A Twisted Derived Torelli Theorem for K3 Surfaces

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Brauer groups

We will always consider smooth projective varieties X.

Definition

The Brauer group of X is

$$\mathrm{Br}(X) := H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}.$$

Example

By the exponential exact sequence, we get

$$H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^*) \longrightarrow H^3(X, \mathbb{Z}).$$

If X is a curve, then $H^2(X, \mathcal{O}_X) = H^3(X, \mathbb{Z}) = 0$.
Hence $\mathrm{Br}(X) = \{0\}$

Brauer groups

Example

The same calculation shows that $\mathrm{Br}\,(\mathbb{P}^n) = 0$.

K3 surfaces

A K3 surface is a complex smooth projective surface X such that

- $H^1(X, \mathbb{Z}) = 0$;
- the canonical bundle is trivial.

In this case, the Universal Coefficient Theorem, yields a nice description of $\mathrm{Br}\,(X)$:

$$\mathrm{Br}\,(X) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$

Brauer groups

Due to the previous remark, for any $\alpha \in \mathrm{Br}(X)$ we put

$$T(X,\alpha) := \ker(\alpha) \subseteq T(X).$$

It inherits a weight-two Hodge structure from $H^2(X,\mathbb{Z})$.

Any $\alpha \in \mathrm{Br}(X)$ is determined by some $B \in H^2(X,\mathbb{Q})$ and vice-versa. (Actually α is determined by $B \in T(X)^\vee \otimes \mathbb{Q}/\mathbb{Z}$.)

In this case we write $\alpha_B := \alpha$.

Any $B \in H^2(X,\mathbb{Q})$ is called B-field.

Brauer groups

Definition

A pair (X, α) where X is a smooth projective variety and $\alpha \in \mathrm{Br}(X)$ is a twisted variety.

Represent $\alpha \in \mathrm{Br}(X)$ as a Čech 2-cocycle

$$\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover $X = \bigcup_{i \in I} U_i$.

Twisted sheaves

An α-twisted coherent sheaf $\mathcal{E}$ is a collection of pairs $(\{\mathcal{E}_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ where

- $\mathcal{E}_i$ is a coherent sheaf on the open subset U_i;
- $\varphi_{ij} : \mathcal{E}_j|_{U_i\cap U_j} \to \mathcal{E}_i|_{U_i\cap U_j}$ is an isomorphism

such that

1. $\varphi_{ii} = \mathrm{id}$,
2. $\varphi_{ji} = \varphi_{ij}^{-1}$ and
3. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

Twisted derived categories

- In this way we get the abelian category **Coh**(X, α).
- Pass to the category of bounded complexes.
- Localize: require that any quasi-isomorphism is invertible.
- We get the bounded derived category $\mathrm{D}^{\mathrm{b}}(X, \alpha)$.

Not all functors with geometric meaning are exact in **Coh**(X, α).

Procedure to produce from them exact functors in $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ (not abelian but triangulated).

We get left and right derived functors.

All "geometric functors" can be derived.

Why twists?

There are two order of problems which requires twists.

Mirror Symmetry (Kontsevich)

This conjecture predicts a nice relationship between a Calabi-Yau manifold X_1 and its mirror X_2.

In particular it "cross relates" the following categories:

- the bounded derived category of the Fukaya category of X_i (Lagrangian submanifolds);
- the bounded derived categories $\mathrm{D}^{\mathrm{b}}(X_i)$ (sheaves).

If one allows B-fields then on the derived categories level one has to consider twists!

We will mainly ignore this problem. (Not completely settled.)

Why twists?

Moduli spaces (Mukai)

If X is a K3 surface and M is a fine moduli space of stable sheaves on X with suitable properties, then M is a K3 surface.

- there exists an equivalence

$$\Phi : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(M)$$

  induced by the universal family (Mukai).
- There is a Hodge isometry $T(X) \cong T(M)$ of the transcendental lattices.

And if M is coarse?

M is a 2-dimensional, irreducible, smooth and projective coarse moduli space of stable sheaves on X.

- Mukai proved that there exists an embedding

$$\varphi : T(X) \hookrightarrow T(M)$$

  which preserves the Hodge and lattice structures.

