Cubic fourfolds, noncommutative K3 surfaces and stability conditions

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Based on the following joint works:
Bayer-Lahoz-Macrì-S.: arXiv:1703.10839
Bayer-Lahoz-Macrì-Nuer-Perry-S.: in preparation
Lecture notes: Macrì-S., arXiv:1807.06169

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The setting

Let X be a **cubic fourfold** (i.e. a smooth hypersurface of degree 3 in $\mathbb{P}^5$). Let H be a hyperplane section.

Most of the time defined over $\mathbb{C}$ but, for some results, defined over a field $\mathbb{K} = \overline{\mathbb{K}}$ with $\mathrm{char}(\mathbb{K}) \neq 2$.

Convince you that, even though X is a **Fano** 4**-fold** ($\Rightarrow$ *ample anticanonical bundle*), it is secretly a **K3 surface** ($\Rightarrow$ *trivial canonical bundle*)!

Hodge theory: Voisin + Hassett

Assume that the base field is $\mathbb{C}$.

Torelli Theorem (Voisin, etc.)

X is determined, up to isomorphism, by its primitive middle cohomology $H^4(X,\mathbb{Z})_{\text{prim}} := (H^2)^{\perp_{H^4}}$.

Cup product

$+$

Hodge structure

(A priori) weight-4 Hodge decomposition:

$$H^4(X,\mathbb{C})$$
$$\|$$
$$H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4}$$
$$\cong$$
$$0 \oplus \mathbb{C} \oplus \mathbb{C}^{21} \oplus \mathbb{C} \oplus 0$$
$$\cong$$

(...not quite right...)

$$H^{2,0}(\mathrm{K3}) \oplus H^{1,1}(\mathrm{K3}) \oplus H^{0,2}(\mathrm{K3})$$
$$\|$$
$$H^2(\mathrm{K3},\mathbb{C}).$$

...a posteriori, $H^4(X,\mathbb{Z})$ has a weight-2 Hodge structure!

Homological algebra

Let us now look at the bounded derived category of coherent sheaves on X:

$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$$

$$\|$$

$$\langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

$$\mathcal{K}u(X)$$

$$\|$$

$$\left\{ E \in \mathrm{D}^{\mathrm{b}}(X) : \begin{array}{l} \mathrm{Hom}\,(\mathcal{O}_X(iH), E[p]) = 0 \\ i = 0, 1, 2 \ \ \forall p \in \mathbb{Z} \end{array} \right\}$$

Kuznetsov component of X

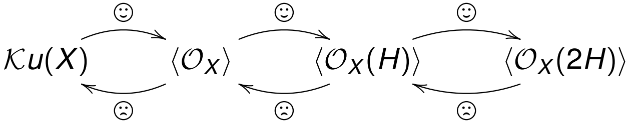
Exceptional objects:

$$\langle \mathcal{O}_X(iH) \rangle \cong \mathrm{D}^{\mathrm{b}}(\mathrm{pt})$$

Homological algebra

Keep in mind that the symbol $\langle \ldots \rangle$ stays for a **semiorthogonal decomposition**:

- $\mathrm{D}^{\mathrm{b}}(X)$ is generated by extensions, shifts, direct sums and summands by the objects in the 4 **admissible** subcategories;
- There are no Homs from right to left between the 4 subcategories:



Homological algebra: properties of $\mathcal{K}u(X)$

Property 1 (Kuznetsov):

The admissible subcategory $\mathcal{K}u(X)$ has a Serre functor $S_{\mathcal{K}u(X)}$ (this is easy!). Moreover, there is an isomorphism of exact functors

$$S_{\mathcal{K}u(X)} \cong [2].$$

Because of this, $\mathcal{K}u(X)$ is called 2**-Calabi-Yau category.**

Remark

If X smooth proj. var., $S_{\mathrm{D^b}(X)}(-) \cong (-) \otimes \omega_X[\dim(X)]$.

$\Downarrow$

Hence $\mathcal{K}u(X)$ could be equivalent to the derived category either of a K3 or of an abelian surface.

