rigid with $\mathrm{End}_{\mathcal{X}_K}(\mathcal{F}) \cong K$, then up to shift $\mathcal{F}$ is a K-rational point.

Ideas from the proof

Definition

A *K-rational point* of $\pi : \mathcal{X} \to \mathrm{Spf}(R)$ is an integral formal subscheme $\mathcal{Z} \subseteq \mathcal{X}$ which is flat of relative dimension zero and such that $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathrm{Spf}(R)$ is an isomorphism.

- One constructs a locally finite stability condition σ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ such that, if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ is σ-stable and semi-rigid with $\mathrm{End}_{\mathcal{X}_K}(\mathcal{F}) \cong K$, then up to shift $\mathcal{F}$ is a K-rational point.

An object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$ is semi-rigid if $\mathrm{Ext}^1_K(\mathcal{F}, \mathcal{F}) \cong K^{\oplus 2}$.

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- Using this stability condition, one proves that there are integers n and m such that the Fourier–Mukai equivalence
$$T^n_{(\mathcal{O}_{\mathcal{X}})_K}\circ\Phi_{\mathcal{E}_K}[m]$$
send K-rational points to K-rational points.

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- One shows that if a Fourier–Mukai equivalence sends K-rational points to K-rational points, then its kernel $\mathcal{G}$ is a sheaf, i.e.
$$\mathcal{G}\in\mathbf{Coh}(\mathcal{X}\times_R\mathcal{X}').$$

Key ingredients

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In the previous proof we use that $(\mathcal{O}_{\mathcal{X}})_K$ is the unique, up to shift, spherical object in $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$.

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In particular, we use that given a locally finite stability condition σ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$, there exists an integer n such that in the stability condition $T^n_{(\mathcal{O}_{\mathcal{X}})_K}(\sigma)$ all K-rational points are stable with the same phase.

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In particular, we use that given a locally finite stability condition σ on $\mathrm{D}^{\mathrm{b}}(\mathcal{X}_K)$, there exists an integer n such that in the stability condition $T^n_{(\mathcal{O}_{\mathcal{X}})_K}(\sigma)$ all K-rational points are stable with the same phase.

Remark

Notice that for our proof we use stability conditions in a very mild form. We just use a specific stability condition in which we can classify all semi-rigid stable objects.

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2. The natural morphism

$$(\Phi_{\mathcal{G}_0})_H : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

is such that $(\Phi_{\mathcal{G}_0})_H = (\Phi_{\mathcal{E}})_H = j$.

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For the second part, we show that $\mathcal{G}_0$ and $\mathcal{E}$ induce the same action on the Grothendieck groups and have the same Mukai vector!

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Warning!

We have not proved that $\mathcal{E}$ is a (shift of a) sheaf! We have just proved that the action in cohomology is the same as the one of a sheaf!

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There exists an explicit description of the first order deformations of the abelian category of coherent sheaves on a smooth projective variety (Toda).

The existence of equivalences between the derived categories of smooth projective K3 surfaces is detected by the existence of special isometries of the total cohomologies.

Question

Can we get the same result for derived categories of first order deformations of K3 surfaces using special isometries between ‘deformations’ of the Hodge and lattice structures on the total cohomologies?

Outline

Hochschild–Kostant–Rosenberg

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One then defines the graded isomorphisms

$$I_K^X = (\mathrm{td}(X)^{1/2} \wedge (-)) \circ I^X_{\mathrm{HKR}} \qquad I_X^K = (\mathrm{td}(X)^{-1/2} \lrcorner (-)) \circ I_X^{\mathrm{HKR}}.$$

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Given a smooth projective variety X and for any $v \in \mathrm{HH}^2(X)$, Toda constructed explicitety the abelian category

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One also have an isomorphism $J : \mathrm{HH}^2(X_1) \to \mathrm{HH}^2(X_1)$ such that

$$(I_{X_1}^{\mathrm{HKR}} \circ J \circ (I_{X_1}^{\mathrm{HKR}})^{-1})(\alpha, \beta, \gamma) = (\alpha, -\beta, \gamma).$$

Outline

The Infinitesimal Derived Torelli Theorem

Theorem (Macrì–S.)

