

Equivalences of K3 Surfaces and Orientation

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Outline

- 1 **Derived Torelli Theorem**
 - Motivations
 - The statement
 - Idea of the proof
 - The main result: Szendroi's conjecture is true

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 - Sketch of the proof

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The problem

Let X be a **K3 surface** (i.e. a smooth complex compact simply connected surface with trivial canonical bundle).

Main problem

Describe the group $\text{Aut}(D^b(X))$ of exact autoequivalences of the triangulated category

$$D^b(X) := D_{\text{Coh}}^b(\mathcal{O}_X\text{-Mod}).$$

Remark (Orlov)

Such a description is available when X is an abelian surface (actually an abelian variety).

Geometric case

Torelli Theorem

Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$$

which maps the class of a Kähler class of X into the Kähler cone of Y . Then there exists a unique isomorphism

$$f : X \cong Y$$

such that $f_* = g$.

Lattice theory + Hodge structures + ample cone

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The derived case

Derived Torelli Theorem (Mukai, Orlov)

Let X and Y be smooth projective K3 surfaces.

- 1 If $\Phi : D^b(X) \cong D^b(Y)$ is an equivalence, then there exists a naturally defined Hodge isometry

$$\Phi_* : \tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(Y, \mathbb{Z}).$$

- 2 Suppose there exists a Hodge isometry $g : \tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(Y, \mathbb{Z})$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : D^b(X) \cong D^b(Y)$ such that $\Phi_* = g$.

It is not symmetric!

Additional structures

Lattice structure: The Mukai pairing (Euler–Poincaré form up to sign). The lattice is denoted $\tilde{H}(X, \mathbb{Z})$.

Orientation: Let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class. Then

$$P(X, \sigma, \omega) := \langle \operatorname{Re}(\sigma), \operatorname{Im}(\sigma), 1 - \omega^2/2, \omega \rangle,$$

is a positive four-space in $\tilde{H}(X, \mathbb{R})$ with a natural orientation.

Hodge structure: The weight-2 Hodge structure on $H^*(X, \mathbb{Z})$ is

$$\tilde{H}^{2,0}(X) := H^{2,0}(X),$$

$$\tilde{H}^{0,2}(X) := H^{0,2}(X),$$

$$\tilde{H}^{1,1}(X) := H^0(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbb{C}).$$

Orientation

- 1 Due to the choice of a basis, the space $P(X, \sigma, \omega)$ comes with a natural orientation.
- 2 The orientation is independent of the choice of σ_X and ω .
- 3 It is easy to see that the isometry

$$j := (\text{id})_{H^0 \oplus H^4} \oplus (-\text{id})_{H^2}$$

is not orientation preserving.

Problem

According to the Derived Torelli Theorem, is the isometry j induced by an autoequivalence?

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Idea of the proof

Definition

$F : D^b(X) \rightarrow D^b(Y)$ is of **Fourier–Mukai type** if there exists $\mathcal{E} \in D^b(X \times Y)$ and an isomorphism of functors

$$F \cong \Phi_{\mathcal{E}} := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \rightarrow Y$ and $q : X \times Y \rightarrow X$ are the natural projections.

The complex \mathcal{E} is the **kernel** of F .

Orlov: Every equivalence is of Fourier–Mukai type.

Ideas form the proof

Using the Chern character one gets the commutative diagram:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\quad \phi \quad} & D^b(Y) \\ \downarrow [-] & & \downarrow [-] \\ K(X) & \xrightarrow{\quad \quad \quad} & K(Y) \\ \downarrow \text{ch}(-) \cdot \sqrt{\text{td}(X)} & & \downarrow \text{ch}(-) \cdot \sqrt{\text{td}(Y)} \\ \tilde{H}(X, \mathbb{Z}) & \xrightarrow{\quad \Phi_* \quad} & \tilde{H}(Y, \mathbb{Z}) \end{array}$$

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The statement

Main Theorem (Huybrechts-Macri-S.)

Let X and Y be smooth projective K3 surfaces. Any equivalence $\Phi : D^b(X) \cong D^b(Y)$ induces naturally a Hodge isometry $\Phi_* : \tilde{H}(X, \mathbb{Z}) \rightarrow \tilde{H}(Y, \mathbb{Z})$ preserving the natural orientation of the four positive directions.

Remark

This result was previously conjectured by Szendroi as a mirror-symmetric analogue of a result of Borcea and Donaldson about the group of diffeomorphisms of a K3 surface.

Equivalent statements

Let $O_+ := O_+(\tilde{H}(X, \mathbb{Z}))$ be the group of orientation preserving Hodge isometries of $\tilde{H}(X, \mathbb{Z})$.

