Algebraic invariants of algebraic varieties: the case of derived categories

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UNIVERSITÀ DEGLI STUDI DI MILANO

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In $\mathbb{P}^5$

The projective space

For any $n > 0$ just consider the quotient

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$

Equations

1 Geometry

In $\mathbb{P}^5$ consider the hypersurface described by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Usually called **Fermat cubic fourfold.**

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It consists of all the points of $\mathbb{P}^5$ whose coordinates satisfy the given equation. In particular:

- It is smooth;
- It has (complex) dimension 4.

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Another similar example is given by the following system of equations in $\mathbb{P}^6$:

$$\begin{cases} x_0 + x_1 + \cdots + x_6 = 0 \\ x_0^3 + x_1^3 + \cdots + x_6^3 = 0 \end{cases}$$

It is usually called **Clebsch–Segre cubic fourfold**.

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Are the Fermat and the Clebsch–Segre cubic fourfolds isomorphic?

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Question 2

How do we decide if two **cubic fourfolds** (i.e. smooth zero loci of a homog. polyn. of deg. 3 in $\mathbb{P}^5$) are isomorphic?

Rationality?

We can be more flexible: can we find an isomorphism

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One constructs the (birational) map g as follows:

$$P_1 \times P_2 \dashrightarrow X \quad (p_1, p_2) \mapsto \ell_{p_1,p_2} \cap X$$

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...in general this is one of the major open problems in algebraic geometry!

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2 Cohomology

Some algebraic/geometric invariants

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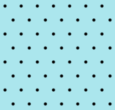
Singular cohomology:

By the **de Rham theorem**:

$$H^4(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^4(X,\mathbb{R}) \cong H^4_{\mathrm{dR}}(X).$$

$H^4(X,\mathbb{Z})$ is a discrete (rank-23 free $\mathbb{Z}$) submod. of the de Rham coho.



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Algebraic classes:

$S = X \cap \mathbb{P}^3 \subseteq \mathbb{P}^5$ is a cubic surface.

E.g. X Fermat, then S given by

$$X \cap \{x_4 = x_5 = 0\}.$$

and eq. $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$.

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Algebraic classes:

S yields a class in $H^4(X, \mathbb{Z})$ by the following procedure:

- It provides an element in $H^4_{\mathrm{dR}}(X)^*$:

$$\int_S : \omega \mapsto \int_S \omega;$$

- Poincaré duality: $H^4_{\mathrm{dR}}(X)^* \cong H^4_{\mathrm{dR}}(X)$;
- It is an integral class.

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Cup product:

With the identification

$$H^4(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^4_{\mathrm{dR}}(X)$$

it is just the the $\wedge$ of forms.

More geometrically: given S_1 and S_2 corresponding to two surfaces in $H^4(X,\mathbb{Z})$, we set:

$$S_1 \cdot S_2 = S_1 \cap S_2 \in H^8(X,\mathbb{Z}) \cong \mathbb{Z}.$$

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- The **Hodge decomposition**

$$H^4(X, \mathbb{C}) = H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X).$$

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The decomposition:

$H^{2,2}(X)$ is generated by algebraic classes as before!

Its orthogonal with respect to the cup product is a 2-dimensional vector space

$$H^{3,1}(X) \oplus H^{1,3}(X)$$

where

$$\overline{H^{1,3}(X)} = H^{3,1}(X).$$

The Torelli problem

2 Cohomology

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C. Voisin (1986)

If X_1 and X_2 are cubic fourfolds with an isometry $H^4(X_1, \mathbb{Z}) \cong H^4(X_2, \mathbb{Z})$ preserving all the structures, then $X_1 \cong X_2$.

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Harris, Hassett (roughly):

A cubic fourfold X is rational if and only if it contains at least another surface S' whose class is diffrent from the one of S with special intersection properties with S in $H^4(X, \mathbb{Z})$.

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...recent progress by Katzarkov–Kontsevich–Pantev–Yu and Iritani!

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- The 4^{th}-cohomology of X (with its additional structures) $\mapsto$ **??**

We are going to make some choices! By no means canonical (but working in most of the examples).

Start with X a cubic fourfold.

More general:

One can take any smooth complex variety Y admitting an embedding in a suitable embedding in a projective space $\mathbb{P}^n$.

Other examples to keep in mind:

- **Quintic threefold:** Given by $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$ in $\mathbb{P}^4$;
- **Quartic surface:** Given by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$.

Examples of **Calabi–Yau threefolds** and **K3 surfaces**.

?=derived categories

Start with X a cubic fourfold.

One can consider **vector bundles** on X.

Examples to keep in mind:

- **Structure sheaf:** If we regard X as a complex analytic variety, we a *sheaf* $\mathcal{O}_X$ such that, for any open subset U,

$$\mathcal{O}_X(U) = \{f \colon U \to \mathbb{C} : f \text{ holomorphic}\};$$

- **(Holomorphic) tangent bundle:** T_X;
- **(Holomorphic) cotangent bundle:** Ω_X;
- **The canonical bundle:** $K_X := \wedge^4 \Omega_X$ (fundamental invariant in our case).

?=derived categories

Start with X a cubic fourfold.