Homological algebra: properties of $\mathcal{K}u(X)$

Property 2 (Addington, Thomas):

$\mathcal{K}u(X)$ comes with an integral cohomology theory in the following sense (here $\mathbb{K} = \mathbb{C}$):

- Consider the $\mathbb{Z}$-module

$$H^*(\mathcal{K}u(X), \mathbb{Z}) := \left\{ e \in K_{\mathrm{top}}(X) : \begin{array}{l} \chi([\mathcal{O}_X(iH)], e) = 0 \\ i = 0, 1, 2 \end{array} \right\}.$$

Remark

$H^*(\mathcal{K}u(X), \mathbb{Z})$ is deformation invariant. So, as a lattice:

$$H^*(\mathcal{K}u(X), \mathbb{Z}) = H^*(\mathcal{K}u(\mathrm{Pfaff}), \mathbb{Z}) = H^*(\mathrm{K3}, \mathbb{Z}) = U^4 \oplus E_8(-1)^2$$

Homological algebra: properties of $\mathcal{K}u(X)$

- Consider the the map $\mathbf{v}\colon K_{\mathrm{top}}(X) \to H^*(X,\mathbb{Q})$ and set

$$H^{2,0}(\mathcal{K}u(X)) := \mathbf{v}^{-1}(H^{3,1}(X)).$$

This defines a **weight-**2 **Hodge structure** on $H^*(\mathcal{K}u(X),\mathbb{Z})$.

Definition

The lattice $H^*(\mathcal{K}u(X),\mathbb{Z})$ with the above Hodge structure is the **Mukai lattice** of $\mathcal{K}u(X)$ which we denote by $\widetilde{H}(\mathcal{K}u(X),\mathbb{Z})$.

$$\Downarrow$$

$\mathcal{K}u(X)$ can only be equivalent
to the derived category of a K3 surface

Homological algebra: properties of $\mathcal{K}u(X)$

$$\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z}) := \widetilde{H}(\mathcal{K}u(X),\mathbb{Z}) \cap \widetilde{H}^{1,1}(\mathcal{K}u(X))$$

$$\cup\!\!\mid \ \text{primitive}$$

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Remark

If X is **very general** (i.e. $H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}H^2$), then

$$\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X),\mathbb{Z}) = A_2.$$

Hence there is no K3 surface S such that $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$!

$\mathcal{K}u(X)$ is a **noncommutative K3 surface.**

Outline

Stability conditions

Bridgeland

If S is a K3 surface, then $\mathrm{D}^{\mathrm{b}}(S)$ carries a stability condition. Moreover, one can describe a connected component $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(S))$ of the space parametrizing all stability conditions.

In the light of what we discussed before, the following is very natural:

Question 1 (Addinston-Thomas, Huybrechts,...)

Is the same true for the Kuznetsov component $\mathcal{K}u(X)$ of any cubic fourfold X?

Stability conditions: a quick recap

Let us start with a quick recall about Bridgeland stability conditions.

- Let **T** be a triangulated category;

- Let Γ be a free abelian group of finite rank with a surjective map $v\colon K(\mathbf{T}) \to \Gamma$.

Example

$\mathbf{T} = \mathrm{D}^{\mathrm{b}}(C)$, for C a smooth projective curve.

$$\Gamma = N(C) = H^0 \oplus H^2$$

with

$$v = (\mathrm{rk}, \deg)$$

A **Bridgeland stability condition** on **T** is a pair $\sigma = (\mathbf{A}, Z)$:

Stability conditions: a quick recap

- $\mathbf{A}$ is the heart of a bounded t-structure on $\mathbf{T}$;
- $Z\colon \Gamma \to \mathbb{C}$ is a group homomorphism

Example

$$\mathbf{A} = \mathrm{Coh}(C)$$

$$Z(v(-)) = -\mathrm{deg} + \sqrt{-1}\mathrm{rk}.$$

such that, for any $0 \neq E \in \mathbf{A}$,

1. $Z(v(E)) \in \mathbb{R}_{>0} e^{(0,1]\pi\sqrt{-1}}$;
2. E has a Harder-Narasimhan filtration with respect to $\lambda_\sigma = -\frac{\mathrm{Re}(Z)}{\mathrm{Im}(Z)}$ (or $+\infty$);
3. Support property (**Kontsevich-Soibelman**): wall and chamber structure with locally finitely many walls.

