A tour on Bridgeland stability

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Outline

1. Moduli spaces and stability
   - Curves
   - Stability
   - Recasting

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1. Moduli spaces and stability
   - Curves
   - Stability
   - Recasting
2. Geometry out of stability
   - Fourier–Mukai transforms
   - Varying stability
3. Bridgeland stability
   - Definition and examples
   - Open problems and results

Motivation

Stability conditions were introduced by Bridgeland to make the notion of Π-stability by Douglas rigorous.

They should provide a generalization of the usual Kähler cone according to String Theory and Mirror Symmetry.

Whereof one cannot speak, thereof one must be silent.
L. Wittgenstein, Tractatus logico-philosophicus

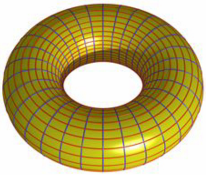
Thus we take a different perspective: we present Bridgeland stability conditions as emerging from the quest of a general approach to the geometry of moduli spaces.

Outline

The baby example

Let E be an elliptic curve. Namely,

1. Topologically: an orientable, compact connected topological surface of genus 1.



The baby example

1. Algebraically: the zero locus in $\mathbb{P}^2$ of a homogeneous polynomial of degree 3.

Example

Consider the homogenous polynomial

$$p(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3.$$

Set

$$E = V(p(x_0, x_1, x_2)) := \{Q \in \mathbb{P}^2 : p(Q) = 0\} \hookrightarrow \mathbb{P}^2.$$

Then X is called Fermat cubic curve.

Sheaves

By looking at E from the second point of view, the torus gains more structure: it is clearly a complex manifold (roughly, E is locally the same as $\mathbb{C}$).

Thus we can define the following sheaves:

- $\mathcal{O}_E$ such that, for any open subset $U \subseteq E$,

$$U \mapsto \mathcal{O}_E(U) := \{f \colon U \to \mathbb{C} : f \text{ is holomorphic}\};$$

- Sheaves of $\mathcal{O}_E$-modules $\mathcal{E}$:

$$U \mapsto \mathcal{E}(U)$$

  and $\mathcal{E}(U)$ is a module over $\mathcal{O}_E(U)$;

Locally free sheaves

- A sheaf $\mathcal{E}$ as above is a locally free sheaf if there exists a positive integer r such that

$$\mathcal{E}|_U \cong (\mathcal{O}_E)|_U^{\oplus r}.$$

The integer r is called rank of $\mathcal{E}$ and it is denoted by $\mathrm{rk}(\mathcal{E})$.

We have another class of sheaves which play a role: torsion sheaves!

Roughly, they are supported at points, with multiplicity.

Moduli spaces

Question 1

Is there another variety X (...or maybe something more refined...) that 'parametrizes' locally free sheaves of a given rank r on E?

If yes, we would (sloppily) call such a geometric object moduli space.

Question 2

How do we study the geometry of these moduli spaces?

Rank = 1

For a locally free sheaf $\mathcal{E}$, we define the following invariants:

- The Euler characteristic:
  $\chi(\mathcal{E}) = \dim_{\mathbb{C}}\mathrm{Hom}(\mathcal{O}_E, \mathcal{E}) - \dim_{\mathbb{C}}\mathrm{Ext}^1(\mathcal{O}_E, \mathcal{E})$, where $\mathrm{Ext}^1(\mathcal{O}_E, \mathcal{E})$ parametrizes extensions

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_E \to 0.$$

- Since E has genus 1, this number is also called degree and denoted $\deg(\mathcal{E})$.

First example

E parametrizes vector bundles of rank 1 and degree 0 on itself. We say that E is self-dual.

Outline

Stability

Idea

If the rank is greater than 1, we cannot hope to have a nice answer to our questions without making further assumptions on the sheaves.

We set

$$\mu(\mathcal{E}) := \begin{cases} \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} & \text{if } \mathcal{E} \text{ is loc. free} \\ +\infty & \text{otherwise.} \end{cases}$$

It is called slope.

Stability

Definition

A sheaf $\mathcal{E}$ is (semi-)stable if, for all proper and non-trivial subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}$ such that $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, we have $\mu(\mathcal{F}) < (\leq)\mu(\mathcal{E})$.

We will refer to this notion of stability as slope or μ stability.

Fix two integers $r > 0$ and $d \in \mathbb{Z}$. We denote by

$$M(r, d)$$

the moduli space of semi-stable sheaves on E with rank r and degree d (...or rather their S-equivalence classes).

Moduli spaces

We denote by $M(r,d)^s$ the open subset of $M(r,d)$ consisting of stable sheaves.

Theorem (Atiyah)

Let r and d be coprime integers as above. Then

- $M(r,d) = M(r,d)^s$;
- $M(r,d)$ is isomorphic to E.