Using the previous result, we would get

$$1 \rightarrow ? \rightarrow \text{Aut}(D^b(X)) \xrightarrow{\Pi} O_+ \rightarrow 1.$$

There is a conjectural description by Bridgeland of the kernel in terms of the geometry of the complex manifold parametrizing stability conditions on $D^b(X)$.

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The statement

Theorem (Huybrechts-Macri-S.)

Let X and Y be generic analytic K3 surfaces (i.e. $\text{Pic}(X) = \text{Pic}(Y) = 0$). If

$$\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(Y)$$

is an equivalence of Fourier-Mukai type, then up to shift

$$\Phi_{\mathcal{E}} \cong T_{\mathcal{O}_Y}^n \circ f_*$$

for some $n \in \mathbb{Z}$ and an isomorphism

$$f : X \xrightarrow{\sim} Y.$$

The functors

Definition

An object $\mathcal{E} \in D^b(X)$ is **spherical** if

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, \mathcal{O}_X is spherical.

The **spherical twist** $T_{\mathcal{O}_X} : D^b(X) \rightarrow D^b(X)$ that sends $\mathcal{F} \in D^b(X)$ to the cone of

$$\bigoplus_i (\mathrm{Hom}(\mathcal{O}_X, \mathcal{F}[i]) \otimes \mathcal{O}_X[-i]) \rightarrow \mathcal{F}$$

is an orientation preserving equivalence.

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Stability conditions (Bridgeland)

For simplicity, we restrict ourselves to the case of stability conditions on derived categories!

Any triangulated category would fit.

A **stability condition** on $D^b(X)$ is a pair $\sigma = (Z, \mathcal{P})$ where

- $Z : \mathcal{N}(X) \otimes \mathbb{C} \rightarrow \mathbb{C}$ is a linear map (the **central charge**; here $\mathcal{N}(X)$ is the sublattice of $\tilde{H}(X, \mathbb{Z})$ orthogonal to $H^{2,0}(X)$.)
- $\mathcal{P}(\phi) \subset D^b(X)$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

The definition

- (a) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.
- (b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all ϕ .
- (c) $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all $\mathcal{E}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.
- (d) Any $0 \neq \mathcal{E} \in \text{D}^b(X)$ admits a **Harder–Narasimhan filtration** given by a collection of distinguished triangles

$$\mathcal{E}_{i-1} \rightarrow \mathcal{E}_i \rightarrow \mathcal{A}_i$$

with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \dots > \phi_n$.

Stability conditions (Bridgeland)

- The non-zero objects in the category $\mathcal{P}(\phi)$ are the **semistable** objects of phase ϕ . The objects \mathcal{A}_i in (d) are the **semistable factors** of \mathcal{E} .
- The minimal objects of $\mathcal{P}(\phi)$ are called **stable** of phase ϕ .
- The category $\mathcal{P}((0, 1])$ is called the **heart** of σ .

Stability conditions (Bridgeland)

To exhibit a stability condition on $D^b(X)$, it is enough to give

- a bounded t -structure on $D^b(X)$ with heart \mathbf{A} ;
- a group homomorphism $Z : K(\mathbf{A}) \rightarrow \mathbb{C}$ such that $Z(\mathcal{E}) \in \mathbb{H}$, for all $0 \neq \mathcal{E} \in \mathbf{A}$, and with the Harder–Narasimhan property ($\mathbb{H} := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}$).

All stability conditions are assumed to be **locally-finite**. Hence every object in $\mathcal{P}(\phi)$ has a finite **Jordan–Hölder filtration**.

$\text{Stab}(D^b(X))$ is the manifold parametrizing locally finite stability conditions.

The group $\text{Aut}(D^b(X))$ of exact autoequivalences of $D^b(X)$ acts on $\text{Stab}(D^b(X))$.

Stability conditions: the generic case

Consider the open subset

$$R := \mathbb{C} \setminus \mathbb{R}_{\geq -1} = R_+ \cup R_- \cup R_0,$$

where the sets are defined in the natural way:

- $R_+ := R \cap (\mathbb{H} \setminus \mathbb{R}_{<0})$,
- $R_- := R \cap (-\mathbb{H} \setminus \mathbb{R}_{<0})$,
- $R_0 := R \cap \mathbb{R}$.

Proposition

For any $z \in R$, there exist an abelian category $\mathcal{A}(z)$ and a linear function Z yielding a stability condition $\sigma_z \in \text{Stab}(D^b(X))$.