Stability conditions: a quick recap

Warning

The example is somehow misleading: it only works in dimension 1!

We denote by

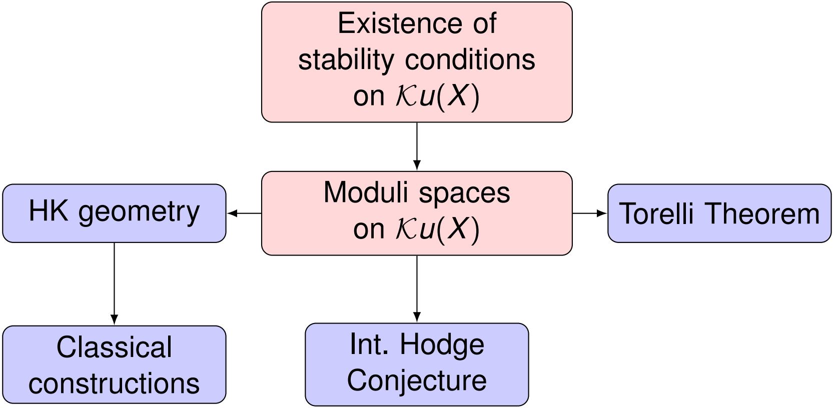
$$\mathrm{Stab}_\Gamma(\mathbf{T}) \quad (\text{or } \mathrm{Stab}_{\Gamma,v}(\mathbf{T}) \text{ or } \mathrm{Stab}(\mathbf{T}))$$

the set of all stability conditions on **T**.

Theorem (Bridgeland)

If non-empty, $\mathrm{Stab}_\Gamma(\mathbf{T})$ is a complex manifold of dimension $\mathrm{rk}(\Gamma)$.

The results


Existence of stability conditions on $\mathcal{K}u(X)$
Moduli spaces on $\mathcal{K}u(X)$
HK geometry
Torelli Theorem
Classical constructions
Int. Hodge Conjecture

The results: existence of stability conditions

Existence of
stability conditions
on $\mathcal{K}u(X)$

The results: existence of stability conditions

Theorem 1 (BLMS, BLMNPS)

1. For any cubic fourfold X, we have $\mathrm{Stab}(\mathcal{K}u(X)) \neq \emptyset$.
2. There is a connected component $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ of $\mathrm{Stab}(\mathcal{K}u(X))$ which is a covering of a period domain $\mathcal{P}_0^+(X)$.

- In (1), $\Gamma = \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$;
- (1) holds over a field $\mathbb{K} = \overline{\mathbb{K}}$, $\mathrm{char}(\mathbb{K}) \neq 2$. (2) holds over $\mathbb{C}$.

The results: existence of stability conditions

The period domain $\mathcal{P}_0^+(X)$ is defined as in Bridgeland's result about K3 surfaces:

- Let $\sigma = (\mathbf{A}, Z) \in \mathrm{Stab}(\mathcal{K}u(X))$. Then $Z(-) = (v_Z, -)$, for $v_Z \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \otimes \mathbb{C}$. Here $(-, -) := -\chi(-, -)$ is the **Mukai pairing** on $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$;
- Let $\mathcal{P}(X)$ be the set of vectors in $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \otimes \mathbb{C}$ whose real and imaginary parts span a positive definite 2-plane;
- Let $\mathcal{P}^+(X)$ be the connected component containing v_Z for the special stability condition in part (1) of Theorem 1;
- Let $\mathcal{P}_0^+(X)$ be the set of vectors in $\mathcal{P}^+(X)$ which are not orthogonal to any (-2)-class in $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$;
- The map $\mathrm{Stab}^\dagger(\mathcal{K}u(X)) \to \mathcal{P}_0^+(X)$ sends $\sigma = (\mathbf{A}, Z) \mapsto v_Z$.

