Derived Torelli Theorem and Orientation

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Joint work with D. Huybrechts and E. Macrì (math.AG/0608430 + work in progress)

Outline

1. **Motivations**
   - The setting
   - The geometric case
   - The derived case

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Describe the group of autoequivalences of the tringulated category

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More generally: Consider the derived category of twisted sheaves!

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Theorem (Torelli Theorem)

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Theorem (Borcea, Donaldson)

Consider the natural map

$$\rho : \mathrm{Diff}(X) \longrightarrow \mathrm{O}(H^2(X,\mathbb{Z})).$$

Then $\mathrm{im}\,(\rho) = \mathrm{O}_+(H^2(X,\mathbb{Z}))$, where $\mathrm{O}_+(H^2(X,\mathbb{Z}))$ is the group of orientation preserving Hodge isometries.

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$$\mathrm{Br}(X) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$$

where $T(X) := \mathrm{NS}(X)^{\perp}$.

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Represent $\alpha \in \mathrm{Br}(X)$ as a Čech 2-cocycle

$$\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

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Remark

The following definitions depend on the choice of the cocycle representing $\alpha \in \mathrm{Br}(X)$.

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3. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

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- Pass to the category of bounded complexes.
- Localize: require that any quasi-isomorphism is invertible.
- We get the bounded derived category $\mathrm{D}^{\mathrm{b}}(X, \alpha)$.

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Proposition (Mukai, Căldăraru)

Let X be a K3 surface and let M be a coarse moduli space of stable sheaves on X as above.

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Proposition (Mukai, Căldăraru)

Let X be a K3 surface and let M be a coarse moduli space of stable sheaves on X as above. Then $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(M, \alpha^{-1})$ (via the twisted universal/quasi-universal family.

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Twisted Derived Torelli Theorem (Huybrechts-S.)

Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$.

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2. Suppose there exists a Hodge isometry $g : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$ that preserves the natural orientation of the four positive directions. Then there exists an equivalence $\Phi : \mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(X', \alpha_{B'})$ such that $\Phi_*^{B,B'} = g$.

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$H^*(X,\mathbb{Z})$ endowed with the Mukai pairing is called Mukai lattice and we write $\widetilde{H}(X,\mathbb{Z})$ for it.

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Hitchin, Huybrechts: complete classification of generalized CY structures on K3 surfaces.

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This gives a generalization of the notion of Picard lattice and transcendental lattice.

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2. It is easy to see that this orientation is independent of the choice of σ_X and ω.
3. The isometry $j := \mathrm{id}_{H^0 \oplus H^4} \oplus (-\mathrm{id})_{H^2}$ is not orientation preserving.

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Fourier-Mukai functors

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$F : \mathrm{D}^{\mathrm{b}}(X, \alpha) \to \mathrm{D}^{\mathrm{b}}(Y, \beta)$ is of Fourier-Mukai type if there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors

$$F \cong \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the natural projections.

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The complex $\mathcal{E}$ is called the kernel of F and a Fourier-Mukai functor with kernel $\mathcal{E}$ is denoted by $\Phi_{\mathcal{E}}$.

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Remark (Bondal-Van den Bergh)

Item (2) is automatic!

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Theorem. (Canonanco-S.)

Let (X, α) and (Y, β) be twisted varieties. Let

$$F : \mathrm{D}^{\mathrm{b}}(X, \alpha) \to \mathrm{D}^{\mathrm{b}}(Y, \beta)$$

be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X, \alpha)$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(Y,\beta)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \text{ if } j < 0.$$

Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$.

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Then there exist $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{E}}$. Moreover, $\mathcal{E}$ is uniquely determined up to isomorphism.

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Theorem. (Canonanco-S.)

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Any equivalence is of Fourier–Mukai type.

The Chern character

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Take $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X, \alpha)$ and its image $\mathcal{E} = [\mathcal{E}]$ in the Grothendieck group

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$$\mathrm{ch}^{B}(\mathcal{E}) \in H^{*}(X, \mathbb{Z})$$

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Remark

The notion of $\mathrm{c}_1(\mathcal{E})$ is well-defined.

A commutative diagram

A commutative diagram

Using the Chern character one gets the commutative diagram:

$$
\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}(X,\alpha) & \xrightarrow[\Phi]{} & \mathrm{D}^{\mathrm{b}}(Y,\beta) \\
\Big\downarrow{\scriptstyle [-]} & & \Big\downarrow{\scriptstyle [-]} \\
K(X,\alpha) & \longrightarrow & K(Y,\beta) \\
\Big\downarrow{\scriptstyle \mathrm{ch}^{B}(-)\cdot\sqrt{\mathrm{td}(X)}} & & \Big\downarrow{\scriptstyle \mathrm{ch}^{B'}(-)\cdot\sqrt{\mathrm{td}(Y)}} \\
\widetilde{H}(X,B,\mathbb{Z}) & \xrightarrow{\Phi_*^{B,B'}} & \widetilde{H}(Y,B',\mathbb{Z}).
\end{array}
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Outline

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Let X and X' be K3 surfaces with large Picard number and let $B \in H^2(X, \mathbb{Q})$ and $B' \in H^2(X', \mathbb{Q})$.

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In the untwisted case, this is always true.

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The ‘final’ Twisted Derived Torelli Theorem

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Theorem (Huybrechts-Macrì-S.)