...the description can be completed in the non-coprime case as well! ...or for any curve.

Filtrations

What can we say of a sheaf which is not semi-stable?

Harder–Narasimhan filtration

Any sheaf $\mathcal{E}$ has a filtration

$$0 = \mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n = \mathcal{E}$$

such that

- The quotient $\mathcal{E}_{i+1}/\mathcal{E}_i$ is semi-stable, for all i;
- $\mu(\mathcal{E}_1/\mathcal{E}_0) > \ldots > \mu(\mathcal{E}_n/\mathcal{E}_{n-1})$.

First question... first answer

Question 1

Is there another variety X (...or maybe something more refined...) that 'parametrizes' locally free sheaves of a given rank r on E?

To get a positive answer to this question

- We have to impose some 'stability (or semi-stability) condition';
- Non semi-stable sheaves can then be filtered by semi-stable ones.

Outline

Recasting 1

An equivalent way to define the slope stability introduced in the previous slides is the following:

(a) We take the category of all (coherent) sheaves on E: locally free sheaves $+$ torsion sheaves.

We spoke about

- Subobjects (definition of slope stability);
- Quotients and extensions (HN filtrations).

We are using that the category is abelian.

Recasting 2

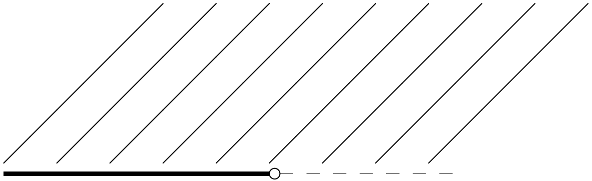
(b) A function Z defined, for all sheaves $\mathcal{E}$, as

$$Z(\mathcal{E}) = -\mathrm{deg}(\mathcal{E}) + \sqrt{-1}\mathrm{rk}(\mathcal{E}) \in \mathbb{C}.$$

Observe that:

- $\mathrm{rk}(\mathcal{E}) \geq 0$ and if $\mathrm{rk}(\mathcal{E}) = 0$, then $\mathrm{deg}(\mathcal{E}) > 0$. Hence, for $\mathcal{E} \neq 0$,

$$Z(\mathcal{E}) \in \mathbb{R}_{>0} e^{(0,1]\sqrt{-1}\pi}.$$



Recasting 3

- Any object in the abelian category has a filtration with respect to the function

$$-\frac{\mathrm{Re}(Z)}{\mathrm{Im}(Z)}(=\mu)$$

Such a filtration is actually unique.

Outline

The problem

Let X_1 be any smooth projective variety (i.e. with an embedding in some projective space). Suppose that M_1 is a moduli space of (semi-)stable sheaves on X_1.

The second question we formulated before is:

Question 2

How do we study the geometry of M_1?

First try: comparing moduli spaces

There is another complex manifold X_2 and a 'functorial association'

$$\Phi : \mathcal{E} \in M_1 \mapsto \Phi(\mathcal{E})$$

such that

- $\Phi(\mathcal{E})$ is a (coherent) sheaf on X_2;
- $\Phi(\mathcal{E})$ is (semi-)stable.

Set M_2 to be the moduli space of (semi-)stable sheaves on X_2 containing $\Phi(\mathcal{E})$.

Hope

Φ is so natural that it induces an isomorphism $M_1 \cong M_2$. Just study M_2! ...which might be simpler if we are smart choosing Φ.

Derived categories

To make this precise, we have to substitute the category of (coherent) sheaves on X_i with $\mathrm{D}^b(X_i)$, where

- The objects in $\mathrm{D}^b(X_i)$ are bounded complexes of coherent sheaves, i.e.

$$\mathcal{E}^\bullet := \{0 \cdots \to \mathcal{E}^{p-1} \xrightarrow{d^{p-1}} \mathcal{E}^p \xrightarrow{d^p} \mathcal{E}^{p+1} \to \cdots \to 0\},$$

  with $d^q \circ d^{q-1} = 0$.

- The morphisms are slightly complicated: they are a localization of the usual morphisms of complexes.
  But we do not need to understand them properly here...

Fourier–Mukai functors 1

We are now in good shape to make the previous construction rigorous:

- Take X_1 and X_2 be smooth projective varieties. Let $p_i : X_1 \times X_2 \to X_i$ be the natural projection. Take $\mathcal{F} \in \mathrm{D}^b(X_1 \times X_2)$.
- For $\mathcal{E} \in \mathrm{D}^b(X_1)$, we set

$$\Phi_{\mathcal{F}}(\mathcal{E}) := (p_2)_*(\mathcal{F} \otimes p_1^*(\mathcal{E}))$$

Definition

A functor isomorphic to one as above is called Fourier–Mukai functor. And $\mathcal{F}$ is its Fourier–Mukai kernel.