The results: moduli spaces

Existence of stability conditions on $\mathcal{K}u(X)$

↓

Moduli spaces on $\mathcal{K}u(X)$

The results: moduli spaces

The construction of moduli spaces of stable objects in $\mathcal{K}u(X)$:

- Let $0 \neq v \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ be a primitive vector;
- Let $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$ be ***v*-generic** (here it means that σ-semistable=σ-stable for objects with Mukai vector v).

Let $M_\sigma(\mathcal{K}u(X), v)$ be the moduli space of σ-stable objects (in the heart of σ) contained in $\mathcal{K}u(X)$ and with Mukai vector v.

Question 2

What is the geometry of $M_\sigma(\mathcal{K}u(X), v)$?

The results: moduli spaces

Theorem 2 (BLMNPS)

1. $M_\sigma(\mathcal{K}u(X), v)$ is non-empty if and only if $v^2 + 2 \geq 0$. Moreover, in this case, it is a smooth projective irreducible holomorphic symplectic manifold of dimension $v^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.

2. If $v^2 \geq 0$, then there exists a natural Hodge isometry

$$\theta\colon H^2(M_\sigma(\mathcal{K}u(X), v), \mathbb{Z}) \cong \begin{cases} v^\perp & \text{if } v^2 > 0 \\ v^\perp/\mathbb{Z}v & \text{if } v^2 = 0, \end{cases}$$

where the orthogonal is taken in $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

The results: moduli spaces

Definition

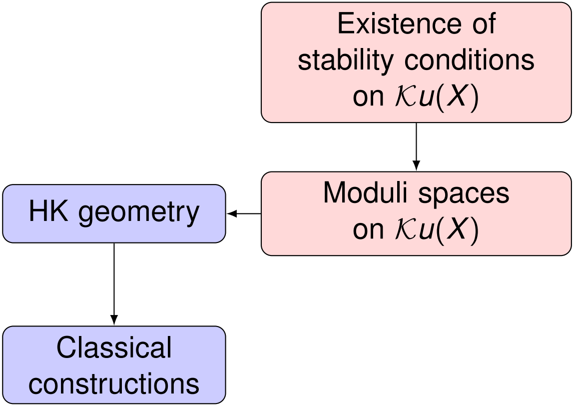
A **hyperkähler manifold** is a simply connected compact kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form.

There are very few examples (up to deformation):

1. K3 surfaces;
2. Hilbert schemes of points on K3 surface (denoted by $\mathrm{Hilb}^n(\mathrm{K3})$;
3. Generalized Kummer varieties (from abelian surfaces);
4. Two sporadic examples by O'Grady.

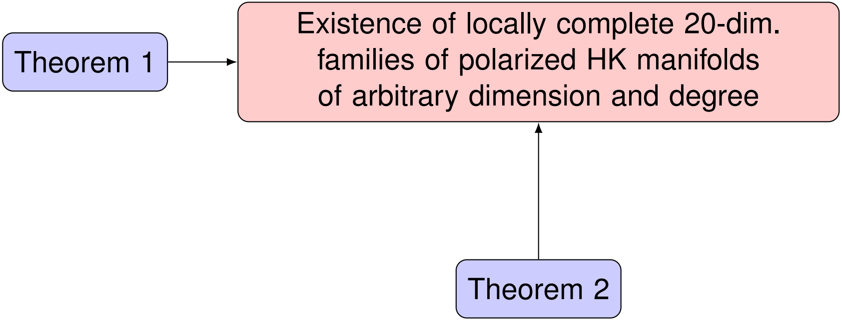
Outline

The general picture



The general picture

The applications of Theorems 1 and 2 motivate the relevance of Question 1:



The precise statement

The setting:

- Let $\mathcal{X} \to S$ be a family of cubic fourfolds;
- Let v be a primitive section of the local system given by $\widetilde{H}(\mathcal{K}u(\mathcal{X}_s), \mathbb{Z})$ such that v stays algebraic on all fibers;
- Assume that, for $s \in S$, there exists a stability condition $\sigma_s \in \mathrm{Stab}^\dagger(\mathcal{K}u(\mathcal{X}_s))$ such that these pointwise stability conditions organize themselves in a family $\underline{\sigma}$. Assume that σ_s is v-generic for very general s (+some invariance of Z...).

