Derived categories and cubic hypersurfaces

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Outline

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The aim of the talk is to propose a 'categorical' treatment for some fundamental (often unknown) geometric properties of smooth (complex) hypersurfaces of degree 3

$$Y \subseteq \mathbb{P}^{n+1}.$$

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- Rationality/irrationality of those varieties;
- Torelli type theorems;
- Geometric description of the Fano varieties of lines of those cubics.

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- Objects:
  $$M^\bullet := \{\cdots \to M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \to \cdots\}$$
- Morphisms: sets of arrows $f^\bullet := \{f^i\}_{i\in\mathbb{Z}}$ making commutative the following diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{M^\bullet}^{i-2}} & M^{i-1} & \xrightarrow{d_{M^\bullet}^{i-1}} & M^i & \xrightarrow{d_{M^\bullet}^{i}} & M^{i+1} & \xrightarrow{d_{M^\bullet}^{i+1}} & \cdots \\
 & & \downarrow{f^{i-1}} & & \downarrow{f^i} & & \downarrow{f^{i+1}} & & \\
\cdots & \xrightarrow{d_{L^\bullet}^{i-2}} & L^{i-1} & \xrightarrow{d_{L^\bullet}^{i-1}} & L^i & \xrightarrow{d_{L^\bullet}^{i}} & L^{i+1} & \xrightarrow{d_{L^\bullet}^{i+1}} & \cdots
\end{array}$$

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- Morphisms: (very) roughly speaking, obtained ‘by inverting qis in $C^b(\mathbf{A})$’.

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Suppose we have a sequence of full triangulated subcategories $\mathbf{T}_1, \ldots, \mathbf{T}_n \subseteq \mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$, where X is smooth projective, such that:

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- $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathbf{T}_i, \mathbf{T}_j) = 0$, for $i > j$,
- For all $K \in \mathrm{D}^{\mathrm{b}}(X)$, there exists a chain of morphisms in $\mathrm{D}^{\mathrm{b}}(X)$

$$0 = K_n \to K_{n-1} \to \ldots \to K_1 \to K_0 = K$$

with $\mathrm{cone}(K_i \to K_{i-1}) \in \mathbf{T}_i$, for all $i = 1, \ldots, n$.

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This is a **semi-orthogonal** decomposition of $\mathrm{D}^{\mathrm{b}}(X)$:

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Derived categories and Fano varieties

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Let X be a smooth projective complex Fano variety and assume that Y is a smooth projective variety such that

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Then $X \cong Y$.

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Question

Does some 'piece' in a semi-orthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(Y)$ behave nicely?

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Use that $J(Y)$ does not decompose as direct sum of Jacobians of curves.

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The subcategory $\mathbf{T}_Y$ is highly non-trivial and cannot be the derived category of a smooth projective variety.

Indeed the Serre functor $S_{\mathbf{T}_Y}$ is such that $S^3_{\mathbf{T}_Y} \cong [5]$. So $\mathbf{T}_Y$ is a so called **Calabi–Yau category of fractional dimension** $\frac{5}{3}$.

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Idea: realize the Fano variety of lines of Y_i as moduli space of stable objects according to a Bridgeland stability condition on $\mathbf{T}_{Y_i}$.

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$\mathrm{Stab}(\mathbf{T})$ is the space parametrizing stability conditions on $\mathbf{T}$.

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Let Y be a cubic 3-fold. As a consequence of the result of Bernardara–Macrì–Mehrotra–S. above, we have that

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Problem

Classify completely all the stability conditions in $\mathrm{Stab}(\mathbf{T}_Y)$.

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1. **Idea:** the irrationality of Y should be related to the presence of gaps in the interval of integers corresponding to the 'generation time' of the objects in $\mathrm{D}^{\mathrm{b}}(Y)$.
2. This is related to a conjecture of Orlov. In this case: the dimension of the category $\mathrm{D}^{\mathrm{b}}(Y)$ is $3 = \dim(Y)$.

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Denote by $T := \langle H^2, P\rangle$ the primitive sublattice (with respect to the intersection form) of $H^4(Y, \mathbb{Z})$ generated by H^2 and P. Then the intersection form is of type

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

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The quadric fibration provides an element

$$\beta \in \mathrm{Br}(S) := H^2(S, \mathcal{O}_S^*)_{\mathrm{tor}}$$

in the **Brauer group** of S.

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- **Torelli theorem (Voisin):** Let Y_1 and Y_2 be two cubic 4-folds and assume that there exists a Hodge isometry

$$\phi : H^4(Y_1, \mathbb{Z}) \to H^4(Y_2, \mathbb{Z})$$

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- **Surjectivity of the period map (Looijenga, Laza):** The period map surjects onto an explicitly described subset of the period domain.

