Cubic hypersurfaces and derived categories: results and open problems

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Outline

1 Setting

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Homological algebra

Let X be a *cubic hypersurface* (i.e. a smooth hypersurface of degree 3 in $\mathbb{P}^n$ over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$) and let H be a hyperplane section:

$$\mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Coh}(X))$$

$$\|$$

$$\langle \mathcal{K}u(X), \mathcal{O}_X, \dots, \mathcal{O}_X((n-3)H) \rangle$$

$$\mathcal{K}u(X)$$

$$\|$$

$$\left\{ E \in \mathrm{D^b}(X) : \begin{array}{l} \mathrm{Hom}\,(\mathcal{O}_X(iH), E[p]) = 0 \\ i = 0, \dots, n-3 \;\; \forall p \in \mathbb{Z} \end{array} \right\}$$

Kuznetsov component of X

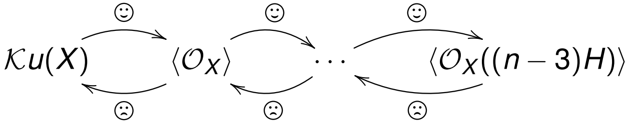
Exceptional objects:

$$\langle \mathcal{O}_X(iH) \rangle \cong \mathrm{D^b}(\mathrm{pt})$$

Homological algebra

Keep in mind that the symbol $\langle \dots \rangle$ stays for a **semiorthogonal decomposition**:

- $\mathrm{D}^{\mathrm{b}}(X)$ is generated by extensions, shifts, direct sums and summands by the objects in the $n - 1$ **admissible** subcategories;
- There are no Homs from right to left between the 4 subcategories:



$n = 4$: cubic threefolds

The admissible subcategory $\mathcal{K}u(X)$ has a Serre functor $S_{\mathcal{K}u(X)}$ (this is easy!). Moreover, there is an isomorphism of exact functors

$$S_{\mathcal{K}u(X)}^{\circ 3} \cong [5].$$

Because of this, $\mathcal{K}u(X)$ is called **fractional Calabi-Yau category** of fractional dimension $\frac{5}{3}$.

$\Downarrow$

Remark

If X smooth proj. var., $S_{\mathrm{D^b}(X)}(-) \cong (-) \otimes \omega_X[\dim(X)]$.

Hence $\mathcal{K}u(X)$ cannot be equivalent to the derived category of a smooth and projective variety.

$n = 5$: cubic fourfolds

Also in this case, the admissible subcategory $\mathcal{K}u(X)$ has a Serre functor $S_{\mathcal{K}u(X)}$ with an isomorphism of exact functors

$$S_{\mathcal{K}u(X)} \cong [2].$$

Hence, $\mathcal{K}u(X)$ is called 2**-Calabi-Yau category**.

$$\Downarrow$$

Hence $\mathcal{K}u(X)$ could be equivalent to the derived category either of a **K3** or of an **abelian surface**.

Recall

K3 and **abelian** surfaces can be distinguished by the fact the former ones do not have odd cohomoloy.

$n = 5$: cubic fourfolds

Addington-Thomas: $\mathcal{K}u(X)$ comes with an integral cohomology theory in the following sense (here $\mathbb{K} = \mathbb{C}$):

- Consider the $\mathbb{Z}$-module

$$H^*(\mathcal{K}u(X), \mathbb{Z}) := \left\{ e \in K_{\mathrm{top}}(X) : \begin{array}{l} \chi([\mathcal{O}_X(iH)], e) = 0 \\ i = 0, 1, 2 \end{array} \right\}.$$

Remark

$H^*(\mathcal{K}u(X), \mathbb{Z})$ is deformation invariant. So, as a lattice:

$$H^*(\mathcal{K}u(X), \mathbb{Z}) = H^*(\mathcal{K}u(\mathrm{Pfaff}), \mathbb{Z}) = H^*(\mathrm{K3}, \mathbb{Z}) = U^4 \oplus E_8(-1)^2$$

Hence $\mathcal{K}u(X)$ is **K3-like**!

$n = 5$: cubic fourfolds

- The Hodge decomposition of $H^4(X, \mathbb{C})$ induces a **weight-2 Hodge structure** on $H^*(\mathcal{K}u(X), \mathbb{Z})$.

Definition

The lattice $H^*(\mathcal{K}u(X), \mathbb{Z})$ with the above Hodge structure is the **Mukai lattice** of $\mathcal{K}u(X)$ which we denote by $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

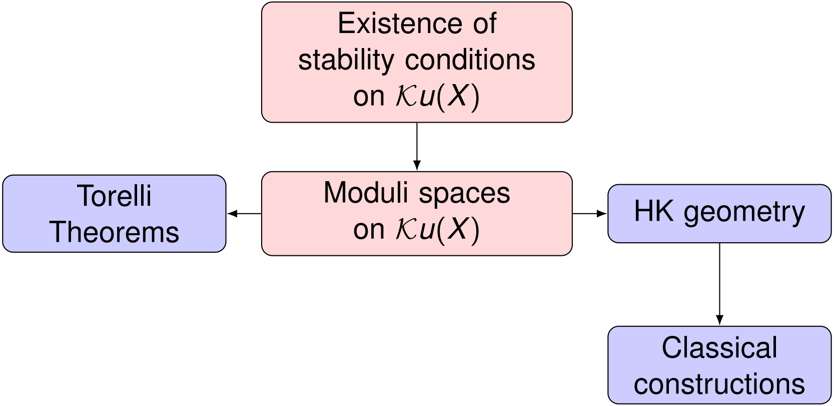
Remark

If X is **very general** (i.e. $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}H^2$), then there is no K3 surface S such that $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(S)$!