The precise statement

Theorem 3 (BLMNPS)

There exists a non-empty open subset $S^0 \subset S$ and a variety $M^0(v)$ with a projective morphism

$$M^0(v) \to S^0$$

that makes $M^0(v)$ a relative moduli space over S^0.

This means that the fiber $M_{\sigma_s}(\mathcal{K}u(\mathcal{X}_s), v)$ of stable objects in the Kuznetsov component of the corresponding cubic fourfold, for $s \in S^0$.

The families of HK manifolds

The construction of the locally complete 20-dimensional families of hyperkähler manifolds goes as follows:

- Take S^0 to be a suitable open subset in the 20-dimensional moduli space $\mathcal{C}$ of cubic fourfolds;
- For every cubic fourfold X, we have a primitive embedding $A_2 \hookrightarrow \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$.
- In A_2 one finds primitive vectors v with arbitrary large v^2.

The families of HK manifolds

We can then apply Theorems 2 and 3:

Corollary 4

For any pair (a, b) of coprime integers, there is a unirational locally complete 20-dimensional family, over an open subset of the moduli space of cubic fourfolds, of polarized smooth projective irreducible holomorphic symplectic manifolds of dimension $2n + 2$, where $n = a^2 - ab + b^2$. The polarization has divisibility 2 and degree either $6n$ if 3 does not divide n, or $\frac{2}{3}n$ otherwise.

...this solves a long standing problem!

$v^2 = 0$: K3 surfaces

Let us start with some easy applications which generalize and complete some existing (very nice!) results:

Corollary 5 (BLMNPS=$\overline{\text{Huybrechts}}$)

Let X be a cubic fourfold. Then there exists a primitive vector $v \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ with $v^2 = 0$ if and only if there is a K3 surface S, $\alpha \in \mathrm{Br}(S)$ and an equivalence $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S, \alpha)$.

Corollary 6 (BLMNPS=$\overline{\text{Addington-Thomas}}$)

Let X be a cubic fourfold. Then there exists a primitive embedding $U \hookrightarrow \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ if and only if there is a K3 surface S and an equivalence $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$.

$v^2 = 0$: K3 surfaces

Let us prove Corollary 4:

- If $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S, \alpha)$, then there is a Hodge isometry $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z}) \cong \widetilde{H}_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$. Take for v the Mukai vector of a skyscraper sheaf.
- Assume we have v. Pick $\sigma \in \mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$ which is v-generic (it exists by the Support Property!).
- $M_\sigma(\mathcal{K}u(X), v)$ is a K3 surface by Theorem 2. Call it S.
- The (quasi-)universal family induces a functor $\mathrm{D}^{\mathrm{b}}(S, \alpha) \to \mathrm{D}^{\mathrm{b}}(X)$ which is fully faithful (because S parametrizes stable objects) and has image in $\mathcal{K}u(X)$ (because S is a moduli space of objects in this category).
- Since $\mathcal{K}u(X)$ is a 2-Calabi-Yau category, we are done.

$v^2 = 0$: K3 surfaces

The conditions in Corollaries 5 and 6:

- having a primitive vector $v \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ with $v^2 = 0$;
- having a primitive embedding $U \hookrightarrow \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$,

are divisorial in the moduli space $\mathcal{C}$ of cubic fourfolds.

Hassett, Huybrechts: they identify countably many Noether-Lefschetz loci in $\mathcal{C}$ which can be completely classified.

