A Twisted Derived Torelli Theorem for K3 Surfaces

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Based on (math.AG/0602399) and on joint works with A. Canonaco (math.AG/0605229), D. Huybrechts (math.AG/0409030, math.AG/0411541) and D. Huybrechts-E. Macrì (math.AG/0608430)

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$$\mathrm{Br}(X) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$

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Any $B \in H^2(X, \mathbb{Q})$ is called B-field.

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Represent $\alpha \in \mathrm{Br}(X)$ as a Čech 2-cocycle

$$\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover $X = \bigcup_{i \in I} U_i$.

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3. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

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All “geometric functors” can be derived.

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There are two order of problems which requires twists.

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$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{Br}(M) \xrightarrow{\varphi^\vee} \mathrm{Br}(X) \longrightarrow 0.$$

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Evidence: Work of Donagi and Pantev about elliptic fibrations.

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$F : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ is of Fourier-Mukai type if there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

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The complex $\mathcal{E}$ is called the kernel of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

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Remark (Bondal, Van den Bergh)

Item (2) is automatic!

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This is known in some geometric cases involving K3 surfaces:

- moduli spaces of stable sheaves on K3 surfaces (Căldăraru);
- K3 surfaces with large Picard number (H.-S.).

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Theorem. (C.-S.)

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$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y,\beta)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \text{ if } j < 0.$$

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Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

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It also simplifies the proof of Kawamata's generalization of Orlov's result to the case of smooth stacks.

More comments

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Proposition

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Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$.

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There is something missing!

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for every $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ in $H^*(X, \mathbb{Z})$.

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$H^*(X, \mathbb{Z})$ endowed with the Mukai pairing is called Mukai lattice and we write $\widetilde{H}(X, \mathbb{Z})$ for it.

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Definition

Let X be a K3 surface with a B-field $B \in H^2(X, \mathbb{Q})$. We denote by $\widetilde{H}(X, B, \mathbb{Z})$ the weight-two Hodge structure on $H^*(X, \mathbb{Z})$ with

$$\widetilde{H}^{2,0}(X, B) := \exp(B) \left(H^{2,0}(X) \right)$$

and $\widetilde{H}^{1,1}(X, B)$ its orthogonal complement with respect to the Mukai pairing.

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For a generic twisted K3 surface (X, α_B) there exists a short exact sequence

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Proposition (H.-S.)

For any positive integer N there exist N pairwise non-isomorphic twisted K3 surfaces

$$(X_1, \alpha_1), \ldots, (X_N, \alpha_N)$$

of Picard number 20 and such that the twisted derived categories $\mathrm{D}^{\mathrm{b}}(X_i, \alpha_i)$, are all Fourier-Mukai equivalent.

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The argument: they notice that, due to the geometric construction of the Kummer surfaces $\mathrm{Km}(A)$ and $\mathrm{Km}(B)$, the transcendental lattices of A and B are Hodge isometric if and only if the transcendental lattices of $\mathrm{Km}(A)$ and $\mathrm{Km}(B)$ are Hodge isometric.

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Furthermore, if one of these two equivalent conditions holds true, then A_1 and A_2 are isogenous.

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Analogue of the second statement!

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Let A_1 and A_2 be abelian surfaces. Then the following two conditions are equivalent:

1. there exist $\alpha_1 \in \mathrm{Br}(\mathrm{Km}(A_1))$ and $\alpha_2 \in \mathrm{Br}(\mathrm{Km}(A_2))$ such that $(\mathrm{Km}(A_1), \alpha_1)$ and $(\mathrm{Km}(A_2), \alpha_2)$ are D-equivalent;
2. there exist $\beta_1 \in \mathrm{Br}(A_1)$ and $\beta_2 \in \mathrm{Br}(A_2)$ such that (A_1, β_1) and (A_2, β_2) are T-equivalent.

Furthermore, if one of these two equivalent conditions holds true, then A_1 and A_2 are isogenous.

Analogue of the second statement!

There are no twisted analogues of the first and third statement!

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This answers an old question of Shioda.

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Proposition (S.)

(i) For any twisted Kummer surface $(\mathrm{Km}(A), \alpha)$, the preimage

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is finite.

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Proposition (S.)

(i) For any twisted Kummer surface $(\mathrm{Km}(A), \alpha)$, the preimage

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is finite.
(ii) For positive integers N and n, there exists a twisted Kummer surface $(\mathrm{Km}(A), \alpha)$ with α of order n in $\mathrm{Br}(\mathrm{Km}(A))$ and such that

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On a twisted K3 surface we can put just a finite number of non-isomorphic twisted Kummer structures.