- We have the short exact sequence

$$0 \longrightarrow T(X) \xrightarrow{\varphi} T(M) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

- Apply $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and get

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{Br}(M) \xrightarrow{\varphi^\vee} \mathrm{Br}(X) \longrightarrow 0.$$

Căldăraru's results

The obstruction

A special generator $\alpha \in \mathrm{Br}(M)$ of the kernel of $\varphi^\vee$ is the obstruction to the existence of a universal family on *M*.

Theorem

Let *X* be a K3 surface and let *M* be a coarse moduli space of stable sheaves on *X* as above. Then

1. $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(M, \alpha^{-1})$ (via the twisted universal/quasi-universal family);
2. there is a Hodge isometry

$$T(X) \cong T(M, \alpha^{-1}).$$

Căldăraru's results

The previous result makes the twisted/coarse setting very similar to the untwisted/fine one!

Conjecture

Let (X, α) and (Y, β) be twisted K3 surfaces. Then the following two conditions are equivalent:

1. $\mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(Y, \beta)$;
2. there exists a Hodge isometry $T(X, \alpha) \cong T(Y, \beta)$.

Evidence: Work of Donagi and Pantev about elliptic fibrations.

Fourier-Mukai functors

Definition

$F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is of Fourier-Mukai type if there exists $\mathcal{E} \in \mathrm{D^b}(X \times Y)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

The complex $\mathcal{E}$ is called the kernel of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

Orlov's result

Theorem (Orlov)

Any exact functor $F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ which

1. is fully faithful
2. admits a left adjoint

is a Fourier-Mukai functor.

Remark (Bondal, Van den Bergh)

Item (2) is automatic!

Twisted case

Question

Are all equivalences between the twisted derived categories of smooth projective varieties of Fourier-Mukai type?

This is known in some geometric cases involving K3 surfaces:

- moduli spaces of stable sheaves on K3 surfaces (Căldăraru);
- K3 surfaces with large Picard number (H.-S.).

The main theorem

Theorem. (C.-S.)

Let (X, α) and (Y, β) be twisted varieties. Let

$$F : \mathrm{D^b}(X, \alpha) \to \mathrm{D^b}(Y, \beta)$$

be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X, \alpha)$,

$$\mathrm{Hom}_{\mathrm{D^b}(Y,\beta)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \quad \text{if } j < 0.$$

Then there exist $\mathcal{E} \in \mathrm{D^b}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

Comments

The previous result covers some interesting cases:

- full functors;
- (as a special case) equivalences.

It also simplifies the proof of Kawamata's generalization of Orlov's result to the case of smooth stacks.

More comments

Proposition

Let (X, α) and (Y, β) be twisted varieties. Then there exists an isomorphism $f : X \cong Y$ such that $f^*(\beta) = \alpha$ if and only if there exists an exact equivalence $\mathbf{Coh}(X, \alpha) \cong \mathbf{Coh}(Y, \beta)$.

The abelian category $\mathbf{Coh}(X, \alpha)$ is a too strong invariant!

Needs:

1. Preserve deep geometric relationships (moduli spaces) (Mukai,...).
2. A good birational invariant. Some kind of "Derived MMP" (Kawamata, Bridgeland, Chen,...).
3. Relevant for physics $\Rightarrow$ Mirror Symmetry (Kontsevich,...).

Geometric case

Theorem (Torelli Theorem)

Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry

$$g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

which maps the class of an ample line bundle on X into the ample cone of Y. Then there exists a unique isomorphism

$$f : X \cong Y$$

such that $f_* = g$.

Lattice theory + Hodge structures + ample cone

Derived case

Derived Torelli Theorem (Orlov+Mukai)

Let X and Y be K3 surfaces. Then the following conditions are equivalent:

1. $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$;
2. there exists a Hodge isometry $f : \widetilde{H}(X, \mathbb{Z}) \to \widetilde{H}(Y, \mathbb{Z})$;
3. there exists a Hodge isometry $g : T(X) \to T(Y)$;
4. Y is isomorphic to a smooth compact 2-dimensional fine moduli space of stable sheaves on X.