Conjecture (Kuznetsov)

X is such that $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$, for a K3 surface S, if and only if X is rational.

$v^2 = 2$: the Fano variety of lines

For a cubic fourfold X, let $F(X)$ be the **Fano variety of lines** in X.

Beauville-Donagi: $F(X)$ is a smooth projective hyperkähler manifold of dimension 4. Moreover, it is deformation equivalent to $\mathrm{Hilb}^2(\mathrm{K3})$.

To see a line $\ell \subseteq X$ as an object in the Kuznetsov component:

$$0 \to F_\ell \to \mathcal{O}_X^{\oplus 4} \to \mathcal{I}_\ell(H) \to 0.$$

Kuznetsov-Markushevich: F_ℓ is in $\mathcal{K}u(X)$ and it is a Gieseker stable sheaf. $F(X)$ is isomorphic to the moduli space of stable sheaves with Mukai vector $v(F_\ell)$.

$v^2 = 2$: the Fano variety of lines

Theorem (Li-Pertusi-Zhao)

For any cubic fourfold X, we have an isomorphism $F(X) \cong M_\sigma(\mathcal{K}u(X), \lambda_1)$, for all **natural** stability conditions σ.

A stability condition σ is **natural** if:

- $\sigma \in \mathrm{Stab}^\dagger(\mathcal{K}u(X))$;
- Under the map $\mathrm{Stab}^\dagger(\mathcal{K}u(X)) \to \mathcal{P}_0^+(X)$, σ is sent to $A_2 \otimes \mathbb{C} \cap \mathcal{P}(X) \subseteq \mathcal{P}_0^+(X)$.

$v^2 = 6$: twisted cubics

Lehn-Lehn-Sorger-van Straten: for X a cubic fourfold not containing a plane,

- Let $M_3(X)$ be the component of the Hilbert scheme $\mathrm{Hilb}^{3t+1}(X)$ containing all twisted cubics which are contained in X. $M_3(X)$ is a smooth projective variety of dimension 10;
- $M_3(X)$ admits a $\mathbb{P}^2$-fibration $M_3(X) \to Z'(X)$, where $Z'(X)$ is a smooth projective variety of dimension 8;
- We can contract a divisor $Z'(X) \to Z(X)$, where $Z(X)$ is a smooth projective hyperkähler manifold of dimension 8 which contains X as a Lagrangian submanifold.

$v^2 = 8$: twisted cubics

Question 3 (M. Lehn):

Is there a modular interpretation for $Z'(X)$ and $Z(X)$?

Theorem (M. Lehn-Lahoz-Macrì-S. and Li-Pertusi-Zhao)

For any cubic fourfold X not containing a plane,

- $Z'(X)$ is isomorphic to a component of a moduli space of Gieseker stable torsion free scheaves of rank 3;
- We have an isomorphism $Z(X) \cong M_\sigma(\mathcal{K}u(X), 2\lambda_1 + \lambda_2)$, for all natural stability conditions σ.

By Theorem 2, $Z(X)$ is automatically (projective and) deformation equivalent to $\mathrm{Hilb}^4(\mathrm{K3})$, which was proved by **Addington-Lehn**.

Concluding remarks

The last two results are stated in a ‘punctual form’ but, in view of Theorem 3, they can be put in families, giving rise to relative moduli spaces of relative dimension 4 and 8.

Question 3

Why do we really care about this alternative description of ‘classical’ hyperkähler manifolds in terms of moduli spaces in the Kuznetsov component?

This is because BLMNPS implies that the Bayer-Macrì machinery can be applied also in this noncommutative setting: all birational models of $F(X)$, $Z(X)$ and all other possible HK from Theorem 2 are isomorphic to moduli spaces of stable objects in the Kuznetsov component (by variation of stability).

Concluding remarks

There several other simple but interesting applications that one can deduce from Theorems 1, 2 and 3:

Exercise (Voisin)

Reprove the Intergral Hodge Conjecture for cubic fourfolds, due to Voisin, by using the same ideas as in the proof of Corollary 4.