$\mathcal{K}u(X)$ is a **noncommutative K3 surface.**

Outline

A quick overview



Stability conditions: a quick recap

Let us start with a quick recall about Bridgeland stability conditions.

- Let $\mathbf{T}$ be a triangulated category;
- Let Γ be a free abelian group of finite rank with a surjective map $v\colon K(\mathbf{T}) \to \Gamma$.

Example

$\mathbf{T} = \mathrm{D}^{\mathrm{b}}(C)$, for C a smooth projective curve.

$$\Gamma = N(C) = H^0 \oplus H^2$$

with

$$v = (\mathrm{rk}, \deg)$$

A **Bridgeland stability condition** on $\mathbf{T}$ is a pair $\sigma = (\mathbf{A}, Z)$:

Stability conditions: a quick recap

- $\mathbf{A}$ is the heart of a bounded t-structure on $\mathbf{T}$;
- $Z\colon \Gamma \to \mathbb{C}$ is a group homomorphism

Example

$\mathbf{A} = \mathrm{Coh}(C)$

$Z(v(-)) = -\deg + \sqrt{-1}\mathrm{rk}.$

such that, for any $0 \neq E \in \mathbf{A}$,

1. $Z(v(E)) \in \mathbb{R}_{>0} e^{(0,1]\pi\sqrt{-1}}$;
2. E has a Harder-Narasimhan filtration with respect to $\lambda_\sigma = -\frac{\mathrm{Re}(Z)}{\mathrm{Im}(Z)}$ (or $+\infty$);
3. Support property (**Kontsevich-Soibelman**): wall and chamber structure with locally finitely many walls.

Stability conditions: a quick recap

Warning

The example is somehow misleading: it only works in dimension 1!

We denote by

$$\mathrm{Stab}_\Gamma(\mathbf{T}) \quad (\text{or } \mathrm{Stab}_{\Gamma,v}(\mathbf{T}) \text{ or } \mathrm{Stab}(\mathbf{T}))$$

the set of all stability conditions on **T**.

Theorem (Bridgeland)

If non-empty, $\mathrm{Stab}_\Gamma(\mathbf{T})$ is a complex manifold of dimension $\mathrm{rk}(\Gamma)$.

The results: existence of stability conditions

Existence of
stability conditions
on $\mathcal{K}u(X)$

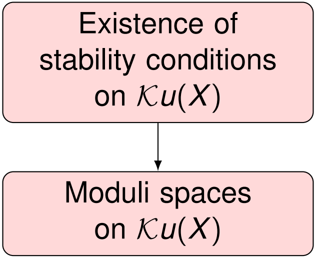
Existence of stability conditions

Theorem 1 (Bayer-Lahoz-Macrì-S., The 4 + Nuer-Perry)

1. For any cubic threefold or fourfold X, we have $\mathrm{Stab}(\mathcal{K}u(X)) \neq \emptyset$.
2. If X is a cubic fourfold ($\mathbb{K} = \mathbb{C}$), we can explicitely describe a connected component $\mathrm{Stab}^\dagger(\mathcal{K}u(X))$ of $\mathrm{Stab}(\mathcal{K}u(X))$.

- This result was conjectured by Addington-Thomas, Kuznetsov and Huybrechts.
- The reason for this conjecure is the wealth of applications that we will discuss.

Moduli spaces



Moduli spaces

The construction of moduli spaces of stable objects in $\mathcal{K}u(X)$:

- Let $0 \neq v$ be a primitive class in the numerical Grothendieck group (when X is a 3-fold) or in $\widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ (when X is a 4-fold);
- Let $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$ (actually in $\mathrm{Stab}^{\dagger}(\mathcal{K}u(X))$) be ***v*-generic** (here it means that σ-semistable=σ-stable for objects with Mukai vector v).

Let $M_\sigma(\mathcal{K}u(X), v)$ be the moduli space of σ-stable objects (in the heart of σ) contained in $\mathcal{K}u(X)$ and with Mukai vector v.

Question

What is the geometry of $M_\sigma(\mathcal{K}u(X), v)$?

Moduli spaces: cubic fourfolds

Theorem 2 (BLMNPS)

Let X be a cubic fourfold ($\mathbb{K} = \mathbb{C}$).

1. $M_\sigma(\mathcal{K}u(X), v)$ is non-empty if and only if $v^2 + 2 \geq 0$. Moreover, in this case, it is a smooth projective irreducible holomorphic symplectic manifold of dimension $v^2 + 2$, deformation-equivalent to a Hilbert scheme of points on a K3 surface.