Lattice theory + Hodge structures

Twisted derived case

Twisted Derived Torelli Theorem (H.-S.)

Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$.

1. If $\Phi : \mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(X', \alpha_{B'})$ is an equivalence, then there exists a naturally defined Hodge isometry $\Phi_*^{B,B'} : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$.
2. Suppose there exists a Hodge isometry $g : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : \mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(X', \alpha_{B'})$ such that $\Phi_*^{B,B'} = g$.

There is something missing!

Lattice structure

Using the cup product, we get the Mukai pairing on $H^*(X, \mathbb{Z})$:

$$\langle \alpha, \beta \rangle := -\alpha_1 \cdot \beta_3 + \alpha_2 \cdot \beta_2 - \alpha_3 \cdot \beta_1,$$

for every $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ in $H^*(X, \mathbb{Z})$.

$H^*(X, \mathbb{Z})$ endowed with the Mukai pairing is called Mukai lattice and we write $\widetilde{H}(X, \mathbb{Z})$ for it.

The Hodge structure

Let $H^{2,0}(X) = \langle \sigma \rangle$ and let B be a B-field on X.

$$\varphi = \exp(B) \cdot \sigma = \sigma + B \wedge \sigma \in H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$$

is a generalized Calabi-Yau structure (Hitchin and Huybrechts).

Definition

Let X be a K3 surface with a B-field $B \in H^2(X, \mathbb{Q})$. We denote by $\widetilde{H}(X, B, \mathbb{Z})$ the weight-two Hodge structure on $H^*(X, \mathbb{Z})$ with

$$\widetilde{H}^{2,0}(X, B) := \exp(B) \left(H^{2,0}(X) \right)$$

and $\widetilde{H}^{1,1}(X, B)$ its orthogonal complement with respect to the Mukai pairing.

Orienatation

Let X be a K3 surface, σ_X be a generator of $H^{2,0}(X)$ and ω be a Kähler class. Then

$$\langle \mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X), 1 - \omega^2/2, \omega \rangle$$

is a positive four-space in $\widetilde{H}(X, \mathbb{R})$.

Remark

It comes, by the choice of the basis, with a natural orientation.

Remark

It is easy to see that this orientation is independent of the choice of σ_X and ω.

Orienatation

The orientation preserving requirement is missing in item (i) of the Twisted Derived Torelli Theorem.

Proposition (H.-S.)

Any known twisted or untwisted equivalence is orientation preserving.

Conjecture

Let X and X' be two algebraic K3 surfaces with B-fields B and B'. If $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{B'})$ is a Fourier-Mukai transform, then $\Phi_*^{B,B'} : \widetilde{H}(X, B, \mathbb{Z}) \to \widetilde{H}(X', B', \mathbb{Z})$ preserves the natural orientation of the four positive directions.

Orienatation

Theorem (H.-M.-S.)

For a generic twisted K3 surface (X, α_B) there exists a short exact sequence

$$1 \to \mathbb{Z}[2] \to \mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X, \alpha_B)) \xrightarrow{\varphi} \mathrm{O}_+ \to 1,$$

where O_+ is the group of the Hodge isometries of $\widetilde{H}(X, B, \mathbb{Z})$ preserving the orientation.

We proved Bridgeland's Conjecture for generic twisted K3 surfaces.

Căldăraru's conjecture is false

Lemma

If $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$ is an equivalence, then there is a Hodge isometry $T(X, \alpha) \cong T(X', \alpha')$.

- Take (X, α) such that $T(X, \alpha) \cong T(X, \alpha^2)$ but $\widetilde{H}(X, B, \mathbb{Z}) \ncong \widetilde{H}(X, 2B, \mathbb{Z})$.
- No twisted Fourier-Mukai transforms $\mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X, \alpha^2)$.
- One implication in Căldăraru's conjecture is false.

Number of Fourier-Mukai partners

Proposition (H.-S.)

Any twisted K3 surface (X, α) admits only finitely many Fourier-Mukai partners up to isomorphisms.

Untwisted $\neq$ Twisted!!

Proposition (H.-S.)

