Bridgeland stability for semiorthogonal decompositions, hyperkähler manifolds and cubic fourfolds

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Based on the following joint works:
Bayer-Lahoz-Macrì-S.: arXiv:1703.10839
Bayer-Lahoz-Macrì-Nuer-Perry-S.: in preparation
Lecture notes: Macrì-S., arXiv:1807.06169

Outline

1 Setting

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2. Results

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3. Applications

The setting

Let X be a **cubic fourfold** (i.e. a smooth hypersurface of degree 3 in $\mathbb{P}^5$).

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Aim of the talk:

Convince you that, even though X is a Fano 4-fold, it is secretly a K3 surface!

Hodge theory: Voisin + Hassett

Torelli Theorem (Voisin, etc.)

X is determined, up to isomorphism, by its primitive middle cohomology $H^4(X, \mathbb{Z})_{\text{prim}}$.
(Cup product + Hodge structure!).

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(A priori) weight-4 Hodge decomposition:

$$H^4(X,\mathbb{C}) = H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4}$$

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...a posteriori, $H^4(X,\mathbb{Z})$ has a weight-2 Hodge structure!

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Exceptional objects:

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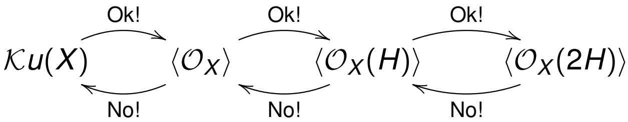
- There are no Homs from right to left between the 4 subcategories:



Homological algebra: properties of $\mathcal{K}u(X)$

Property 1 (Kuznetsov):

The admissible subcategory $\mathcal{K}u(X)$ has a Serre functor $S_{\mathcal{K}u(X)}$ (this is easy!). Moreover, there is an isomorphism of exact functors

$$S_{\mathcal{K}u(X)} \cong [2].$$

Because of this, $\mathcal{K}u(X)$ is called 2**-Calabi-Yau category.**

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Hence $\mathcal{K}u(X)$ could be equivalent
to the derived category either of a K3 surface
or of an abelian surface.

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- Consider the $\mathbb{Z}$-module

$$H^*(\mathcal{K}u(X), \mathbb{Z}) := \left\{ e \in K_{\mathrm{top}}(X) : \begin{array}{l} \chi([\mathcal{O}_X(iH)], e) = 0 \\ i = 0, 1, 2 \end{array} \right\}.$$

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The lattice $H^*(\mathcal{K}u(X), \mathbb{Z})$ with the above Hodge structure is the **Mukai lattice** of $\mathcal{K}u(X)$ which we denote by $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

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$\mathcal{K}u(X)$ can only be equivalent
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If X is **very general** (i.e. $H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}H^2$), then

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Hence there is no K3 surface S such that $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$!

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$\mathcal{K}u(X)$ is a **noncommutative K3 surface.**

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In the light of what we discussed before, the following is very natural:

Question 1 (Addinston-Thomas, Huybrechts,...)

Is the same true for the Kuznetsov component $\mathcal{K}u(X)$ of any cubic fourfold X?

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A **Bridgeland stability condition** on **T** is a pair $\sigma = (\mathbf{A}, Z)$, where:

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3. Support property (**Kontsevich-Soibelman**): wall and chamber structure with locally finitely many walls.

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Theorem (Bridgeland)

If non-empty, $\mathrm{Stab}_\Gamma(\mathbf{T})$ is a complex manifold of dimension $\mathrm{rk}(\Gamma)$.

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- (1) holds over a field $\mathbb{K} = \overline{\mathbb{K}}$, $\mathrm{char}(\mathbb{K}) \neq 2$. (2) holds over $\mathbb{C}$.

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- Let $\sigma = (\mathbf{A}, Z) \in \mathrm{Stab}(\mathcal{K}u(X))$. Then $Z(-) = (v_Z, -)$, for $v_Z \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \otimes \mathbb{C}$. Here $(-,-) := -\chi(-,-)$ is the **Mukai pairing** on $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$;

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- Let $\mathcal{P}^+(X)$ be the connected component containing v_Z for the special stability condition in part (1) of Theorem 1;

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- Let $\sigma = (\mathbf{A}, Z) \in \mathrm{Stab}(\mathcal{K}u(X))$. Then $Z(-) = (v_Z, -)$, for $v_Z \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \otimes \mathbb{C}$. Here $(-,-) := -\chi(-,-)$ is the **Mukai pairing** on $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$;
- Let $\mathcal{P}(X)$ be the set of vectors in $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \otimes \mathbb{C}$ whose real and imaginary parts span a positive definite 2-plane;
- Let $\mathcal{P}^+(X)$ be the connected component containing v_Z for the special stability condition in part (1) of Theorem 1;
- Let $\mathcal{P}_0^+(X)$ be the set of vectors in $\mathcal{P}^+(X)$ which are not orthogonal to any (-2)-class in $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$;

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- The map $\mathrm{Stab}^\dagger(\mathcal{K}u(X)) \to \mathcal{P}_0^+(X)$ sends $\sigma = (\mathbf{A}, Z) \mapsto v_Z$.

The results: moduli spaces

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Warning

$M_\sigma(\mathcal{K}u(X), v)$ has a weird geometry, in general!

The results: moduli spaces

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1. $M_\sigma(\mathcal{K}u(X), v)$ is non-empty if and only if $v^2 + 2 \geq 0$. Moreover, in this case, it is a smooth projective irreducible holomorphic symplectic manifold of dimension $v^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.