Stability conditions: the generic case

Proposition

For any $\sigma \in \text{Stab}(D^b(X))$, there is $n \in \mathbb{Z}$ such that $T_{\mathcal{O}_X}^n(\mathcal{O}_p)$ is stable in σ , for any closed point $p \in X$.

Definition

An object $\mathcal{E} \in D^b(X)$ is **semi-rigid** if $\text{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{E}[1]) \cong \mathbb{C}^{\oplus 2}$.

Lemma

If $z \in \mathbb{R}_{<0}$, then the only semi-rigid stable objects in $\mathcal{A}(z)$ are the skyscraper sheaves.

The proof

Consider an equivalence of Fourier–Mukai type
 $\Phi : D^b(X) \rightarrow D^b(Y)$.

- (a) Take the distinguished stability condition

$$\sigma = \sigma_{Z=(u,v=0)}$$

constructed before. Let

$$\tilde{\sigma} := \Phi_{\mathcal{E}}(\sigma).$$

- (b) We have seen that, there exists an integer n such that all skyscraper sheaves \mathcal{O}_p are stable of the same phase in the stability condition $T_{\mathcal{O}_Y}^n(\tilde{\sigma})$.

The proof

- (c) The composition $\Psi := T_{\mathcal{O}_Y}^n \circ \Phi_{\mathcal{E}}$ has the properties:
- 1 It sends the stability condition σ to a stability condition σ' for which all skyscraper sheaves are stable of the same phase.
 - 2 Up to shifting the kernel \mathcal{F} of Ψ sufficiently, we can assume that $\phi_{\sigma'}(\mathcal{O}_y) \in (0, 1]$ for all closed points $y \in Y$.

Thus, the heart $\mathcal{P}'((0, 1])$ of the t -structure associated to σ' (identified with $\mathcal{A}(z)$) contains as stable objects the images $\Psi(\mathcal{O}_p)$ of all closed points $p \in X$ and all skyscraper sheaves \mathcal{O}_y .

The proof

- (d) We observed that the only semi-rigid stable objects in $\mathcal{A}(z)$ are the skyscraper sheaves. Hence, for all $p \in X$ there exists a point $y \in Y$ such that $\Psi(\mathcal{O}_p) \cong \mathcal{O}_y$. Therefore Ψ is a composition of f_* , for some isomorphism

$$f : X \xrightarrow{\sim} Y,$$

and the tensorization by a line bundle.

- (e) But there are no non-trivial line bundles on Y .

Concluding remarks

There are some important features in the proof:

Proposition

Up to shifts, \mathcal{O}_X is the only spherical object in the category $D^b(X)$.

Theorem (Huybrechts-Macri-S.)

The manifold parametrizing numerical stability conditions on $D^b(X)$ is connected and simply-connected.

This proves a conjecture by Bridgeland in the generic analytic case.

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The non-orientation Hodge isometry

Take any projective K3 surface X .

We have already remarked that the isometry

$$j := (\text{id})_{H^0 \oplus H^4} \oplus (-\text{id})_{H^2}$$

is not orientation preserving.

Since any orientation preserving Hodge isometry lifts to an equivalence $\Phi : D^b(X) \rightarrow D^b(X)$ (due to HLOY and Huybrechts-S.), to prove the main theorem, it is enough to prove that j is not induced by a Fourier–Mukai equivalence.

We proceed by contradiction assuming that there exists $\mathcal{E} \in D^b(X \times X)$ such that $(\Phi_{\mathcal{E}})_* = j$.

The twistor space

Definition

A Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ is called **very general** if there is no non-trivial integral class $0 \neq \alpha \in H^{1,1}(X, \mathbb{Z})$ orthogonal to ω , i.e. $\omega^\perp \cap H^{1,1}(X, \mathbb{Z}) = 0$.

Take the twistor space $\mathbb{X}(\omega)$ of X determined by the choice of a very general Kähler class $\omega \in \mathcal{K}_X \cap \text{Pic}(X) \otimes \mathbb{R}$. Hence we get a complex deformation

$$\pi : \mathbb{X}(\omega) \rightarrow \mathbb{P}(\omega).$$

Take $R := \mathbb{C}[[t]]$ to be the ring of power series in t with residue field $K := \mathbb{C}((t))$.

The twistor space

If $R_n := k[[t]]/(t^{n+1})$, then the infinitesimal neighbourhoods

$$\mathcal{X}_n := \mathbb{X}(\omega) \times \text{Spec}(R_n),$$

form an inductive system and give rise to a formal R -scheme

$$\pi : \mathcal{X} \rightarrow \text{Spf}(R),$$

which is the **formal neighbourhood of X** in $\mathbb{X}(\omega)$.