Fourier–Mukai functors 2

1. Fourier: these are sheafifications of the usual Fourier transform

$$(p_2)_*$$

Fourier–Mukai functors 2

1. Fourier: these are sheafifications of the usual Fourier transform

$$\begin{array}{lcl} (p_2)_* & \Longrightarrow & \int \\ \mathcal{F}\otimes & \Longrightarrow & \text{multiplication by the Fourier kernel.} \end{array}$$

2. Mukai: Used by Mukai to study moduli spaces on abelian varieties (i.e. higher dimensional analogues of elliptic curves).

3. Hodge: 'Categorification' of the usual notion of correspondence.

Disadvantages

These functors are (essentially) always a natural choice and they make our first try work in several interesting examples.

But, in general,

1. FM functors do not send sheaves to sheaves.
2. FM functors do not preserve stability, in the sense we explained before.

Outline

Second try: varying stability 1

Suppose that X carries many different types of stability (stability conditions) and that all these stability conditions are nicely parametrized by a geometric object S.

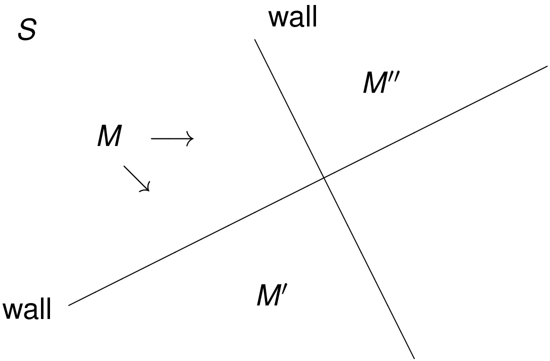
Then one may start with a moduli space M of μ-stable sheaves and begin changing stability inside S.

Second try: varying stability 2

- There might be regions (chambers) of S where M does not change even if stability is changing.
- But passing through a different region (wall) of S, all sheaves in M get destabilized and M has to be replaced by a different moduli space M' of stable sheaves.

We call this wall–crossing phenomenon.

Second try: varying stability 3



During this process, we might get M' and M'' birational to M: this means that M and M' (or M'') are isomorphic just along open subsets.

Second try: varying stability 4

To make this more precise, one should consider (twisted) Gieseker stability.

Variation of this stability means then variation of the corresponding polarization.

This, in turn, is related to variations of GIT quotients: Thaddeus, Matsuki–Wentworth, ...

Hope and bad news

Question 2'

By varying stability, can we get all birational models of M?

Again, variations of the usual stability cannot be sufficient to get such a complete picture.

Outline

Main idea

The two methods we described:

1. Apply FM functors and change the model;
2. Vary stability and look for all birational models

are very simple and promising ...but they do not fit nicely with the usual notion of stability...

Hence...

Change perspective on stability!

Bridgeland definition (very roughly)

Simply: axiomatize and make general the recasting of μ-stability discussed before!

A (Bridgeland) stability condition on $\mathrm{D}^b(X)$, for X a smooth projective variety, is a pair

$$\sigma = (\mathbf{A}, Z)$$

where

1. **A** is an abelian category (...with some technical assumptions...);

Bridgeland definition (very roughly)

2. Z is a group homomorphism such that
   - $Z(\mathcal{E}) \in \mathbb{R}_{>0} e^{(0,1]\sqrt{-1}\pi}$, for $0 \neq \mathcal{E} \in \mathbf{A}$;
   - Any $0 \neq \mathcal{E} \in \mathbf{A}$ has a Harder–Narasimhan filtration with respect to the slope
   $$-\frac{\mathrm{Re}(Z)}{\mathrm{Im}(Z)}$$
3. Kontsevich–Soibelman: support property (ensuring that, if we have one stability condition, then we get an entire open subset).

Properties (Bridgeland)

- Bridgeland stability is preserved under Fourier–Mukai equivalences;
- The space $\mathrm{Stab}(\mathrm{D}^b(X))$, parametrizing Bridgeland stability conditions, is actually a complex manifold of finite dimension. Moreover $\mathrm{Stab}(\mathrm{D}^b(X))$ has a wall and chamber structure.

Hence, in this setup, we can apply our two methods.

Wall crossing 1

Warning

The usual μ-stability is a stability condition in the sense of Bridgeland if and only if the dimension of X is 1.

Thus, in general, given a moduli space M of stable sheaves on X, we first need to find a Bridgeland stability condition

$$\sigma \in \mathrm{Stab}(\mathrm{D}^b(X))$$

such that

$$M \cong \widetilde{M},$$

where $\widetilde{M}$ is a moduli space of σ-stable objects.