One can consider **vector bundles** on X.

We can further take **bounded complexes** of vector bundles.

Bounded complexes:

They are just (infinite) sequences:

$$E^{\bullet} \quad \cdots \to 0 \to E^i \xrightarrow{d^i} \ldots \xrightarrow{d^{i+n-1}} E^{i+n} \to 0 \to \ldots$$

where

- Each E^j is a vector bundle;
- Only finitely many of them are non-trivial;
- If we compose the maps in the diagram we get

$$d^{j+1} \circ d^j = 0;$$

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Such complexes are the objects of the **derived category**

$$\mathrm{D}^b(X).$$

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Morphisms:

A morphism between the complexes $E^\bullet$ and $F^\bullet$ is a sequence of vertical morphisms in the diagram:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & E^i & \longrightarrow & \dots E^{i+n} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & F^i & \longrightarrow & \dots F^{i+n} & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

All squares commute!

?=derived categories

Morphisms in $\mathrm{D}^b(X)$:

- For $E^\bullet$ and $F^\bullet$ take morphisms of complexes;
- We can then single out *quasi-isos*;

Cohomologies & quasi-isomorphisms:

Give a complex $E^\bullet$, we can compute its cohomologies

$$H^j(E^\bullet) = \frac{\ker(d^j)}{\mathrm{Im}(d^{j-1})}.$$

A morphism of complexes is a *quasi-iso* if it induces isomorphisms on all cohomologies.

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Morphisms in $\mathrm{D}^b(X)$:

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 {}^{\text{quis}}\swarrow & & \searrow & & & {}^{\text{quis}}\swarrow & & \searrow & \\
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3 Derived categories

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Warning/take-home message:

Confused? ...you are not the only one!
$\mathrm{D}^b(X)$ is complicated. And its complexity grows according to two factors:

- The dimension of X;
- How close the canonical bundle is to be trivial (i.e. close to $\mathcal{O}_X$).

Easy cases

Note: We have a natural operations on complexes $E \mapsto E[1]$ (shift to the left!).

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Finding a classification in higher dimension is out of reach.

Good news

Bondal and Orlov (special case):

If X_1 and X_2 are cubic fourfold, then $\mathrm{D}^b(X_1) \cong \mathrm{D}^b(X_2)$ if and only if $X_1 \cong X_2$.

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More general: Let X be a smooth projective variety such that K_X is either ample or antiample. Let Y be a smooth projective variety such that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$. Then $X \cong Y$.

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More general: Let X be a smooth projective variety such that K_X is either ample or antiample. Let Y be a smooth projective variety such that $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$. Then $X \cong Y$.

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??=semiorthogonal components

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where $\mathcal{O}_X(1)$ is the (rank-1) vector bundle associated to the algebraic class of a hyperplane section ($X \cap \mathbb{P}^4$) and $\mathcal{O}_X(2)$ is 'twice' $\mathcal{O}_X(1)$.

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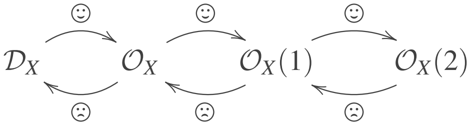

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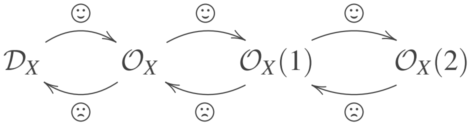

$\mathcal{D}_X$ is the important **semiorthogonal** block $\leftrightsquigarrow H^4(X, \mathbb{Z})$

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Derived Torelli problem:

What about the converse? Does $\mathcal{D}_X$ determine X?

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The celebrated case of cubic 3-folds is due to **Clemens–Griffiths**.

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Enough to reprove easily Voisin's (cohomological) Torelli theorem!

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Solution: Note that $\mathcal{D}_X$ carries more structure! It carries a *stability condition* with respect to which $F(X)$ parametrizes stable objects. And equivalences preserve stability.

Table of Contents

A wealth of categories

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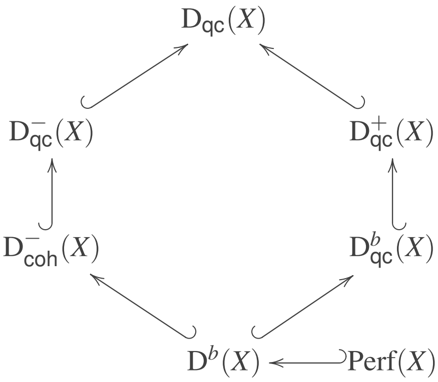

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Classical question 1:

Could other choices better to do geometry?

A wealth of categories

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- The quotient $\mathrm{D}^{b}_{\mathsf{coh}}(X)/\mathrm{Perf}(X)$ is called *singularity category* and measures how singular X is.

Classical question 2:

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Example: injective resolutions

Let X be a smooth projective scheme. Take $\mathbf{Inj}(X)$ to be the category such that

- *Objects*: bounded below complexes of injective objects with bounded coherent cohomology;
- *Morphisms*: morphisms of complexes.

Then:

$$H^0(\mathbf{Inj}(X)) = \mathrm{D}^b(X).$$