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For a positive integer d, define $\mathcal{C}_d$ to be the set of all $Y \in \mathcal{C}$ such that

- There is a rank-2 lattice K_d with $\det(K_d) = d$.
- There is a primitive embedding $K_d \hookrightarrow H^4(Y, \mathbb{Z})$.
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Hassett: $\mathcal{C}_d$ is an irreducible divisor as soon as $d > 6$ and $d \equiv 0, 2 \pmod 6$.

Outline

1 The geometric setting

2 3-folds
- Geometry
- Derived categories
- Bridgeland stability conditions
- Irrationality

3 4-folds
- Geometry
- **Derived categories**
- Categorical Torelli theorem
- Fano varieties of lines
- Rationality

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Recall that a triangulated category $\mathbf{T}$ is a **2-Calabi–Yau category** if $\mathbf{T}$ has a Serre functor which is isomorphic to the shift by 2.

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Remark

If Y is generic with the above properties (i.e. $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y) = \langle H^2, P \rangle$), then there is no smooth projective K3 surface S' such that

$$\mathbf{T}_Y \cong \mathrm{D}^{\mathrm{b}}(S').$$

Twisted sheaves

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on an analytic open cover $S = \bigcup_{i \in I} U_i$.

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A β-twisted coherent sheaf $\mathcal{E}$ is a collection of pairs $(\{\mathcal{E}_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ where

- $\mathcal{E}_i$ is a coherent sheaf on the open subset U_i;
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such that

1. $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ji} = \varphi_{ij}^{-1}$;
2. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \beta_{ijk} \cdot \mathrm{id}$.

Twisted sheaves

Represent $\beta \in \mathrm{Br}(S)$ as a Čech 2-cocycle

$$\{\beta_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover $S = \bigcup_{i \in I} U_i$.

A β-twisted coherent sheaf $\mathcal{E}$ is a collection of pairs $(\{\mathcal{E}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$ where

- $\mathcal{E}_i$ is a coherent sheaf on the open subset U_i;
- $\varphi_{ij} : \mathcal{E}_j|_{U_i \cap U_j} \to \mathcal{E}_i|_{U_i \cap U_j}$ is an isomorphism

such that

1. $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ji} = \varphi_{ij}^{-1}$;
2. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \beta_{ijk} \cdot \mathrm{id}$.

In this way we get the abelian category $\mathbf{Coh}(S, \beta)$.

Outline

Results and questions

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Theorem (Bernardara–Macrì–Mehrotra–S.)

Given a cubic fourfold Y containing a plane P and such that C is smooth, there exist only finitely many isomorphism classes of cubic 4-folds $Y_1 = Y, Y_2, \ldots, Y_n$ containing a plane and with smooth plane sextics such that $\mathbf{T}_Y \cong \mathbf{T}_{Y_j}$, with $j \in \{1, \ldots, n\}$. Moreover, if Y is generic, then $n = 1$.

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Questions

1. Can we prove a similar result for any possible cubic 4-fold (with a plane or not)?
2. Can the number n be arbitrarily large?

Outline

Classical results

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Assume that $d = 2(n^2 + n + 1)$ for $n \geq 2$. Then the generic cubic 4-fold Y contained in $\mathcal{C}_d$ is such that $F(Y) \cong \mathrm{Hilb}^2(S)$ for some K3 surface S.

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When there is a plane, the twist cannot be avoided...

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If Y is a generic cubic fourfold containing a plane, then $F(Y)$ is isomorphic to a moduli space of stable objects in the derived category $\mathrm{D}^{\mathrm{b}}(S, \beta)$ of bounded complexes of β-twisted coherent sheaves on S.

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Theorem (Macrì–S.)

For all cubic fourfolds Y containing a plane, the Fano variety $F(Y)$ is birational to a smooth projective moduli space of twisted sheaves on a K3 surface. Moreover, if Y is generic, then such a birational map is either an isomorphism or a Mukai flop.

Outline

Hodge theoretical results

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Notice that the presence of such a section implies that the Brauer class β in $\mathrm{Br}(S)$ is automatically trivial.

The categorical approach

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A cubic 4-fold Y is rational if and only if there exists a K3 surface S' and an exact equivalence $\mathbf{T}_Y \cong \mathrm{D}^{\mathrm{b}}(S')$.

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Problem

Use categorical methods to prove that the generic cubic 4-fold with a plane is not rational.