Cubic fourfolds, noncommutative K3 surfaces and stability conditions

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Based on the following joint works:
Bayer-Lahoz-Macrì-S.: arXiv:1703.10839
Bayer-Lahoz-Macrì-Nuer-Perry-S.: in preparation
Lecture notes: Macrì-S., arXiv:1807.06169

Outline

1 Setting

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2 Results

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3 Applications

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The setting

Let X be a **cubic fourfold** (i.e. a smooth hypersurface of degree 3 in $\mathbb{P}^5$). Let H be a hyperplane section.

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Convince you that, even though X is a **Fano 4-fold** ($\Rightarrow$ *ample anticanonical bundle*), it is secretly a **K3 surface** ($\Rightarrow$ *trivial canonical bundle*)!

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Torelli Theorem (Voisin, etc.)

X is determined, up to isomorphism, by its primitive middle cohomology $H^4(X, \mathbb{Z})_{\text{prim}} := (H^2)^{\perp_{H^4}}$.

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(A priori) weight-4 Hodge decomposition:

$$\begin{array}{c} H^4(X,\mathbb{C}) \\ \| \\ H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4} \end{array}$$

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$$\begin{array}{c} H^{2,0}(\mathrm{K3}) \oplus H^{1,1}(\mathrm{K3}) \oplus H^{0,2}(\mathrm{K3}) \\ \| \\ H^2(\mathrm{K3},\mathbb{C}). \end{array}$$

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...a posteriori, $H^4(X,\mathbb{Z})$ has a weight-2 Hodge structure!

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Exceptional objects:

$$\langle \mathcal{O}_X(iH) \rangle \cong \mathrm{D^b}(\mathrm{pt})$$

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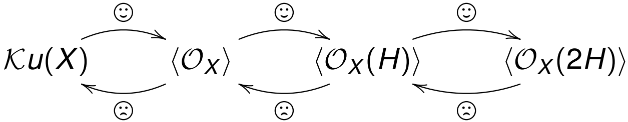
- $\mathrm{D}^{\mathrm{b}}(X)$ is generated by extensions, shifts, direct sums and summands by the objects in the 4 **admissible** subcategories;
- There are no Homs from right to left between the 4 subcategories:



Homological algebra: properties of $\mathcal{K}u(X)$

Property 1 (Kuznetsov):

The admissible subcategory $\mathcal{K}u(X)$ has a Serre functor $S_{\mathcal{K}u(X)}$ (this is easy!). Moreover, there is an isomorphism of exact functors

$$S_{\mathcal{K}u(X)} \cong [2].$$

Because of this, $\mathcal{K}u(X)$ is called 2**-Calabi-Yau category.**

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If X smooth proj. var., $S_{\mathrm{D^b}(X)}(-) \cong (-) \otimes \omega_X[\dim(X)]$.

Hence $\mathcal{K}u(X)$ could be equivalent to the derived category either of a K3 or of an abelian surface.

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$$H^*(\mathcal{K}u(X), \mathbb{Z}) := \left\{ e \in K_{\mathrm{top}}(X) : \begin{array}{l} \chi([\mathcal{O}_X(iH)], e) = 0 \\ i = 0, 1, 2 \end{array} \right\}.$$

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- Consider the the map $\mathbf{v}\colon K_{\text{top}}(X) \to H^*(X, \mathbb{Q})$ and set

$$H^{2,0}(\mathcal{K}u(X)) := \mathbf{v}^{-1}(H^{3,1}(X)).$$

This defines a **weight-**2 **Hodge structure** on $H^*(\mathcal{K}u(X), \mathbb{Z})$**.**

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Definition

The lattice $H^*(\mathcal{K}u(X), \mathbb{Z})$ with the above Hodge structure is the **Mukai lattice** of $\mathcal{K}u(X)$ which we denote by $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

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If X is **very general** (i.e. $H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}H^2$), then

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$\mathcal{K}u(X)$ is a **noncommutative K3 surface.**

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Stability conditions

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In the light of what we discussed before, the following is very natural:

Question 1 (Addinston-Thomas, Huybrechts,...)

Is the same true for the Kuznetsov component $\mathcal{K}u(X)$ of any cubic fourfold X?

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A **Bridgeland stability condition** on **T** is a pair $\sigma = (\mathbf{A}, Z)$:

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3. Support property (**Kontsevich-Soibelman**): wall and chamber structure with locally finitely many walls.