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The first order deformation

The equivalence $\Phi_{\mathcal{E}}$ induces a morphism

$$\Phi_{\mathcal{E}}^{HH} : HH^2(X) \rightarrow HH^2(X).$$

Proposition

Let $v_1 \in H^1(X, \mathcal{T}_X)$ be the Kodaira–Spencer class of first order deformation given by a twistor space $\mathbb{X}(\omega)$ as above. Then

$$v'_1 := \Phi_{\mathcal{E}}^{HH}(v_1) \in H^1(X, \mathcal{T}_X).$$

The first order deformation

Let \mathcal{X}'_1 be the first order deformation corresponding to v'_1 .

Using results of Toda one gets the following conclusion

Proposition (Toda)

For v_1 and v'_1 as before, there exists $\mathcal{E}_1 \in D^b(\mathcal{X}_1 \times_{R_1} \mathcal{X}'_1)$ such that

$$i_1^* \mathcal{E}_1 = \mathcal{E}_0 := \mathcal{E}.$$

Here $i_1 : \mathcal{X}_0 \times_{\mathbb{C}} \mathcal{X}_0 \hookrightarrow \mathcal{X}'_1 \times_{R_1} \mathcal{X}'_1$ is the natural inclusion.

Hence there is a first order deformation of \mathcal{E} .

Higher order deformations

More generally

We construct, at any order n , an analytic deformation \mathcal{X}'_n such that there exists $\mathcal{E}_n \in D^b(\mathcal{X}_n \times_{\mathbb{R}_n} \mathcal{X}'_n)$, with

$$i_n^* \mathcal{E}_n = \mathcal{E}_{n-1}.$$

Main difficulties

- 1 Rewrite Lieblich-Lowen's obstruction for deforming complexes in terms of Atiyah–Kodaira classes.
- 2 Show that the obstruction is zero.

Our approach imitates the first order case (using relative Hochschild homology).

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Equivalences go to equivalences

There exists a sequence

$$\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}') \hookrightarrow \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}') \rightarrow \mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K),$$

where $\mathbf{Coh}_0(\mathcal{X} \times_R \mathcal{X}')$ is the abelian category of sheaves on $\mathcal{X} \times_R \mathcal{X}'$ supported on $X \times X$.

Proposition

Let $\tilde{\mathcal{E}} \in D^b(\mathcal{X} \times_R \mathcal{X}')$ be such that $\mathcal{E} = i^* \tilde{\mathcal{E}}$ (here $i: X \times X \rightarrow \mathcal{X} \times_R \mathcal{X}'$ is the inclusion). Then $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_K$ are kernels of Fourier–Mukai equivalences.

Here $\tilde{\mathcal{E}}_K$ is the image via the natural functor in

$$D^b((\mathcal{X} \times_R \mathcal{X}')_K) := D^b(\mathbf{Coh}((\mathcal{X} \times_R \mathcal{X}')_K)).$$

The generic fiber

Proposition

The triangulated category $D^b(\mathcal{X}_K) := D^b(\mathbf{Coh}(\mathcal{X}_K))$ is a generic K3 category, i.e. $[2]$ is the Serre functor and $(\mathcal{O}_X)_K$ is, up to shifts, the unique spherical object.

Use the generic analytic case

Hence, reasoning as the analytic generic case, one can compose $\Phi_{\mathcal{E}_K}$ with some power of the spherical twist by $(\mathcal{O}_X)_K$ getting a Fourier–Mukai equivalence $\Phi_{\mathcal{G}_K}$ where $\mathcal{G} \in \mathbf{Coh}(\mathcal{X} \times_R \mathcal{X}')$.

The conclusion

Properties of \mathcal{G}

- 1 $\mathcal{G}_0 := i^*\mathcal{G}$ is a sheaf in $\mathbf{Coh}(X \times X)$.
- 2 The natural morphism

$$(\Phi_{\mathcal{G}_0})_* : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$$

is such that $(\Phi_{\mathcal{G}_0})_* = (\Phi_{\mathcal{E}})_* = j$.

Notice that \mathcal{G}_0 and \mathcal{E} have the same Mukai vector!

The conclusion

The contradiction is now obtained using the following lemma:

Lemma

If $\mathcal{F} \in \mathbf{Coh}(X \times X)$, then $(\Phi_{\mathcal{F}})_* \neq j_*$.

Open question

Which is the kernel of the map $\mathrm{Aut}(D^b(X)) \rightarrow \mathrm{O}_+(\tilde{H}(X, \mathbb{Z}))$?