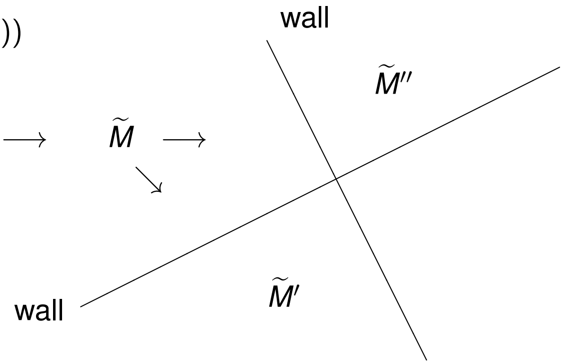
Wall crossing 2

$\mathrm{Stab}(\mathrm{D}^b(X))$

∞

$M \cong \widetilde{M} \quad \longrightarrow \quad \widetilde{M} \quad \longrightarrow$

$\searrow$



Wall crossing 3

These techniques have been successfully exploited for moduli spaces of (Gieseker) stable sheaves on smooth projective complex surfaces.

In particular, just to mention some:

- Arcara–Bertram–Coskun–Huizenga: Hilbert scheme of points on the projective plane (i.e. stable sheaves with very special topological invariants);

Wall crossing 4

- Bayer–Macrì: Moduli spaces of (Gieseker) stable sheaves on K3 surfaces (e.g. zero locus in $\mathbb{P}^3$ of $x_0^4 + x_1^4 + x_2^4 + x_3^4$);

- Minamide–Yanagida–Yoshioka: Moduli spaces of (Gieseker) stable sheaves on abelian surfaces.

- Nuer: Moduli spaces of (Gieseker) stable sheaves on Enriques surfaces (i.e. quotients of special K3 surfaces under the action of a free involution).

Outline

Open problems 1

Main problem

Is $\mathrm{Stab}(\mathrm{D}^b(X))$ non empty, for a smooth projective variety X?

- $\dim(X) = 1$ (Bridgeland): $\mathrm{Stab}(\mathrm{D}^b(X)) \neq \emptyset$. Namely, μ-stability is THE example.
- $\dim(X) = 2$ (Bridgeland and others): one can describe connected components of $\mathrm{Stab}(\mathrm{D}^b(X)) \neq \emptyset$.

Open problems 1

The really challenging case is the one of smooth projective varieties of dimension 3.

Even more precisely, we really need to know if $\mathrm{Stab}(\mathrm{D}^b(X)) \neq \emptyset$, for a smooth projective Calabi–Yau 3-fold X or a variety with trivial canonical bundle:

- Applications to string theory and mathematical physics;
- Counting invariants.

Results

Theorem (Bayer–Macrì–S.)

If X is any abelian 3-fold or some Calabi–Yau 3-folds (of quotient type), then $\mathrm{Stab}(\mathrm{D}^b(X)) \neq \emptyset$.

We prove much more: we describe a connected component as in the surface case!

An example of Calabi–Yau 3-folds (not covered by our result) is the Fermat quintic, i.e. zero locus in $\mathbb{P}^4$ of the homogeneous polynomial

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5.$$

Results

The special Calabi–Yau's we study are obtained by one of the following two constructions

- Quotients of an abelian 3-fold A by the free action of a finite group G (Type A Calabi-Yau's);
- Quotients of an abelian 3-fold A by the action of a finite group G such that the quotient A/G has a crepant resolution of Calabi–Yau type.

Example

For an example of the last set of CY's, one can take the product $E \times E \times E$, where E is an elliptic curve, and quotient by the diagonal action of $\mathbb{Z}/3\mathbb{Z}$.

Results

Thus the result for Calabi–Yau 3-folds is deduced by the one for abelian 3-folds by inducing stability conditions.

Let A be an abelian 3-fold and let G be a finite group acting on A. Let $\mathrm{Stab}(\mathrm{D}^b(A))^G$ denote G-invariant stability conditions.

Macrì–Mehrotra–S.

There is a closed embedding

$$\mathrm{Stab}(\mathrm{D}^b(A))^G \hookrightarrow \mathrm{Stab}(\mathrm{D}^b(Y)),$$

where Y is a crepant resolution of A/G.

Open problems 2

The Main Problem is still open in its complete generality but the techniques developed to treat the case of abelian 3-folds seem promising, for several other 3-folds.

Indeed, the non-emptiness result is known in other cases:

- 3-dimensional projective space: Macrì, Bayer–Macrì–Toda;
- 3-dimensional quadrics: Schmidt;
- Generic ppav: Maciocia–Piyaratne (special case of our result).

Open problems 3

Problem 2

Study the birational geometry of moduli spaces of stable sheaves on 3-folds.

This is certainly a difficult problem. But it could work in several interesting cases: special Hilbert schemes on $\mathbb{P}^3$.