For any positive integer *N* there exist *N* pairwise non-isomorphic twisted K3 surfaces

$$(X_1, \alpha_1), \ldots, (X_N, \alpha_N)$$

of Picard number 20 and such that the twisted derived categories $\mathrm{D}^{\mathrm{b}}(X_i, \alpha_i)$, are all Fourier-Mukai equivalent.

The untwisted case: HLOY

- Given two abelian surfaces A and B,

$$\mathrm{D}^{\mathrm{b}}(A) \cong \mathrm{D}^{\mathrm{b}}(B)$$

  if and only if

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(B)).$$

The argument: they notice that, due to the geometric construction of the Kummer surfaces $\mathrm{Km}(A)$ and $\mathrm{Km}(B)$, the transcendental lattices of A and B are Hodge isometric if and only if the transcendental lattices of $\mathrm{Km}(A)$ and $\mathrm{Km}(B)$ are Hodge isometric. Then, they apply the Derived Torelli Theorem.

The untwisted case: HLOY

Can be reformulated in the following way:

- Given two abelian surfaces A and B,

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Km}(A)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Km}(B))$$

  if and only if there exists a Hodge isometry between the transcendental lattices of A and B.

Due to a result of Mukai, equivalent to:

- Given two abelian surfaces A and B, $\mathrm{D}^{\mathrm{b}}(A) \cong \mathrm{D}^{\mathrm{b}}(B)$ if and only if $\mathrm{Km}(A) \cong \mathrm{Km}(B)$.

The twisted case

Definition

Let (X_1, α_1) and (X_2, α_2) be twisted K3 or abelian surfaces.

1. They are *D*-equivalent if there exists a twisted Fourier-Mukai transform
$$\Phi : \mathrm{D^b}(X_1, \alpha_1) \to \mathrm{D^b}(X_2, \alpha_2).$$
2. They are *T*-equivalent if there exist $B_i \in H^2(X_i, \mathbb{Q})$ such that $\alpha_i = \alpha_{B_i}$ and a Hodge isometry
$$\varphi : T(X_1, \alpha_{B_1}) \to T(X_2, \alpha_{B_2}).$$

The twisted case

Theorem (S.)

Let A_1 and A_2 be abelian surfaces. Then the following two conditions are equivalent:

1. there exist $\alpha_1 \in \mathrm{Br}(\mathrm{Km}(A_1))$ and $\alpha_2 \in \mathrm{Br}(\mathrm{Km}(A_2))$ such that $(\mathrm{Km}(A_1), \alpha_1)$ and $(\mathrm{Km}(A_2), \alpha_2)$ are D-equivalent;
2. there exist $\beta_1 \in \mathrm{Br}(A_1)$ and $\beta_2 \in \mathrm{Br}(A_2)$ such that (A_1, β_1) and (A_2, β_2) are T-equivalent.

Furthermore, if one of these two equivalent conditions holds true, then A_1 and A_2 are isogenous.

Analogue of the second statement!

There are no twisted analogues of the first and third statement!

The number of Kummer structures

By the previous theorem, we have a surjective map

$$\Psi : \{\text{Tw ab surf}\}/\cong \longrightarrow \{\text{Tw Kum surf}\}/\cong .$$

The main result of Hosono, Lian, Oguiso and Yau proves that

1. the preimage of $[(\mathrm{Km}(A), 1)]$ is finite, for any abelian surface A and $1 \in \mathrm{Br}(A)$ the trivial class.
2. The cardinality of the preimages of Ψ can be arbitrarily large.

This answers an old question of Shioda.

The number of Kummer structures

This picture can be completely generalized to the twisted case.

Proposition (S.)

(i) For any twisted Kummer surface $(\mathrm{Km}(A), \alpha)$, the preimage

$$\Psi^{-1}([(\mathrm{Km}(A), \alpha)])$$

is finite.

(ii) For positive integers N and n, there exists a twisted Kummer surface $(\mathrm{Km}(A), \alpha)$ with α of order n in $\mathrm{Br}(\mathrm{Km}(A))$ and such that

$$|\Psi^{-1}([(\mathrm{Km}(A), \alpha)])| \geq N.$$

On a twisted K3 surface we can put just a finite number of non-isomorphic twisted Kummer structures.