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$$1 \to \mathbb{Z}[2] \to \mathrm{Aut}\left(\mathrm{D}^{\mathrm{b}}(X, \alpha_B)\right) \xrightarrow{\varphi} \mathrm{O}_+ \to 1,$$

where O_+ is the group of the Hodge isometries of $\widetilde{H}(X, B, \mathbb{Z})$ preserving the orientation.

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where O_+ is the group of the Hodge isometries of $\widetilde{H}(X, B, \mathbb{Z})$ preserving the orientation.

A twisted K3 surface (X, α) is generic meaning that it is generic in the moduli space of twisted K3 surfaces.

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A generic K3 surface (X, α) is, in the rest of this talk, characterized by the fact that:

1. The underlying K3 surface X is generic and projective (i.e. Picard number 1).
2. The twist α is generic (i.e. restrictions on the order of α).

Warning

We have no idea about how to use the generic case to prove the conjecture for any twisted K3 surface (X, α).

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Take an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y, \beta)$, of generic twisted K3 surfaces (X, α) and (Y, β). Then the natural map Φ_* induced on cohomology is an orientation preserving Hodge isometry.

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Remark

We proved Bridgeland's Conjecture for generic twisted K3 surfaces.

Outline

The genericity hypothesis

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Definition

An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X, \alpha)$ is spherical if

$$\mathrm{Hom}\,(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

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Proposition

There exists a dense subset of twisted K3 surfaces (X, α) such that $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ does not contain spherical objects.

In the untwisted case, $\mathrm{D}^{\mathrm{b}}(X)$ always contains spherical objects.

Stability conditions: Bridgeland

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A stability condition on $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ is a pair $\sigma = (Z, \mathcal{P})$ where

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- $\mathcal{P}(\phi) \subset \mathrm{D}^{\mathrm{b}}(X, \alpha)$ are full additive subcategories for each $\phi \in \mathbb{R}$

satisfying the following conditions:

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with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

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All stability conditions are assumed to be locally-finite. Hence every object in $\mathcal{P}(\phi)$ has a finite Jordan–Hölder filtration. $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ is the set of locally finite stability conditions.

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The group $\mathrm{Aut}\,(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ acts on $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X, \alpha))$.

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1. For each connected component $\Sigma \subseteq \mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ there is a linear subspace $V(\Sigma) \subseteq (\mathcal{N}(X,\alpha) \otimes \mathbb{C})^{\vee}$ with a well-defined linear topology such that the natural map

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is a local homeomorphism.
2. The manifold $\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ is finite dimensional.

Stability conditions: examples

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Denote by

$$P(X,\alpha) \subset \mathcal{N}(X,\alpha) \otimes \mathbb{C}$$

the open subset of vectors φ such that real part and imaginary part of φ generate a positive plane in $\mathcal{N}(X,\alpha) \otimes \mathbb{R}$.

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We shall denote by $P^+(X,\alpha)$ the one that contains

$$\varphi = \exp(B + i\omega)$$

with B rational B-field and ω rational Kähler class.

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2. The full additive subcategory $\mathcal{T}(\varphi) \subset \mathbf{Coh}(X, \alpha)$ of the non-trivial objects $\mathcal{E} \in \mathrm{D}^\mathrm{b}(X, \alpha)$ such that every non-trivial torsion free quotient $\mathcal{E} \twoheadrightarrow \mathcal{E}'$ satisfies $\mathrm{im}(Z_\varphi(\mathcal{E}')) > 0$.

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3. The full additive subcategory $\mathcal{F}(\varphi) \subset \mathbf{Coh}(X, \alpha)$ of the non-trivial objects $\mathcal{E} \in \mathrm{D}^\mathrm{b}(X, \alpha)$ which are torsion free and are such that every non-zero subsheaf $\mathcal{E}'$ satisfies $\mathrm{im}(Z_\varphi(\mathcal{E}')) \leq 0$.

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Define the abelian category

$$\mathcal{A}(\varphi) := \left\{ \mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X, \alpha) : \begin{array}{l} \bullet\ \mathcal{H}^i(\mathcal{E}) = 0 \text{ for } i \notin \{-1, 0\}, \\ \bullet\ \mathcal{H}^{-1}(\mathcal{E}) \in \mathcal{F}(\varphi), \\ \bullet\ \mathcal{H}^0(\mathcal{E}) \in \mathcal{T}(\varphi) \end{array} \right\}.$$

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For good choices of B and ω, the pair $(Z_\varphi, \mathcal{A}(\varphi))$ defines a stability condition.

Stability conditions: the connected component

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Following Bridgeland one defines a connected component

$$\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha)) \subseteq \mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$$

containing the stability conditions previously described.

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Theorem (Huybrechts-Macrì-S.)

If (X,α) is a generic twisted K3 surface, then $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ is the unique connected component of maximal dimension in $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$. Such a component is also simply-connected.

Concluding the proof

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$$\pi : \mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha)) \to P^{+}(X,\alpha)$$

which is a covering map.

2. Since $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ is the unique connected component of maximal dimension, the action of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ on $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ preserves it.

Concluding the proof

1. There exists a continuous map
$$\pi : \mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha)) \to P^{+}(X,\alpha)$$
which is a covering map.

2. Since $\mathrm{Stab}^{\dagger}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ is the unique connected component of maximal dimension, the action of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ on $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ preserves it.

3. This action corresponds to the action of deck transformations, and $P^{+}(X,\alpha)$ corresponds to the positive directions. One gets that any $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ is orientation preserving.