2. If $v^2 \geq 0$, then there exists a natural Hodge isometry

$$\theta \colon H^2(M_\sigma(\mathcal{K}u(X), v), \mathbb{Z}) \cong \begin{cases} v^\perp & \text{if } v^2 > 0 \\ v^\perp / \mathbb{Z}v & \text{if } v^2 = 0, \end{cases}$$

where the orthogonal is taken in $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$.

Moduli spaces: cubic fourfolds

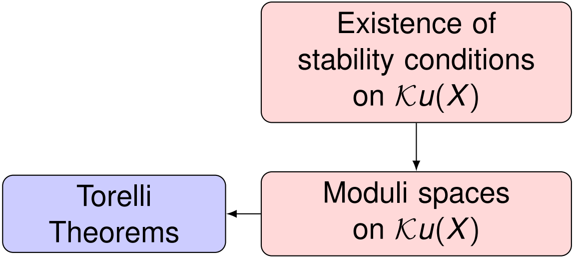
Definition

A **hyperkähler manifold** is a simply connected compact kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form.

There are very few examples (up to deformation):

1. K3 surfaces;
2. Hilbert schemes of points on K3 surface (denoted by $\mathrm{Hilb}^n(\mathrm{K3})$);
3. Generalized Kummer varieties (from abelian surfaces);
4. Two sporadic examples by O'Grady.

A quick overview



A Derived Torelli Theorem for cubic threefolds

Let X be a cubic threefold ($\mathbb{K} = \mathbb{C}$).

Theorem 3 (Bernardara-Macrì-Merhotra-S., Yang-Pertusi)

There is an isomorhism (of polarized surfaces)

$$F(X) := \{\text{lines} \subseteq X\} \cong M_\sigma(\mathcal{K}u(X), v),$$

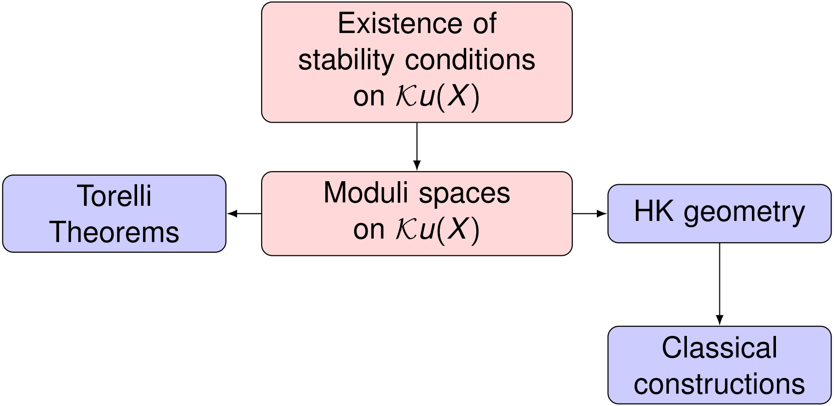
where v is the class of the ideal sheaf of a line.

From this we deduce the following Refined Derived Torelli Theorem:

Theorem 4 (Bernardara-Macrì-Merhotra-S.)

If X_1 and X_2 are cubic threefolds, then $X_1 \cong X_2$ if and only if $\mathcal{K}u(X_1) \cong \mathcal{K}u(X_2)$.

A quick overview



Families of HK manifolds

One can make Theorem 2 work relative over a base and get the following striking application:

Theorem 5 (BLMNOPS)

For any pair (a, b) of coprime integers, there is a unirational locally complete 20-dimensional family, over an open subset of the moduli space of cubic fourfolds, of polarized smooth projective irreducible holomorphic symplectic manifolds of dimension $2n + 2$, where $n = a^2 - ab + b^2$. The polarization has divisibility 2 and degree either $6n$ if 3 does not divide n, or $\frac{2}{3}n$ otherwise.

...this solves a long standing problem!

Outline

Gushel-Mukai fourfolds

Definition

A **Gushel-Mukai fourfold** is a smooth intersection of the cone in $\mathbb{P}^{10}$ over $\mathrm{Gr}(2,5) \subseteq \mathbb{P}^9$ with $\mathbb{P}^8 \subseteq \mathbb{P}^{10}$ and a quadric $Q \subseteq \mathbb{P}^{10}$.

Kuznetsov, Perry: $\mathrm{D}^{\mathrm{b}}(X)$ has a semiorthogonal decomposition with a componet $\mathcal{K}u(X)$ of K3 type and 4 exceptional objects.

Problem 1

Show that $\mathcal{K}u(X)$ carries stability conditions.

Some progress by Perry-Pertusi-Zhao. This would yield many new geometric results.

Debarre-Voisin fourfolds

Definition

A **Debarre-Voisin fourfold** is a smooth linear section of $\mathrm{Gr}(3, 10)$.

Debarre, Voisin, Fonarev, Kuznetsov: $\mathrm{D}^{\mathrm{b}}(X)$ has a semiorthogonal decomposition with a componet $\mathcal{K}u(X)$ of K3 type and 108 exceptional objects.

Problem 2

Show that $\mathcal{K}u(X)$ carries stability conditions.

More difficult than the previous case!