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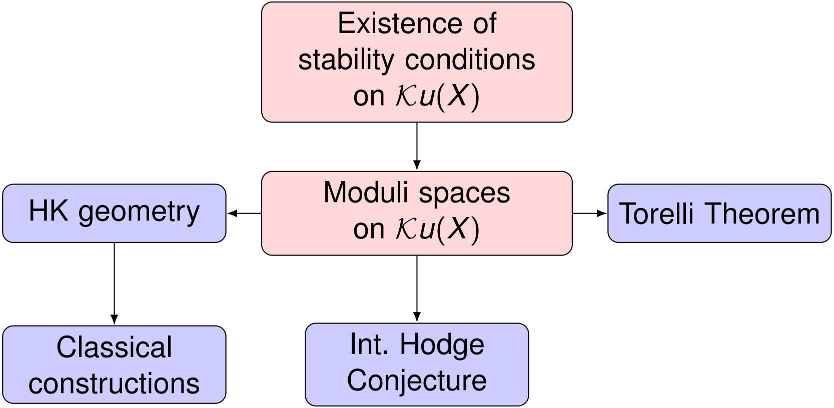
Theorem (Bridgeland)

If non-empty, $\mathrm{Stab}_\Gamma(\mathbf{T})$ is a complex manifold of dimension $\mathrm{rk}(\Gamma)$.

The results


Existence of stability conditions on $\mathcal{K}u(X)$
Moduli spaces on $\mathcal{K}u(X)$
HK geometry
Torelli Theorem
Classical constructions
Int. Hodge Conjecture

The results: existence of stability conditions

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- (1) holds over a field $\mathbb{K} = \overline{\mathbb{K}}$, $\mathrm{char}(\mathbb{K}) \neq 2$. (2) holds over $\mathbb{C}$.

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- Let $\sigma = (\mathbf{A}, Z) \in \mathrm{Stab}(\mathcal{K}u(X))$. Then $Z(-) = (v_Z, -)$, for $v_Z \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \otimes \mathbb{C}$. Here $(-,-) := -\chi(-,-)$ is the **Mukai pairing** on $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$;

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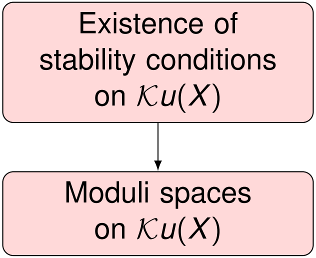
The results: moduli spaces



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Question 2

What is the geometry of $M_\sigma(\mathcal{K}u(X), v)$?

The results: moduli spaces

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Theorem 2 (BLMNPS)

1. $M_\sigma(\mathcal{K}u(X), v)$ is non-empty if and only if $v^2 + 2 \geq 0$. Moreover, in this case, it is a smooth projective irreducible holomorphic symplectic manifold of dimension $v^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.

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2. If $v^2 \geq 0$, then there exists a natural Hodge isometry

$$\theta \colon H^2(M_\sigma(\mathcal{K}u(X), v), \mathbb{Z}) \cong \begin{cases} v^\perp & \text{if } v^2 > 0 \\ v^\perp/\mathbb{Z}v & \text{if } v^2 = 0, \end{cases}$$

where the orthogonal is taken in $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

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Definition

A **hyperkähler manifold** is a simply connected compact kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form.

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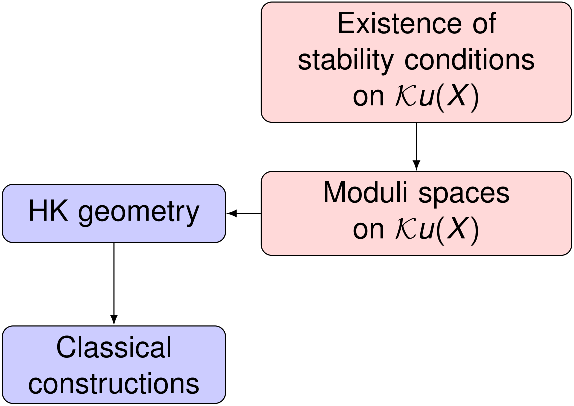
Outline

1 Setting

2 Results

3 **Applications**

The general picture



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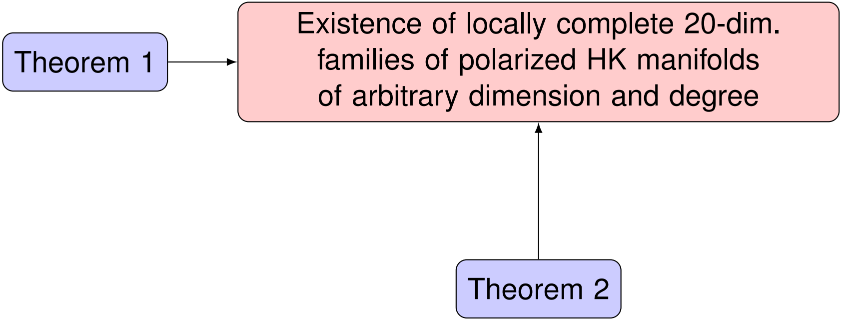