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2. If $v^2 \geq 0$, then there exists a natural Hodge isometry

$$\theta \colon H^2(M_\sigma(\mathcal{K}u(X), v), \mathbb{Z}) \cong \begin{cases} v^\perp & \text{if } v^2 > 0 \\ v^\perp / \mathbb{Z}v & \text{if } v^2 = 0, \end{cases}$$

where the orthogonal is taken in $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

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3. The most intriguing part in the proof is the non-emptiness statement!

Outline

1. Setting
2. Results
3. **Applications**

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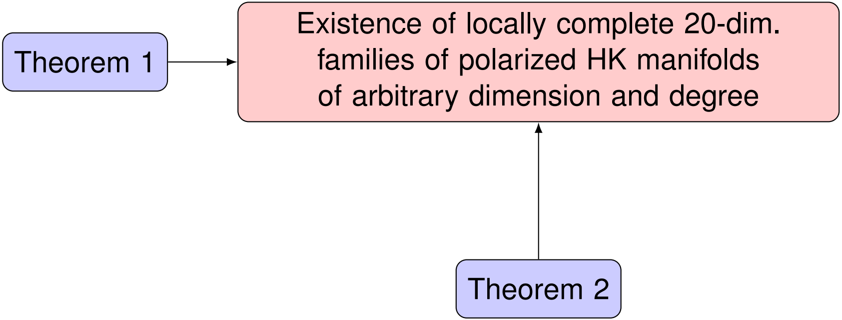

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- Let v be a primitive section of the local system given by $\widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$ such that v stays algebraic on all fibers;
- Assume that, for $s \in S$, there exists a stability condition $\sigma_s \in \mathrm{Stab}^\dagger(\mathcal{K}u(\mathcal{X}_s))$ such that these pointwise stability conditions organize themselves in a family $\underline{\sigma}$. Assume that σ_s is v-generic for very general s (+some invariance of Z...).

The precise statement

Theorem 3 (BLMNPS)

1. There exists a finite cover $\widetilde{S} \to S$, an algebraic space $\widetilde{M}(v)$, and a proper morphism $\widetilde{M}(v) \to \widetilde{S}$ that makes $\widetilde{M}(v)$ a **relative moduli space over** $\widetilde{S}$ (i.e. the fiber $M_{\sigma_s}(\mathcal{K}u(\mathcal{X}_s), v)$ of stable objects in the Kuznetsov component of the corresponding cubic fourfold).

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2. There exists a non-empty open subset $S^0 \subset S$ and a variety $M^0(v)$ with a projective morphism $M^0(v) \to S^0$ that makes $M^0(v)$ a relative moduli space over S^0.

The families of HK manifolds

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Remark

These families have polarization of arbitrary large degree. The family we construct are automatically unirational.

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Let X be a cubic fourfold. Then there exists a primitive vector $v \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ with $v^2 = 0$ if and only if there is a K3 surface S, $\alpha \in \mathrm{Br}(S)$ and an equivalence $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S, \alpha)$.

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Let X be a cubic fourfold. Then there exists a primitive embedding $U \hookrightarrow \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ if and only if there is a K3 surface S and an equivalence $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$.

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- Since $\mathcal{K}u(X)$ is a 2-Calabi-Yau category, we are done.

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Question 2

What's the geometric meaning of having $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S, \alpha)$?

$v^2 = 2$: the Fano variety of lines

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Kuznetsov-Markushevich: F_ℓ is in $\mathcal{K}u(X)$ and it is a Gieseker stable sheaf. $F(X)$ is isomorphic to the moduli space of stable sheaves with Mukai vector $v(F_\ell)$.

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Theorem (Li-Pertusi-Zhao)

For any cubic fourfold X, we have an isomorphism
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A stability condition σ is **natural** if:

- $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$;
- Under the map $\mathrm{Stab}^\dagger(\mathcal{K}u(X)) \to \mathcal{P}_0^+(X)$, σ is sent to $A_2 \otimes \mathbb{C} \cap \mathcal{P}(X) \subseteq \mathcal{P}_0^+(X)$.

$v^2 = 6$: twisted cubics

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- We can contract a divisor $Z'(X) \to Z(X)$, where $Z(X)$ is a smooth projective hyperkähler manifold of dimension 8 which contains X as a Lagrangian submanifold.

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Question (M. Lehn):

Is there a modular interpretation for $Z'(X)$ and $Z(X)$?

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For any cubic fourfold X not containing a plane,

- $Z'(X)$ is isomorphic to a component of a moduli space of Gieseker stable torsion free scheaves of rank 3;
- We have an isomorphism $Z(X) \cong M_\sigma(\mathcal{K}u(X), 2\lambda_1 + \lambda_2)$, for all natural stability conditions σ.

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By Theorem 2, $Z(X)$ is automatically (projective and) deformation equivalent to $\mathrm{Hilb}^4(\mathrm{K3})$, which was proved by **Addington-Lehn**.

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This is because BLMNPS implies that the Bayer-Macrì machinery can be applied also in this noncommutative setting: all birational models of $F(X)$, $Z(X)$ and all other possible HK from Theorem 2 are isomorphic to moduli spaces of stable objects in the Kuznetsov component (by variation of stability).

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Exercise (Voisin)

Reprove the Intergral Hodge Conjecture for cubic fourfolds, due to Voisin, by using the same ideas as in the proof of Corollary 4.

Concluding remarks

There several other simple but interesting applications that one can deduce from Theorems 1, 2 and 3:

Exercise (Voisin)

Reprove the Intergral Hodge Conjecture for cubic fourfolds, due to Voisin, by using the same ideas as in the proof of Corollary 4.

Corollaries 4 and 5 allow us to extend recent results by Sheridan-Smith about Mirror Symmetry of K3 surfaces appearing as Kuznetsov components of cubic fourfolds.