Let X_1 and X_2 be smooth complex projective K3 surfaces and let $v_i \in \mathrm{HH}^2(X_i)$, with $i = 1, 2$. Then the following are equivalent:

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1. There exists a Fourier–Mukai equivalence

$$\Phi_{\widetilde{\mathcal{E}}} : \mathrm{D}^{\mathrm{b}}(X_1, v_1) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X_2, v_2)$$

with $\widetilde{\mathcal{E}} \in \mathrm{D}_{\mathrm{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$.

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with $\widetilde{\mathcal{E}} \in \mathrm{D}_{\mathrm{perf}}(X_1 \times X_2, -J(v_1) \boxplus v_2)$.
2. There exists an orientation preserving effective Hodge isometry
$$g : \widetilde{H}(X_1, v_1, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X_2, v_2, \mathbb{Z}).$$

The structures

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For X a K3, $v \in \mathrm{HH}^2(X)$ and σ_X is a generator for $\mathrm{HH}_2(X)$, let

$$w := I_K^X(\sigma_X) + \epsilon I_K^X(\sigma_X \circ v) \in \widetilde{H}(X, \mathbb{Z}) \otimes \mathbb{Z}[\epsilon]/(\epsilon^2).$$

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1. The $\mathbb{Z}[\epsilon]/(\epsilon^2)$-linear extension of the generalized Mukai pairing $\langle -, - \rangle_M$.
2. A weight-2 decomposition on $\widetilde{H}(X,\mathbb{Z}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$

$$\widetilde{H}^{2,0}(X,v) := \mathbb{C}[\epsilon]/(\epsilon^2)\cdot w \qquad \widetilde{H}^{0,2}(X,v) := \overline{\widetilde{H}^{2,0}(X,v)}$$

and $\widetilde{H}^{1,1}(X,v) := (\widetilde{H}^{2,0}(X,v) \oplus \widetilde{H}^{0,2}(X,v))^{\perp}$.

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An effective isometry is orientation preserving if g_0 preserves the orientation of the four-space.

Outline

Commutativity I

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The first key ingredient (of independent interest) is the following:

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Theorem (Macrì–S.)

Let X_1 and X_2 be smooth complex projective varieties and let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X_1 \times X_2)$. Then the following diagram

$$\begin{tikzcd}
\mathrm{HH}_*(X_1) \arrow[r, "(\Phi_{\mathcal{E}})_{\mathrm{HH}}"] \arrow[d, "I_K^{X_1}"'] & \mathrm{HH}_*(X_2) \arrow[d, "I_K^{X_2}"] \\
\widetilde{H}(X_1, \mathbb{C}) \arrow[r, "(\Phi_{\mathcal{E}})_H"] & \widetilde{H}(X_2, \mathbb{C})
\end{tikzcd}$$

commutes.

Commutativity II

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Using that for K3 surfaces $H^{0,2}$ is 1-dimensional and the previous result, one get the following commutative diagram (for a Fourier–Mukai equivalence $\Phi_{\mathcal{E}}$):

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\mathrm{HH}^*(X_1) \arrow[r, "(\Phi_{\mathcal{E}})^{\mathrm{HH}}"] \arrow[d, "(-)\circ\sigma_{X_1}"'] & \mathrm{HH}^*(X_2) \arrow[d, "(-)\circ(\Phi_{\mathcal{E}})_{\mathrm{HH}}(\sigma_{X_1})"] \\
\mathrm{HH}_*(X_1) \arrow[r, "(\Phi_{\mathcal{E}})_{\mathrm{HH}}"] \arrow[d, "I_K^{X_1}"'] & \mathrm{HH}_*(X_2) \arrow[d, "I_K^{X_2}"] \\
\widetilde{H}(X_1,\mathbb{C}) \arrow[r, "(\Phi_{\mathcal{E}})_H"] & \widetilde{H}(X_2,\mathbb{C}),
\end{tikzcd}$$

where σ_{X_1} is a generator of $\mathrm{HH}_2(X_1)$.

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In particular, one could simplify the hypothesis about the choice of the Kähler class giving rise to the twistor space.