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The setting:

- Let $\mathcal{X} \to S$ be a family of cubic fourfolds;
- Let v be a primitive section of the local system given by $\widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$ such that v stays algebraic on all fibers;
- Assume that, for $s \in S$, there exists a stability condition $\sigma_s \in \mathrm{Stab}^\dagger(\mathcal{K}u(\mathcal{X}_s))$ such that these pointwise stability conditions organize themselves in a family $\underline{\sigma}$. Assume that σ_s is v-generic for very general s (+some invariance of Z...).

The precise statement

Theorem 3 (BLMNPS)

There exists a non-empty open subset $S^0 \subset S$ and a variety $M^0(v)$ with a projective morphism

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This means that the fiber $M_{\sigma_s}(\mathcal{K}u(\mathcal{X}_s), v)$ of stable objects in the Kuznetsov component of the corresponding cubic fourfold, for $s \in S^0$.

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- In A_2 one finds primitive vectors v with arbitrary large v^2.

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For any pair (a, b) of coprime integers, there is a unirational locally complete 20-dimensional family, over an open subset of the moduli space of cubic fourfolds, of polarized smooth projective irreducible holomorphic symplectic manifolds of dimension $2n+2$, where $n = a^2 - ab + b^2$. The polarization has divisibility 2 and degree either $6n$ if 3 does not divide n, or $\frac{2}{3}n$ otherwise.

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- Since $\mathcal{K}u(X)$ is a 2-Calabi-Yau category, we are done.

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Conjecture (Kuznetsov)

X is such that $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, for a K3 surface S, if and only if X is rational.

$v^2 = 2$: the Fano variety of lines

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Kuznetsov-Markushevich: F_ℓ is in $\mathcal{K}u(X)$ and it is a Gieseker stable sheaf. $F(X)$ is isomorphic to the moduli space of stable sheaves with Mukai vector $v(F_\ell)$.

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Theorem (Li-Pertusi-Zhao)

For any cubic fourfold X, we have an isomorphism $F(X) \cong M_\sigma(\mathcal{K}u(X), \lambda_1)$, for all **natural** stability conditions σ.

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A stability condition σ is **natural** if:

- $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$;
- Under the map $\mathrm{Stab}^\dagger(\mathcal{K}u(X)) \to \mathcal{P}_0^+(X)$, σ is sent to $A_2 \otimes \mathbb{C} \cap \mathcal{P}(X) \subseteq \mathcal{P}_0^+(X)$.

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- We can contract a divisor $Z'(X) \to Z(X)$, where $Z(X)$ is a smooth projective hyperkähler manifold of dimension 8 which contains X as a Lagrangian submanifold.

$v^2 = 8$: twisted cubics

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Is there a modular interpretation for $Z'(X)$ and $Z(X)$?

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Theorem (M. Lehn-Lahoz-Macrì-S. and Li-Pertusi-Zhao)

For any cubic fourfold X not containing a plane,

- $Z'(X)$ is isomorphic to a component of a moduli space of Gieseker stable torsion free scheaves of rank 3;
- We have an isomorphism $Z(X) \cong M_\sigma(\mathcal{K}u(X), 2\lambda_1 + \lambda_2)$, for all natural stability conditions σ.

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Theorem (M. Lehn-Lahoz-Macrì-S. and Li-Pertusi-Zhao)

For any cubic fourfold X not containing a plane,

- $Z'(X)$ is isomorphic to a component of a moduli space of Gieseker stable torsion free scheaves of rank 3;
- We have an isomorphism $Z(X) \cong M_\sigma(\mathcal{K}u(X), 2\lambda_1 + \lambda_2)$, for all natural stability conditions σ.

By Theorem 2, $Z(X)$ is automatically (projective and) deformation equivalent to $\mathrm{Hilb}^4(\mathrm{K3})$, which was proved by **Addington-Lehn**.

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This is because BLMNPS implies that the Bayer-Macrì machinery can be applied also in this noncommutative setting: all birational models of $F(X)$, $Z(X)$ and all other possible HK from Theorem 2 are isomorphic to moduli spaces of stable objects in the Kuznetsov component (by variation of stability).

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Exercise (Voisin)

Reprove the Intergral Hodge Conjecture for cubic fourfolds, due to Voisin, by using the same ideas as in the proof of Corollary 4.